

Extensions of Koenderink’s formula

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ABSTRACT. We describe Koenderink type theorems for Whitney umbrellas which hold along special curves. We also show that the shape (convexity and concavity) of images of orthogonal projections of Whitney umbrellas relates to the Gaussian curvature, even though it is unbounded near the singularity.

1. Introduction

In 1984 and 1990, J. J. Koenderink showed theorems that relate to how one actually sees a surface. Let $\xi \in \mathbb{R}^3$ be a unit vector and $\pi_\xi : \mathbb{R}^3 \rightarrow \xi^\perp$ the orthogonal projection onto $\xi^\perp$, where $\xi^\perp$ is a plane perpendicular to ξ . Let $f : U \rightarrow \mathbb{R}^3$ be a regular surface in $\mathbb{R}^3$, where $U \subset \mathbb{R}^2$ is an open subset. We denote the singular set of $\pi_\xi \circ f$ by $S(\pi_\xi \circ f)$. The set $\pi_\xi \circ f(S(\pi_\xi \circ f))$ is the (*apparent*) *contour* or *profile* of f as viewed from the direction ξ , and $f(S(\pi_\xi \circ f))$ is the *rim* or *contour generator* of f as viewed from ξ . Koenderink showed the following theorem.

Theorem 1.1 ([12, Appendix], [13, page 433]). *Suppose $p \in S(\pi_\xi \circ f)$, namely, ξ is in $T_p f(U)$, and $\pi_\xi \circ f(S(\pi_\xi \circ f))$ is a regular curve near p . Let κ_c be the curvature of the plane curve $\pi_\xi \circ f(S(\pi_\xi \circ f)) \subset \xi^\perp$, and κ_s the curvature of the normal section of M at p by the plane that contains ξ . Then*

$$K = \kappa_c \kappa_s = \kappa_c \kappa_n(\xi)$$

holds at p , where K is the Gaussian curvature of $f(U)$, and $\kappa_n(\xi)$ is the normal curvature of $f(U)$ with respect to ξ .

This means that the Gaussian curvature is the product of the curvature of the normal section with the curvature of the contour (or the profile). In particular, this theorem claims that convex (resp. concave, inflection) points of the contour of a smooth regular surface correspond to elliptic (resp. hyperbolic, parabolic) points on the surface.

A Koenderink type formula holds when the contour of a regular surface is $(2, 3)$ -cusp (Proposition 3.1). In [19] it is shown that a Koenderink type formula also holds for

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cuspidal edges. Does any Koenderink type formula hold for other singular surfaces? Whitney umbrellas are the only stable singularities of mappings from $\mathbb{R}^2$ to $\mathbb{R}^3$, and its differential geometry is recently discussed by several geometers ([2, 3, 4, 6, 7, 8, 11, 16, 17, 23, 24, 27]). In [9, 10, 19], Koenderink type formulas in several other space forms are obtained. It is a natural problem to study a Koenderink type formula for Whitney umbrellas. The purpose of this paper is to extend the theorem above when f is the Whitney umbrella. However, there is no well-defined unit normal vector field at Whitney umbrellas and the Gaussian curvature is unbounded near a Whitney umbrella. Considering a curve $\gamma(t)$ emanating Whitney umbrella, one obtain a Laurent series of the Gaussian curvature $K(\gamma(t))$. The principal part of the series depends on the curve. We show that Koenderink type theorems do hold along special curves near Whitney umbrellas, and do not hold along generic curves (Theorems 4.4 and 4.6).

All maps considered here are of class C^∞ .

2. Preliminaries

2.1. Geometric properties of Whitney umbrella

A map-germ $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ is a *Whitney umbrella* if f is $\mathcal{A}$ -equivalent to the map germ defined by $(u, v) \mapsto (u, v^2, uv)$ at the origin. Two map-germs f_1, f_2 are $\mathcal{A}$ -equivalent if there exist diffeomorphism-germs $\sigma : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ and $\tau : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ such that

$$\tau \circ f_1 \circ \sigma = f_2$$

holds. Let $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ be a Whitney umbrella. At the singular point 0, the tangent plane degenerates to a line, that is, the image of df_0 is a line, which is called the *tangent line*. The plane passing through the singular point 0 perpendicular to the tangent line is called the *normal plane*. There exists a non-zero vector $\eta \in T_0\mathbb{R}^2$ such that $df_0(\eta) = 0$. We call η the *null vector* (cf. [14]). The plane passing through $f(0)$ spanned by the tangent line and $\eta\eta f(0)$ is called the *principal plane*, where $\eta\eta f$ is the second-order directional derivative of f with respect to η , and it is not in the tangent line since f is a Whitney umbrella (cf. [8]). We remark that the definitions of the tangent line, normal plane and principal plane are independent of the choice of coordinate systems in the source space (cf. [8]). A regular curve $\gamma(t) = (u(t), v(t)) : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ whose tangent vector is transverse (respectively tangent) to η is called a *tangential curve* (respectively *null curve*). Clearly, if γ is a tangential curve, then $f \circ \gamma$ is tangent to the tangent line of f at $f(0)$. On the other hand, if γ is a null curve, then $f \circ \gamma$ is singular at 0.

It is shown in [24] that Whitney umbrellas are generically classified into two types: the elliptic Whitney umbrella, whose parabolic line, in the source space, has an A_1^- singularity (locally two transversally intersecting curves); and the hyperbolic Whitney umbrella, whose parabolic line has an A_1^+ singularity (locally an isolated point). At the transition between two types, there is a Whitney umbrella whose parabolic line is a (2,3)-cusp, and such a Whitney umbrella is called a parabolic Whitney umbrella (see [16, 23]). Here, a map-germ $\gamma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ is (l_1, l_2) -cusp if γ is $\mathcal{A}$ -equivalent to

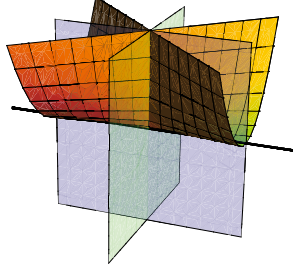


FIGURE 1. The tangent line, normal plane and principal plane of the standard Whitney umbrella $(u, v) \mapsto (u, uv, v^2)$.

$t \mapsto (t^{l_1}, t^{l_2})$. It is well-known that if $j^{2n+1}\gamma(0)$ is $\mathcal{A}$ -equivalent to (t^2, t^{2n+1}) , then γ is a $(2, 2n+1)$ -cusp.

The following proposition is well-known.

Proposition 2.1 ([24]). *Let the map $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ be a Whitney umbrella. Then there exists a diffeomorphism-germ $\varphi_s : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ and an isometry-germ $\varphi_{isom} : (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$ such that*

$$\begin{aligned} & \varphi_{isom} \circ f \circ \varphi_s(u, v) \\ &= \left(u, uv + \sum_{i=3}^k \frac{b_i}{i!} v^i + O(u, v)^{k+1}, \sum_{i+j=2}^k \frac{a_{ij}}{i!j!} u^i v^j + O(u, v)^{k+1} \right) \quad (a_{02} > 0) \end{aligned} \quad (2.1)$$

holds, where $O(u, v)^n$ consists of the terms whose degrees are greater than n .

A proof of this proposition can also be found in [3] (see also [8, Introduction]).

Let $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ be a Whitney umbrella as in (2.1). Then the tangent line is the x -axis and the null vector can be chosen as $\eta = \partial/\partial v$. Furthermore, the normal plane is the yz -plane and the principal plane is the xz -plane. Let

$$\gamma(t) = \left(p_1 t + \frac{1}{2} p_2 t^2 + O(t^3), q_1 t + \frac{1}{2} q_2 t^2 + O(t^3) \right) \quad ((p_1, q_1) \neq (0, 0)) \quad (2.2)$$

be a curve, where $O(t^n)$ consists of the terms whose degrees are greater than n . Because of $(p_1, q_1) \neq (0, 0)$ and $a_{02} \neq 0$, the limiting normal vector of f along γ is

$$\lim_{t \rightarrow 0} \nu(\gamma(t)) = \pm \frac{1}{\sqrt{p_1^2 + (a_{11}p_1 + a_{02}q_1)^2}} (0, -a_{11}p_1 - a_{02}q_1, p_1)$$

and this vector lies in the normal plane. Furthermore, if γ is a null curve, then $p_1 = 0$, and thus $\lim \nu(\gamma(t)) = (0, \pm 1, 0)$. This vector does not depend on the null curve up to ± 1 (cf. [8]). We denote this vector ν_n . Let us assume that $\gamma(t)$ is a null curve. Then $p_1 = 0$

and $q_1 \neq 0$ and the limiting tangent vector of $\hat{\gamma} = f \circ \gamma$ is

$$\lim_{t \rightarrow 0} \frac{\hat{\gamma}'(t)}{|\hat{\gamma}'(t)|} = \pm \frac{1}{\sqrt{p_2^2 + a_{02}^2 q_1^4}} (p_2, 0, a_{02} q_1^2),$$

and this vector lies in the principal plane. Let us further assume that $\gamma(t)$ is a double curve, namely, $\hat{\gamma} = f \circ \gamma$ satisfies $\hat{\gamma}(t) = \hat{\gamma}(-t)$. Then we have

$$\gamma(t) = \left(-\frac{1}{6} b_3 q_1^2 t^2 + O(t^3), q_1 t + \frac{(-a_{03} + a_{11} b_3) q_1^2}{6 a_{02}} t^2 + O(t^3) \right)$$

and

$$\hat{\gamma}(t) = \left(-\frac{1}{6} b_3 q_1^2 t^2 + O(t^4), O(t^4), \frac{1}{2} a_{02} q_1^2 t^2 + O(t^4) \right).$$

Thus

$$\lim_{t \rightarrow 0} \frac{\hat{\gamma}'(t)}{|\hat{\gamma}'(t)|} = \pm \frac{1}{\sqrt{9 a_{02}^2 + b_3^2}} (-b_3, 0, 3 a_{02})$$

holds. We call the direction of this vector the *tangential direction of the double curve*.

The Gaussian curvature $K(u, v)$ of f of the form (2.1) is computed in [4, p44] by setting $(u, v) = (r \cos \theta, r \sin \theta)$:

$$K(r \cos \theta, r \sin \theta) = \frac{1}{r^2} \left(\frac{a_{02}(a_{20} \cos^2 \theta - a_{02} \sin^2 \theta)}{(\cos^2 \theta + (a_{11} \cos \theta + a_{02} \sin \theta)^2)^2} + O(r) \right).$$

This formula tells us the limit $\lim_{r \rightarrow 0} r^2 K$ of the Gaussian curvature of (2.1) depends only on θ . We consider a curve $\gamma(t)$ as in (2.2) and consider the Laurent expansion

$$K(\gamma(t)) = \frac{1}{t^2} \left(\frac{a_{02}(a_{20} p_1^2 - a_{02} q_1^2)}{(p_1^2 + (a_{11} p_1 + a_{02} q_1)^2)^2} + O(t) \right). \quad (2.3)$$

Thus considering $\lim_{t \rightarrow 0} t^2 |d\gamma/dt(0)|^2 K(\gamma(t))$, it can be regarded as the principal part of the diverging Gaussian curvature and it is determined only by the direction $v = \gamma'(0)$. Thus we denote it by K_v .

2.2. Geometric properties of cusps in the plane

Different from the case of regular surfaces, a section of a Whitney umbrella by a plane has a singular point. It is well known that a section of a Whitney umbrella by a plane is generically a $(2, 3)$ -cusp. A map-germ $g : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$ is called a *fold* (respectively, a *cusp*) if it is $\mathcal{A}$ -equivalent to $(u, v) \mapsto (u, v^2)$ (respectively, $(u, v) \mapsto (u, v^3 + uv)$) at 0. In this section, we introduce an invariant of $(2, 3)$ -cusps given in [21].

A map-germ $\gamma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ is a $(2, 3)$ -cusp if and only if it holds that $\gamma'(0) = 0$ and $\det(\gamma''(0), \gamma'''(0)) \neq 0$. Then the *cuspidal curvature* μ_γ of γ at 0 is defined by

$$\mu_\gamma = \frac{\det(\gamma''(t), \gamma'''(t))}{|\gamma''(t)|^{5/2}} \Big|_{t=0}. \quad (2.4)$$

See [21] for details. We have the following proposition.

Proposition 2.2 (cf., [22, Theorem 1.1]). *Let $\gamma(t) : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ be a $(2, 3)$ -cusp. Then*

$$\mu_\gamma = 2\sqrt{2} \lim_{t \rightarrow 0} \sqrt{|s(t)|} \kappa(t) \quad \text{and} \quad |\mu_\gamma| = 2 \sqrt{\left| \frac{d^2 \gamma}{dt^2}(0) \right|} \lim_{t \rightarrow 0} |t \kappa(t)|$$

hold, where $s(t) = \int_0^t |\gamma'| dt$ is the arclength.

Proof. For the first equality, see [22, Theorem 1.1]. Since the Taylor expansion of $s(t)$ is $s(t) = \sqrt{(d^2 \gamma / dt^2)(0)} / 2t + O(t^2)$, we have the second equality. $\square$

3. Koenderink type formulas for smooth surfaces

In this section, we deal with a Koenderink type theorem for regular surfaces whose contour has a singular point. Let $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ be a regular surface whose Gaussian curvature K is $K \leq 0$ at the origin. Take a vector $\xi_s \in T_0 \mathbb{R}^2$ of f , and set $g = \pi_\xi \circ f$, where $\xi = df(\xi_s)$. Then one can easily see that g at 0 is a cusp if and only if $II(\xi_s, \xi_s) = 0$, $d(II(\xi_s, \xi_s)) \neq (0, 0)$ and $\xi_s(II(\xi_s, \xi_s)) \neq 0$ hold at 0, where $II(\xi, \xi) = L\xi_1^2 + 2M\xi_1\xi_2 + N\xi_2^2$ for $\xi = \xi_1\partial_u + \xi_2\partial_v$ and $\xi_s(II(\xi_s, \xi_s))$ stands for the directional derivative of $II(\xi_s, \xi_s)$ by ξ_s . Here (u, v) is a coordinate system on $(\mathbb{R}^2, 0)$, and L, M, N are given by

$$L = f_{uu} \cdot \nu, \quad M = f_{uv} \cdot \nu, \quad N = f_{vv} \cdot \nu, \quad (3.1)$$

where ν is a unit normal vector to f .

Under the above condition, the contour of f as viewed from ξ is the $(2, 3)$ -cusp at 0, and the section of f by the plane $\langle \xi, \nu(0) \rangle_{\mathbb{R}}$ is a regular curve. Let $\alpha(t_c)$ be the parameterization of $S(g)$ and $\beta(t_s)$ the parameterization of the set $\{\det(f, \xi, \nu(0)) = 0\}$ with $\alpha(0) = \beta(0) = 0$. Then $\hat{\alpha}(t_c) = g \circ \alpha(t_c)$ gives a parameterization for the contour of f as viewed from ξ , and $\hat{\beta}(t_s) = f \circ \beta(t_s)$ gives a parameterization for the section of f by $\langle \xi, \nu(0) \rangle_{\mathbb{R}}$ at 0. Let $\kappa_c(t_c)$ and $\kappa_s(t_s)$ be the curvatures of $\hat{\alpha}(t_c)$ and $\hat{\beta}(t_s)$, respectively. We remark that $\alpha(t_c)$ and $\beta(t_s)$ are tangent to each other at 0. Under the above setting, we have the following proposition.

Proposition 3.1. *It holds that*

$$\lim_{\substack{t_c \rightarrow 0 \\ t_s \rightarrow 0}} \left| \frac{\kappa_c(t_c) \kappa_s(t_s)}{l} \right| = \left| \frac{1}{l} \frac{\mu_{\hat{\alpha}}}{2\sqrt{(d^2 \hat{\alpha} / dt_c^2)(0)}} \frac{d\kappa_s}{dt_s}(0) \right| = |K(0, 0)|,$$

where $l(\neq 0) \in \mathbb{R}$ is determined by $|d\alpha/dt_c| = l |d\beta/dt_s|$ at 0.

Proof. Without loss of generality we may assume that

$$f(u, v) = \left(u, v, \frac{1}{2}(k_1^2 u^2 - k_2^2 v^2) + \sum_{m=3}^k c_m(u, v) + O(u, v)^{k+1} \right),$$

where

$$k_1 > 0, \quad k_2 > 0, \quad \text{and} \quad c_m(u, v) = \sum_{i+j=3} \frac{a_{ij}}{i!j!} u^i v^j.$$

Then we see that $\nu(0, 0) = (1, 0, 0)$ and $K(0, 0) = -k_1^2 k_2^2$. Choose $\xi_s = k_2 \partial_u + \varepsilon k_1 \partial_v$, which are parallel to the asymptotic direction at the origin, and set $\xi = (k_2, \varepsilon k_1, 0)/(k_1^2 + k_2^2)^{1/2}$, where $\varepsilon = \pm 1$.

Firstly, we consider the curvature of the contour of f as viewed from ξ . We have

$$\begin{aligned} \det(f_u, f_v, \tilde{\xi}) &= -k_1^2 k_2 u + \varepsilon k_1 k_2^2 v \\ &\quad - \frac{\varepsilon a_{21} k_1 + a_{30} k_2}{2} u^2 - (\varepsilon a_{12} k_1 + a_{21} k_2) uv - \frac{\varepsilon a_{03} k_1 + a_{12} k_2}{2} v^2 + O(u, v)^3, \end{aligned}$$

where $\tilde{\xi} = (k_2, \varepsilon k_1, 0)$. Since $k_1 k_2 \neq 0$, $\alpha(t_c)$ can be expressed as

$$\alpha(t_c) = \left(t_c, \frac{\varepsilon k_1}{k_2} t_c + \frac{3\varepsilon c_3(k_2, \varepsilon k_1)}{k_1 k_2^4} t_c^2 + O(t_c^3) \right),$$

and thus

$$\hat{\alpha}(t_c) = \left(-\frac{3c_3(k_2, \varepsilon k_1)}{k_2^3(k_1^2 + k_2^2)} t_c^2, \frac{3\varepsilon c_3(k_2, \varepsilon k_1)}{k_1 k_2^2(k_1^2 + k_2^2)} t_c^2, 0 \right) + O(t_c^3). \quad (3.2)$$

Since

$$\begin{aligned} \det(d^2 \hat{\alpha}(0)/dt_c^2, d^3 \hat{\alpha}(0)/dt_c^3, \xi) &= -\frac{72\varepsilon c_3(k_2, \varepsilon k_1)^2}{k_1 k_2^6 \sqrt{k_1^2 + k_2^2}} \\ &= -\frac{\varepsilon(\xi_s \Pi(\xi_s, \xi_s))^2 (k_1^2 + k_2^2)^{5/2}}{k_1 k_2^6} \neq 0, \end{aligned} \quad (3.3)$$

we observe that $\hat{\alpha}$ is a $(2, 3)$ -cusp. The curvature of $\hat{\alpha}$ as a plane curve is given by

$$\begin{aligned} \kappa_c &= \frac{\det(\hat{\alpha}', \hat{\alpha}'', \xi)}{|\hat{\alpha}'|^3} = \frac{-\frac{36\varepsilon c_3(k_2, \varepsilon k_1)^2}{k_1 k_2^6 \sqrt{k_1^2 + k_2^2}} t_c^2 + O(t_c^3)}{\left(\frac{36c_3(k_2, \varepsilon k_1)^2}{k_1^2 k_2^6 (k_1^2 + k_2^2)} t_c^2 + O(t_c^3) \right)^{3/2}} \\ &= -\frac{\varepsilon k_1^2 k_2^3 (k_1^2 + k_2^2) (t_c^2 + O(t_c^3))}{6|c_3(k_2, \varepsilon k_1)| (t_c^2 + o(t_c^3))^{3/2}}. \end{aligned} \quad (3.4)$$

Next, we consider the curvature of the section of f with the plane $\langle \xi, \nu(0, 0) \rangle_{\mathbb{R}}$ at the origin. Since $\det(f, \tilde{\xi}, \nu(0, 0)) = \varepsilon k_1 u - k_2 v$ holds, $\beta(t_s)$ can be expressed as $\beta(t_s) = (lt_s, \varepsilon k_1/k_2 lt_s)$ for some $l (\neq 0) \in \mathbb{R}$. Hence, we obtain

$$\hat{\beta}(t_s) = \left(lt_s, \frac{\varepsilon k_1}{k_2} lt_s, \frac{c_3(k_2, \varepsilon k_1)}{k_2^3} l^3 t_s^3 + O(t_s^4) \right). \quad (3.5)$$

The curvature of $\hat{\beta}$ as a plane curve is given by

$$|\kappa_s| = \left| \frac{\det(\hat{\beta}', \hat{\beta}'', \xi \times \nu(0, 0))}{|\hat{\beta}'|^3} \right| = \left| \frac{6c_3(k_2, \varepsilon k_1)}{k_2(k_1^2 + k_2^2)} lt_s + O(t_s^2) \right|. \quad (3.6)$$

Since $d\alpha/dt_c = l d\beta/dt_s$ at 0, $t_s = \pm t_c + O(t_c^2)$ holds. Thus it follows from (3.4) and (3.6) that

$$\lim_{\substack{t_c \rightarrow 0 \\ t_s \rightarrow 0}} \left| \frac{\kappa_c(t_c) \kappa_s(t_s)}{l} \right| = k_1^2 k_2^2.$$

By Proposition 2.2, we have the assertion. $\square$

We remark that (3.2) and (3.5) imply that the assumption $d\alpha/dt_c = l d\beta/dt_s$ at 0 means

$$\left| \frac{d^2 \hat{\alpha}}{dt_c^2}(0) \right| : \left| \frac{d\hat{\beta}}{dt_s}(0) \right| = 6 \left| \frac{c_3(k_2, \varepsilon k_1)}{\kappa_1 \kappa_2^2 (\kappa_1^2 + \kappa_2^2)} \right| : l$$

where $t_c^2/2$ is the arclength of $\hat{\alpha}$, and t_s is the arclength of $\hat{\beta}$. Let $\gamma : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ be a curve which is $(2, 3)$ -cusp. In [22], a parameterization near a $(2, 3)$ -cusp is introduced. Let $s(t)$ be a usual arc-length function. Then the function $\text{sgn}(t)\sqrt{|s(t)|}$ is C^∞ differentiable ([22, Lemma 2.1]) and it is called *half-arclength*. This implies that t^2 is the arclength. Then it can be seen that the parameter t satisfies that $t^2/2$ is an arc-length of γ if and only if $|\gamma| = |t|$.

4. Koenderink type formulas for Whitney umbrellas

In this section, we give Koenderink type formulas for orthogonal projections of Whitney umbrellas.

4.1. Koenderink type formulas at a Whitney umbrella

We fix our setting as follows.

Notation 4.1. Let $f : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^3, 0)$ be a Whitney umbrella with a unit normal vector ν defined on $\mathbb{R}^2 \setminus 0$, and $\xi^\perp \subset \mathbb{R}^3$ a plane perpendicular to $\xi = (\xi_1, \xi_2, \xi_3)$ ($|\xi| = 1$), and $\pi_\xi : \mathbb{R}^3 \rightarrow \xi^\perp$ the orthogonal projection. Denote $g = \pi_\xi \circ f : (\mathbb{R}^2, 0) \rightarrow (\xi^\perp, g(0))$. Let $\alpha(t_c) : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ be a parameterization of the singular set $S(g)$, and $\hat{\alpha} = g \circ \alpha$. Let $\beta(t_s) : (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, 0)$ denote a parameterization of the section of image f by the plane $\langle \xi, \nu_n \rangle_{\mathbb{R}}$ at 0, namely, $\text{image}(f \circ \beta) = \text{image } f \cap \langle \xi, \nu_n \rangle_{\mathbb{R}}$, and $\hat{\beta} = f \circ \beta$. We remark that $\lim_{t_c \rightarrow 0} \nu(\alpha(0)) = \nu_n$. Let κ_c be the curvature of $\hat{\alpha}$ and κ_s be the curvature of $\hat{\beta}$ as plane curves.

Under this notation, $\hat{\alpha}$ gives a parameterization of the contour of f as viewed from ξ . We have the following lemma.

Lemma 4.2. *Under the setting as in Notation 4.1,*

- (1) *If ξ does not lie in the principal plane of f , then $\hat{\alpha}$ is a regular curve,*
- (2) *If ξ is in the principal plane of f but ξ is neither the tangential direction of f nor the tangential direction of the double curve of f , then $\hat{\alpha}$ and $\hat{\beta}$ are both $(2, 3)$ -cusps.*

Remark 4.3. If f is a normal form of Whitney umbrella as in (2.1), then the above conditions are equivalent to the following conditions respectively:

- (1) $\xi_2 \neq 0$.
- (2) $\xi_2 = 0, \xi_3 \neq 0$ and $3a_{02}\xi_1 + b_3\xi_3 \neq 0$.

Proof of Lemma 4.2. Without loss of generality, we may assume that f is of the form (2.1).

(1) We assume $\xi_2 \neq 0$. Then since $g_u = (\pi_\xi \circ f)_u = f_u - (f_u \cdot \xi)\xi$ holds, $S(g) = \{(u, v) \mid J(u, v) = 0\}$, where we set $J = \det(f_u, f_v, \xi)$. Since $f_v(0, 0) = 0$, we have $J_v(0, 0) = \det(f_u, f_{vv}, \xi)(0, 0) = -a_{02}\xi_2 \neq 0$. Therefore, α can be given in $\alpha(u) = (u, v(u))$. Since $f_v(0, 0) = 0$, we have

$$\hat{\alpha}'(0) = f_u - (f_u \cdot \xi)\xi = (\xi_2^2 + \xi_3^2, -\xi_1\xi_2, -\xi_1\xi_3) \neq (0, 0, 0),$$

and thus (1) is proved.

(2) We assume the conditions of (2). Since $\xi_2 = 0$ and $f_v(0) = (0, 0, 0)$, we show that

$$J_u(0, 0) = \det(f_u, f_{uv}, \xi)(0, 0) = \xi_3,$$

$$J_v(0, 0) = \det(f_u, f_{vv}, \xi)(0, 0) = 0,$$

$$J_{vv}(0, 0) = 2\det(f_{uv}, f_{vv}, \xi)(0, 0) + \det(f_u, f_{vvv}, \xi)(0, 0) = 2a_{02}\xi_1 + b_3\xi_3.$$

It follows that α can be given by $\alpha(v) = (u(v), v)$, where $u'(0) = 0$ and $u''(0)$ is given by $-(2a_{02}\xi_1 + b_3\xi_3)/\xi_3$. Since $u'(0) = 0$ and $f_v(0, 0) = (0, 0, 0)$, we show that $\hat{\alpha}'(0) = (0, 0, 0)$. Moreover, we have

$$\begin{aligned} (f \circ \alpha)''(0) &= f_u(0, 0)u''(0) + f_{vv}(0, 0) \\ &= \left(-(2a_{02}\xi_1 + b_3\xi_3)/\xi_3, 0, a_{02} \right) \\ (f \circ \alpha)'''(0) &= f_u(0, 0)u'''(0) + 3f_{uv}(0, 0)u''(0) + f_{uuu}(0, 0). \end{aligned} \tag{4.1}$$

Hence, we have

$$\begin{aligned} \det(\hat{\alpha}''(0), \hat{\alpha}'''(0), \xi) &= \det\left((f \circ \alpha)''(0), (f \circ \alpha)'''(0), \xi\right) \\ &= \begin{vmatrix} u''(0) & u'''(0) & \xi_1 \\ 0 & 3u''(0) + b_3 & 0 \\ a_{02} & a_{03} + 3a_{11}u''(0) & \xi_3 \end{vmatrix} = \frac{2(3a_{02}\xi_1 + b_3\xi_3)^2}{\xi_3} \neq 0, \end{aligned} \tag{4.2}$$

and thus we conclude that $\hat{\alpha}$ is a $(2, 3)$ -cusp. Here, $|\cdot| = \det(\cdot)$. Now we turn to $\hat{\beta}$. Since $u'(0) = 0$, that is, α is a null curve, we have $\nu_n = \pm(0, 1, 0)$. We set $\Sigma(u, v) = \det(f(u, v), \nu_n, \xi)$. The intersection of f by $\langle \nu_n, \xi \rangle_{\mathbb{R}}$ at 0 is given by $f(\{\Sigma = 0\})$. Since $f_v(0, 0) = (0, 0, 0)$, we have $\Sigma_u(0, 0) = \pm\xi_3$ and $\Sigma_v(0, 0) = 0$. It follows that β can be given by $\beta(v) = (\tilde{u}(v), v)$. Moreover, we have $\tilde{u}'(0) = 0$ and $\tilde{u}''(0) = -\Sigma_{vv}(0, 0)/\Sigma_u(0, 0) = a_{02}\xi_1/\xi_3$. Since $\tilde{u}'(0) = 0$, we have

$$\hat{\beta}''(0) = f_u(0, 0)\tilde{u}''(0) + f_{vv}(0, 0) = \left(a_{02}\xi_1/\xi_3, 0, a_{02}\right), \tag{4.3}$$

$$\hat{\beta}'''(0) = f_u(0, 0)\tilde{u}'''(0) + 3f_{uv}\tilde{u}''(0) + f_{vvv}(0, 0).$$

Note that $\nu_n \times \xi = \pm(\xi_3, 0, -\xi_1)$. We see that

$$\det(\hat{\beta}''(0), \hat{\beta}'''(0), \nu_n \times \xi) = \begin{vmatrix} u''(0) & u'''(0) & \pm\xi_3 \\ 0 & 3u''(0) + b_3 & 0 \\ a_{02} & 3a_{11}u''(0) + a_{03} & \mp\xi_1 \end{vmatrix} = \pm \frac{a_{02}(3a_{02}\xi_1 + b_3\xi_3)}{\xi_3^2} \neq 0,$$

and (2) is proved. $\square$

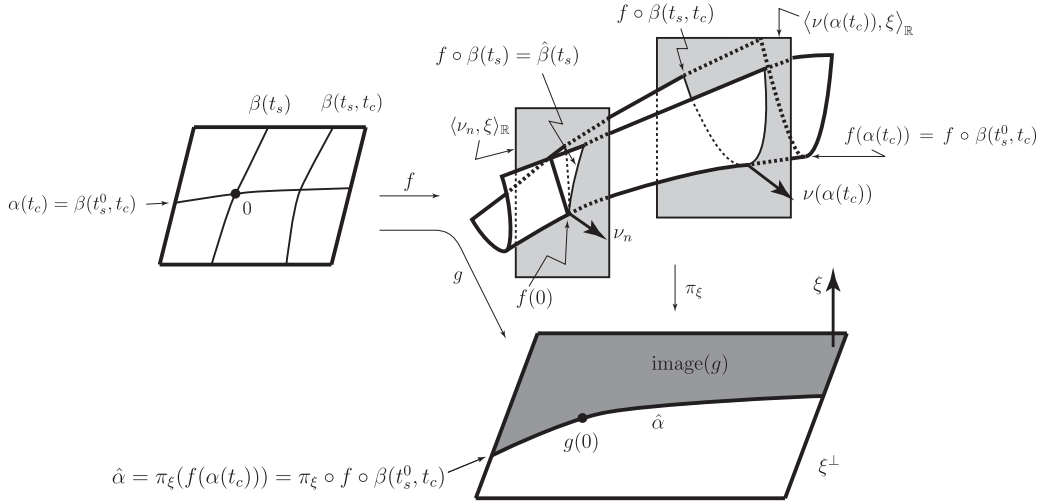


FIGURE 2. A situation of the projection

Under the setting as in Notation 4.1, let us consider a parameter family of sections of f by the planes spanned by $\nu(\alpha(t_c))$ and ξ passing through $f(\alpha(t_c))$. See Figure 2 for this situation. Set

$$\Sigma^{t_c}(u, v) = \Sigma(u, v, t_c) = \det(f(u, v) - f(\alpha(t_c)), \nu(\alpha(t_c)), \xi). \quad (4.4)$$

Then $f(\{\Sigma^{t_c} = 0\})$ represents the section of f by the plane spanned by $\nu_n = \nu(\alpha(t_c))$ and ξ passing through $f(\alpha(t_c))$. If $t_c \neq 0$, then $\{\Sigma^{t_c} = 0\}$ is non-singular at $\alpha(t_c)$. When $t_c = 0$, if ξ is not the tangential direction of f then $\{\Sigma^{t_c} = 0\}$ is non-singular at $\alpha(0)$. Therefore, if ξ is not the tangential direction of f then it holds that the family $\{\Sigma^{t_c} = 0\}$ can be parameterized as $\beta(t_s, t_c) : (\mathbb{R}, t_s^0) \times (\mathbb{R}, 0) \rightarrow (\mathbb{R}^2, \alpha(t_c))$ and a member of the family can be parameterized as $\beta^{t_c}(t_s) = \beta(t_s, t_c)$, where t_s^0 satisfies $\beta(t_s^0, t_c) = \alpha(t_c)$. We remark that $\lim_{t_c \rightarrow 0} \beta(t_s, t_c) = \beta^0(t_s) = \beta(t_s)$, where $\beta(t_s)$ is defined as in Notation 4.1. Let $\kappa_s(t_s, t_c)$ be the curvature of $\hat{\beta}^{t_c} = f \circ \beta^{t_c}$ as a plane curve. If the condition (2) of Lemma 4.2 is satisfied, then the contour and the section are both (2, 3)-cusps. From Proposition 2.2,

it is expected that $\lim_{t_c \rightarrow 0} t_c^2 K(\alpha(t_c))$ relates to $\lim_{t_s \rightarrow 0, t_c \rightarrow 0} t_c \kappa_c(t_c) t_s \kappa_s(t_s, t_c)$ and the product of $\mu_{\hat{\alpha}}$ and $\mu_{\hat{\beta}}$. We have Theorem 4.4. For the sake of simplicity, we assume $|d\alpha/dt_c(0)| = |d\beta^0/dt_s(0)|$.

Theorem 4.4. *Under the setting as in Notation 4.1 and above, if ξ satisfies the conditions (2) of Lemma 4.2, and $|d\alpha/dt_c(0)| = |d\beta^0/dt_s(0)|$, then the following holds:*

$$\left| \lim_{t_c \rightarrow 0} t_c^2 K(\alpha(t_c)) \right| = 2 \lim_{\substack{t_s \rightarrow 0 \\ t_c \rightarrow 0}} |t_c \kappa_c(t_c) t_s \kappa_s(t_s, t_c)| = 2 \lim_{\substack{t_s \rightarrow 0 \\ t_c \rightarrow 0}} |t_c \kappa_c(t_c) t_s \kappa_n(t_s, t_c)|,$$

where $\kappa_n(t_s, t_c)$ is the normal curvature along $\beta^{t_c}(t_s)$ in the direction $d\beta^{t_c}(t_s)/dt_s$.

For a regular surface $f : U \rightarrow \mathbb{R}^3$, the normal curvature of f at p in the tangent direction $\xi = \xi_1 f_u(p) + \xi_2 f_v(p)$ ($\xi_1, \xi_2 \in \mathbb{R}$) is defined by

$$\kappa_n(\xi) = \frac{L \xi_1^2 + 2M \xi_1 \xi_2 + N \xi_2^2}{E \xi_1^2 + 2F \xi_1 \xi_2 + G \xi_2^2},$$

where (u, v) is a coordinate system on U , ν is the unit normal vector to f , and $E = f_u \cdot f_u$, $F = f_u \cdot f_v$, $G = f_v \cdot f_v$, and L, M, N are as in (3.1). We remark that by (4.1), we see that $|\hat{\beta}''(0)| = |3a_{02}\xi_1 + b_3\xi_3|$. This formula and (4.3) imply that the assumption $|d\alpha/dt_c(0)| = |d\beta^0/dt_s(0)|$ means

$$\left| \frac{d^2 \hat{\alpha}}{dt_c^2}(0) \right| : \left| \frac{d^2 \hat{\beta}}{dt_s^2}(0) \right| = |3a_{02}\xi_1 + b_3\xi_3| : \left| \frac{a_{02}}{\xi_3} \right|$$

where $t_c^2/2$ is the arclength of $\hat{\alpha}$, and $t_s^2/2$ is the arclength of $\hat{\beta}$.

Corollary 4.5. *It holds that*

$$|K_\eta| = \frac{|\mu_{\hat{\alpha}} \mu_{\hat{\beta}}|}{2 \sqrt{\left| \frac{d^2 \hat{\alpha}}{dt_c^2}(0) \right|} \sqrt{\left| \frac{d^2 \hat{\beta}}{dt_s^2}(0) \right|}},$$

where η is a null vector defined as in Section 2.1.

Proof. Theorem 4.4, Proposition 2.2 and the Definition of K_η . We remark that under the condition (2) of Lemma 4.2, $\alpha'(0) = \eta$ holds. $\square$

4.2. Limit of the usual Koenderink theorem along the contour

Let us consider an orthogonal projection of a Whitney umbrella. Since Whitney umbrella is an isolated singular point, usual Koenderink formulas hold at each point on contour away from the Whitney umbrella. Here we consider a Koenderink type theorem for Whitney umbrella as a limit of the usual Koenderink theorem along the contour.

Theorem 4.6. *Under the setting as in Notation 4.1 and above, if ξ satisfies the condition (1) of Lemma 4.2, then the following equalities hold:*

$$\begin{aligned} \lim_{t_c \rightarrow 0} t_c^2 K(\alpha(t_c)) &= \lim_{t_c \rightarrow 0} t_c^2 \kappa_c(t_c) \kappa_s(t_s^0, t_c), \\ \left| \lim_{t_c \rightarrow 0} t_c^2 K(\alpha(t_c)) \right| &= \lim_{t_c \rightarrow 0} \left| t_c^2 \kappa_c(t_c) \kappa_n(t_s^0, t_c) \right|. \end{aligned} \quad (4.5)$$

Here, $\hat{\alpha}$ and $\hat{\beta}^{t_c}(t_s)$ are oriented so that

$$\frac{(d\hat{\alpha}/dt_c)(0)}{|(d\hat{\alpha}/dt_c)(0)|} = - \lim_{t_c \rightarrow +0} \nu(\alpha(t_c)) \times \xi \quad \text{and} \quad \lim_{t_c \rightarrow +0} \frac{(\partial \hat{\beta} / \partial t_s)(t_s^0, t_c)}{|(\partial \hat{\beta} / \partial t_s)(t_s^0, t_c)|} = \xi, \quad (4.6)$$

respectively. Moreover, planes $\xi^\perp$ and $\langle \nu(\alpha(t_c)), \xi \rangle_{\mathbb{R}}$ are oriented so that the vectors ξ and $\nu(\alpha(t_c)) \times \xi$ are normal vectors to each plane.

If ξ satisfies the condition (2) of Lemma 4.2 and $|d\alpha/dt_c(0)| = |d\beta/dt_s(0)|$, then the following holds:

$$\left| \lim_{t_c \rightarrow 0} t_c^2 K(\alpha(t_c)) \right| = \lim_{t_c \rightarrow 0} \left| t_c \kappa_c(t_c) t_s^0 \kappa_s(t_s^0, t_c) \right| = \lim_{t_c \rightarrow 0} \left| t_c \kappa_c(t_c) t_s^0 \kappa_n(t_s^0, t_c) \right|. \quad (4.7)$$

Since $f(\alpha(t_c))$ belongs to the contour generator, and f at $\alpha(t_c)$ is regular, the usual Koenderink theorem tells the equality

$$K(\alpha(t_c)) = \kappa_c(t_c) \kappa_s(t_s^0, t_c).$$

If ξ satisfies the condition (1) of Lemma 4.2, then since $\pi_\xi(f(\alpha(t_c)))$ is regular, one can obtain the Koenderink theorem by taking limit along $\alpha(t_c)$. On the other hand, if ξ satisfies the condition (2) of Lemma 4.2, then since $\pi_\xi(f(\alpha(t_c)))$ has (2, 3)-cusp at $t_c = 0$, the Koenderink theorem holds after an adjustment as in (4.7).

Proofs of Theorems 4.4 and 4.6. Without loss of generality, we may assume that f is of the form (2.1). The singular set $S(g)$ is given by

$$\begin{aligned} S(g) &= \{(u, v) \mid \det(f_u(u, v), f_v(u, v), \xi) = 0\} \\ &= \left\{ (u, v) \mid (a_{11}\xi_2 + \xi_3)u - a_{02}\xi_2v \right. \\ &\quad \left. + \frac{1}{2}((-2a_{20}\xi_1 - a_{21}\xi_2)u^2 - 2a_{12}uv + (2a_{02}\xi_1 - a_{03}\xi_2)v^2) + O(u, v)^3 \right\}. \end{aligned} \quad (4.8)$$

Firstly, we assume that the condition (2) of Lemma 4.2, namely, $\xi_2 = 0$, $\xi_3 \neq 0$ and $3a_{02}\xi_1 + b_3\xi_3 \neq 0$ hold. By (4.8), it holds that $S(g)$ is locally parameterized by $\alpha(t_c) = (u(t_c), t_c)$, where $u(t_c) = -(2a_{02}\xi_1 + b_3\xi_3)t_c^2/(2\xi_3) + O(t_c^3)$. Then we have

$$\nu(\alpha(t_c)) = \varepsilon \left(t_c, -1, -\frac{\xi_1}{\xi_3}t_c \right) + O(t_c^2) \quad \text{and} \quad K(\alpha(t_c)) = \frac{1}{t_c^2} \left(-\frac{1}{a_{02}^2} + O(t_c) \right), \quad (4.9)$$

where $\varepsilon = +1$ (resp. -1) if $t_c > 0$ (resp. < 0). Since

$$\Sigma = \varepsilon \left(-\xi_3 u + \frac{1}{2} (a_{20}\xi_1 u^2 + 2a_{11}\xi_1 uv + a_{02}\xi_1 v^2 - (3a_{02}\xi_1 - b_3\xi_3)t_c^2) + O(u, v, t_c)^3 \right),$$

we have $\beta^{t_c}(t_s) = \beta(t_s, t_c) = (\hat{u}(t_s, t_c), t_s)$, where

$$\hat{u}(t_s, t_c) = \frac{a_{02}\xi_1}{2\xi_3} t_s^2 - \frac{3a_{02}\xi_1 + b_3\xi_3}{2\xi_3} t_c^2 + O(t_s, t_c)^3.$$

We remark that $d\alpha/dt_c(0) = d\beta^0/dt_s(0)$ and $t_s^0 = t_c$. On the other hand, by straightforward calculations, we have

$$\kappa_c(t_c) = \frac{\det(d\alpha/dt_c(t_c), d^2\alpha/dt_c^2(t_c), \xi)}{|d\alpha/dt_c(t_c)|^3} = \frac{(3a_{02}\xi_1 + b_3\xi_3)^2 t_c^2 / \xi_3 + O(t_c^3)}{((3a_{02}\xi_1 + b_3\xi_3)^2 t_c^2 + O(t_c^3))^{3/2}}, \quad (4.10)$$

$$\begin{aligned} \kappa_s(t_s, t_c) &= \frac{\det(\partial\hat{\beta}/\partial t_s(t_s, t_c), \partial^2\hat{\beta}/\partial t_s^2(t_s, t_c), \nu(\alpha(t_c)) \times \xi)}{|\partial\hat{\beta}/\partial t_s(t_s, t_c)|^3} \\ &= \frac{\det(\partial\hat{\beta}/\partial t_s(t_s, t_c), \partial^2\hat{\beta}/\partial t_s^2(t_s, t_c), (f_u \times f_v)(\alpha(t_c)) \times \xi)}{|(f_u \times f_v)(\alpha(t_c))| |\partial\hat{\beta}/\partial t_s(t_s, t_c)|^3} \\ &= \frac{a_{02}^2(3a_{02}\xi_1 + b_3\xi_3)t_c(t_s^2 + t_c^2)/(2\xi_3^2) + O(t_s, t_c)^4}{\sqrt{a_{02}^2 t_c^2 + O(t_c^3)} (a_{02}^2 t_s^2 / \xi_3^2 + O(t_s^3))^{3/2}}, \end{aligned} \quad (4.11)$$

$$\begin{aligned} \kappa_n(t_s, t_c) &= \frac{L(\beta(t_s, t_c))(\partial\hat{u}/\partial t_s)^2 + 2M(\beta(t_s, t_c))(\partial\hat{u}/\partial t_s) + N(\beta(t_s, t_c))}{E(\beta(t_s, t_c))(\partial\hat{u}/\partial t_s)^2 + 2F(\beta(t_s, t_c))(\partial\hat{u}/\partial t_s) + G(\beta(t_s, t_c))} \\ &= \frac{-a_{02}(3a_{02}\xi_1 + b_3\xi_3)(t_s^2 + t_c^2)/(2\xi_3) + O(t_s, t_c)^3}{\sqrt{a_{02}^2 t_s^2 + O(t_s^3)} (a_{02}^2 t_s^2 / \xi_3^2 + O(t_s^3))}. \end{aligned} \quad (4.12)$$

It follows from (4.10) and (4.11) that

$$\begin{aligned} \lim_{\substack{t_s \rightarrow 0 \\ t_c \rightarrow 0}} |t_c \kappa_c(t_c) t_s \kappa_s(t_s, t_c)| &= \left| \frac{1}{\xi_3 |3a_{02}\xi_1 + b_3\xi_3|} \frac{|\xi_3|(3a_{02}\xi_1 + b_3\xi_3)}{2a_{02}^2} \right| \\ &= \frac{1}{2a_{02}^2} = \frac{1}{2} \lim_{t_c \rightarrow 0} |t_c^2 K(\alpha(t_c))|. \end{aligned}$$

Similarly, using (4.10) – (4.12), we show that

$$\begin{aligned} \lim_{\substack{t_s \rightarrow 0 \\ t_c \rightarrow 0}} |t_c \kappa_c(t_c) t_s \kappa_n(t_s, t_c)| &= \frac{1}{2} \lim_{t_c \rightarrow 0} |t_c^2 K(\alpha(t_c))|, \\ \lim_{t_c \rightarrow 0} |t_c \kappa_c(t_c) t_s^0 \kappa_s(t_s^0, t_c)| &= \lim_{t_c \rightarrow 0} |t_c^2 K(\alpha(t_c))|, \\ \lim_{t_c \rightarrow 0} |t_c \kappa_c(t_c) t_s^0 \kappa_n(t_s^0, t_c)| &= \lim_{t_c \rightarrow 0} |t_c^2 K(\alpha(t_c))|. \end{aligned}$$

We remark that if $t_c = t_c(r_c)$ and $t_s = t_s(r_s)$, then by same calculations obtaining above relations we have

$$\begin{aligned} \lim_{r_c \rightarrow 0} r_c^2 K(\alpha(t_c(r_c))) &= \frac{1}{(dt_c/dr_c(0))^2} \lim_{t_c \rightarrow 0} t_c^2 K(\alpha(t_c)), \\ \lim_{\substack{r_s \rightarrow 0 \\ r_c \rightarrow 0}} \left| r_c \kappa_c(t_c(r_c)) r_s \kappa_s(t_s(r_s), t_c(r_c)) \right| &= \frac{1}{\left| \frac{dt_c}{dr_c}(0) \right| \left| \frac{dt_s}{dr_s}(0) \right|} \lim_{\substack{t_s \rightarrow 0 \\ t_c \rightarrow 0}} \left| t_c \kappa_c(t_c) t_s \kappa_s(t_s, t_c) \right|, \\ \lim_{\substack{r_s \rightarrow 0 \\ r_c \rightarrow 0}} \left| r_c \kappa_c(t_c(r_c)) r_s \kappa_n(t_s(r_s), t_c(r_c)) \right| &= \frac{1}{\left| \frac{dt_c}{dr_c}(0) \right| \left| \frac{dt_s}{dr_s}(0) \right|} \lim_{\substack{t_s \rightarrow 0 \\ t_c \rightarrow 0}} \left| t_c \kappa_c(t_c) t_s \kappa_n(t_s, t_c) \right|, \\ \lim_{r_c \rightarrow 0} \left| r_c \kappa_c(t_c(r_c)) r_s^0 \kappa_s(t_s(r_s^0), t_c(r_c)) \right| &= \frac{1}{\left| \frac{dt_c}{dr_c}(0) \right| \left| \frac{dt_s}{dr_s}(0) \right|} \lim_{t_c \rightarrow 0} \left| t_c \kappa_c(t_c) t_s^0 \kappa_s(t_s^0, t_c) \right|, \\ \lim_{r_c \rightarrow 0} \left| r_c \kappa_c(t_c(r_c)) r_s^0 \kappa_n(t_s(r_s^0), t_c(r_c)) \right| &= \frac{1}{\left| \frac{dt_c}{dr_c}(0) \right| \left| \frac{dt_s}{dr_s}(0) \right|} \lim_{t_c \rightarrow 0} \left| t_c \kappa_c(t_c) t_s^0 \kappa_n(t_s^0, t_c) \right|, \end{aligned}$$

where r_s^0 satisfies the condition $\beta(t_s(r_s^0), t_c(r_c)) = \alpha(t_c(r_c))$. These relations show the necessity of the assumption $|d\alpha/dt_c(0)| = |d\beta^0/dt_s(0)|$, and by the same arguments in the proof of Theorem 4.4, one can obtain the geometric meaning of this assumption. We complete the proofs of Theorem 4.4 and the second assertion of Theorem 4.6.

Secondly, we assume the condition (1) of Lemma 4.2. By (4.8), it holds that $S(g)$ is locally parameterized by $\alpha(t_c) = (t_c, v(t_c))$, where

$$\begin{aligned} v(t_c) &= \frac{\xi_3 - a_{11}\xi_2}{a_{02}\xi_2} t_c \\ &\quad + \frac{1}{2a_{02}^3\xi_2^3} \left((2a_{02}\xi_1 - a_{03}\xi_2)(\xi_3 - a_{11}\xi_2)^2 \right. \\ &\quad \left. - 2a_{12}a_{02}\xi_2^2(\xi_3 - a_{11}\xi_2) - a_{02}^2\xi_2^2(a_{21}\xi_2 + 2a_{02}\xi_1) \right) t_c^2 + O(t_c^3). \end{aligned}$$

We note that $\hat{\alpha}(t_c)$ is a tangential curve, which is defined as in Section 2.1. By straightforward calculations we show that $\nu(\alpha(t_c))$ is expressed as

$$\begin{aligned} \nu(\alpha(t_c)) &= \frac{\varepsilon \operatorname{sgn}(\xi_2)}{\sqrt{\xi_2^2 + \xi_3^2}} \left(\frac{(a_{11}\xi_2 - \xi_3)^2}{a_{02}\xi_2} t_c, -\xi_3 - \frac{\xi_1(a_{11}\xi_2 - \xi_3)^2}{a_{02}(\xi_2^2 + \xi_3^2)} t_c, \xi_2 - \frac{\xi_1\xi_3(a_{11}\xi_2 - \xi_3)^2}{a_{02}(\xi_2^2 + \xi_3^2)} t_c \right) + O(t_c^2), \end{aligned}$$

where $\varepsilon = 1$ (resp. -1) if $t_c > 0$ (resp. < 0), and that $K(\alpha(t))$ is expressed as

$$K(\alpha(t_c)) = \frac{1}{t_c^2} \left(-\frac{\xi_2^2((a_{11}\xi_2 - \xi_3)^2 - a_{20}a_{02}\xi_2^2)}{(\xi_2^2 + \xi_3^2)^2} + O(t_c) \right). \quad (4.13)$$

On the other hand, we have

$$\kappa_c(0) = \frac{\det(d\hat{\alpha}/dt_c(0), d^2\hat{\alpha}/dt_c^2(0), \xi)}{|d\hat{\alpha}/dt_c(0)|^3} = \frac{(a_{11}\xi_2 - \xi_3)^2 - a_{02}a_{20}\xi_2^2}{a_{02}\xi_2(\xi_2^2 + \xi_3^2)^{3/2}}. \quad (4.14)$$

Since we have

$$\begin{aligned} \Sigma &= \operatorname{sgn}(\xi_2) \sqrt{\xi_2^2 + \xi_3^2} (-u + t_c) + \frac{\operatorname{sgn}(\xi_2)}{2\sqrt{\xi_2^2 + \xi_3^2}} \left(a_{20}\xi_1\xi_3u^2 + 2\xi_1(\xi_2 + a_{11}\xi_3)uv \right. \\ &\quad \left. + a_{02}\xi_1\xi_3v^2 + \frac{\xi_1(2a_{11}\xi_2^3 - 2\xi_2^2\xi_3 + a_{11}^2\xi_2^2\xi_3 - a_{20}a_{02}\xi_2^2\xi_3 - \xi_3^3)}{a_{02}\xi_2^2} t_c^2 \right) + O(u, v, t_c)^3, \end{aligned}$$

we have $\beta(t_s, t_c) = (u(t_s, t_c), t_s)$ and $u(t_c, t_s)$ is locally written as

$$\begin{aligned} u(t_s, t_c) &= t_c + \frac{1}{2} \left(\frac{a_{02}\xi_1\xi_3}{\xi_2^2 + \xi_3^2} t_s^2 - \frac{2\xi_1(\xi_2 + a_{11}\xi_3)}{\xi_2^2 + \xi_3^2} t_s t_c \right. \\ &\quad \left. + \frac{\xi_1(a_{11}\xi_2 - \xi_3)(2\xi_1^2 + a_{11}\xi_2\xi_3 + \xi_3^2)}{a_{02}\xi_2^2(\xi_2^2 + \xi_3^2)} t_c^2 \right) + O(t_s, t_c)^3. \end{aligned} \quad (4.15)$$

We remark that $\beta(v(t_c), t_s) = (u(v(t_c), t_s), v(t_c)) = \alpha(t_c)$, that is, $t_s^0 = v(t_c)$. Straightforward calculations show that

$$\begin{aligned} \kappa_s(t_s^0, t_c) &= \frac{\det(\partial\hat{\beta}/\partial t_s(t_s, t_c), \partial^2\hat{\beta}/\partial t_s^2(t_s, t_c), \nu(\alpha(t_c)) \times \xi)}{|\partial\hat{\beta}/\partial t_s(t_s, t_c)|^3} \Big|_{t_s=t_s^0} \\ &= \frac{-a_{02}/\xi_2 t_c^2 + O(t_c^3)}{\sqrt{(\xi_2^2 + \xi_3^2)t_c^2/\xi_2^2 + O(t_c^3)}(t_c^2/\xi_2 + O(t_c^3))^{3/2}}, \end{aligned} \quad (4.16)$$

$$\begin{aligned} \kappa_n(t_s^0, t_c) &= \frac{L(\beta(t_s, t_c))(\partial\hat{u}/\partial t_s)^2 + 2M(\beta(t_s, t_c))(\partial\hat{u}/\partial t_s) + N(\beta(t_s, t_c))}{E(\beta(t_s, t_c))(\partial\hat{u}/\partial t_s)^2 + 2F(\beta(t_s, t_c))(\partial\hat{u}/\partial t_s) + G(\beta(t_s, t_c))} \Big|_{t_s=t_s^0} \\ &= \frac{a_{02}t_c + O(t_c^2)}{\sqrt{(\xi_2^2 + \xi_3^2)t_c^2/\xi_2^2 + O(t_c^3)}(tc^2/\xi_2^2 + O(t_c^3))}. \end{aligned} \quad (4.17)$$

From (4.13) – (4.16), we have

$$\lim_{t_c \rightarrow 0} t_c^2 \kappa_c(t_c) \kappa_s(t_s^0, t_c) = \frac{(a_{11}\xi_2 - \xi_3)^2 - a_{02}a_{20}\xi_2^2}{a_{02}\xi_2(\xi_2^2 + \xi_3^2)^{3/2}} \left(-\frac{a_{02}\xi_2^3}{\sqrt{\xi_2^2 + \xi_3^2}} \right) = \lim_{t_c \rightarrow 0} t_c^2 K(\alpha(t_c)).$$

Similarly, we have

$$\lim_{t_c \rightarrow 0} |t_c^2 \kappa_c(t_c) \kappa_n(t_s^0, t_c)| = \left| \lim_{t_c \rightarrow 0} t_c^2 K(\alpha(t_c)) \right|.$$

We remark that since now we have

$$\frac{d\hat{\alpha}/dt_c(0)}{|d\hat{\alpha}/dt_c(0)|} = -\text{sgn}(\xi_2) \lim_{t_c \rightarrow +0} \nu(\alpha(t_c)) \times \xi, \quad \lim_{t_c \rightarrow +0} \frac{d\hat{\beta}/dt_s(t_s, t_c)}{|d\hat{\beta}/dt_s(t_s, t_c)|} \Big|_{t_s=t_s^0} = \text{sgn}(\xi_2)\xi,$$

when $\xi_2 < 0$ the equality (4.6) holds by replacing ξ by $-\xi$ while holding the equality (4.5). $\square$

4.3. Convexity and concavity of Whitney umbrellas

One can easily see that the image of the orthogonal projection of a hyperbolic Whitney umbrella in any direction is concave at a point corresponding to the singular point. On the other hand, the image of the orthogonal projection of an elliptic Whitney umbrella in a direction is concave, convex or has an inflection at a point corresponding to the singular point. It is natural to expect that the above argument relates to the Gaussian curvature along the contour. In this section we study the convexity and concavity of Whitney umbrellas.

Let us consider under the setting as in Section 4. The section of f at 0 with the plane spanned by $\nu(\alpha(0))$ and ξ is given by $f(\beta^0(t_s))$. The image $\pi_\xi(f(\beta^0(t_s)))$ is a half-line with initial point 0. We denote the direction vector of $\pi_\xi(f(\beta^0(t_s)))$ by $\mathbf{v}_s$ and denote the (left-hand) unit normal vector to the contour $\hat{\alpha}(t_c)$ by N_c .

Theorem 4.7. *Under the setting as above and Notation 4.1, assume the contour $\hat{\alpha}(t_c)$ is not singular at $t_c = 0$, then*

$$\text{sgn}((\mathbf{v}_s \cdot N_c(0)) \kappa_c(0)) = \text{sgn} \left(\lim_{t_c \rightarrow 0} t^2 K(\alpha_c(t_c)) \right).$$

If $\text{sgn}((\mathbf{v}_s \cdot N_c(0)) \kappa_c(0)) > 0$ (resp. < 0), then $\hat{\alpha}(0)$ is the convex (resp. concave) point of the image of g (Figure 3 and 4). Furthermore, if $\text{sgn}((\mathbf{v}_s \cdot N_c(0)) \kappa_c(0)) = 0$ then $\hat{\alpha}(0)$ is a inflection point of the contour $\hat{\alpha}$.

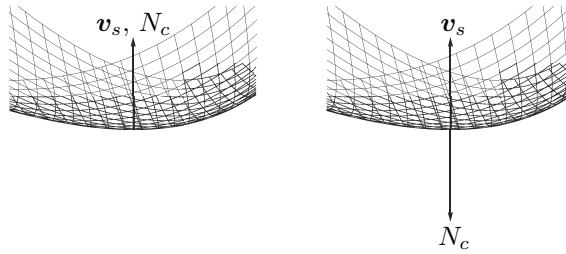
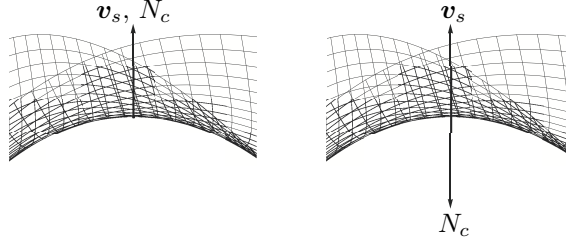


FIGURE 3. The case $\text{sgn}((\mathbf{v}_s \cdot N_c(0)) \kappa_c(0)) > 0$.


 FIGURE 4. The case $\text{sgn}((\mathbf{v}_s \cdot N_c(0)) \kappa_c(0)) < 0$.

Proof of Theorem 4.7. Without loss of generality, we may assume that f is of the form (2.1). Now we have $\xi_2 \neq 0$. It follows from (4.15) that

$$g(\beta^0(t_s)) = \left(0, -\frac{a_{02}\xi_2\xi_3}{2(\xi_2^2 + \xi_3^2)}t_s^2, \frac{a_{02}\xi_2^2}{2(\xi_2^2 + \xi_3^2)}t_s^2 \right) + O(t_s^3). \quad (4.18)$$

Therefore, $\mathbf{v}_s$ is given by

$$\mathbf{v}_s = \left(0, -\frac{\text{sgn}(\xi_3)\xi_2}{\sqrt{\xi_2^2 + \xi_3^2}}, \frac{\text{sgn}(\xi_2)\xi_2}{\sqrt{\xi_2^2 + \xi_3^2}} \right). \quad (4.19)$$

Since $\hat{\alpha}$ can be considered as a plane curve in the plane $\xi^\perp$ with normal vector ξ ,

$$N_c(0) = \frac{d\hat{\alpha}}{dt_c}(0) \times \xi = \left(0, \frac{\xi_3}{\sqrt{\xi_2^2 + \xi_3^2}}, -\frac{\xi_2}{\sqrt{\xi_2^2 + \xi_3^2}} \right). \quad (4.20)$$

It follows from (4.14), (4.19) and (4.20) that

$$\text{sgn}((\mathbf{v}_s \cdot N_c(0)) \kappa_c(0)) = -\text{sgn}((a_{11}\xi_2 - \xi_3)^2 - a_{20}a_{02}).$$

On the other hand, from (4.13)

$$\text{sgn} \left(\lim_{t_c \rightarrow 0} t_c^2 K(\alpha(t_c)) \right) = -\text{sgn}((a_{11}\xi_2 - \xi_3)^2 - a_{20}a_{02}),$$

and the proof is complete. $\square$

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Extensions of Koenderink's formula

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