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Lectures on Khovanov homology, the Lee spectral sequence and the s-invariant

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Abstract. These are the lecture notes for a course on Khovanov homology held at Gökova Geometry Topology Institute in Summer 2022. In these notes, we introduce Khovanov homology, the Lee spectral sequence, Lee homology and Rasmussen's s-invariant after providing sufficient background in knot theory and spectral sequences. We tried to provide a self-contained and short overview of the topics. Most of the lectures are geared towards graduate students with basic knowledge in low-dimensional topology and algebraic topology. We end the notes with applications in low-dimensional topology.

Contents

Key words and phrases. Khovanov homology, s-invariant, Lee spectral sequence.

1. Introduction

Khovanov homology, defined in [1], is an invariant of oriented links in S^3, that categorifies the Jones polynomial. For a link $L \subset S^3$, the Khovanov homology of L takes the form of a bi-graded vector space, and we use the notation $Kh^{i,j}(L)$ where i is the homological grading and j is the quantum grading, or the q-grading for short.

In Lecture 1, we give the necessary background in knot theory most of which is used throughout the notes. This contains the Seifert genus, knot 4-ball genus, the concordance and more.

Lecture 2 gives a detailed construction of the Khovanov chain complex associated to a link diagram and shows it is a link invariant, i.e. does not depend on the choice of the diagram. The long exact sequence associated to a skein triple, a very useful computational tool as seen in the last lectures, is also given.

Lecture 3 is an introduction to spectral sequences of bi-graded chain complexes. We consider a situation in which there is another boundary map on the Khovanov chain groups whose inclusion to the Khovanov chain complex gives rise to a bi-complex. The total complex of this bi-complex is, however, no longer q-graded. We show that the total complex is now filtered by q and the filtration induces a spectral sequence with the 0-page the Khovanov complex and converging to the homology of the total complex.

In Lecture 4, we define this new boundary map and, in light of Lecture 3, we obtain the Lee spectral sequence and Lee homology, denoted by $\mathrm{Kh}_{Lee}(L)$. Lee homology turns out to be surprisingly simple: It is the direct sum of $|L|$ copies of $\mathbb{Q}\oplus\mathbb{Q}$ with generators located in homological grading 0, where $|L|$ is the number of components of L. The remainder of Lecture 4 is dedicated to proving this fact: We describe explicit generators for $\mathrm{Kh}_{\mathrm{Lee}}(L)$ and complete the proof by giving an upper bound on the rank.

For filtered chain complexes there is a grading induced by the filtration. For a knot K, $\mathrm{Kh}_{Lee}(K)$ is simply $\mathbb{Q} \oplus \mathbb{Q}$. Rasmussen [2] shows that the filtration-induced gradings of the generators of $\mathrm{Kh}_{Lee}(K)$ are both knot invariants and always differ by 2. This leads him to define the s-invariant by taking the average of these two gradings. Furthermore, s gives a lower bound on the 4-ball genus and is a concordance invariant. This is done in Lecture 5.

Lecture 6 contains applications of Khovanov homology, the Lee spectral sequence and s-invariant to problems in low-dimensional topology such as the slice-Bennequin inequality, Milnor's conjecture and a few more.

2. Lecture 1: An elementary introduction to knot theory

This chapter contains some fundamental concepts in knot theory that will be used in later lectures. A reader that has familiarity with knot theory may skip the chapter.

Definition 2.1. *A knot K is a smooth embedding of S^1 into S^3. A link L is a smooth embedding of $S^1 \sqcup \cdots \sqcup S^1$ into S^3.*

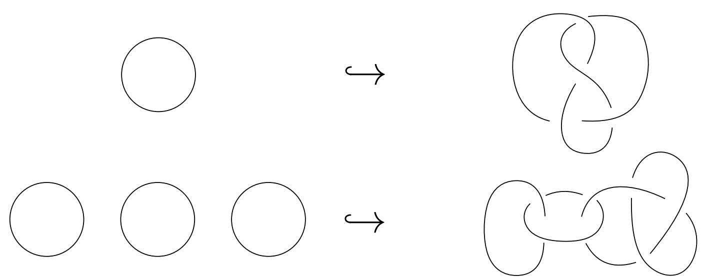


FIGURE 1.

Two links are "the same" if they are isotopic in S^3. To denote the isotopy of knots we will use the symbol "$\simeq$". We study knots or links via their diagrams, i.e. their projections on the plane consisting of only smooth curves and crossings. Multiple diagrams may represent the same knot or link. There are three "Reidemeister moves" which act locally as in Figure 2 and it is obvious that if two diagrams are related by a Reidemeister move they represent the same link. It turns out the converse is also true and the following fundamental theorem of Reidemeister determines exactly when two diagrams represent the same link.

Theorem 2.1 ([3]). *Two link diagrams represent isotopic links in S^3 if and only if they are related by a finite number of Reidemeister moves and their inverses, and planar isotopies.*

Once L is given an orientation (give a direction to the each connected component), we can assign a sign to a crossing following the right-hand rule:

For a given oriented diagram D, we define

$$\begin{aligned} n_- &:= \text{ number of negative crossings in } D \\ n_+ &:= \text{ number of positive crossings in } D. \end{aligned}$$

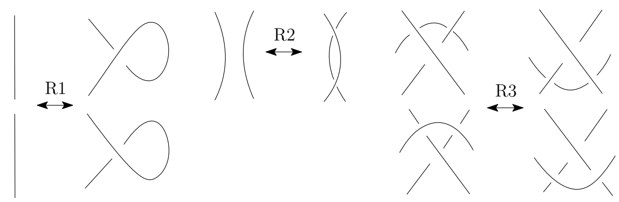


FIGURE 2. Reidemeister moves

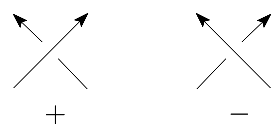


FIGURE 3. Positive and negative crossings

The number of crossings of D is $c(D) := n_+ + n_-$, and the writhe of D is $w(D) = n_+ - n_-$. The crossing number of L is defined to be

$$c(L) := \min\{c(D) \mid D \text{ is a diagram of } L\}.$$

Exercise 1. *Show that* R1 *increases or decreases* $w(D)$ *by 1, while* R2 *and* R3 *preserve it.*

2.1. The linking number

Let D be an oriented diagram for a link L. We denote the number of components of L by $|L|$. Suppose $|L| > 1$. We define *the linking number of* L, an invariant of links with more than one component as follows.

Definition 2.2. *Let D_1 and D_2 be two distinct components of D. The linking number of D_1 and D_2, $lk(D_1, D_2)$, is the signed sum of the crossings between D_1 and D_2 in D. Let C denote the crossings between D_1 and D_2, then*

$$lk(D_1, D_2) = \frac{1}{2}\sum_{c \in C} sign(c).$$

The linking number of D, $lk(D)$, is the sum of linking numbers of distinct components of D

$$lk(D) = \sum_{i<j} lk(D_i, D_j).$$

Note that, if we change the orientation on D_1, then $lk(\overline{D}_1, D_2) = -lk(D_1, D_2)$.

Exercise 2. *Let K_1 and K_2 be the knots represented by D_1 and D_2, respectively. Prove that $lk(D_1, D_2)$ is an invariant of $K_1 \cup K_2$. Similarly, prove that $lk(D)$ is an invariant of L. (Hint: Check the behavior under* R1, R2, *and* R3.*)*

In light of the above exercise, we can denote the linking numbers by $lk(K_1, K_2)$ or $lk(L)$.

Exercise 3. *Show that L_1 and L_2 in Figure 4 are not isotopic but their components are.*

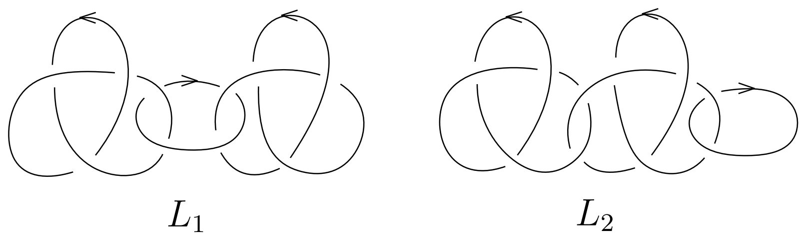


FIGURE 4.

2.2. Some basic operations on knots

Definition 2.3. *The connected sum of oriented knots K_1 and K_2, written as $K_1 \# K_2$, is a knot obtained by cutting K_1 and K_2 at some point and pasting them together in an orientation preserving way. See Figure 5 for an illustration of the operation.*

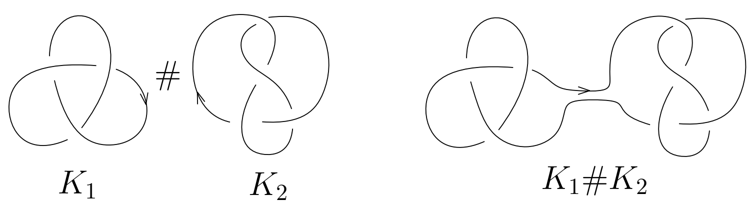


FIGURE 5.

The connected sum is a well-defined binary operation on (oriented) isotopy classes of knots. It is commutative, associative, and the unknot is the identity element. However, there is no inverse – we cannot obtain the unknot as a connected sum of two non-trivial knots. The connected sum, on the other hand, is not well-defined on links with more than one component. This is because cutting and gluing of distinct components may yield non-isotopic links.

Definition 2.4. *The mirror image of L, written $-L$, is the image of L under an orientation-reversing homeomorphism of S^3.*

As there is a unique orientation reversing homeomorphism of S^3 up to isotopy, this operation is well-defined. Given a diagram D of L, reflecting D across the plane it is drawn in changes all crossings of D while leaving the rest of it the same. The resulting diagram is denoted by $-D$ and represents $-L$.

Definition 2.5. *L is said to be amphichiral if $L \simeq -L$.*

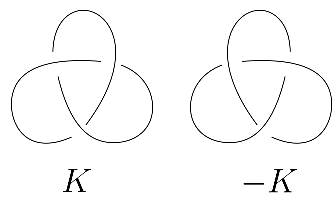


FIGURE 6. The right-handed trefoil and the left-handed trefoil

Exercise 4. *Show that the Figure-8 knot given in Figure 7 is amphichiral.*

Definition 2.6. *Fixing n points p_i in a disk D^2, an n-braid β is an embedding of n arcs $\gamma_i : [0,1] \to D^2 \times [0,1]$ so that $\gamma_i(t) \in D^2 \times \{t\}$ and the endpoints of the γ_i corresponding to 0 (resp. 1) as a set map to $\{p_i\}_{i=1}^n = 1$ in $D^2 \times \{0\}$ (resp. $D^2 \times \{1\}$).*

A braid can be converted into a link by joining the starting and the ending points as in Figure 7. The result is called a closed braid and denoted by $\widehat{\beta}$.

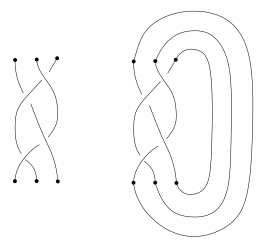


FIGURE 7. A braid with index 3 and its closure, which is the Figure-8 knot

The number of strands n is said to be the *braid index* $b(\beta)$. By a theorem of Alexander [4], we know that every knot or link can be represented by a closed braid. We define *the braid index* of L as follows:

$$b(L) := \min\{n \mid n \text{ is the braid index of } \beta \text{ with } \widehat{\beta} = L\}.$$

Note that although for an n-stranded braid β its braid index $b(\beta) = n$, $b(\widehat{\beta})$ is not necessarily n. In fact, $b(\widehat{\beta}) \leq b(\beta)$, which is evident by the definition.

2.3. Surfaces in S^3 bounded by links

Definition 2.7. *An orientable surface with a given link as its boundary is called a Seifert surface for the link.*

The existence of such a surface for a given link is proved by Seifert [5], who introduced an algorithm to find a Seifert surface for any link in S^3. This is called the *Seifert algorithm* and is given below:

- Fix an orientation on D.
- Resolve each crossing as in Figure 8. This yields a set of oriented circles on the plane, which we call *the Seifert circles* of D.

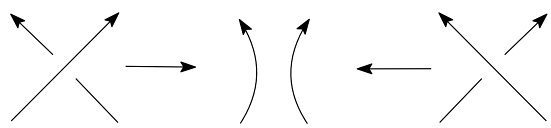


FIGURE 8. The oriented resolution of a crossing

- Fill the Seifert circles by disks, and at each crossing connect the disks to one another by a twisted band.

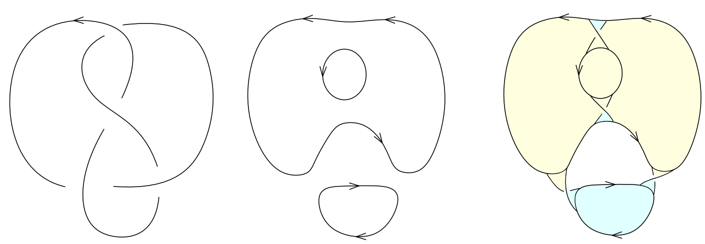


FIGURE 9. A Seifert surface for the Figure-8 knot obtained by the Seifert algorithm

Let F_D denote the surface obtained by applying the Seifert algorithm to D and f be the number of the Seifert circles of F_D. See Figure 9 for an illustration of the algorithm.

Exercise 5. *Convince yourself that F_D is indeed orientable.*

Exercise 6. *Show that $g(F_D) = \frac{2-f+c(D)-|L|}{2}$ where $g(F_D)$ is the genus of the surface F_D.*

Now that we know every link bounds an orientable surface, we then define the *Seifert genus* or *3-genus* of L as follows:

$$g(L) := \min\{g(F) \,|\, F \text{ is an orientable surface in } S^3 \text{ with } \partial F = L\}$$

Let $\chi(L)$ denote the Euler characteristic of a Seifert surface with genus $g(L)$.

Exercise 7. *Show that $\chi(F) \leq \chi(L)$ for any Seifert surface F of L (Hint: Use the minimality of $g(L)$).*

As all the strands in a braid diagram are oriented the same way (either upwards or downwards) the braid index becomes the number of Seifert circles. By Exercise 7 we immediately obtain an upper bound on $-\chi(L)$ as follows:

$$c(\beta) - b(\beta) \geq -\chi(\widehat{\beta})$$

On the other hand, a powerful lower bound is given on $-\chi(L)$ in terms of diagrammatic properties of any braid β with $\widehat{\beta} = L$. This is the celebrated Bennequin inequality.

Theorem 2.2 (Bennequin inequality [6]). *Let β be a braid with closure $\widehat{\beta}$. Then*

$$w(\beta) - b(\beta) \leq -\chi(\widehat{\beta}).$$

Another way to construct a surface bounded by D is by coloring the plane in the checkerboard fashion. This yields two distinct surfaces – the white surface S_W and the black surface S_B – bounded by D. However, it is possible that both of these surfaces can be non-orientable. See Figure 10.

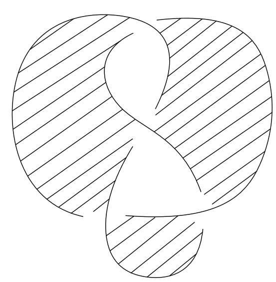


FIGURE 10. The checkerboard surfaces for the Figure-8 knot

Exercise 8. *Show that if D is a knot diagram with odd crossing number $c(D)$, then either S_B or S_W is non-orientable. (Hint: Use Euler characteristic.)*

In fact, for any knot diagram D if $c(D) \geq 1$, then one of the checkerboard surfaces is always non-orientable.

2.4. Surfaces in 4-dimensions bounded by links

Let us now consider surfaces properly embedded in $\mathbb{D}^4$ that bound a link in S^3. These are called the slice surfaces of L. We define the *4-genus* (or the *slice genus*) of L to be

$$g_4(L) := \min\{\, g(F) \mid F \hookrightarrow \mathbb{D}^4 \text{ such that } \partial F = L \text{ and } F \text{ is orientable and connected}\},$$

and $\chi_4(L)$ to be the Euler characteristic of a slice surface of L with genus $g_4(L)$.

It is evident that the 4-genus is at most the Seifert genus, as one can push any Seifert surface of L into $\mathbb{D}^4$ and get a properly embedded surface in $\mathbb{D}^4$ bounding L. Thus, $g_4(L) \leq g(L)$ and hence $\chi_4(L) \geq \chi(L)$.

One can improve the Bennequin inequality by replacing $\chi(L)$ with $\chi_4(L)$, yielding the *slice-Bennequin inequality.*

Theorem 2.3 (The slice-Bennequin inequality)**.** *Let β be a braid with closure $\widehat{\beta}$. Then*

$$w(\beta) - b(\beta) \leq -\chi_4(\widehat{\beta}).$$

In Subsection 7.1 we will outline a proof of Theorem 2.3 for knots using Khovanov homology (the original proof – which works not only for knots but links with any number of components – is due to Rudolph [7]).

Definition 2.8. *A cobordism between links L_0 and L_1 is a properly embedded and oriented surface Σ with boundary in $S^3 \times [0,1]$ such that $\Sigma \cap (S^3 \times 0) = L_0$, and $\Sigma \cap (S^3 \times 1) = L_1$.*

Link cobordisms are generally described combinatorially by movie presentations.

Definition 2.9. *A movie presentation of a link cobordism is a finite sequence of link diagrams $D_0 = D^0, D^1, D^2, \ldots, D^m = D_1$ such that to get from D^i to D^{i+1} either a Reidemeister move, an isotopy or a Morse move is performed. The local pictures of the Morse moves are given in Figure 11.*

A cobordism of genus-0 between knots K_0 and K_1 is called a *concordance* and we write $K_1 \sim K_2$ if they are concordant.

Exercise 9. *Show that the concordance is an equivalence relation on the set of knots.*

Definition 2.10. *A knot K is slice if $g_4(K) = 0$ or, equivalently $K \sim U$.*

The connected sum $K\# -K$ is always slice for any knot K. Figure 12 demonstrates a movie of a concordance from $3_1\# -3_1$ to U. This proves that $3_1\# -3_1$ is slice, in fact ribbon as the movie in Figure 12 consists of only 1- and 2-handles.

Theorem 2.4. *The set of concordance classes of knots, denoted by $Conc(S^3)$, forms an abelian group with its operation induced by connected sum. Moreover, the equivalence class of the unknot consists of the slice knots and represents the identity.*

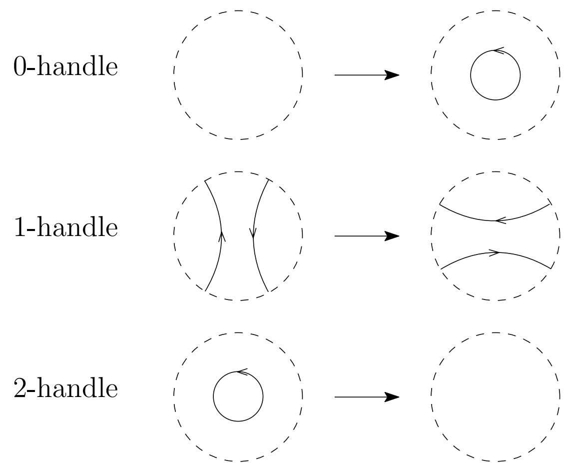


FIGURE 11. Morse moves

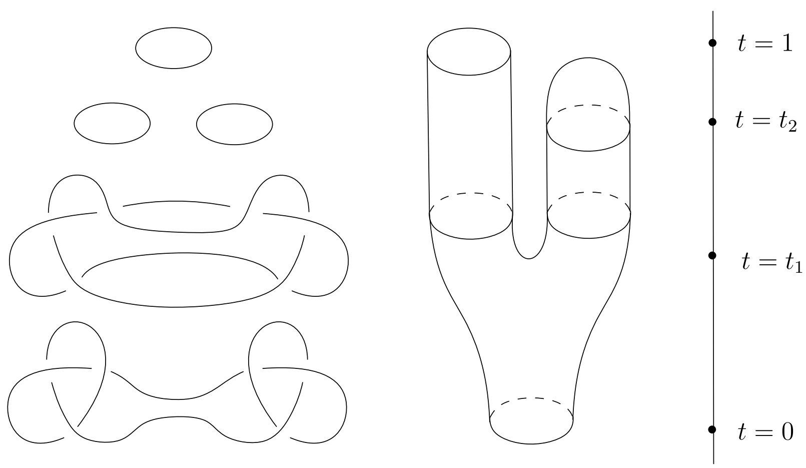


FIGURE 12. A movie presenting the square knot $3_1\# - 3_1$ is slice

We finish off this section with the *the unknotting number* –an invariant defined by using diagrams yet tied to 4-dimensional invariants.

Definition 2.11. *The unknotting number of a knot K, denoted by $u(K)$, is the minimum number of crossing changes required to convert a diagram of K into a diagram of the unknot, taken over all possible diagrams of K.*

Exercise 10. *Show that $g_4(K) \leq u(K)$.*

3. Lecture 2: Introduction to Khovanov homology

Main references we follow for this lecture are [1] and [8]. Khovanov homology is a machinery which assigns to each link diagram D a bi-graded chain complex $(\mathrm{CKh}^{i,j}(D), d)$, and hence an abelian group $\mathrm{Kh}^{i,j}(D)$ for each (i,j) by employing the usual homology argument

$$\{... \to \mathrm{CKh}^{i,j}(D) \xrightarrow{d^i} \mathrm{CKh}^{i+1,j}(D) \to ...\}.$$

It is clear from the writing that i is the homology grading. On the other hand, j is called the *quantum* grading, or q-grading for short.

We introduce the machinery with an example. Let L be an oriented link diagram and D be a diagram of L in $\mathbb{R}^2$ with n-crossings.

Step 1. We associate an n-dimensional cube $\{0,1\}^n$ to D. Take any order on the crossings of D and call it $(c_1, \dots, c_n)$. Resolve each crossing c_i of D in one of two possible ways.

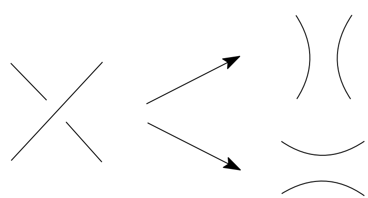


FIGURE 13. Two possible crossing resolutions

There are 2^n possible resolutions hence all possible resolutions gives an n-dimensional *cube* diagram with 2^n vertices. Each vertex corresponds to a resolution consisting of disjoint union of circles. Note we have yet to make use of the orientation.

Figure 14 demonstrates the cube of resolutions of the (negative) Hopf link, where 0-resolutions are labeled by red arcs and 1-resolutions are labeled by blue arcs.

Step 2. Assign a graded abelian group to each vertex α of the cube of resolutions

$$\alpha \in \{0,1\}^n \mapsto V_\alpha = V^{\otimes k}$$

where k is the number of connected components in the resolution α, and $V = \mathbb{Q}x \oplus \mathbb{Q}\mathbb{1}$ is a graded vector space over $\mathbb{Q}$ of rank 2 with $\deg x = -1$ and $\deg \mathbb{1} = 1$. See Figure 15.

Step 3. We define the boundary maps of a complex as follows: Put a directed edge from vertex α_i to vertex α_j if α_i and α_j have the same resolution, except at a single crossing, where α_i has a 0-resolution, and α_j has a 1-resolution. See Figure 15 for an illustrative example.

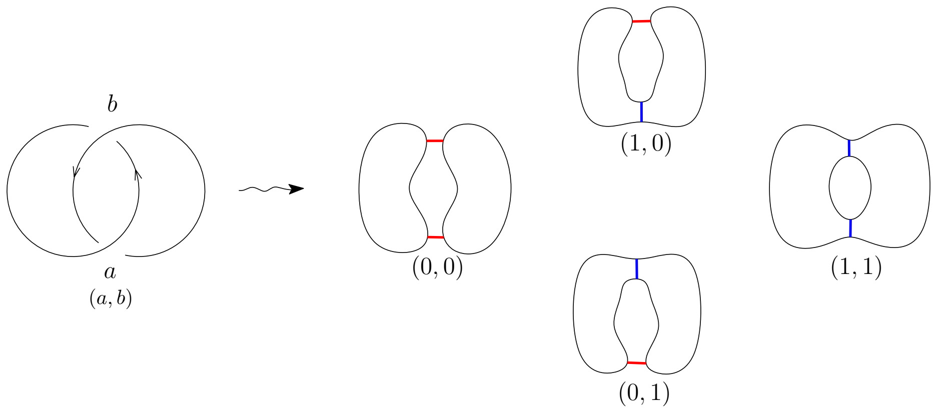


Figure 14. Cube of resolutions for the Hopf link

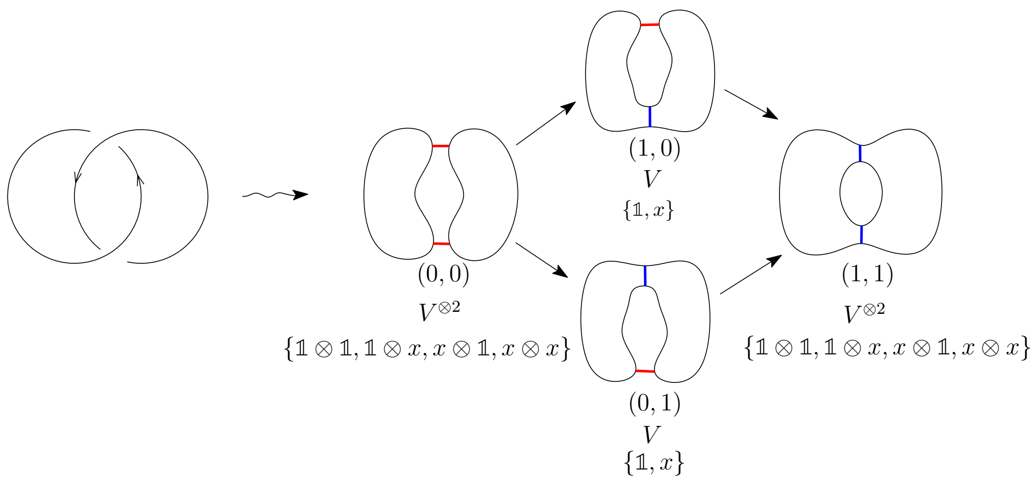


Figure 15. Chain groups and boundary maps

Note that each vertex of the cube consists of a disjoint union of circles, and the directed arrows (edges) between vertices correspond to Morse one moves, i.e. cobordisms: Either merges two circles into one or splits one into two. We associate morphisms to each cobordism, and these maps are given in Figure 16.

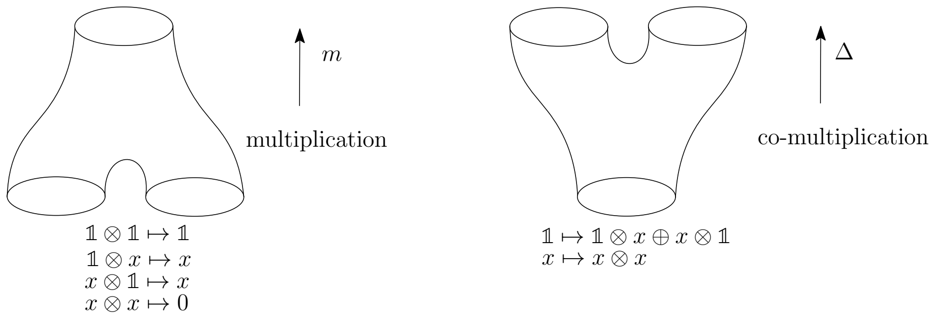


FIGURE 16. Boundary maps

The co-boundary maps between chain groups are induced from m and Δ. Before we give the formal definitions we discuss the gradings.

Step 4. In this step, we will focus on establishing gradings and other relevant concepts. Let $W = \underset{m}{\oplus} W_m$ be a graded abelian group with homogeneous components W_m. The degree-shift operation is defined as $W\{l\} := \underset{m}{\oplus} W\{l\}_m$ where $W\{l\}_m = W_{m-l}$.

Similarly on a chain complex $C = \{\cdots \to C^r \to C^{r+1} \to \dots\}$ the height-shift operation is defined as $C[s] := \{\cdots \to C[s]^r \to C[s]^{r+1} \to \dots\}$ where $C[s]^r := C^{r-s}$.

Note that the degree shift operation $\{\ \}$ is a shift in q-grading, and the height shift $[\ ]$ is a shift in homological grading.

Let $V = \underset{\alpha\in\{0,1\}^n}{\oplus} V_\alpha$ be the graded abelian group associated to the diagram D, define

$$\begin{aligned}
r_\alpha &:= \text{ number of 1-resolutions in } \alpha \text{ (the height of } \alpha) \\
n_- &:= \text{ number of negative crossings in } D \\
n_+ &:= \text{ number of positive crossings in } D.
\end{aligned}$$

We define the bi-graded chain groups now. Define

$$\overline{C}^i(D) := \underset{r_\alpha = i}{\oplus} V_\alpha\{i\}$$

and the subgroup of $\overline{C}^i(D)$ consisting of elements of degree $j-i$ is denoted by $\overline{C}^{i,j}(D)$. Bi-graded Khovanov chain groups denoted by $\mathrm{CKh}^{i,j}(D)$, are defined as follows

$$\mathrm{CKh}(D) := \overline{C}[-n_-]\{n_+ - 2n_-\}$$
$$\text{i.e. } \mathrm{CKh}^{i,j}(D) = \overline{C}^{i+n_-, j-n_++2n_-}(D).$$

Hence, for a given element $v \in \mathrm{CKh}^{i,j}(D)$, we can express its homological grading i and q-grading j as

$$i = r_\alpha - n_-$$
$$j = \deg(v) + i + n_+ - n_-.$$

In Figure 17 the (i,j) grading of each element is labeled by a red pair below the generators. The vertical sums over the same j degrees gives bi-graded Khovanov chain groups.

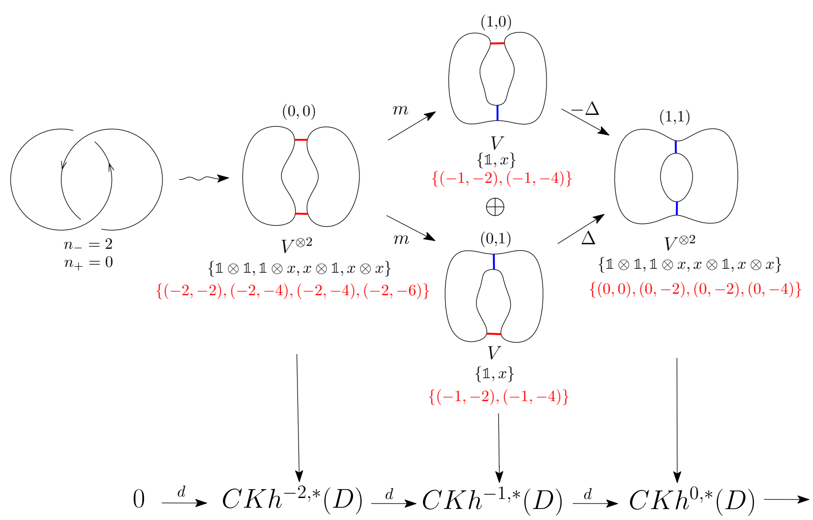


FIGURE 17. The bi-graded Khovanov chain groups, generators and bi-gradings

The following step will cover the formal definition of the boundary maps.

Step 5. It was already mentioned that the boundary maps are induced from the cobordisms between the vertices where associated maps are defined over the generators in Figure 15 by m and Δ. The only missing part is writing the boundary map precisely with the sign convention. We use the notation $\alpha < \beta$ if there is a directed arrow from the vertex α to the vertex β, and $d_{\alpha\beta}$ for the associated map (m or Δ) from α to β

$$\alpha = (..., * = 0, ...) \xrightarrow{d_{\alpha\beta}} \beta = (..., * = 1, ...).$$

Then the boundary map is defined as

$$\begin{aligned} d^i : & \mathrm{CKh}^{i,j}(D) \to \mathrm{CKh}^{i+1,j}(D) \\ d(v) = & \sum_{\substack{\alpha<\beta \\ \beta\in\{0,1\}^n}} (-1)^{n_{\alpha\beta}} d_{\alpha\beta}(v) \end{aligned}$$

where $v \in V_\alpha$, and $n_{\alpha\beta}$ is the number of 1-resolutions in α before the slot "$*$".

Exercise 11. *Show that d increases the homology grading i by 1, and preserves the j grading in Khovanov chain groups* $\mathrm{CKh}(D)$.

Before we see d is indeed a chain map we finish the calculation of the Hopf link. We continue from Figure 17.

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{CKh}^{-2,-6}(D) & \to & \mathrm{CKh}^{-1,-6}(D) & \to & \mathrm{CKh}^{0,-6}(D) & \to & 0 \\
0 & \to & \langle x\otimes x\rangle & \to & 0 & \to & 0 & \to & 0 \\
\\
0 & \to & \mathrm{CKh}^{-2,-4}(D) & \to & \mathrm{CKh}^{-1,-4}(D) & \to & \mathrm{CKh}^{0,-4}(D) & \to & 0 \\
0 & \to & \langle \mathbb{1}\otimes x, x\otimes \mathbb{1}\rangle & \to & \langle (x,0),(0,x)\rangle & \to & \langle x\otimes x\rangle & \to & 0 \\
\\
0 & \to & \mathrm{CKh}^{-2,-2}(D) & \to & \mathrm{CKh}^{-1,-2}(D) & \to & \mathrm{CKh}^{0,-2}(D) & \to & 0 \\
0 & \to & \langle \mathbb{1}\otimes \mathbb{1}\rangle & \to & \langle (\mathbb{1},0),(0,\mathbb{1})\rangle & \to & \langle \mathbb{1}\otimes x, x\otimes \mathbb{1}\rangle & \to & 0 \\
\\
0 & \to & \mathrm{CKh}^{-2,0}(D) & \to & \mathrm{CKh}^{-1,0}(D) & \to & \mathrm{CKh}^{0,0}(D) & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & \langle \mathbb{1}\otimes \mathbb{1}\rangle & \to & 0
\end{array}$$

A simple calculation gives the Khovanov homology groups $\mathrm{Kh}^{i,j}(D)$ as in Table 1.

$j \diagdown i$	0	−1	−2
0	$\mathbb{Q}$	0	0
−2	$\mathbb{Q}$	0	0
−4	0	0	$\mathbb{Q}$
−6	0	0	$\mathbb{Q}$

TABLE 1. The Khovanov homology of the Hopf link. The homological grading i is on the horizontal axis, and the quantum grading j is on the vertical axis. The entry corresponding to column i and row j is $\mathrm{Kh}^{i,j}(D)$.

To see $(\mathrm{CKh}^{i,j}(D), d)$ is a chain complex we need the following lemma.

Lemma 3.1. *Let* a, b, $c \in V = \mathbb{Q}x \oplus \mathbb{Q}\mathbb{1}$.

(i) *m is commutative and associative:*

$$m(a \otimes b) = m(b \otimes a), \text{ and } m(m(a \otimes b) \otimes c) = m(a \otimes m(b \otimes c))$$

(ii) *Δ is co-commutative and co-associative:*

$$\sigma(\Delta(c)) = \Delta(c), \text{ and } (\Delta \otimes id)(\Delta(c)) = (id \otimes \Delta)(\Delta(c)) \text{ where } \sigma(a \otimes b) = b \otimes a$$

(iii) $\Delta(m(a \otimes b))) = (m \otimes id)((id \otimes \Delta)(a \otimes b))$

Proof. It is enough to check these properties for the generators. We will only write it explicitly for part (i), and leave the rest to the reader.

(i) Commutativity follows from the definition of m,

$$m(x \otimes \mathbb{1}) = x = m(\mathbb{1} \otimes x).$$

Similarly, the following table shows the associativity.

$a \otimes b \otimes c$	$m(m(a \otimes b) \otimes c)$	$m(a \otimes m(b \otimes c))$
$\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$
$\mathbb{1} \otimes x \otimes \mathbb{1}$	x	x
$\mathbb{1} \otimes \mathbb{1} \otimes x$	x	x
$x \otimes \mathbb{1} \otimes \mathbb{1}$	x	x
$x \otimes x \otimes \mathbb{1}$	0	0
$x \otimes \mathbb{1} \otimes x$	0	0
$\mathbb{1} \otimes x \otimes x$	0	0
$x \otimes x \otimes x$	0	0

□

Corollary 3.2. $(\mathrm{CKh}^{i,j}(D), d)$ *is a chain complex, i.e.* $d^2 = 0$.

Proof. For the notations we follow Figure 18. By Lemma 3.1 every face of the cube diagram commutes. That gives $d^2(v) = 0$ for any generator $v \in V_\alpha$ in $\mathbb{Z}/2\mathbb{Z}$ coefficients. Now consider the sign where $n_{\alpha\beta}$ as before.

$$\begin{aligned}
d^2(v) &= d((-1)^{n_{\alpha\beta_1}} d_{\alpha\beta_1}(v) + (-1)^{n_{\alpha\beta_2}} d_{\alpha\beta_2}(v)) \\
&= (-1)^{n_{\beta_1\gamma}}(-1)^{n_{\alpha\beta_1}} d_{\beta_1\gamma}(d_{\alpha\beta_1}(v)) + (-1)^{n_{\beta_2\gamma}}(-1)^{n_{\alpha\beta_2}} d_{\beta_2\gamma}(d_{\alpha\beta_2}(v)) \\
&= (-1)^{n_{\alpha\beta_2}+1}(-1)^{n_{\alpha\beta_1}} d_{\beta_1\gamma}(d_{\alpha\beta_1}(v)) + (-1)^{n_{\alpha\beta_2}}(-1)^{n_{\alpha\beta_1}} d_{\beta_2\gamma}(d_{\alpha\beta_2}(v)) \\
&= (-1)^{n_{\alpha\beta_2}}(-1)^{n_{\alpha\beta_1}}(-d_{\beta_1\gamma}(d_{\alpha\beta_1}(v)) + d_{\beta_2\gamma}(d_{\alpha\beta_2}(v))) = 0
\end{aligned}$$

In the third step we use $n_{\beta_2\gamma} = n_{\alpha\beta_1}$, and $n_{\beta_1\gamma} = n_{\alpha\beta_2} + 1$ which is clear from Figure 18. In the last step commutativity of the faces is used.

□

$$
\begin{tikzcd}
 & \beta_1 = (*1 * 0*) \arrow[dr, "d_{\beta_1\gamma}"] & \\
\alpha = (*0 * 0*) \arrow[ur, "d_{\alpha\beta_1}"] \arrow[dr, "d_{\alpha\beta_2}"'] & & \gamma = (*1 * 1*) \\
 & \beta_2 = (*0 * 1*) \arrow[ur, "d_{\beta_2\gamma}"'] &
\end{tikzcd}
$$

$$d_{\beta_1\gamma}\, d_{\alpha\beta_1} = d_{\beta_2\gamma}\, d_{\alpha\beta_2}$$

FIGURE 18. A face of the cube

3.1. Link invariance

In the proof of link invariance, the *mapping cone* construction is used repeatedly. For the sake of being self-contained, we will remind the reader the definition, and some basic facts related to it without proofs.

Definition 3.1. *Let $\mathcal{A} = (A_n, d_A^n)$ and $\mathcal{B} = (B_n, d_B^n)$ be cochain complexes and $f : \mathcal{A} \to \mathcal{B}$ be a chain map, i.e. family of homomorphisms $f_n = A^n \to B^n$ such that the following diagram commutes*

$$
\begin{tikzcd}
A^n \arrow[r, "d_A^n"] \arrow[d, "f_n"'] & A^{n+1} \arrow[d, "f_{n+1}"] \\
B^n \arrow[r, "d_B^n"'] & B^{n+1}
\end{tikzcd}
\qquad\qquad d_B^n f_n = f_{n+1} d_A^n.
$$

The mapping cone of f, cone(f) is a cochain complex $\mathcal{C} = (C^n, d_C^n)$ where

$$C^n := A^{n+1} \oplus B^n = A[-1]^n \oplus B^n, \qquad d_C^n := \begin{pmatrix} d_A & 0 \\ f & -d_B \end{pmatrix}.$$

Indeed $d_C^2 = \begin{pmatrix} d_A^2 & 0 \\ f d_A - d_B f & d_B^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Lemma 3.3. *The mapping cone construction gives the following short exact sequence of cochain complexes*

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{B} & \xrightarrow[inc]{i} & cone(f) & \xrightarrow[proj]{p} & \mathcal{A}[-1] & \longrightarrow 0 \\
 & & & \mathcal{A}[-1] \oplus \mathcal{B} & & & \\
 & b & \mapsto & (0,b) & & & \\
 & & & (a,b) & \mapsto & a &
\end{array}$$

and this induces a long exact sequence of cohomology groups

$$\cdots \to H^n(\mathcal{B}) \xrightarrow{i^*} H^n(\mathcal{C}) \xrightarrow{p^*} H^n(\mathcal{A}[-1])) \xrightarrow{\delta} H^{n+1}(\mathcal{B}) \to \ldots.$$

Lemma 3.4. *Let $f : \mathcal{A} \to \mathcal{B}$ be a cochain map between two cochain complexes. Then, $f^* : H^*(\mathcal{A}) \to H^*(\mathcal{B})$ is an isomorphism if and only if $(\mathcal{C}, d) = (cone(f), d)$ is acyclic, i.e. all homology groups vanish.*

Corollary 3.5. *Let $(\mathcal{C}, d)$ be a cochain complex and $(\mathcal{C}', d)$ be a subcomplex, that is, $C'^n \subset C^n$ and $d(C'^n) \subset C'^{n+1}$. Then*

(i) *If $\mathcal{C}'$ is acyclic, then $H^*(\mathcal{C}) \cong H^*(\mathcal{C}/\mathcal{C}')$*
(ii) *If $\mathcal{C}/\mathcal{C}'$ is acyclic, then $H^*(\mathcal{C}') \cong H^*(\mathcal{C})$*

Proof. The following short exact sequence

$$0 \longrightarrow \mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{p} \mathcal{C}/\mathcal{C}' \longrightarrow 0$$

gives the long exact sequence

$$\cdots \longrightarrow H^n(\mathcal{C}') \xrightarrow{i^*} H^n(\mathcal{C}) \xrightarrow{p^*} H^n(\mathcal{C}/\mathcal{C}') \xrightarrow{\delta} H^{n+1}(\mathcal{C}') \longrightarrow \ldots.$$

Then,

(i) If $\mathcal{C}'$ is acyclic then $H^n(C') = 0$ for all n, hence p^* gives the desired isomorphism.
(ii) Similarly if $\mathcal{C}/\mathcal{C}'$ is acyclic then $H^n(\mathcal{C}/\mathcal{C}') = 0$ hence i^* is an isomorphism.

□

Now we can state the main theorem of the section.

Theorem 3.6. $\mathrm{Kh}^{i,j}(D)$ *remains unchanged under Reidemeister moves.*

Proof. We will not write the complete proof of Theorem 3.6 but rather give the idea of the proof for the R1 moves. It is worth it to mention here that the shifts in the definition of Khovanov chain complex $[-n_-]\{n_+ - 2n_-\}$ are necessary in the proof.
Case 1.
To show the invariance for the left handed R1 move consider Figure 19. The diagrams D, D_0, and D_1 are understood to be identical outside of the crossing region. We can

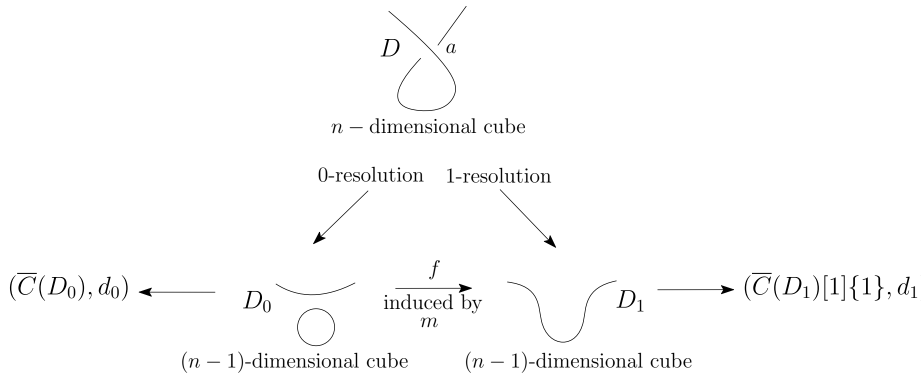


FIGURE 19. Left handed R1 move

think of D_0 and D_1 as independent diagrams, and let $f : \overline{C}^{i,j}(D_0) \to \overline{C}^{i+1,j}(D_1)[1]\{1\}$ be a map induced by the boundary map d of the cochain complex $(\overline{C}(D), d)$.

$$\begin{tikzcd}
\dots \arrow[r] & \overline{C}^{i,j}(D_0) \arrow[r, "d_0"] \arrow[dr, "f"] & \overline{C}^{i+1,j}(D_0) \arrow[r, "d_0"] \arrow[dr, "f"] & \overline{C}^{i+2,j}(D_0) \arrow[r, "d_0"] & \dots \\
 & \oplus & \oplus & \oplus & \\
\dots \arrow[r] & \overline{C}^{i,j}(D_1)[1]\{1\} \arrow[r, "d_1"] & \overline{C}^{i+1,j}(D_1)[1]\{1\} \arrow[r, "d_1"] & \overline{C}^{i+2,j}(D_1)[1]\{1\} \arrow[r, "d_1"] & \dots
\end{tikzcd}$$

We know that f commutes with m and Δ. We only need to check the signs to see it is a chain map. Take a monomial $v \in \overline{C}^{i,j}(D)$ with resolution $0 * \cdots * 0 * \cdots *$ and label the resolution corresponding to the a-crossing with red. Then the following diagram checks the sign where k is the number of 1-resolutions before the slot at which the Morse move is applied. Notice that d_1 doesn't see the red labeled 1-resolution.

$$\begin{tikzcd}
0 * \cdots * 0 * \cdots * \arrow[r, "d_0"] \arrow[d, "f"'] & (-1)^k 0 * \cdots * 1 * \cdots * \arrow[d, "f"] \\
1 * \cdots * 0 * \cdots * \arrow[r, "d_1"'] & (-1)^k 1 * \cdots * 1 * \cdots *
\end{tikzcd} \qquad fd_0 = d_1 f$$

This construction allows us to write

$$\begin{array}{ccccc}
\overline{C}(D) & = & \overline{C}(D_0) & \oplus & \overline{C}(D_1)[1]\{1\} \\
 & & & \uplus & \\
 & & (x & , & y)
\end{array}$$

$$d(x,y) = (d_0(x), f(x) - d_1(y)),$$

$$\mathcal{C} = (\overline{C}(D), d) = (cone(f), d) = \left(\overline{C}(D_0) \oplus \overline{C}(D_1)[1]\{1\}, \begin{pmatrix} d_0 & 0 \\ f & -d_1 \end{pmatrix}\right).$$

One can write $\overline{C}(D_0) = \overline{C}(D') \otimes V$ where the component V corresponds to the disjoint circle in the diagram D_0 in Figure 19. By marking the disjoint circle by $\mathbb{1}$ we obtain a subcomplex $\overline{C}(D') \otimes \langle \mathbb{1} \rangle$ of $\overline{C}(D_0) = \overline{C}(D') \otimes V$ where $\langle \mathbb{1} \rangle$ represents the subspace of V generated by $\mathbb{1}$. Similarly, we can write the subcomplex $\mathcal{C}'$ as:

$$\begin{array}{ccccc} \mathcal{C} & = & cone(f) & = & cone((\overline{C}(D') \otimes V, d_0) \xrightarrow{m} (\overline{C}(D_1)[1]\{1\}, d_1)) \\ \mathcal{C}' & = & cone(f') & = & cone((\overline{C}(D') \otimes \langle \mathbb{1} \rangle, d_0) \xrightarrow{m} (\overline{C}(D_1)[1]\{1\}, d_1)) \end{array}.$$

Remember that f' is induced from m, and

$$\begin{array}{ccc} * \otimes x \otimes \mathbb{1} & \overset{m}{\mapsto} & * \otimes x \\ * \otimes \mathbb{1} \otimes \mathbb{1} & \overset{m}{\mapsto} & * \otimes \mathbb{1} \end{array}$$

then $f'(v \otimes \mathbb{1}) = v$ for all $v \in \overline{C}(D') \otimes \langle \mathbb{1} \rangle$, hence f' is an isomorphism. By Lemma 3.4, $\mathcal{C}'$ is acyclic and so Corollary 3.5 implies

$$H^*(\mathcal{C}/\mathcal{C}') \cong H^*(\mathcal{C}).$$

It is an exercise for the reader to check that

$$\mathcal{C}/\mathcal{C}' = cone((\overline{C}(D') \otimes V/\langle \mathbb{1} \rangle, d_0) \to 0) \cong (\overline{C}(D')\{-1\}, d_0).$$

The last shift which is a shift in q-grading together with the shifting in Khovanov chain complexes $[-n_-]\{n_+ - 2n_-\}$ gives the result

$$\mathrm{Kh}^{i,j}(D) \cong Kh^{i,j}(D').$$

To see this notice that number of positive crossings (say $(n_+ - 1)$) in D' is one less than the number of positive crossings (say n_+) in D, and number of negative crossings are the same for both D and D'. Using the grading definitions from Step 4 we can compare the gradings in the chain level:

$$\begin{aligned} \mathrm{CKh}^{i,j}(D) &= \overline{C}^{i+n_-, j-n_++2n_-}(D) \text{ and,} \\ \mathrm{CKh}^{i,j}(D') &= \overline{C}^{i+n_-, j-n_++1+2n_-}(D') = \overline{C}^{i+n_-, j-n_++2n_-}(D')\{-1\} \end{aligned}$$

which finishes the proof.

Case 2.

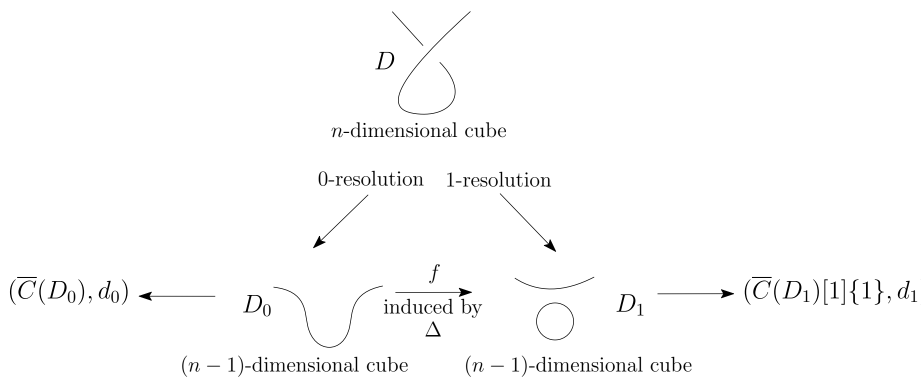


FIGURE 20. Right handed R1 move

Using a similar convention $\overline{C}(D_1)[1]\{1\} \cong (\overline{C}(D') \otimes V)[1]\{1\}$ we have

$$\begin{array}{ccccccc} \mathcal{C} & = & cone(f) & = & cone(& (\overline{C}(D_0), d_0) & \xrightarrow{\Delta} & ((\overline{C}(D') \otimes V)[1]\{1\}, d_1)) \\ \mathcal{C}' & = & cone(f') & = & cone(& 0 & \xrightarrow{\Delta} & ((\overline{C}(D') \otimes \langle \mathbb{1} \rangle)[1]\{1\}, d_1)) \end{array}$$

which gives

$$\mathcal{C}/\mathcal{C}' = cone(\overline{C}(D_0) \to (\overline{C}(D') \otimes V/\langle \mathbb{1} \rangle)[1]\{1\}).$$

$\mathcal{C}/\mathcal{C}'$ is acyclic since $[\Delta(v)] = [v \otimes x]$ for all $v \in \overline{C}(D_0)$. By Lemma 3.4 we have $H^*(\mathcal{C}) \cong H^*(\mathcal{C}')$. Combining with $\mathcal{C} = (\overline{C}(D), d)$ and $\mathcal{C}' = ((\overline{C}(D') \otimes \langle \mathbb{1} \rangle)[1]\{1\}, d_1) \cong (\overline{C}(D')[1]\{2\}, d_1)$ we get the result.

Similar to the argument given in Case 1, the shifts $[1]\{2\}$ gives the correct grading when we pass to the Khovanov chain complexes. Consider the shifts $[-n_-]\{n_+ - 2n_-\}$ in the definition of Khovanov chain complexes, and notice that number of positive crossings are the same for both diagrams D and D' while the number of negative crossings is one less in diagram D' than D. Therefore, we have

$$\begin{aligned} \mathrm{CKh}^{i,j}(D) &= \overline{C}^{i+n_-, j-n_++2n_-}(D), \text{ and} \\ \mathrm{CKh}^{i,j}(D') &= \overline{C}^{i+n_--1, j-n_++2n_--2}(D') = \overline{C}^{i+n_-, j-n_++2n_-}(D')[1]\{2\} \end{aligned}$$

which gives $\mathrm{Kh}^{i,j}(D) \cong \mathrm{Kh}^{i,j}(D')$. □

3.2. Long exact sequence

Suppose the diagrams D, D_0 and D_1 are identical outside of the region depicted below.

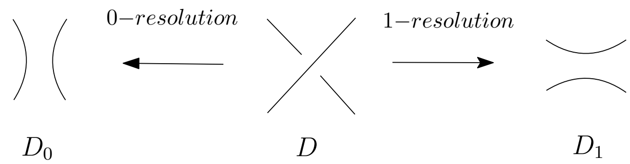


FIGURE 21.

As a vector space (ignoring the gradings) $\mathrm{CKh}(D)$ decomposes as

$$\mathrm{CKh}(D) = \mathrm{CKh}(D_0) \oplus \mathrm{CKh}(D_1).$$

With the correct grading shifts $(\mathrm{CKh}(D_1), d_1)$ can be seen as a subcomplex of $(\mathrm{CKh}(D), d)$ which gives rise to a short exact sequence

$$0 \to \mathrm{CKh}(D_1) \to \mathrm{CKh}(D) \to \mathrm{CKh}(D_0) \to 0.$$

Next we will investigate the gradings. Start with a generator $v \in V_\alpha$ where α represents a vertex in the cube of resolutions of D. Let $v \in \mathrm{CKh}^{i,j}(D)$, so we have

$$\begin{aligned} i &= r_\alpha - n_- \\ j &= \deg(v) + i + n_+ - n_-. \end{aligned}$$

If the crossing of D in Figure 21 is negative, then D_1 inherits an orientation from D. Hence we know the number of negative and positive crossings of D_1 in terms of the number of negative and positive crossings of D. On the other hand, we do not have any control on the orientation of D_0, so we define the quantity k as

$$k = \#\text{negative crossings of } D_0 - \#\text{negative crossings of } D.$$

Using Figure 22 we have either $v \in \mathrm{CKh}^{i-k,j-3k-1}(D_0)$ or $v \in \mathrm{CKh}^{i,j+1}(D_1)$ depending on the resolution of α at the specified crossing in Figure 21. Therefore, we have the short exact sequence

$$0 \to \mathrm{CKh}^{i,j+1}(D_1) \to \mathrm{CKh}^{i,j}(D) \to \mathrm{CKh}^{i-k,j-3k-1}(D_0) \to 0$$

induces

$$\cdots \to \mathrm{Kh}^{i,j+1}(D_1) \to \mathrm{Kh}^{i,j}(D) \to \mathrm{Kh}^{i-k,j-3k-1}(D_0) \xrightarrow{\delta} \mathrm{Kh}^{i+1,j+1}(D_1) \to \dots$$

where the connecting homomorphism is induced by the boundary map of $(\mathrm{CKh}(D), d)$.

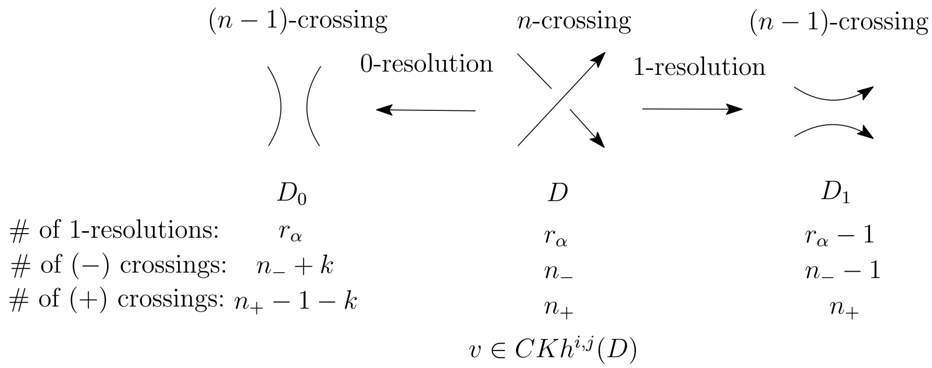


Figure 22.

Similarly, if the crossing of D is positive as in Figure 23, then we have either $v \in \mathrm{CKh}^{i,j-1}(D_0)$ or $v \in \mathrm{CKh}^{i-k-1,j-3k-2}(D_1)$ where

$$k = \#\text{negative crossings of } D_1 - \#\text{negative crossings of } D.$$

Therefore, we have a short exact sequence

$$0 \to \mathrm{CKh}^{i-k-1,j-3k-2}(D_1) \to \mathrm{CKh}^{i,j}(D) \to \mathrm{CKh}^{i,j-1}(D_0) \to 0$$

which induces

$$\cdots \to \mathrm{Kh}^{i-k-1,j-3k-2}(D_1) \to \mathrm{Kh}^{i,j}(D) \to \mathrm{Kh}^{i,j-1}(D_0) \xrightarrow{\delta} \mathrm{Kh}^{i-k,j-3k-2}(D_1) \to \cdots.$$

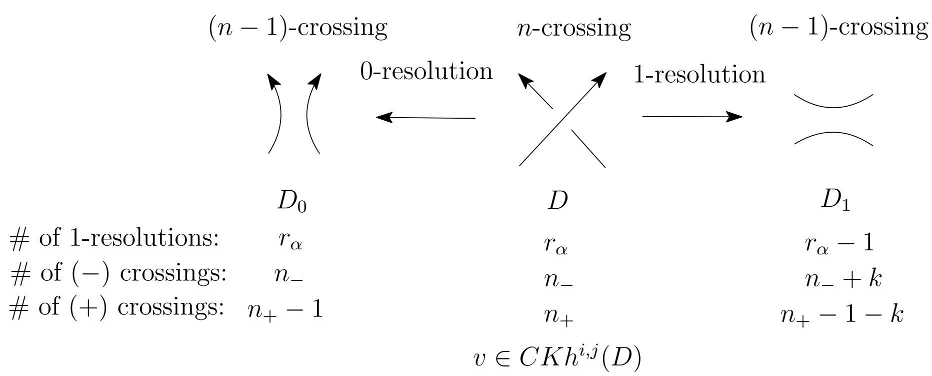


Figure 23.

Exercise 12. *Use the long exact sequence associated to a crossing of the Hopf link in Figure 14 to recompute its Khovanov homology. (Hint: For either choice of the crossing, D_0 and D_1 are the unknot.)*

Using the Khovanov homology of the Hopf link, we can now easily compute the Khovanov homology of the trefoil.

Exercise 13. *Use the long exact sequence associated to a crossing of -3_1 in Figure 6 to compute* $\mathrm{Kh}(-3_1)$*, and show* $\mathrm{Kh}(-3_1)$ *is isomorphic to:*

$j \diagdown i$	-2	-1	0
-1	0	0	$\mathbb{Q}$
-3	0	$\mathbb{Q}$	$\mathbb{Q}$
-5	0	0	0
-7	$\mathbb{Q}$	0	0

(Hint: For either choice of the crossing, D_0 is the unknot and D_1 is the Hopf link.)

4. Lecture 3: Introduction to spectral sequences

For this lecture we follow [9] and [10].

4.1. Bi-complex and filtration

Definition 4.1. *A* bi-complex *(or a double complex) is an ordered triple $(\mathcal{C}, d', d'')$ where $(\mathcal{C}, d')$ and $(\mathcal{C}, d'')$ are (co)chain complexes of bi-graded modules so that d' and d'' anti commute*

$$d'd'' + d''d' = 0.$$

For a given bi-complex $(\mathcal{C}, d', d'')$ we have a lattice where lattice points are bi-graded chain groups, horizontal lines form chain complexes using d', and vertical lines form chain complexes using d''.

Definition 4.2. Total complex $(\mathrm{Tot}(\mathcal{C}), D)$ *of a bi-graded bi-complex $(\mathcal{C}, d', d'')$ is a chain complex where the chain groups are the sum of diagonal entries of the lattice, and the boundary map is given by*

$$D = d' + d''.$$

Example 4.1. Let $(\mathrm{CKh}(L), d)$ be the Khovanov chain complex of a link L, and $(\mathrm{CKh}(L), \Phi)$ be another chain complex obtained using a degree $(1, 4)$ chain map Φ. Then the chain groups of the total complex $(\mathcal{C}, D) = (\mathrm{Tot}(\mathrm{CKh}), d + \Phi)$ are direct sum of the diagonal entries as in the following diagram.

$$\begin{tikzcd}
 & \vdots \arrow[d, "\Phi"] & \vdots \arrow[d, "\Phi"] & \vdots \arrow[d, "\Phi"] & \\
\cdots \arrow[r] & C^{n-2,m-4} \arrow[r, "d"] \arrow[d, "\Phi"] & C^{n-1,m-4} \arrow[r, "d"] \arrow[d, "\Phi"] & C^{n,m-4} \arrow[r, "d"] \arrow[d, "\Phi"] \arrow[dl, dash, "\oplus"] & \cdots \\
\cdots \arrow[r] & C^{n-1,m} \arrow[r, "d"] \arrow[d, "\Phi"] & C^{n,m} \arrow[r, "d"] \arrow[d, "\Phi"] \arrow[dl, dash, "\oplus"] & C^{n+1,m} \arrow[r, "d"] \arrow[d, "\Phi"] & \cdots \\
\cdots \arrow[r] & C^{n,m+4} \arrow[r, "d"] \arrow[d, "\Phi"] & C^{n+1,m+4} \arrow[r, "d"] \arrow[d, "\Phi"] & C^{n+2,m+4} \arrow[r, "d"] \arrow[d, "\Phi"] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Definition 4.3. *A* filtration *of a chain complex* $(\mathcal{C}, d)$ *is a sequence of graded submodules* $(F^p\mathcal{C})_{p\in\mathbb{Z}}$ *of* $\mathcal{C}$ *so that*

$$\cdots \subset F^{p+1}\mathcal{C} \subset F^p\mathcal{C} \subset F^{p-1}\mathcal{C} \subset \dots$$

and satisfies

$$d : (F^p\mathcal{C})^n \to (F^p\mathcal{C})^{n+1}$$

for all n, p*. To simplify notation, we denote* $(F^p\mathcal{C})^n$ *as* F^p_n*.*

Example 4.2. A filtration of $(\mathcal{C}, D) = (\mathrm{Tot}(\mathrm{CKh}(L), d + \Phi)$ can be given using the q-grading

$$(F^j(\mathcal{C}))^i = \bigoplus_{t\geq 0} \mathrm{CKh}^{i,j+4t}(L).$$

It is a finite filtration, meaning there exists k and k' so that

$$0 = F_i^{k'} \subset \cdots \subset F_i^{j+4} \subset F_i^j \subset F_i^{j-4} \subset \cdots \subset F_i^k = \bigoplus_{t\in\mathbb{Z}} \mathrm{CKh}^{i,j+4t}(L)$$

$$\begin{array}{rccc}
D = d + \Phi : & F_i^j & \to & F_{i+1}^j \\
 & \cup & & \cup \\
 & \alpha \in \mathrm{CKh}^{i,j}(L) & \mapsto & D(\alpha) \in \mathrm{CKh}^{i+1,j}(L) \oplus \mathrm{CKh}^{i+1,j+4}(L)
\end{array}$$

In the following diagram the vertical lines gives the filtration and the horizontal lines gives the cohomology of the corresponding complex.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & F_i^k & \xrightarrow{d+\Phi} & F_{i+1}^k & \xrightarrow{d+\Phi} & F_{i+2}^k & \longrightarrow & \cdots \\
 & & \cup & & \cup & & \cup & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \cup & & \cup & & \cup & & \\
\cdots & \longrightarrow & F_i^{k'} & \xrightarrow{d+\Phi} & F_{i+1}^{k'} & \xrightarrow{d+\Phi} & F_{i+2}^{k'} & \longrightarrow & \cdots
\end{array}$$

4.2. Exact couples and spectral sequences

Definition 4.4. *An* exact couple *of bi-graded modules D and E is an exact triangle, i.e.* $\beta \circ \alpha = 0,\ \gamma \circ \beta = 0,\ \alpha \circ \gamma = 0$

$$\begin{tikzcd} D \arrow[rr, "\alpha"] & & D \arrow[dl, "\beta"] \\ & E \arrow[ul, "\gamma"] & \end{tikzcd}$$

where α, β, γ are bi-graded maps with degrees (a, a'), (b, b'), and (c, c') respectively.

Define $d : E \to E$ by $d := \beta \circ \gamma$. Then

$$d^2 = \beta \circ (\gamma \circ \beta) \circ \gamma = \beta \circ 0 \circ \gamma = 0$$

so the homology of the exact couple is defined to be

$$E_1 := H(E, d) = (\ker\, d)/(\mathrm{im}\ d)$$

We can construct a new exact couple, called *derived couple* as follows

$$\begin{tikzcd} D_1 \arrow[rr, "\alpha_1"] & & D_1 \arrow[dl, "\beta_1"] \\ & E_1 \arrow[ul, "\gamma_1"] & \end{tikzcd}$$

where $D_1 := im(\alpha)$,

(1) $\alpha_1 : D_1 \to D_1;\ \alpha_1 = \alpha_{|D_1}$,
(2) $\beta_1 : D_1 \to E_1;\ \beta_1(x) = [\beta \circ \alpha^{-1}(x)]$, and
(3) $\gamma_1 : E_1 \to D_1;\ \gamma_1[e] = \gamma(e)$.

If $(a, a'),\ (b, b'),$ and (c, c') denote the bi-gradings of α, β, and γ, then the bi-gradings of α_1, β_1, and γ_1 are $(a, a'),\ (b - a, b' - a'),$ and (c, c') respectively.

Exercise 14. *Check that $\alpha_1, \beta_1,$ and γ_1 are well defined, and the derived couple is exact.*

By repeating this process, we obtain a sequence of exact couples

$$\begin{tikzcd} D_n \arrow[rr, "\alpha_n"] & & D_n \arrow[dl, "\beta_n"] \\ & E_n \arrow[ul, "\gamma_n"] & \end{tikzcd}.$$

Definition 4.5. *In this construction the sequence $E_n = H(E_{n-1}, d_{n-1})$ is called the* spectral sequence *of the exact couple.*

Proposition 4.1. *Every filtration $(F^p\mathcal{C})_{p\in\mathbb{Z}}$ of a (co)chain complex $(\mathcal{C}, d)$ determines an exact couple whose bi-graded maps look like*

$$\begin{tikzcd} D_1 \arrow[rr, "\alpha_1(1,-1)"] & & D_1 \arrow[dl, "\beta_1(0,0)"] \\ & E_1 \arrow[ul, "\gamma_1(0,1)"] & \end{tikzcd}$$

Proof. The short exact sequence of cochain complexes

$$0 \to F^{j+1} \xrightarrow{\iota} F^j \xrightarrow{p} F^j/F^{j+1} \to 0$$

induces a long exact sequence in cohomology

$$\cdots \to H^n(F^{j+1}) \xrightarrow{\iota^*} H^n(F^j) \xrightarrow{p^*} H^n(F^j/F^{j+1}) \xrightarrow{\delta} H^{n+1}(F^{j+1}) \to \ldots.$$

For each $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ define

$$D_1^{i,j} := H^{i+j}(F^j) \text{ and } E_1^{i,j} := H^{i+j}(F^j/F^{j+1})$$

Set $i + j = n$. Using the new notation rewrite the LES, and define α_1, β_1, γ_1 as follows

$$\cdots \to D_1^{i-1,j+1} \xrightarrow[\alpha_1^{i-1,j+1}]{\iota^*} D_1^{i,j} \xrightarrow[\beta_1^{i,j}]{p^*} E_1^{i,j} \xrightarrow[\gamma_1^{i,j}]{\delta} D_1^{i,j+1} \xrightarrow[\alpha_1^{i,j+1}]{} D_1^{i+1,j} \xrightarrow[\beta_1^{i+1,j}]{} E_1^{i+1,j} \to \ldots.$$

Then (D_1, E_1), along with α, β, γ is the desired exact couple. □

By iteration we get

$$\begin{tikzcd} D_r \arrow[rr, "\alpha_r(1,-1)"] & & D_r \arrow[dl, "\beta_r(-r+1,r-1)"] \\ & E_r \arrow[ul, "\gamma_r(0,1)"] & \end{tikzcd}$$

$d_r = \beta_r \circ \gamma_r$ has bi-degree $(1 - r, r)$.

Thus, the corresponding spectral sequence is

$$E_{r+1}^{i,j} := \ker\ d_r^{i,j}/\mathrm{im}\ d_r^{i+r-1,j-r}$$

and

$$E_0^{i,j} := F_n^j/F_n^{j+1}.$$

The boundary maps of $(E_0^{i,j}, d_0^{i,j})$ are induced by the chain maps of the total complex.

Example 4.3. Before we continue Example 4.2 we change the filtration from $F^{j+1} \subset F^j$ to $F^{j+4} \subset F^j$. Define $\mathbb{F}^{j+k} := F^{j+4k}$, and note that working with $\mathbb{F}$ preserves all the indexing of $D_r^{i,j}$, $E_r^{i,j}$ and $d_r^{i,j}$.

We first show that, the first page $E_1^{i,j}$ of the spectral sequence is the Khovanov homology.

By definitions $E_0^{i,j} = F_n^j / F_n^{j+4} = \mathrm{CKh}^{n,j}(L)$, and $d_0^{i,j} : E_0^{i,j} \to E_0^{i+1,j}$ is induced by the chain map of the total complex $(\mathrm{Tot}(\mathrm{CKh}(L)), D)$

$$\begin{tikzcd}
0 \arrow[r] & F_n^{j+4} \arrow[r, "inc"] \arrow[d, "D"] & F_n^j \arrow[r, "proj"] \arrow[d, "D"] & F_n^j/F_n^{j+4} \arrow[r] \arrow[d, "d_0^{i,j}"] & 0 \\
0 \arrow[r] & F_{n+1}^{j+4} \arrow[r, "inc"] & F_{n+1}^j \arrow[r, "proj"] & F_{n+1}^j/F_{n+1}^{j+4} \arrow[r] & 0.
\end{tikzcd}$$

Remember $D = d + \Phi$, and since $\Phi(a) \in F_{n+1}^{j+4}$ for any $a \in F_n^j$ then

$$d_0^{i,j}(a_j) = d(a_j)$$

where a_j represents the image of a under projection. Hence,

$$E_1^{i,j} = H^{i+j}(E_0, d_0) = \frac{\ker d_0^{i,j}}{\operatorname{im} d_0^{i-1,j}} = \frac{\ker d^{n,j}}{\operatorname{im} d^{n-1,j}} = \mathrm{Kh}^{i+j,j}(L).$$

Next, we show that the second page of the spectral sequence is

$$E_2^{i,j} = \frac{\ker\ (\Phi^{n,j} : \mathrm{Kh}^{n,j}(L) \to \mathrm{Kh}^{n+1,j+4}(L))}{\operatorname{im}\ (\Phi^{n-1,j-4} : \mathrm{Kh}^{n-1,j-4}(L) \to \mathrm{Kh}^{n,j}(L))}.$$

The boundary map $d_1^{i,j} : E_1^{i,j} \to E_1^{i,j+1}$ is given by the composition

$$E_1^{i,j} \xrightarrow{\gamma_1(0,1)} D_1^{i,j+1} \xrightarrow{\beta_1(0,0)} E_1^{i,j+1}$$

where γ is the connecting homomorphism and β is induced by the projection. Using the same convention $i + j = n$, and keeping $\mathbb{F}^{j+k} = F^{j+4k}$ in mind, we can write this as

$$H^n(F^j/F^{j+4}) \xrightarrow{\delta} H^{n+1}(F^{j+4}) \xrightarrow{p_*} H^{n+1}(F^{j+4}/F^{j+8}).$$

Take $[a_j] \in H^n(F^j/F^{j+4}) = E_1^{i,j} = \frac{\ker\ d^{n,j}}{\mathrm{im}\, d^{n-1,j}}$. We can think of $a_j \in \mathrm{CKh}^{n,j}(L)$, and $d^{n,j}(a_j) = 0$. The connecting homomorphism δ sends $[a_j]$ to $[b] \in H^{n+1}(F^{j+4})$ where $b \in F_{n+1}^{j+4}$. By definition of the connecting homomorphism we have $D(a_j) = i(b) = b \in F_{n+1}^{j+4}$

$$\begin{tikzcd} & & F_n^j \arrow[r, "proj"] \arrow[d, "D"] & F_n^j/F_n^{j+4} \arrow[r] & 0 \\ 0 \arrow[r] & F_{n+1}^{j+4} \arrow[r, "inc"] & F_{n+1}^j & & \end{tikzcd}$$

$$\gamma_1([a_j]) = [D(a_j)].$$

This implies the intended outcome

$$d_1^{i,j}([a_j]) = \beta_1\gamma_1([a_j]) = \beta_1([D(a_j)]) = [D(a_j)_{j+4}] = [\Phi^{n,j}(a_j)],$$
$$E_2^{i,j} = \frac{\ker\ (\Phi^{n,j} : \mathrm{Kh}^{n,j}(L) \to \mathrm{Kh}^{n+1,j+4}(L))}{\mathrm{im}\ (\Phi^{n-1,j-4} : \mathrm{Kh}^{n-1,j-4}(L) \to \mathrm{Kh}^{n,j}(L))}.$$

Definition 4.6. *Let $(F^p\mathcal{C})_{p\in\mathbb{Z}}$ be a filtration of a (co)chain complex $(\mathcal{C}, d)$, and $\iota_p : F^p\mathcal{C} \to \mathcal{C}$ be inclusions. We define the* induced filtration $({}_{(p)}H^n(\mathcal{C}))_{p\in\mathbb{Z}}$ *on $H^n(\mathcal{C})$ by*

$${}_{(p)}H^n(\mathcal{C}) := Im(H^n(F^p) \xrightarrow{\iota_p^*} H^n(\mathcal{C}))\ .$$

Exercise 15. *If $F^p\mathcal{C}$ is a bounded filtration, then the induced filtration ${}_{(p)}H^n(\mathcal{C})$ on (co)homology is also bounded.*

Definition 4.7. *We say a spectral sequence $(E_r, d_r)_{r\geq 1}$ converges to a graded module H if there is some bounded filtration ${}_{(p)}H$ of H such that*

$$E_\infty^{q,p} \cong {}_{(p)}H^n/{}_{(p-1)}H^n,$$

and denoted by

$$E_2^{q,p} \underset{p}{\Rightarrow} H^n.$$

Theorem 4.2. *Let $(\mathrm{Tot}(\mathcal{C}), D)$ be the total complex of a bi-graded bi-complex $(\mathcal{C}, d', d'')$ and (E_r, d_r) be the spectral sequence of a finite filtration in the second grading. Then*

$$E_2^{i,j} \underset{j}{\Rightarrow} H^n(\mathrm{Tot}(\mathcal{C}))$$

Proof. Consider the long exact sequence of r^{th} derived couple in Proposition 4.1

$$\cdots \to D_r^{i+r-2,j-r+2} \xrightarrow{\alpha_r^{i+r-2,j-r+2}} D_r^{i+r-1,j-r+1} \xrightarrow{\beta_r^{i+r-1,j-r+1}} E_r^{i,j} \xrightarrow{\gamma_r^{i,j}} D_r^{i,j+1} \to \ldots$$

Using the definition of D_1 and α_1 we write

$$\begin{array}{cccc} \alpha_1^{i-1,j+1}: & D_1^{i-1,j+1} & \to & D_1^{i,j} \\ & \shortparallel & & \shortparallel \\ & H^{i+j}(F^{j+1}) & \to & H^{i+j}(F^j) \end{array}$$

which is induced by the inclusion map $\iota^{j+1}: F^{j+1} \to F^j$, and $D_2^{i,j} = \alpha_1^{i-1,j+1}(D_1^{i-1,j+1})$. Hence,

$$\begin{aligned} D_3^{i,j} &= \alpha_2^{i-1,j+1}(D_2^{i-1,j+1}) \\ &= \alpha_2^{i-1,j+1}\alpha_1^{i-2,j+2}(D_1^{i-1,j+1}) \\ &= \iota_*^{j+1}\iota_*^{j+2}(D_1^{i-1,j+1}) \\ &= (\iota^{j+1}\iota^{j+2})_*(D_1^{i-1,j+1}). \end{aligned}$$

By a similar argument we can write for any r,

$$D_r^{i,j} = (\iota^{j+1} \dots \iota^{j+r-1})_*(D_1^{i-1,j+1}) \text{ is induced by } F^{j+r-1} \hookrightarrow \cdots \hookrightarrow F^j.$$

Since the filtration $F^p\mathcal{C}$ is finite, by working with a large enough r, we have $D_r^{i,j+1} = 0$. Moreover, by changing the variable $j \mapsto j - r + 1$, and $j + 1 \mapsto j - r + 2$ we get

$$D_r^{\#,j-r+1} = (\iota^{j-r+2} \dots \iota^{j})_*(D_1^{\#,j-r+2}) \text{ is induced by } F^{j} \hookrightarrow \cdots \hookrightarrow F^{j-r+1}$$

$$D_r^{\#,j-r+2} = (\iota^{j-r+3} \dots \iota^{j+1})_*(D_1^{\#,j-r+3}) \text{ is induced by } F^{j+1} \hookrightarrow \cdots \hookrightarrow F^{j-r+2}$$

again for a large r we have $F_n^{j-r+1} = F_n^{j-r+2} = \mathrm{Tot}(\mathcal{C})^n$, hence,

$$D_r^{\#,j-r+1} =_{(j)} H^n(\mathrm{Tot}(\mathcal{C})), \text{ and } D_r^{\#,j-r+2} =_{(j+1)} H^n(\mathrm{Tot}(\mathcal{C})).$$

Now the long exact sequence of r^{th} derived couple becomes

$$\cdots \to_{(j+1)} H^n(\mathrm{Tot}(\mathcal{C})) \to_{(j)} H^n(\mathrm{Tot}(\mathcal{C})) \to E_r^{i,j} \to D_r^{i,j+1} = 0 \to \dots$$

by the first isomorphism theorem and exactness

$$E_r^{i,j} =_{(j)} H^n(\mathrm{Tot}(\mathcal{C}))/_{(j+1)}H^n(\mathrm{Tot}(\mathcal{C})).$$

□

5. Lecture 4: The Lee spectral sequence

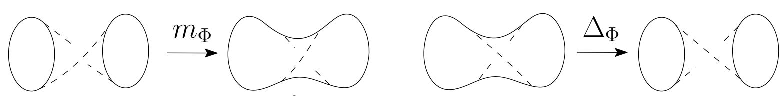
In this section we will define the Lee homology $\mathrm{Kh}_{Lee}(L)$ and outline the proofs of the following theorems, both of which are due to Lee [11].

Theorem 5.1. *There is a spectral sequence with E_1-term $\mathrm{Kh}(L)$ which converges to $\mathrm{Kh}_{Lee}(L)$. The E_1 and higher terms of this spectral sequence are invariants of the link L.*

Theorem 5.2. *$Kh_{Lee}(L)$ has rank 2^m, where m is the number of components of L.*

The spectral sequence in Theorem 5.1 is often referred to as *the Lee Spectral Sequence*.

Proof of Theorem 5.1. Let us consider the map Φ on $\mathrm{CKh}(L)$ whose assignment is as follows:



$$
\begin{array}{lcl}
\mathbb{1}\otimes\mathbb{1} & \longmapsto & 0 \\
\mathbb{1}\otimes x & \longmapsto & 0 \\
x\otimes\mathbb{1} & \longmapsto & 0 \\
x\otimes x & \longmapsto & \mathbb{1}
\end{array}
\qquad\qquad
\begin{array}{lcl}
\mathbb{1} & \longmapsto & 0 \\
x & \longmapsto & \mathbb{1}\otimes\mathbb{1}
\end{array}
$$

Figure 24.

Exercise 16. *Show that Φ is a map of bi-degree $(1,4)$ on $\mathrm{CKh}(L)$.*

It is tedious, as in the cases of m and Δ, to show that m_Φ and Δ_Φ are (co)commutative and (co)associative. One also needs to check

$$\Delta_\Phi \circ m_\Phi = (m_\Phi \otimes Id)\circ(Id\otimes\Delta_\Phi),$$

which ensures $\Phi^2 = 0$. Then, by Example 4.3, we have a spectral sequence whose first pages are as follows:

$$
\begin{aligned}
0^{th}\text{ page} : E_0^{i,j} &= \mathrm{CKh}^{i+j,j}(L)\\
1^{st}\text{ page} : E_1^{i,j} &= \frac{\ker\ d^{i+j,j}}{\mathrm{im}\ d^{i+j-1,j}} = \mathrm{Kh}^{i+j,j}(L)\\
2^{nd}\text{ page} : E_2^{i,j} &= \frac{\ker\ (\Phi^{i+j,j} : \mathrm{Kh}^{i+j,j}(L)\to \mathrm{Kh}^{i+j+1,j+4}(L))}{\mathrm{im}\ (\Phi^{i+j-1,j-4} : \mathrm{Kh}^{i+j-1,j-4}(L)\to \mathrm{Kh}^{i+j,j}(L))}.
\end{aligned}
$$

We observe the following:

Exercise 17. *Show that Φ anticommutes with d, i.e. $\Phi\circ d = -d\circ\Phi$.*

From Exercise 17 we conclude $(\mathrm{CKh}(L), d, \Phi)$ is a double complex. Now, let $Kh_{Lee}(L) := H^*(\mathrm{Tot}(\mathrm{CKh}(L)), d + \Phi)$. Then the proof of Theorem 5.1 follows immediately from Example 4.2 and Theorem 4.2. See Section 4 in [11] for the proof of the invariance under the Reidemeister moves. $\square$

Let D denote $d + \Phi$. The assignment of D on the chain level is as follows:

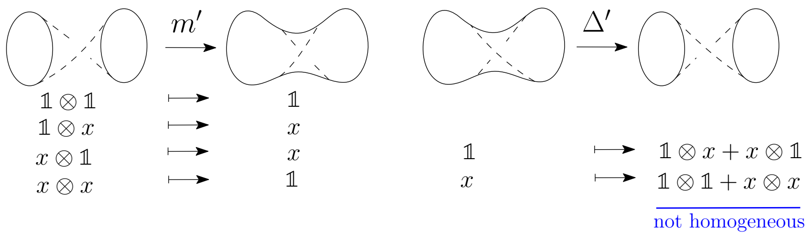


FIGURE 25.

Observe that $\Delta'(x)$ is not homogeneous. We will introduce a new basis $\{a, b\}$ for V where $a = x + \mathbb{1}$ and $b = x - \mathbb{1}$. Then the maps m' and Δ' are given by:

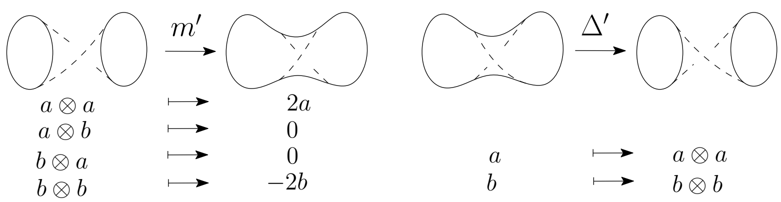


FIGURE 26.

Now we are ready to prove Theorem 5.2.

Proof of Theorem 5.2 . We will first show that the rank of $\mathrm{Kh}_{Lee}(L)$ is at least 2^m. To do this, we will describe a set of generators for $\mathrm{Kh}_{Lee}(L)$, each of which corresponds to an orientation on L.

Fix an orientation o of L. Observe that the 0-resolution of a positive crossing and the 1-resolution of a negative crossing both preserve orientation. Check this using Figure 27. This resolution indeed coincides with the resolution in Seifert algorithm, and hence, it is often referred to as *the Seifert resolution* or *the o-oriented resolution.* We may simply call it the oriented resolution if the orientation does not need to be specified.

Let $\alpha_o \in \{0, 1\}^n$ denote the o-oriented resolution of L. Then the circles of α_o are Seifert circles and the crossings correspond to the bands of the Seifert surface. To preserve the orientation, a band occurs between circles only in the following two situations as in Figure

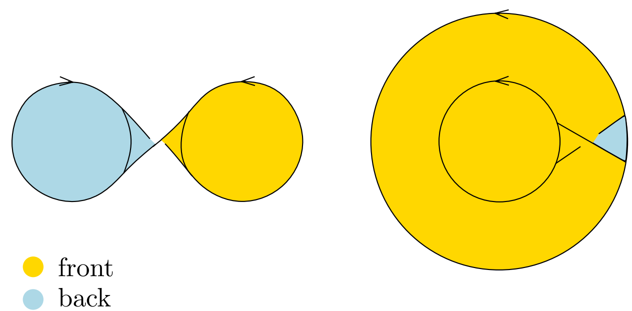


Figure 27. The possible local pictures of Seifert bands

27. As the bands connect two distinct circles, we see that the boundary map D at α_o merges two circles that are connected by positive crossings. Similarly, D mapping to α_o splits a circle into two that are connected by negative crossings.

As seen in Figure 26, $a \otimes b$ and $b \otimes a$ generate $ker(m')$ and $coker(\Delta')$. Then, if we label the circles of α_o with our basis elements $\{a, b\}$ such that any two circles connected by a band have different labels, then the resulting state will be in the kernel but not in the image of D, and hence, it represents a nontrivial element of $\mathrm{Kh}_{Lee}(L)$.

To make the above argument rigorous, we construct the corresponding state $s_o \in \bigcup_{j\in\mathbb{Z}} \mathrm{CKh}^{0,j}(L)$.

Let p denote the point at infinity. For each circle C in α_o, draw a ray r_C connecting C and p. Then we obtain a map from the set of circles in α_o to $\mathbb{Z}_2$ by counting the signed transverse intersection of r_C and other circles in α_o.

The map r distinguishes the cases in Figure 27. On the left, the merging circles have the same r value but different orientations, and on the right they have different r values but the same orientation. Let $o(C) = +$ if the induced orientation of o on C is clockwise, otherwise we say $o(C) = -$. Then we make an assignment of a or b for s_C as follows:

$$s_C = \begin{cases} a & \text{if } r(C) = 0 \text{ and } o(C) = + \\ b & \text{if } r(C) = 0 \text{ and } o(C) = - \\ b & \text{if } r(C) = 1 \text{ and } o(C) = + \\ a & \text{if } r(C) = 1 \text{ and } o(C) = - \end{cases}.$$

This proves $\mathrm{rank}(\mathrm{Kh}_{Lee}(L)) \geq 2^m$. Observe that in case of a knot K, $s_{\bar{o}}$, the element corresponding to the opposite orientation $\bar{o}$, is simply obtained by s_o by switching each a to b and each b to a.

We now pause the proof and ask the reader to work on an explicit example of our construction.

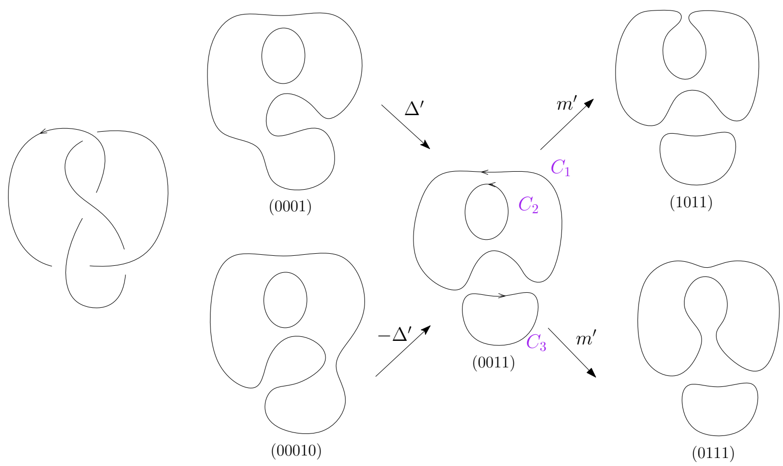


FIGURE 28. The oriented resolution of Figure-8 knot and the boundary maps at this resolution

Exercise 18. *Let o be the orientation of the Figure-8 knot as in Figure 28. First show that $s_o = b \otimes a \otimes a$ and $s_{\bar{o}} = a \otimes b \otimes b$. Then show that $[s_o]$ and $[s_{\bar{o}}]$ are nontrivial elements of $\mathrm{Kh}_{Lee}(4_1)$.*

It remains to show $\mathrm{rank}(\mathrm{Kh}_{Lee}(L)) \leq 2^m$. As in the case of $\mathrm{Kh}(L)$, there is a long exact sequence for the Lee homology associated to an unoriented skein relation:

$$\longrightarrow \mathrm{Kh}^{i-1}_{Lee}(D_0) \longrightarrow \mathrm{Kh}^{i-1}_{Lee}(D_1) \longrightarrow \mathrm{Kh}^{i}_{Lee}(D) \longrightarrow \mathrm{Kh}^{i}_{Lee}(D_0) \longrightarrow \mathrm{Kh}^{i}_{Lee}(D_1) \longrightarrow .$$

We will first show the bound for knots and links with 2 components by inducting on the number of crossings. The base case is as follows:

$$\mathrm{Kh}_{Lee}(\bigcirc) \cong \mathrm{Kh}(\bigcirc) \cong \mathbb{Q} \oplus \mathbb{Q}$$
$$\mathrm{Kh}_{Lee}(\bigcirc\,\bigcirc) \cong \mathrm{Kh}(\bigcirc\,\bigcirc) \cong \mathbb{Q} \oplus \mathbb{Q}^2 \oplus \mathbb{Q}.$$

Now suppose the statement holds for knots and 2-component links with less than c crossings. Let D be a minimal diagram of a knot K with c crossings. Pick a positive crossing in D (if there is no such crossing we work with $-D$). Then D_0 is a link with 2 components and D_1 is a knot.

We need the following proposition to complete the proof for knots.

Proposition 5.3. *Two of the canonical generators of D_0 are paired with two of those of D_1.*

Proof. Fix an orientation o_1 on D_1. This will induces an orientation o_0 on D_0, that is the same as o_1 outside the resolution. Then, on the chain level, δ' maps the states at the o_0-oriented resolution to those at the o_1-oriented resolution.

Let us first assume δ' is merging circles. Let C_1 and C_2 denote the circles in the o_0-oriented resolution of D_0 merged by δ'. There are two possibilities: C_1 and C_2 either bound disjoint disks, i.e. $r(C_1) = r(C_2)$, or are cocentric, i.e. $|r(C_1) - r(C_2)| = 1$. In the first case $o_0(C_1) = o_0(C_2)$, and for the latter case $o_0(C_1)$ and $o_0(C_2)$ are not the same as seen in Figure 29. Thus, in either case we have $s_{C_1} = s_{C_2}$. Without loss of generality, assume that the corresponding label assigned to s_{C_1} for s_{o_0} is a (then it is b for $s_{\bar{o}_0}$). Then $\delta'([s_{o_0}]) = 2[s_{o_1}]$ and $\delta'([s_{\bar{o}_0}]) = -2[s_{\bar{o}_1}]$.

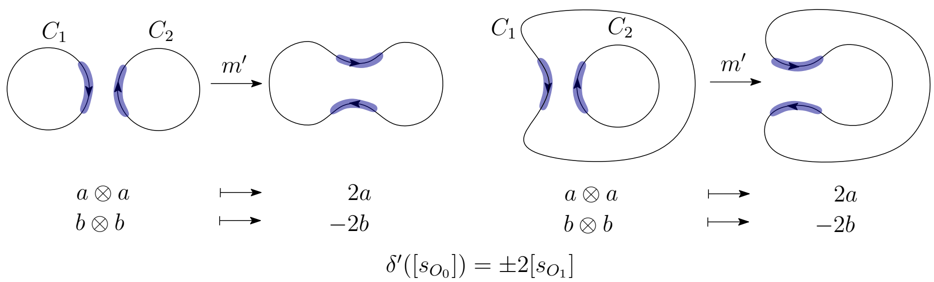


FIGURE 29.

Now assume δ' splits a circle. Let C_1 and C_2 denote the circles in the o_1-oriented resolution of D_1 splitted by δ'. Similarly as before, there are two possibilities: C_1 and C_2 either bound disjoint disks, i.e. $r(C_1) = r(C_2)$, or are cocentric, i.e. $|r(C_1) - r(C_2)| = 1$. In the first case $o_1(C_1) = o_1(C_2)$, and for the latter case $o_1(C_1)$ and $o_1(C_2)$ are not the same as seen in Figure 29. Thus, in either case we have $s_{C_1} = s_{C_2}$, and hence, $\delta'([s_{o_0}]) = [s_{o_1}]$ and $\delta'([s_{\bar{o}_0}]) = [s_{\bar{o}_1}]$. This completes the proof.

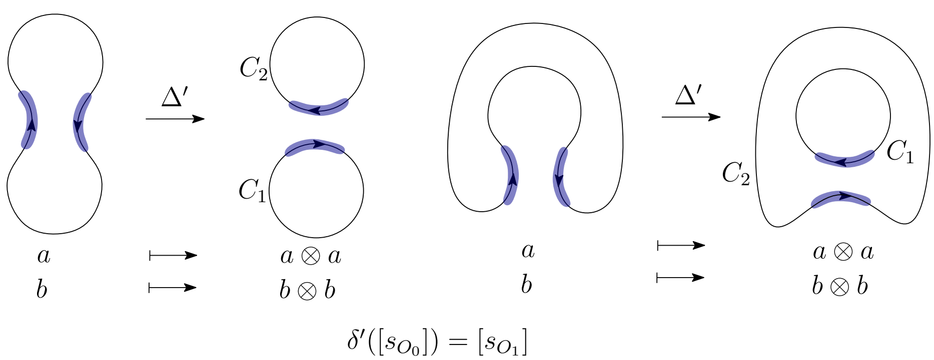


FIGURE 30.

□

By the induction hypothesis and Proposition 5.3 we obtain the following:

$$\text{rank Kh}_{Lee}(D) \leq \text{rank Kh}_{Lee}(D_0) + \text{rank Kh}_{Lee}(D_1) - 4 = 2.$$

This proves the statement for knots. Now suppose D is a minimal diagram of a 2-component link L with crossing number c. If D is a disjoint union of two knots with diagrams D_1 and D_2, then the result is obvious as $\mathrm{Kh}_{Lee}(D) = \mathrm{Kh}_{Lee}(D_1) \otimes \mathrm{Kh}_{Lee}(D_2)$. Otherwise, let us pick a crossing of D whose strands belong to distinct components of L. Then both D_0 and D_1 will be knots as seen in Figure 31, and hence, the result follows from the rank inequality.

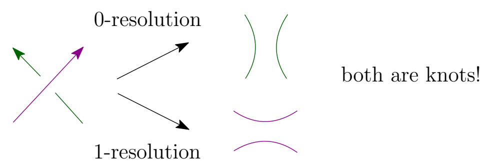


FIGURE 31. The crossing with strands belonging to different components of L and its resolutions

To complete the proof, we induct on the number of components of L. Any diagram of an n-component link L is either a disjoint union of links with fewer components or has a crossing whose 0- and 1-resolutions give rise to a link with $n-1$ components. The rest of the proof goes as before.

□

Example 5.1. Recall $\mathrm{Kh}(-3_1)$ from Exercise 13. We see that the only possible differential of bi-degree $(1,4)$ for the Lee spectral sequence is $d_1 : \mathrm{Kh}^{-2,-7}(-3_1) \to \mathrm{Kh}^{-1,-3}(3_1)$, and so we find $\mathrm{Kh}_{Lee}(-3_1) = \mathbb{Q} \oplus \mathbb{Q}$.

$j \diagdown i$	-2	-1	0
-1	0	0	$\mathbb{Q}$
-3	0	$\mathbb{Q}$	$\mathbb{Q}$
-5	0	0	0
-7	$\mathbb{Q}$	0	0

TABLE 2. The generators in the orange cells do not survive in E_2-page and the ones in the pink cells are the generators of $\mathrm{Kh}_{Lee}(-3_1)$.

6. Lecture 5: The s-invariant

In this section, we will define Rasmussen's s-invariant for knots and prove the following.

Theorem 6.1. $|s(K)| \leq 2g_4(K)$.

Theorem 6.2. *s induces a homomorphism from Conc(S^3) to $\mathbb{Z}$.*

In the previous lecture we saw that for a knot K, $\mathrm{Kh}_{Lee}(K) = \mathbb{Q} \oplus \mathbb{Q}$. Although q is no longer a grading on $(\mathrm{Tot}(\mathrm{CKh}(K)), D)$ but a filtration, it is still possible to define a grading using this filtration. We use the associated grading for the generators of $\mathrm{Kh}_{Lee}(K)$ to define the s-invariant of K, $s(K)$. Then we will study the behaviour of the s-invariant under cobordism, and prove Theorems 6.1 and 6.2.

We start with a few technical definitions. For a filtered cochain complex, there is a grading induced by the filtration defined as follows.

Definition 6.1. *Let $\mathcal{C}$ be a filtered cochain complex with filtration $(F^p\mathcal{C})_{p\in\mathbb{Z}}$. $x \in \mathcal{C}^i$ has a filtration-induced grading j if and only if $x \in F_i^j$ but $x \notin F_i^{j+1}$.*

Definition 6.2. *A filtration $F^p\mathcal{C}$ of $\mathcal{C}$ induces a filtration $F^pH^*(\mathcal{C})$ on $H^*(\mathcal{C})$:*

A class $[x] \in H^*(\mathcal{C})$ is in $F^pH^*(\mathcal{C})$ if and only if it has a representative in $F^p\mathcal{C}$.

Recall from Example 4.2 that the q-grading on $\mathrm{CKh}(K)$ induces a filtration on the Lee chain complex $(\mathrm{Tot}(\mathrm{CKh}(K)), D)$. Let us abuse the notation and call the grading on $(\mathrm{Tot}(\mathrm{CKh}(K)), D)$ induced by the q-filtration *the q-grading*. Then we have the following.

Definition 6.3. *The s-grading on $\mathrm{Kh}_{Lee}(K)$ is the grading induced by the q-grading on $(\mathrm{Tot}(\mathrm{CKh}(K)), D)$.*

In light of Theorem 5.2 and Definition 6.3, we define the following.

Definition 6.4.

$$s_{\min}(K) = \min\{s(x) | x \in \mathrm{Kh}_{Lee}(K) \text{ and } x \neq 0\}$$
$$s_{\max}(K) = \max\{s(x) | x \in \mathrm{Kh}_{Lee}(K) \text{ and } x \neq 0\}.$$

Recall $\mathrm{Kh}_{Lee}(\bigcirc) \cong \mathrm{Kh}(\bigcirc) = \mathbb{Q}_{(-1)} \oplus \mathbb{Q}_{(1)}$, then $s_{\min}(\bigcirc) = -1$ and $s_{\max}(\bigcirc) = 1$.

It turns out that s_{min} and s_{max} differ by 2.

Proposition 6.3. $s_{\max}(K) = s_{\min}(K) + 2$.

This justifies the following definition.

Definition 6.5. $s(K) = s_{\max}(K) - 1 = s_{\min}(K) + 1$.

The last definition we need for is the following.

Definition 6.6. *If $f : \mathcal{C} \to \mathcal{C}'$ is a map between two filtered chain complexes, then we say f preserves the filtration if $f(F^p\mathcal{C}) \subseteq F^p\mathcal{C}'$. More generally, we say f is a filtered map of degree k if $f(F^p\mathcal{C}) \subseteq F^{p+k}\mathcal{C}'$.*

Example 6.1. We will show that $\mathrm{CKh}^{i,j}(D) = 0$ if $j \not\equiv |L| \mod 2$. Let α be an oriented resolution, and $v \in V_\alpha$. Then $i = r_\alpha - n_- = 0$ hence

$$j(v) = r_\alpha - n_- + \deg(v) + n_+ - n_- = \deg(v) + n_+ - n_-$$

Let $|\alpha|$ represents the number of circles in α. Then $\deg(v) \equiv |\alpha| \mod 2$, and $n_+ - n_- \equiv n \mod 2$. Also, since every oriented resolution on D changes the number of components by ± 1, $|\alpha| \equiv |L| + n \mod 2$. That gives

$$j(v) \equiv |L| \mod 2.$$

Now take any resolution β, and a generator $w \in V_\beta$. One can go from β to α by some number of resolution changes, say k, and any resolution change also changes the number circles, hence, $|\beta|$ and $|\alpha| + k$ have the same parity. Similarly r_β and $r_\alpha + k$ have the same parity. Which gives

$$j(w) = r_\beta - n_- + \deg(w) + n_+ - n_- \equiv |L| \mod 2.$$

Hence, for a knot K, $\mathrm{Kh}^{i,j}(K)$ is supported in odd q-degrees.

In order to prove Proposition 6.3 we need the following lemma:

Lemma 6.4. *$\mathrm{Kh}_{Lee}(K)$ decomposes as $\mathrm{Kh}^e_{Lee}(K) \oplus \mathrm{Kh}^o_{Lee}(K)$, where $\mathrm{Kh}^e_{Lee}(K)$ is generated by all states with q-grading conguent to $1 \mod 4$, and $\mathrm{Kh}^o_{Lee}(K)$ is generated by all states with q-grading conguent to $3 \mod 4$. Furthermore, $[s_o + s_{\bar{o}}]$ is contained in one of the summands and $[s_o - s_{\bar{o}}]$ is contained in the other, and they generate the corresponding summands.*

Proof. Recall Example 6.1 that $\mathrm{CKh}^{i,j}(K)$ is supported in odd j-degrees. Then the decomposition is immediate as d preserves the q-grading while Φ raises it by 4.

Let $\iota : (\mathrm{Tot}(\mathrm{CKh}(K)), D) \to (\mathrm{Tot}(\mathrm{CKh}(K)), D)$ be defined by

$$\iota(v) = \begin{cases} -v & \text{if } [v] \in \mathrm{Kh}^e_{Lee}(K) \\ v & \text{if } [v] \in \mathrm{Kh}^o_{Lee}(K) \end{cases}.$$

For each v with $[v] \in \mathrm{Kh}^e_{Lee}(K)$ (or $\mathrm{Kh}^o_{Lee}(K)$), $v = \sum_k v_k$ where each v_k is a state and the number of x (or $\mathbb{1}$) in each v_k have the same parity. Thus, $\iota(v) = \pm \sum_i i^{\otimes n_k}(v_k)$ where n_k is the number of components in v_k and $i : V \to V$ is the map given by $i(\mathbb{1}) = -\mathbb{1}$ and $i(x) = x$.

Since $i(a) = i(x + \mathbb{1}) = x - \mathbb{1} = b$ and $i(b) = i(x - \mathbb{1}) = x + \mathbb{1} = a$, $\iota(s_o) = \pm s_{\bar{o}}$ and $\iota(s_{\bar{o}}) = \pm s_o$. Then, $\iota(s_o + s_{\bar{o}}) = \pm(s_o + s_{\bar{o}})$ while $\iota(s_o - s_{\bar{o}}) = \mp(s_o - s_{\bar{o}})$, which completes the proof.

□

Note that $[s_o + s_{\bar{o}}]$ and $[s_o - s_{\bar{o}}]$ generate $\mathrm{Kh}_{Lee}(K)$. Since they belong to distinct direct summands of $\mathrm{Kh}_{Lee}(K)$, $s_{\min}$ and $s_{\max}$ are given by the s-gradings of these elements and so they are distinct.

Corollary 6.5. *$s_{\max}(K) > s_{\min}(K)$.*

Exercise 19. *Show that* $s[s_o] = s[s_{\bar{o}}] = s_{\min}(K)$.

To prove Proposition 6.3, it remains to show the following.

Lemma 6.6. $s_{\max}(K) \leq s_{\min}(K) + 2$.

Proof. Consider the map $f : \mathrm{Kh}_{Lee}(K \sqcup U) \to \mathrm{Kh}_{Lee}(K)$, which is locally m' depicted as:

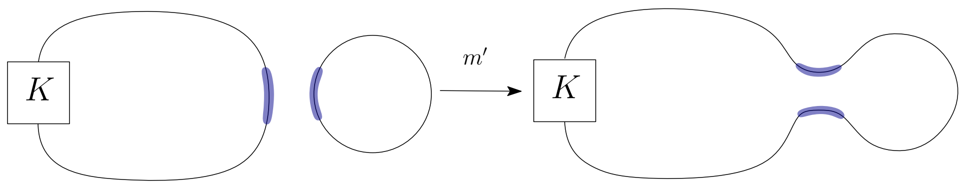


FIGURE 32.

Exercise 20. *Show that f is a filtered map of degree -1 with respect to the s-grading.*

Let s_a denote one of the s_o or $s_{\bar{o}}$ whose circle component containing the blue arc is labeled a. Similarly, denote the other one by s_b. Without loss of generality, we may assume $s(s_a + s_b) = s_{\max}(K)$. Then $f((s_a + s_b) \otimes a) = 2s_a$. Then, by Exercise 19, and Exercise 20 we obtain

$$\begin{aligned} s(f([(s_a + s_b) \otimes a])) &\geq s((s_a + s_b) \otimes a) - 1 \\ s(s_a) &\geq s_{\max}(K) - 1 - 1 \\ s_{\min}(K) &\geq s_{\max}(K) - 2. \end{aligned}$$

□

Now Proposition 6.3 follows from Corollary 6.5 and Lemma 6.6. See Section 6 in [2] for the invariance under Reidemeister moves. This completes that s is a knot invariant.

The invariant s behaves nicely with respect to the mirror image and connected sum.

Proposition 6.7. $s(-K) = -s(K)$.

Proof. $(\mathrm{Tot}(\mathrm{CKh}(K)), D)$ is isomorphic to the dual of $(\mathrm{Tot}(\mathrm{CKh}(K)), D)$ via the map sending $\mathbb{1} \mapsto x$ and $x \mapsto \mathbb{1}$. Then the filtration-induced q-gradings of dual elements of $(\mathrm{Tot}(\mathrm{CKh}(K)), D)$ and $(\mathrm{Tot}(\mathrm{CKh}(K)), D)$ are negative of each other. Then the result follows from the fact that the surviving generators are dual of each other as the associated spectral sequences are also dual.

□

Exercise 21. $s(K_1 \# K_2) = s(K_1) + s(K_2)$. *(Hint: Consider the multiplication map from* $\mathrm{Kh}_{Lee}(K_1 \coprod K_2)$ *to* $\mathrm{Kh}_{Lee}(K_1 \# K_2)$*, a generalization of f in Lemma 6.6. Follow the proof strategy of Lemma 6.6 and use Proposition 6.7.)*

6.1. Behaviour under cobordism

Let S be a cobordism from L_0 to L_1. From Morse theory, we know that S can be decomposed into a series of elementary cobordisms, i.e. Morse moves

$$S = S_1 \cup S_2 \cup \cdots \cup S_k$$

where each S_i is a cobordism from l_{i-1} to l_i with $l_0 = L_0$ and $l_k = L_1$ obtained by the addition of a 0, 1, or 2-handle. Recall the local picture of the elementary cobordisms are depicted in Figure 11.

Since Kh_{Lee} describes a $1+1-$dimensional TQFT, given a cobordism S from L_0 to L_1 there is an induced map $\phi_S : \mathrm{Kh}_{Lee}(L_0) \to \mathrm{Kh}_{Lee}(L_1)$, that is invariant under smooth isotopies of S relative to the boundary. Furthermore, this assignment is functorial. Hence, the map $\phi_S : \mathrm{Kh}_{Lee}(L_0) \to \mathrm{Kh}_{Lee}(L_1)$ decomposes as

$$\phi_S = \phi_{S_k} \circ \cdots \circ \phi_{S_2} \circ \phi_{S_1}.$$

Note that $\phi_\emptyset = \mathbb{Q}$. Similar to Khovanov homology, the cobordism maps are defined as follows:

- S_i is the addition of a 0-handle: Let $\iota' : \mathbb{Q} \to V$ be a map given by $\iota(1) = \frac{a-b}{2}$. Then

$$\begin{aligned}\phi_{S_i} = id \otimes \iota' : \mathrm{Kh}_{Lee}(l_{i-1}) &\to \mathrm{Kh}_{Lee}(l_{i-1} \sqcup U).\\ [v] &\mapsto [v \otimes \frac{a-b}{2}] = [v \otimes \mathbb{1}]\end{aligned}$$

  In this case, ϕ_{S_i} is a filtered map of degree 1.

- S_i is the addition of a 1-handle: Then

$$\phi_{S_i} : \mathrm{Kh}_{Lee}(l_{i-1}) \to \mathrm{Kh}_{Lee}(l_i).$$

  Here we apply either $id \otimes m'$ or $id \otimes \Delta'$, depending on whether the move results in a merge or a split at the vertex of the cube. When ϕ_{S_i} behaves as $id \otimes m'$, as seen in Exercise 20, it is a filtered map of degree -1.

Exercise 22. *Show that when ϕ_{S_i} behaves as $id \otimes \Delta'$, it also has degree -1.*

- S_i is the addition of a 2-handle: Let $\epsilon' : V \to \mathbb{Q}$ be a map given by $\epsilon'(a) = \epsilon'(b) = 1$. Then

$$\begin{aligned}\phi_{S_i} = id \otimes \epsilon' : \mathrm{Kh}_{Lee}(l_{i-1} = l_i \sqcup U) &\to \mathrm{Kh}_{Lee}(l_i).\\ [v \otimes a] &\mapsto [v]\\ [v \otimes b] &\mapsto [v]\end{aligned}$$

  Here ϕ_{S_i} is also a filtered map of degree 1.

Recall that $\chi(S) = \#\{\text{0-handles}\} - \#\{\text{1-handles}\} + \#\{\text{2-handles}\}$. Then we find that ϕ_S is a filtered map of degree $\chi(S)$. If S is an unknotted 2-sphere S^2, then $\phi_{S^2} = \epsilon' \circ \iota'$, which will be 0. In particular:

Exercise 23. *Let Σ_g be an unknotted closed genus-g surface. Show that if g is even, then $\phi_{\Sigma_g} = 0$, otherwise $\phi_{\Sigma_g}(1) = 2^g$.*

Therefore, we may focus on cobordisms which do not contain a closed connected component. This and our interest in the 4-genus motivate the following definition.

Definition 6.7. *An oriented cobordism S from L_0 to L_1 is said to be weakly connected if every connected component of S has a boundary in L_0.*

Fix an orientation o on S. This will induce an orientation o_i on each l_i. Let $[s_{o_i}]$ denote the generator of $\mathrm{Kh}_{Lee}(l_i)$ determined by o_i. Recall $l_k = L_1$. Then, we have the following:

Proposition 6.8. *Suppose S is a weakly connected cobordism from L_0 to L_1. Then, $\phi([s_{o_0}])$ is a nonzero multiple of $[s_{o_k}]$.*

Proof. To prove the proposition, it is sufficient to show the statement for each elementary cobordism. Note that a 0-handle should be followed by a 1-handle as S is weakly connected. Thus, we handle this case last.

1-handle move: First consider when ϕ_{S_i} behaves as $id \otimes m'$. Let C_1 and C_2 denote the circles that are merged and C the resulting circle. Then we have two cases:

i. $r(C_1) = r(C_2)$. To preserve the orientation it should be that $r(C_1) = r(C_2) = r(C)$ and $o_{i-1}(C_1) = o_{i-1}(C_2) = o_i(C)$. Then the labels of $[s_{o_{i-1}}]$ corresponding to C_1 and C_2 and the label of s_{o_i} for C are the same. Because $m'(a \otimes a) = 2a$ and $m'(b \otimes b) = 2b$, we have the desired result that $\phi_{S_i}([s_{o_{i-1}}]) = 2[s_{o_i}]$.

ii. $r(C_1) - r(C_2) = \pm 1$. Then, $o_{i-1}(C_1)$ and $o_{i-1}(C_2)$ are not the same, which implies that the labels of $[s_{o_{i-1}}]$ corresponding to C_1 and C_2 are still the same. The rest follows as above because $r(C) = r(C_2)$ and $o_i(C) = o_{i-1}(C_2)$.

We leave it as an exercise to prove the statement when ϕ_{S_i} behaves as $id \otimes \Delta'$.

Exercise 24. *Show that if ϕ_{S_i} behaves as $id \otimes \Delta'$, then $\phi_{S_i}([s_{o_{i-1}}]) = [s_{o_i}]$.*

2-handle move: In this case, $l_{i-1} = l_i \sqcup U$. Then $[s_{o_{i-1}}]$ is either $[s_{o_i} \otimes a]$ or $[s_{o_i} \otimes b]$. Since $\epsilon'(a) = \epsilon'(b) = 1$, in either case we get $\phi_{S_i}([s_{o_i} \otimes a]) = \phi_{S_i}([s_{o_i} \otimes b]) = [s_{o_i}]$.

Now, we are ready to prove the statement for 0-handle attachment.

0-handle move: Here, $l_i = l_{i-1} \sqcup U$. Then, $[s_{o_i}]$ is either $[s_{o_{i-1}} \otimes a]$ or $[s_{o_{i-1}} \otimes b]$. Since $\iota'(1) = \frac{a-b}{2}$, $\phi_{S_i}([s_{o_{i-1}}]) = \frac{1}{2}([s_{o_{i-1}} \otimes a] - [s_{o_{i-1}} \otimes b])$. Well, this is not quite $\phi_{S_i}([s_{o_{i-1}}]) = a_i[s_{o_i}]$ for some nonzero a_i. However, we observe that since S is weakly connected, any 0-handle move is eventually followed by a 1-handle move S_j merging the unknotted component created by this 0-handle and another circle in the oriented resolution of l_{j-1}. For simplicity, assume $j = i + 1$. Then we see from above that ϕ_{S_j} maps the linear summand of $\phi_{S_i}([s_{o_{i-1}}])$ which is not $[s_{o_i}]$ to 0. This completes the proof and finally we have the desired result.

□

Corollary 6.9. *If S is a connected cobordism from K_0 to K_1, then ϕ_S is an isomorphism.*

Proof of Theorem 6.1. Let S be a cobordism from K to U with minimal genus, i.e. $g(S) = g_4(K)$. Then ϕ_S is a map of degree $\chi(S) = 2 - 2g_4 - 2 = -2g_4$. Let v be a nonzero element in $\mathrm{Kh}_{Lee}(K)$ with $s(v) = s(K) + 1$. From Proposition 6.8 $\phi_S(v) \neq 0$. Because 1 is the highest s-grading in $\mathrm{Kh}_{Lee}(U)$, we have the following:

$$\begin{aligned} s(v) - 2g_4(K) &\leq s(\phi_S(v)) \leq 1 \\ s(K) + 1 - 2g_4(K) &\leq 1 \\ s(K) &\leq 2g_4(K). \end{aligned}$$

We may apply the same argument to $-K$ to obtain $s(K) \geq -2g_4(K)$, which finally yields the desired inequality $|s(K)| \leq 2g_4(K)$.

□

Corollary 6.10. *If K is slice, then $s(K) = 0$.*

Now we are ready to prove s is a concordance homomorphism.

Proof of Theorem 6.2. We already have $s(-K) = -s(K)$ and $s(K_1 \# K_2) = s(K_1) + s(K_2)$, which is Proposition 6.7 and Example 21, respectively. It remains to show that if two knots are concordant, then they have the same s-invariant. Let $K_1 \sim K_2$. Then, $K_1 \# -K_2$ is slice. By Corollary 6.10 and the above two properties of the s-invariant we have

$$s(K_1 \# -K_2) = 0 \quad \Rightarrow \quad s(K_1) - s(K_2) = 0 \quad \Rightarrow \quad s(K_1) = s(K_2),$$

which completes the proof. □

Example 6.2. Recall Example 5.1. We see that the q-gradings of the surviving generators are -3 and -1. We record this by writing $\mathrm{Kh}_{Lee}(-3_1) = \mathbb{Q}_{(-3)} \oplus \mathbb{Q}_{(-1)}$. Then we find $s(-3_1) = -2$ and hence $s(3_1) = 2$. This gives another proof that 3_1 is not slice.

7. Applications

7.1. The slice-Bennequin inequality, the 4-genus of positive knots and the Milnor conjecture

A knot is said to be *positive* if it has a diagram in which all crossings are positive. One of the most important and the earliest application of the Khovanov homology and the s-invariant is determining the 4-ball genus of positive knots to be the same as the 3-genus, in fact, the genus of the Seifert surface obtained by a positive diagram of K. Because torus knots are positive, the Milnor conjecture follows immediately as a corollary. The proof we outline in this subsection is due to Rasmussen [2] (the original proof of the Milnor conjecture is due to Kronheimer and Mrowka using the gauge theory [12]).

We consider a distinguished element in $\mathrm{Kh}(K)$, which becomes particularly interesting for positive knots as seen below. Fix a positive diagram of K. Let f denote the number of Seifert circles and w the writhe. We use Plamenevskaya's notation [13] and denote this chain element by $\tilde{\psi}(K)$ and its homology class by $\psi(K)$. Let

$$\tilde{\psi}(K) = x \otimes x \otimes \cdots \otimes x \in V_{\alpha_o} = V^{\otimes f} \in \mathrm{CKh}^0(K)$$

be the state in the oriented resolution α_o obtained by labeling each circle by x. Recall the boundary maps at α_o correspond to merging circles and $m(x \otimes x) = 0$, so we immediately observe $\tilde{\psi}(K)$ is a cycle in $(\mathrm{CKh}(K), d)$.

Lemma 7.1. *$\psi(K) \in \mathrm{Kh}^{0,-f+w}(K)$, and hence, $-f + w \leq s(K) - 1$.*

Proof. $\tilde{\psi}(K)$ is a homogenous element and $q(\psi)$ can be computed as follows:

$$q(\psi) = \deg(\tilde{\psi}(K)) + i + n_+ - n_- = -f + w.$$

As $\tilde{\psi}(K)$ has the lowest degree among the elements in V_{α_o}, $q(\psi(K)) \leq s_{\min}(K) \leq s(K) - 1$, and hence, we have the latter statement.

□

We are now ready to prove the *slice-Bennequin inequality.*

Proof of Theorem 2.3. We prove this by inducting on the number of components of L. The base case – the inequality for the knots – follows by Lemma 7.1, Theorem 6.1 and the fact that $f = b$:

$$w - b \leq s(K) - 1 \leq 2g_4(K) - 1 = -\chi_4(K),$$

and the inductive step is handled by the following exercise.

Exercise 25. *Let L' be a link with $|L'| > 1$. Pick a crossing of L' such that its oriented resolution gives L with $|L| = |L'| - 1$. Show that*

$$|\chi_4(L') - \chi_4(L)| \leq 1$$

(Consider the slice surface of L which is a union of the cobordism obtained by resolving the crossing and a minimal slice surface of L').

□

Note $\psi(K)$ can be trivial, as in the case of the Figure-8 knot, see Figure 28.

Exercise 26. *Let's take $(x \otimes x, 0) \in \mathrm{CKh}^{-1}(4_1)$. Show that $d((x \otimes x, 0)) = \Delta(x) \otimes x = x \otimes x \otimes x$. Thus, conclude that $\psi(4_1) = 0$.*

However, as the oriented resolution occurs at $(0, \ldots, 0)$ in a positive diagram, $\psi(K)$ is non-zero for a positive knot K.

Lemma 7.2. *Let K be a positive knot. Then $\psi(K) \neq 0$.*

Now we are ready to prove that the Seifert surface of a positive diagram is both 3- and 4-genus minimizing.

Theorem 7.3. *Suppose K is a positive knot, then $g_4(K) = g(K)$. Furthermore, if D is a positive diagram of K, then $g_4(K) = g(K) = g(F_D)$.*

Proof. This follows from $w = c$ in D, Exercise 6, Lemma 7.1 and Theorem 6.1. □

Corollary 7.4 (the Milnor conjecture)**.** *Let $T_{p,q}$ be the (p,q)-torus knot. Then $g_4(T_{p,q}) = g(T_{p,q}) = \frac{(p-1)(q-1)}{2}$.*

Remark 7.1. Working with braid diagrams, the braid index b is equal to f and the self-linking number of L $sl(K)$ is $-f + w$. Then the above results can be stated in terms of $sl(L)$. Transverse links L, that are everywhere transverse to the contact planes in the standard contact 3-sphere (S^3, ξ_{std}), are represented by closed braids and sl is an invariant of transverse links. Plamenevskaya [13] proves that $\psi(K)$ is an invariant of transverse links in (S^3, ξ_{std}). She also extends Lemma 7.2 to the closures of quasipositive braids.

7.2. The Knight Move conjecture

Because the differentials of the Lee spectral sequence are of bi-degree $(1, 4n)$, we have a decomposition of Khovanov homology of a knot into a single pawn pair

$$\mathbb{Q}\{0, s-1\} \oplus \mathbb{Q}\{0, s+1\}$$

and several, possibly "longer" knight moves, of the form

$$\mathbb{Q}\{i, j\} \oplus \mathbb{Q}\{i+1, j+4n\},$$

where $\mathbb{Q}\{i, j\}$ denotes a summand of $\mathrm{Kh}(K)$ in bi-grading (i, j).

Earlier, the computations of Khovanov and Bar-Natan produced only examples whose Khovanov homology decomposes into pairs with $n = 1$. This motivated the following conjecture.

The Knight Move Conjecture ([1], [8])*.* The Khovanov homology of a knot K decomposes into pawn move

$$\mathbb{Q}\{0, s-1\} \oplus \mathbb{Q}\{0, s+1\}$$

together with a set of knight moves

$$\mathbb{Q}\{i, j\} \oplus \mathbb{Q}\{i+1, j+4\}.$$

The conjecture indicates that the Lee spectral sequence collapses at the E_2 page, and hence all the Lee differentials d_n for $n \geq 2$ vanish. This was confirmed for alternating knots [14], quasi-alternating knots [15], knots with $u(K) = 1$ [16]. However, it was proven to be false in general: Manolescu and Marengon [17] gave the first example of a knot with a non-trivial Lee differential d_2.

Exercise 27. *Let K be the knot in Figure 33 with* $\mathrm{Kh}(K)$ *given in Table 7.2. Show that the second Lee differential d_2 of bi-degree $(1, 8)$ is non-vanishing.*

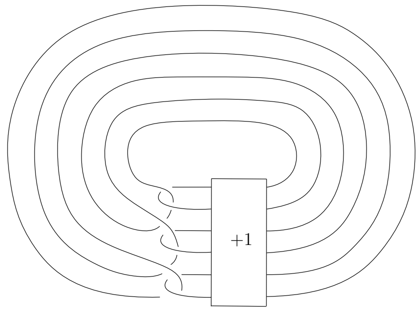


FIGURE 33. The knot K in [17]

$j\diagdown i$	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
13																							1
11																						1	
9																				1	2	1	
7																			3	4	2		
5																		5	4	1			
3																1	6	6	4	1			
1															3	9	10	4	1	1			
-1														3	9	8	3	1	1				
-3													3	10	12	6	1						
-5											1	5	10	10	2		1						
-7									1	2	4	6	7	3	1								
-9								2	1	3	7	8	2										
-11							2	2	4	5	3												
-13						3	4	2	2	2	1												
-15					3	3	1	2	1														
-17				2	3	2	1																
-19			2	3	1																		
-21		1	2																				
-23		2																					
-25	1																						

TABLE 3. The entry corresponding to column i and row j is the rank of $\mathrm{Kh}^{i,j}(K)$. The generator in the blue cell is not cancelled by d_1 differential.

7.3. Detection of knots

One of the most essential topological questions one may ask about a knot invariant is which knots it detects. In 2011, Kronheimer and Mrowka [18] proved Khovanov homology detects the unknot. They construct a spectral sequence with E_1-page the (reduced) Khovanov homology and converging to a homological knot invariant (to be precise: instanton Floer homology of the sutured knot complement) already known to detect the unknot.

Since then, many other detection results of Khovanov homology have been obtained: It is now known that Khovanov homology detects the unlink [19], the trefoil [20], the Hopf link [21], the split links [22], the Figure-8 knot [23] and more.

However, there are many knots that Khovanov homology cannot distinguish. Watson [24] gave the first examples of distinct knots with the identical Khovanov homology. These are some infinite families of Kanenobu knots, which are also the first examples of distinct knots with the same Jones polynomial [25].

Although it is known Khovanov homology detects the unknot, it remains unknown whether the Jones polynomial detects the unknot. In the remainder of this section, we describe a way to construct knots with trivial Jones polynomial and then discuss how Khovanov homology can be used to show they are trivial.

Definition 7.1. *A symmetric union of J is a knot obtained by inserting crossings along the axis of the symmetry in $J\# -J$ and is denoted by $(D_J \cup -D_J)(n_1, \ldots, n_k)$ where k is the number of twist regions and $n_i \neq 0$ is the signed number of crossings inserted in the i^{th} twist region.*

This construction is due to Kinoshita and Terasaka [26]. On the left of Figure 34 is the Kinoshita-Terasaka knot presented as $(D_U \cup -D_U)(2)$. For a crossing in the i^{th} twist region, D_1 is a symmetric union $(D_J \cup -D_J)(n_1, \ldots, n_i - 1, \ldots, n_k)$ and D_0 is a link with 2-components. See Figure 34 for such a triple. Kanenobu knots are symmetric unions of 4_1 with two twist regions and for either choice of a region the resolution D_1 is the 2-component unlink. Having D_1 to be the 2-component unlink — the simplest link with 2-components — the skein relation and the long exact sequence associated to a crossing along the symmetry axis yields the results in the work of Kanenobu and Watson.

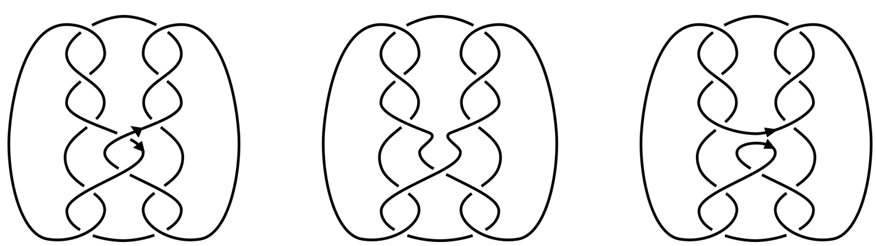


FIGURE 34. The skein triple (D, D_0, D_1) resulting from the resolutions of a (negative) crossing of a symmetric union diagram of the Kinoshita-Terasaka knot along the axis of mirror symmetry

Consider the following formula for the Jones Polynomial of a symmetric union with one twist region in terms of the Jones Polynomial of J and the number of inserted crossings given by Tanaka.

Theorem 7.5 ([27]). *Let K be a symmetric union $(D_J \cup -D_J)(n)$, then*

$$t^n V_K(t) + (-1)^n V_K(t^{-1}) = (t^n + (-1)^n) V_J(t) V_J(t^{-1}).$$

In particular, if K is amphichiral, then $V_K(t) = V_J(t)V_J(t^{-1})$.

Let $J = U$. Then, if K is amphichiral, it follows from the above theorem that $V_K(t) = 1$. Thus, if there is such a nontrivial K, there would be a nontrivial knot with trivial Jones polynomial. However, using Khovanov homology the first author ([28]) showed that such a knot is always trivial.

Exercise 28. *Let K_n be a symmetric union $(D_U \cup -D_U)(n)$. Use Khovanov homology's unknot detection result, the Lee spectral sequence and the long exact sequence of the triple to show that if K_n is nontrivial, then there exists a bi-grading (a, b) such that $\mathrm{Kh}^{a,b}(K_n) \ncong \mathrm{Kh}^{-a,-b}(K_n)$, and hence K_n cannot be amphichiral.*

7.4. Obstructing sliceness

An important application of Theorem 6.1 is to obstruct sliceness. Although there are plenty of such obstructions, Piccirillo's proof of the non-sliceness of Conway knot showcases s as a powerful invariant to obstruct sliceness.

The Conway knot is a (positive) mutant of the Kinoshita-Terasaka knot, which is a slice knot — in fact a symmetric union of U. The mutation preserves many properties of knots such as the Alexander polynomial, the Jones polynomial, the homeomorphism type of the double branched cover, hyperbolic volume, even Khovanov homology and more. Therefore, most of the obstructions become useless when it comes to determine whether a mutants of a slice knot is slice or not. The first examples of non-slice knots that are mutant of slice knots are given by Kirk and Livingston [29] via a detailed analysis of metabelian sliceness obstructions. These metabelian invariants also vanish for topologically slice knots. Because the Conway knot is topologically slice as it has Alexander polynomial 1, these methods do not apply to determine the sliceness status of the Conway knot.

The 0-knot trace $X_0(K)$ is a 4-manifold obtained by attaching a 0-framed 2-handle to B^4 with attaching sphere K. A topological property of $X_0(K)$ determines the sliceness of K:

Lemma 7.6. *K is slice if and only if $X_0(K)$ smoothly embeds into S^4.*

So, if two knots K and K' have diffeomorphic 0-knot traces $X_0(K) \cong X_0(K')$, then either both of K and K' are slice or not slice.

There are a few techniques to construct pairs of knots with the same trace such as the dualizable patterns construction, developed by Akbulut in [30]. Following this construction, Piccirillo builds K' with the same trace as the Conway knot, see [31, Figure 2]. Since K' has Alexander polynomial 1, the techniques used in [29] are not applicable to determine the sliceness of K', but fortunately s is.

Exercise 29. *Use Table 1 in [31] to show $s(K') = 2$. Hence, conclude that K' is not slice, and so the Conway knot is not slice.*

7.5. Distinguishing slice disks

Any link cobordism $\Sigma : D_0 \to D_1$ gives rise to a chain map

$$\mathrm{CKh}(\Sigma) : \mathrm{CKh}^{i,j}(D_0) \to \mathrm{CKh}^{i,j+\chi(\Sigma)}(D_1)$$

which induce a homomorphism

$$\mathrm{Kh}(\Sigma) : \mathrm{Kh}(D_0) \to \mathrm{Kh}(D_1).$$

The following theorem is (almost) conjectured by Khovanov and proved by Jacobsson.

Theorem 7.7 ([32])**.** *For oriented links L_0 and L_1, presented by diagrams D_0 and D_1, and oriented link cobordism Σ from L_0 to L_1, the induced homomorphism*

$$\mathrm{Kh}(\Sigma) : \mathrm{Kh}(D_0) \to \mathrm{Kh}(D_1)$$

is invariant up to multiplication by -1 under ambient isotopy of Σ fixing $\partial\Sigma$ setwise.

Theorem 7.8 ([33])**.** *The knot K in Figure 35 bounds two ribbon disks Σ and Σ' which are labeled by red and blue dotted arcs respectively. Σ and Σ' are diffeomorphic but are not isotopic rel boundary.*

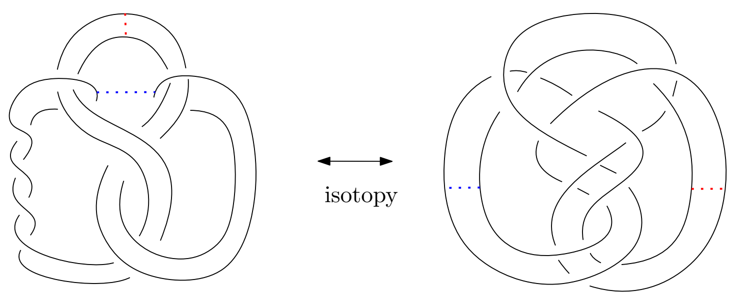


FIGURE 35. Two distinct diagrams of the Akbulut knot K

The diagram on the left in Figure 35 is how K appears in [33]. Akbulut distinguishes the disks in Theorem 7.8 using Donaldson theory — if these two ribbons were isotopic relative to the boundary then the Akbulut cork wouldn't be a cork [34].

Recently, in [35], Hayden and Sundberg distinguished two ribbon disks bounded by the Akbulut knot K using Khovanov homology, and the diagram on the right in Figure 35. Here we will write a proof using the diagram on the left following their strategy, which relies on Theorem 7.7.

Assume two disks Σ and Σ' which are isotopic relative to the boundary. Then by Theorem 7.7 the induced maps must be the same up to multiplication by -1.

$$\mathrm{Kh}(\Sigma) = \pm\mathrm{Kh}(\Sigma') : \mathrm{Kh}^{i,j}(K) \to \mathrm{Kh}^{i,j+1}(\emptyset)$$

Notice that the only non trivial chain group in $\mathrm{CKh}(\emptyset)$ is supported in degree $(0,0)$, and $\chi(\Sigma)=1$. Here, to prove Σ and Σ' are exotic we will find a cycle $\sigma \in \mathrm{CKh}^{0,-1}(K)$ so that $\mathrm{CKh}(\Sigma)(\sigma) = 0 \in \mathrm{CKh}^{0,0}(\emptyset)$ and $\mathrm{CKh}(\Sigma')(\sigma) = 1 \in \mathrm{CKh}^{0,0}(\emptyset)$. The Seifert resolution is always supported in degree $i=0$, so it is logical to start with the Seifert resolution and modify it to get a cycle with quantum grading at $j=-1$.

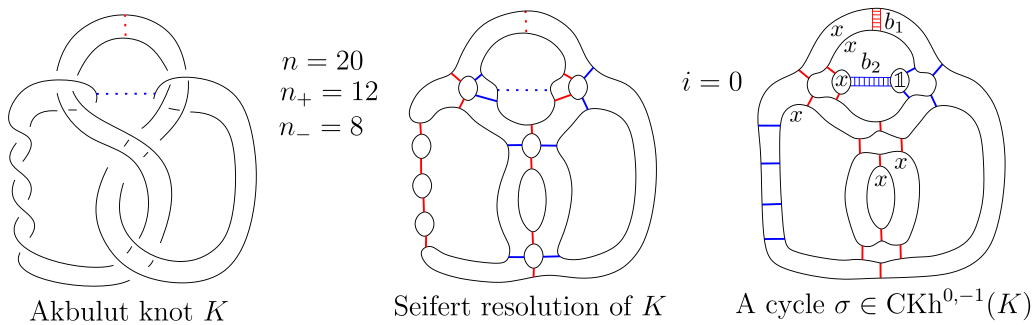


FIGURE 36.

In Figure 36 in the middle we see the Seifert resolution of K. To make it a cycle one can use Elliot's algorithm [36] which translates here as: A state σ is a cycle if and only if every red arc (0-resolution) connects two different x-labeled circles and it is merging when it is changed to 1-resolution. The rest of the argument is just following the image of the cycle σ under the maps $\mathrm{CKh}(\Sigma)$ and $\mathrm{CKh}(\Sigma')$. These maps are a composition of chain maps induced by Morse moves and Reidemeister moves. We only recall the maps we need to follow the proof. In [35] all the maps are explicitly given in tables.

Morse move induced chain maps:

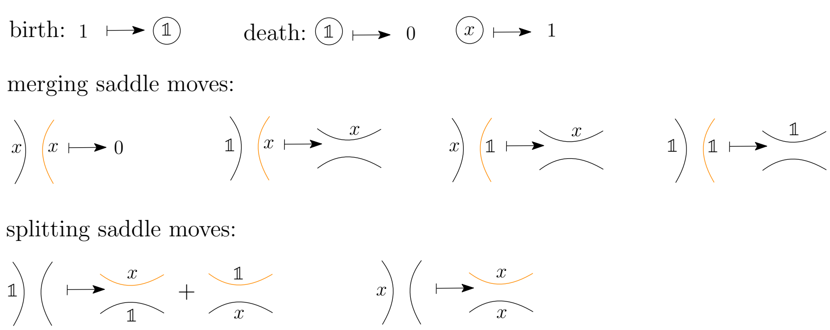


FIGURE 37.

Reidemeister move induced chain maps: We will be using only the followings.

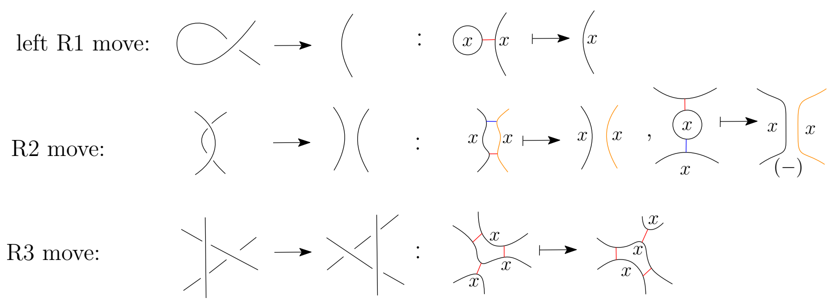


Figure 38.

Now we can finish the proof. $\mathrm{CKh}(\Sigma)(\sigma) = 0$ since the red band in the right picture of Figure 36 merging two x-labeled circles. Now, Figure 39, Figure 40, and Figure 41 show $\mathrm{CKh}(\Sigma')(\sigma) = 1$, hence complete the proof.

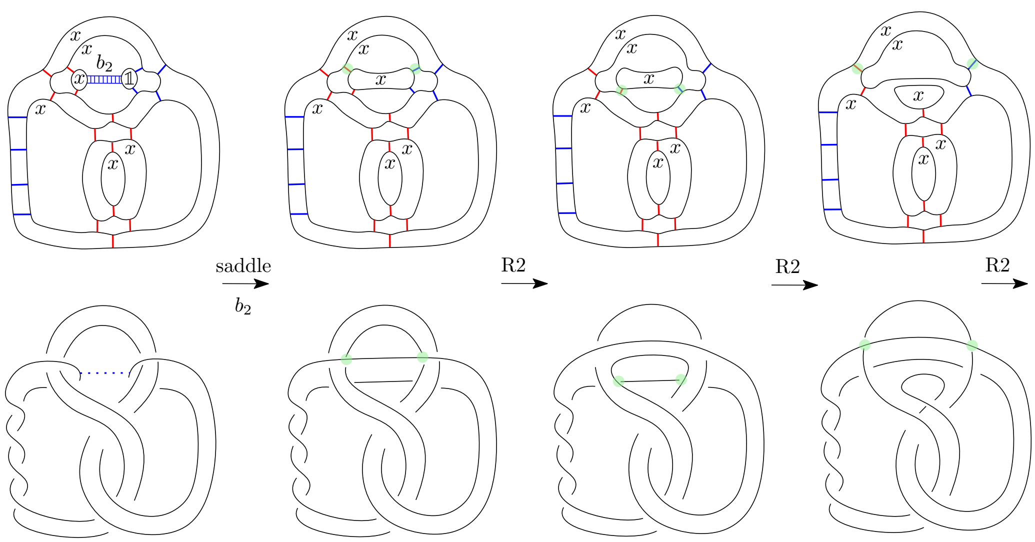


Figure 39.

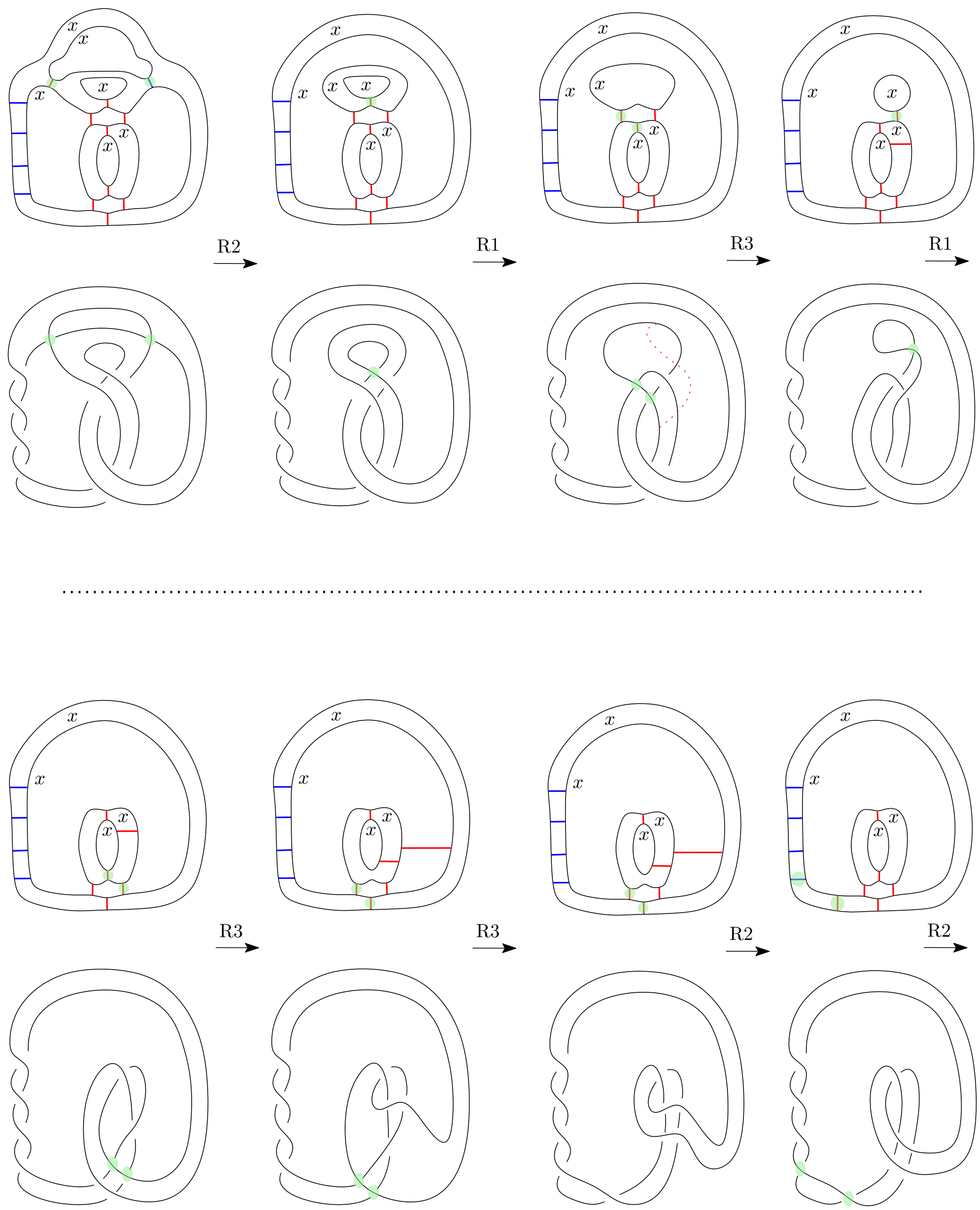
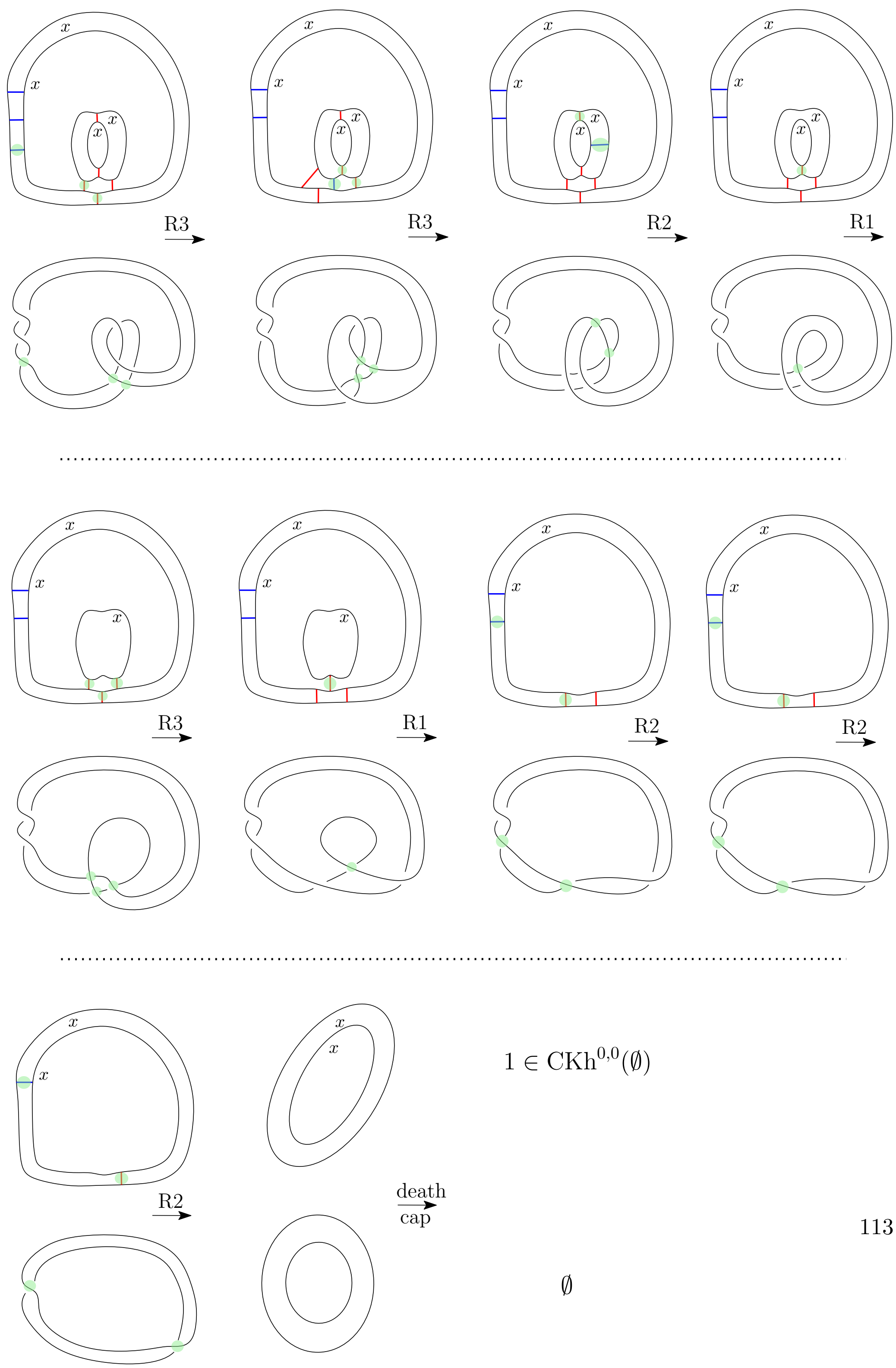


FIGURE 40.



R3
R3
R2
R1
R3
R1
R2
R2
R2
death
cap
$1 \in \mathrm{CKh}^{0,0}(\emptyset)$
$\emptyset$

FIGURE 41.

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