Proceedings of 27^{th} Gökova
Geometry-Topology Conference
pp. 57 – 163

Algebraic groups in action

Mahir Bilen Can

Abstract. These are the expanded lecture notes of the author's webinar series at the Gökova Geometry Topology Institute between the dates August 23 - August 26, 2021. The main goal of these lecture series was to familiarize the audience with some of the major developments in the theory algebraic group actions that took place in the last several decades. Most of the lectures were geared towards non-experts. While introducing some basic notions of Lie groups and algebraic groups in a down-to-earth manner, we tried to provide a short but self-contained overview of the classification of the reductive group actions of complexity 0.

Contents

Key words and phrases. Algebraic group actions, spherical varieties, toric varieties, symmetric spaces, linear algebraic monoids.

1. Introduction

In any topological investigation of manifolds, compactness[1] is an ultimate tool to have since it enables combinatorial methods. For algebraic varieties[2], the most natural topology is the Zariski topology, which is usually not Hausdorff. Therefore, the ordinary compactness notion should be replaced with a notion of "completeness". A variety X is called *complete* if for every variety Y, the second projection $X \times Y \to Y$ is a closed mapping in the Zariski topology. In [77] (Géométrie algébrique et géométrie analytique, 1955), Serre shows that an algebraic variety X defined over $\mathbb{C}$ is compact in the classical Hausdorff topology if and only if X is complete[3]. Important examples of complete varieties include all projective varieties[4]. In particular every projective complex algebraic variety is compact. It turns out that all smooth complete algebraic surfaces are projective varieties. However, there exist singular nonprojective complete algebraic surfaces as well as smooth nonprojective complete three-folds; see the references in [36, Remark 4.10.2]. For a simple example that is close to the spirit of this text, see [26, Example 4.2.13]. Later we will see that there are compact complex Lie groups which are not projective.

Many interesting manifolds (resp. algebraic varieties) are naturally equipped with the Lie group (resp. algebraic group) actions on them. An orbit of a Lie group or an algebraic group action is called a *homogeneous space*. Clearly, the group itself is a homogeneous space under each of its left-, right-, or two-sided actions. Other examples of homogeneous spaces include all Grassmann manifolds, all complex tori, all projective spaces, all spaces of constant curvature, such as the euclidean spaces, the spheres, as well as certain finite quotients of (pseudo)orthogonal groups, all symmetric spaces. While some of these orbit spaces are compact in the Hausdorff topology many of them are not complete. The theory of compactifications of homogeneous spaces of Lie groups is a vast area of geometry with connections to almost all known areas of mathematics. In these notes, we will be concerned with a small combinatorial corner of the theory of equivariant completions for only a certain type of algebraic group actions. In fact, most of our discussion will not be specifically about completions of orbits but about their "equivariant embeddings".

Definition 1.1. Let G be a Lie or an algebraic group, and let $H \subseteq G$ be a closed subgroup. A *G-equivariant embedding of G/H* is an irreducible G-variety X with an open orbit isomorphic to G/H.

At the heart of the theory of equivariant embeddings of orbits are the equivariant embeddings of the group itself. If the group itself is compact then it may look at first sight that no new information would be gained from its equivariant embeddings. This is

[1] A locally compact Hausdorff topological space is compact if and only if for every topological space Y, the projection map $X \times Y \to Y$ is a closed map.

[2] For us, an algebraic variety is a reduced and separated scheme of finite type defined over an algebraically closed field.

[3] Very roughly summarized, [77] states that the analytification functor $X \rightsquigarrow X^{\mathrm{an}}$ from proper $\mathbb{C}$-schemes to compact Hausdorff $\mathbb{C}$-analytic spaces is fully faithful.

[4] The vanishing set of a collection of homogeneous polynomials in $\mathbb{P}^n$ is called a projective variety.

correct to some degree for algebraic groups. Nevertheless, many complex algebraic groups are obtained from compact Lie groups as we will explain later. We note that any complex algebraic group has a unique structure of a complex Lie group. The converse of this statement is not true as it is easily seen from the following example: Let A be denote the image in $\mathbb{C}^2$ of the holomorphic map $s \mapsto (e^s, e^{is})$, $s \in \mathbb{C}$. Then A is a one-dimensional complex Lie group since it is biholomorphically isomorphic to $(\mathbb{C}, +)$. However it is easy to check that there is no nonzero polynomial in two (or more) variables $f(x, y) \in \mathbb{C}[x, y]$ such that $f(e^s, e^{\sqrt{-1}s}) = 0$ for all $s \in \mathbb{C}$. Therefore, A is not a complex algebraic variety. Since not every (complex) Lie group is algebraic, we want to know where the complete Lie groups stand in relation with the algebraic groups. While answering this question, we will explain why the existence of a faithful finite dimensional linear representation of a Lie group is important for algebraicity. Let us proceed with a real example.

The three dimensional unit sphere in $\mathbb{S}^3 \subset \mathbb{R}^4$ can be viewed as the special unitary group via the identification,

$$\mathbf{SU}_2(\mathbb{C}) := \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} :\ a\bar{a} + b\bar{b} = 1,\ a, b \in \mathbb{C} \right\} \longrightarrow \mathbb{S}^3 := \{x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\}$$

$$\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \longmapsto (\mathrm{re}(a), \mathrm{im}(a), \mathrm{re}(b), \mathrm{im}(b)).$$

Hence, it admits the two-sided multiplication action of $\mathbf{SU}_2(\mathbb{C})$. Since $\mathbb{S}^3$ is a compact subset of $\mathbb{R}^4$, the Lie group $\mathbf{SU}_2(\mathbb{C})$ is compact as a real Lie group. But notice that the equation $a\bar{a} + b\bar{b} = 1,\ a, b \in \mathbb{C}$ that defines $\mathbf{SU}_2(\mathbb{C})$ as a matrix group is not really a polynomial equation. Therefore we cannot regard $\mathbf{SU}_2(\mathbb{C})$ as a complex Lie group or as a complex algebraic variety[5] although it is still a "real algebraic group".

An *algebraic variety in* $\mathbb{R}^n$[6] is a subset of the form

$$X_S := \{x \in \mathbb{R}^n :\ f(x) = 0 \ \text{ for every } f \in S\},$$

where S is a set of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$. Clearly, any real polynomial, that is to say a polynomial with real coefficients, is also a complex polynomial. Therefore, regarding S as a subset of $\mathbb{C}[x_1, \ldots, x_n]$ we obtain an affine complex algebraic variety,

$$X_S(\mathbb{C}) := \{x \in \mathbb{C}^n :\ f(x) = 0 \ \text{ for every } f \in S\},$$

called the *complex points of the variety* X_S, or the $\mathbb{C}$-*form of* X_S. Conversely, let $X \subseteq \mathbb{C}^n$ be an affine complex algebraic variety. A *real form of* X (or, an $\mathbb{R}$-*form of* X) is a real

[5]A group G equipped with the structure of a differentiable manifold (resp. complex manifold, resp. complex algebraic variety) is said to be a *real Lie group* (resp. a complex Lie group, resp. a complex algebraic group) if the underlying group operations are smooth maps (resp. holomorphic maps, resp. algebraic morphisms). Equivalently, a real Lie group (resp. a complex Lie group, resp. a complex algebraic group) is a group object in the category of differentiable manifolds (resp. category of complex manifolds, resp. category of algebraic varieties).

[6]Strictly speaking, this is the set of "$\mathbb{R}$-rational points" of the complex affine scheme defined by the polynomials in S.

algebraic variety X_0 in $\mathbb{R}^n$ such that the $\mathbb{C}$-form of X_0 is isomorphic to X as a complex algebraic variety. For the convenience of reader, in Appendix 12, we included a brief discussion about the forms of an algebraic set defined over an arbitrary field.

Let $\mathbb{C}[X]$ denote the $\mathbb{C}$-algebra of $\mathbb{C}$-valued polynomial functions on X. Let $z_1, \dots, z_n$ denote the restriction to X of the coordinate functions of $\mathbb{C}^n$. Then $\mathbb{C}[X]$ is generated by $z_1, \dots, z_n$ over $\mathbb{C}$. Let $\overline{\mathbb{C}[X]}$ denote the $\mathbb{C}$-algebra $\mathbb{C}[\bar{z}_1, \dots, \bar{z}_n]$, where $\bar{z}_i$ $(i \in \{1, \dots, n\})$ is the complex conjugate of the polynomial function z_i $(i \in \{1, \dots, n\})$. Let Y be another complex affine algebra. A continuous map of complex affine varieties $f : X \to Y$ is called an *anti-holomorphic morphism* if $f^*\mathbb{C}[Y] \subseteq \overline{\mathbb{C}[X]}$. In this case, evidently, $\bar{f} : X \to Y$ is a morphism of affine varieties. For every real affine variety $X_0 \subset \mathbb{R}^n$ there exists a unique anti-holomorphic automorphism of the complexification $\tau : X_0(\mathbb{C}) \to X_0(\mathbb{C})$ such that $\tau(x_0) = x_0$ for all $x_0 \in X_0$ and $\tau^2(x) = x$ for all $x \in X_0(\mathbb{C})$. It turns out that with a small restriction, the converse of this observation holds as well: For every anti-holomorphic involutory morphism $\tau : X \to X$, where X is a complex affine algebraic variety, if the fixed point set X_0 of τ contains a smooth point of X, then X_0 is a real form of X. Our first example, the special unitary group $\mathbf{SU}_2(\mathbb{C})$, is a complete (compact) real form of $\mathbf{SL}_2(\mathbb{C})$, which is a real hypersurface in $\mathbb{R}^4$. A *compact group* is a compact Hausdorff topological group. We proceed with an additional assumption that our compact group is locally euclidean. It is well-known that such a group has the structure of a Lie group[7]. A compact group called a *linear compact group* if it admits an injective topological group homomorphism into $\mathbf{GL}(V)$, where V is a finite dimensional complex vector space. Thanks to Stone-Weierstrass theorem, it can be shown without much difficulty that every linear compact group has the structure of a real algebraic variety in $\mathbb{R}^n$ for some $n \in \mathbb{Z}^+$. In particular, every linear compact group has the structure of a linear real algebraic group. Thus, every linear compact group K has a complexification $K(\mathbb{C})$, which is a complex affine algebraic group[8]. It turns out that $K(\mathbb{C})$ is a *reductive group*, that is, an affine algebraic group which does not contain any nontrivial normal unipotent subgroup. Furthermore, the assignment $K \rightsquigarrow K(\mathbb{C})$ defines an equivalence of categories between the category of linear compact groups and the category of complex reductive algebraic groups. In summary, we learn from this equivalence of categories that

(1) for every reductive complex algebraic group G there exists a unique complete (compact) real form K of G such that $K(\mathbb{C}) \cong G$,
(2) every reductive complex algebraic group G is *linear*, that is to say, it admits a faithful algebraic group representation on some finite dimensional complex vector space.

[7]Hilbert's fifth problem asked if every locally euclidean topological group is a Lie group or not. The answer is yes. For compact groups this was first proved by von Neumann in [86]. A detailed explanation of the general case is given by Montgomery and Zippin [55, Chapter IV, §4.10].

[8]As far as we can tell from the literature on this subject, it was first Chevalley [25, Chapter VI, §VIII] who constructed a complex affine algebraic group for each linear compact group. An easily accessible treatment of the "complexifications of real analytic groups" can be found in Hochschild's book [38, Chapter XVII].

The above discussion shows that the representation theory of linear compact groups is related to the theory of reductive groups in a fundamental way. We note that reductive groups are not complete unless they are zero dimensional. Next, we will consider complete (compact) complex Lie groups. The completeness assumption on such a group puts a very severe restriction on the multiplicative structure.

Let G be a connected complex Lie group, let $e \in G$ be the identity element. For every vector v in the tangent space of G at e, there exists a unique holomorphic one-parameter semigroup homomorphism $\psi_v : \mathbb{C} \to G$ such that $d(\psi_v)_0 : T_0\mathbb{C} \to T_eG$ maps $1 \in T_0\mathbb{C}$ to $v \in T_eG$. Consequently, we get the *exponential map* $\exp : T_eG \to G$ by $\exp(v) := \psi_v(1)$. In particular, we have $\psi_v(t) = \exp(tv)$ (by the uniqueness of ψ_v). Thanks to implicit function theorem, we know that the exponential map exp is a local diffeomorphism. This fact immediately implies that the kernel of exp is a discrete subgroup of the vector space T_eG. We will mention another important property of the exponential map. The uniqueness property of ψ_v also implies that the differential of any complex Lie group automorphism $f : X \to X$ commutes with exp:

$$f(\exp(v)) = \exp(df_e(v)) \qquad (v \in T_eG). \tag{1.2}$$

Let $C_g : G \to G$ $(g \in G)$ denote the conjugation map $x \mapsto gxg^{-1}$, $x \in G$. Then we see that $C_g(\exp(v)) = \exp(d(C_g)_e(v))$ for every $v \in T_eG$.

We now proceed with the assumption that G is a complete connected complex Lie group. It follows that G does not admit any non-constant holomorphic map to an affine space, hence, its adjoint representation on T_eG is the trivial representation. In other words the differentials of the conjugation maps, dC_g $(g \in G)$, are the identity maps. In particular by (1.2) we have $C_g(\exp(v)) = \exp(v)$ for every $v \in T_eG$. This means that $\exp(v)$ is contained in the center of G for every $v \in T_eG$. But G is connected, so, the image of exp generates G as a group. We conclude from these observations that every complete connected complex Lie group has the structure of an abelian group.

Furthermore, G is isomorphic to a quotient of a vector space T_eG by a discrete subgroup, $\ker\exp$. The induced map $T_eG/\ker\exp \to G$ is a holomorphic map between two equal dimensional complex manifolds. Therefore, it is an isomorphism of complex Lie groups. Also, since lattices are the only discrete subgroups of vector spaces with compact quotient, $\ker\exp$ is a lattice.

Notice that, since a complete complex Lie group G cannot be a compact *linear* group, we cannot apply the easy 'complexification' method to conclude that G is an algebraic group. In fact, this is not true beyond dimension one. The crucial point here is to know whether the complete complex Lie group admits an embedding into a projective space or not. Indeed, the Theorem of Chow states that if X is a complete algebraic variety, and $Y \subseteq X$ is a closed analytic subset of the analytic space structure on X, then Y is Zariski closed in X. In [56], Mumford shows that for a complete complex Lie group of the form $X := V/L$, where V is a g-dimensional complex vector space, $L \subseteq V$ a lattice, the following statements are equivalent:

(1) The underlying variety of X is a projective variety.
(2) There exist g algebraically independent meromorphic functions on X.
(3) There is a positive definite hermitian form H on V such that the image of H on $L \times L$ is integral.

A complete algebraic group is called an *abelian variety*. We just saw that the complete complex Lie groups are closely related to the abelian varieties although there are subtle differences (the existence of a projective embedding, also known as a *polarization*). We also know that the complete real Lie groups lead to the theory of reductive groups, which are affine algebraic groups. Of course, there are many algebraic groups which are neither affine nor projective. But we will not discuss the completions of non-affine algebraic groups beyond this introduction. The reason is a classical theorem of Chevalley (1953), which states that a connected algebraic group is an extension of an abelian variety by a connected affine algebraic group. We will restrict our attention mostly to the equivariant embeddings of the orbits of connected affine algebraic groups. Our aim is to show that this study not only produces new geometric and combinatorial structures related to the actions of affine groups but also reveals new facts about these groups. Furthermore, surprisingly, some equivariant affine embeddings of affine groups appear rather naturally as homogeneous vector bundles on abelian groups see [8, 21]. This already indicates the strong categorical relationship between affine equivariant embeddings and the category of homogeneous vector bundles on abelian varieties.

Some highlights for the reader: The study of the compact linear real Lie groups is equivalent to the study of the complex reductive algebraic groups. A complex reductive algebraic group is not complete unless it is a finite group. The compact complex Lie groups are precisely the quotients of vector spaces by their sublattices. A compact complex Lie group is an abelian variety if it is polarized, which means that it has an embedding into a projective space by some meromorphic functions. Now an outstanding question that motivates our other lectures is: how do we describe the completions of reductive groups?

2. Preliminaries

Most of the results that we review in this section can be found in the standard resources, [6, 39, 80] (for algebraic groups) and [36] (for algebraic varieties). However, in many places we use our notation.

Throughout our text, we work with algebraic varieties that are defined over an algebraically closed field. We fix the notation k for such a field. In some places we will assume that $k = \mathbb{C}$. To work over algebraically closed fields different than $\mathbb{C}$, it is convenient to use the following definition of an algebraic variety: An *algebraic variety* is a reduced separated k-scheme of finite type. Notice that, unlike the definition used in [36], our definition does not require that an algebraic variety be irreducible as a topological set. The reason behind our convention is that, algebraic groups may have more than one connected components. An *algebraic group* is a group scheme over k whose underlying scheme is a smooth algebraic variety. If $k = \mathbb{C}$, then an algebraic group defined over k will be called a *complex algebraic group*.

A morphism of algebraic groups is a regular map which is also a group homomorphism. For example, the map $\mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*$, $z \mapsto (1/z, z^2)$ is a morphism of complex algebraic groups. If the underlying variety of an algebraic group G is affine, then G is called an *affine (or linear) algebraic group*. In the rest of this text we will be concerned with the affine algebraic groups only. Note that every finite group is both affine and complete at the same time. These are precisely the zero-dimensional algebraic groups. Also, the only connected, zero-dimensional algebraic group is the trivial group.

The multiplicative group structure on $k\backslash\{0\}$ is often denoted by $\mathbb{G}_m$ while the additive group structure on k is occasionally denoted by $\mathbb{G}_a$. Both of these algebraic groups are connected, one-dimensional, and affine. Any affine algebraic group having these properties is isomorphic to either $\mathbb{G}_m$ or $\mathbb{G}_a$.

2.1. Some algebraic geometry terminology

We continue to review basic notions from algebraic geometry.

Let X be an irreducible variety. First we assume that X is a quasi-affine variety in $\mathbb{A}^n$. This means that X is the intersection of a Zariski-open set with an irreducible subvariety in $\mathbb{A}^n$. A function $f : X \to k$ is called *regular at a point* $p \in X$ if the exists an open neighborhood U of p, where f is a rational function of the form $f = h/g$, for some polynomials $h, g \in k[x_1, \dots, x_n]$ where g is nowhere zero on U. If f is regular at every point of X, then f is said to be a *regular function on* X. Next, we assume that X is a quasi-projective variety in $\mathbb{P}^{n-1}$ (where $n \geqslant 1$). This means that X is the intersection of a Zariski-open set with an irreducible projective variety in $\mathbb{P}^{n-1}$. The notion of a regular function f on $U \subseteq X$ is defined as before but with the additional requirement that the polynomials h and g are homogeneous and they are of the same degree. If U is an open subset of X, then the set of regular functions on U will be denoted by $\mathcal{O}_X(U)$. It is easy to check that $\mathcal{O}_X(U)$ is a ring with respect to point-wise addition and multiplication of regular functions. The *field of rational functions on* X, denoted by $k(X)$, is defined as follows: an element of $k(X)$ is an equivalence class of pairs $\langle U, f\rangle$ where U is a nonempty open subset of X, and $f \in \mathcal{O}_X(U)$. Two pairs $\langle U, f\rangle$ and $\langle V, g\rangle$ are identified if $f = g$ on $U \cap V$. The elements of $k(X)$ are called *rational functions* on X. To ease our notation, whenever it is convenient, an element $\langle U, f\rangle$ from $k(X)$ will be denoted by f. The *local ring of an irreducible subvariety* $Z \subseteq X$, denoted by $\mathcal{O}_{Z,X}$, is the subring of elements $\langle V, f\rangle \in k(X)$ such that $V \cap Z \neq \varnothing$. In other words, we have $\mathcal{O}_{Z,X} := \{f \in k(X) : \ f \text{ is defined on an open subset } V \subseteq X \text{ such that } V \cap Z \neq \varnothing\}$.

Exercise: 1) Show that $\mathcal{O}_{Z,X}$ is indeed a local ring. 2) Show that the residue field of $\mathcal{O}_{Z,X}$ is the field of rational functions on Z. Hint: Assume that X is an affine variety of the form $X := \operatorname{Spec} R$, where R is an integral domain. Let $Z \subseteq X$ be an irreducible subvariety defined by a prime ideal $P \subseteq R$. Show that the localization of R at P, denoted by R_P, is $\mathcal{O}_{Z,X}$. Show that the residue field of R_P is isomorphic to the field of fractions of R/P.

The assignment of the rings $\mathcal{O}_X(U)$ to each open subset $U \subseteq X$ defines a sheaf of rings on X. It is called the *sheaf of regular functions of* X, or the *structure sheaf of* X.

Exercise: Let p be a point in X. Show that the stalk of the sheaf $\mathcal{O}_X$ at p[9] agrees with the local ring of the point p viewed as a subvariety of X.

For the biggest open set, $U = X$, there are several different names and notations for $\mathcal{O}_X(X)$. Sometimes it is denoted by $k[X]$, and sometimes it is denoted by $H^0(X, \mathcal{O}_X)$. In the former notation it is often called the *k-algebra of regular functions of* X; if X is affine, then it is called the *coordinate ring of* X. If the notation $H^0(X, \mathcal{O}_X)$ is used, then it is customarily called the *k-algebra of global sections of the sheaf* $\mathcal{O}_X$. Finally, let us mention that the notation $\mathcal{O}(X)^*$ stands for the group of invertible elements of the multiplicative semigroup of the ring of regular functions $\mathcal{O}(X)$. For any irreducible variety X, we will denote the quotient group $\mathcal{O}(X)^*/k^*$ by $E(X)$.

2.2. Basic abstract representation theory terminology

Let V be a *k-vector space*, that is, a vector space defined over k. Let S be a group. An *(abstract) k-representation* of S is a group homomorphism $\rho : S \to GL(V)$. If such a homomorphism exists, then V is sometimes called an *S-module*. In practice, when a k-representation is mentioned one usually writes V or (ρ, V) instead of $\rho : S \to GL(V)$. The *dimension* of a k-representation is the vector space dimension of V.

Example 2.1. Let S be an abelian group. Let k be an algebraically closed field. Let (ρ, V) be a finite dimensional k-representation of S. Since S is abelian, there is a basis $\{v_1, \dots, v_n\}$ of V such that, for each $i \in [n]$, v_i is a joint eigenvector for all $g \in S$. By using this basis we identify $GL(V)$ with $\mathbf{GL}_n(k)$. Then the image of S under $\rho : S \to \mathbf{GL}_n(k)$ consists of diagonal matrices.

Example 2.2. Let S_n denote the symmetric group of permutations of the set $\{1, \dots, n\}$. We will view S_n as the set of $n \times n$ 0/1 matrices with exactly one 1 in each row and each column. More precisely, if σ is an element from S_n, then we will "represent" it by the matrix $\rho(\sigma) = (\sigma_{i,j})_{i,j=1}^n$, where

$$\sigma_{i,j} := \begin{cases} 1 & \text{if } \sigma(i) = j; \\ 0 & \text{otherwise.} \end{cases}$$

[9]Let $\mathscr{F}$ be a (pre)sheaf of groups on a topological space X. Let p be a point of X. The *stalk of $\mathscr{F}$ at p* is the direct limit of the groups $\mathscr{F}(U)$ for all open sets U containing p via the restriction maps of the (pre)sheaf.

For example, for $S_3 = \{123, 213, 132, 231, 312, 321\}$, we have $\{\rho(\sigma) : \sigma \in S_3\} =$

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}.$$

It is easy to check that the assignment $\sigma \mapsto \rho(\sigma)$, where $\sigma \in S_n$, defines a k-representation of S_n.

Example 2.3. Let H_3 denote the following complex affine algebraic group:

$$H_3 := \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : \ a, b, c \in \mathbb{C} \right\}.$$

In a certain context, H_3 is called the *complex Heisenberg group of* 3×3 *matrices*. We consider the conjugate-matrix multiplication action of H_3 on $\mathbb{C}^3$,

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + \bar{a}y + \bar{b}z \\ y + \bar{c}z \\ z \end{bmatrix}.$$

This is a three dimensional $\mathbb{C}$-representation of H_3.

Example 2.4. Let G denote the $\mathbf{GL}_2(\mathbb{C})$. Let R denote the polynomial ring $R := \mathbb{C}[X, Y]$. Clearly, R is an infinite dimensional vector space. Let A be an element from G such that $A^{-1} := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. It is easy to check that the map $\varphi_A : R \to R$ defined by

$$\varphi_A(f(X, Y)) := f(aX + cY, bX + dY) \qquad (f(X, Y) \in R)$$

is a linear automorphism of R. Furthermore, it is easy to verify that $\varphi_A \circ \varphi_B = \varphi_{AB}$ for every $A, B \in G$. Therefore, the map $\varphi : G \to GL(R)$, $A \mapsto \varphi_A$ defines an infinite dimensional $\mathbb{C}$-representation of G on R.

2.3. Terminology of reductive groups

An abstract group G is called 'simple' if it does not possess any nontrivial normal subgroup. For algebraic groups, this definition is too restrictive in the following sense. A big part of the theory of algebraic groups depends on the theory of Lie algebras. By definition, the *Lie algebra of an algebraic group* G, denoted $\mathrm{Lie}(G)$, is the tangent space of G at the identity element $e \in G$ together with the bilinear product $[,] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$, called the *Lie bracket*, which is induced from the multiplication on G. However, the passage from G to $\mathrm{Lie}(G)$ does not depend on the finite central subgroups of G since it uses the differentials of one-parameter subgroups. For example, consider the special linear group $G := \mathbf{SL}_n(\mathbb{C})$. The center $Z(G)$ of G is isomorphic to the cyclic group of order n. For any subgroup $Z' \subset Z(G)$, the Lie algebra of the quotient G/Z' is equal to the space of traceless $n \times n$ matrices with complex entries. But this is a *simple Lie algebra*, that is, a nonabelian Lie algebra without a nontrivial ideal. Among the quotient groups of G, the only abstract simple group is the one that is given by the full quotient, $G/Z(G) = \mathbf{PSL}_n(\mathbb{C})$. In this regard, it is more natural to call an algebraic group G a *simple algebraic group* (or even better, a *quasi-simple algebraic group*, or an *almost-simple algebraic group*) if it has no nontrivial closed connected normal subgroups. For example, $\mathbf{SL}_2(\mathbb{C})$ is a simple group but not $\mathbf{SL}_2(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C})$. The problem with the second example is that it has more than one "simple factors". Later we will see that the simple algebraic groups are parametrized by certain combinatorial objects called "indecomposable, reduced root systems".

A positive dimensional algebraic group G is called *semisimple* if it has no closed connected commutative normal subgroups except e. For example, $\mathbf{SL}_2(\mathbb{C}) \times \mathbf{SL}_2(\mathbb{C})$ is semisimple but not $\mathbf{GL}_2(\mathbb{C})$. The problem with the second example is that it has a one dimensional center. A Lie algebra $\mathfrak{g}$ is said to be a *semisimple Lie algebra* if $\mathfrak{g}$ has no nontrivial abelian ideals. If $\mathfrak{g}$ is the Lie algebra of G, then it is a semisimple Lie algebra if and only if G is a semisimple algebraic group.

Let G be an algebraic group, and let $\{G_i : \ i \in I\}$ be the set of all minimal, closed, connected, and nonabelian subgroups of G such that $\dim G_i > 0$ for $i \in I$. Then G_i's are called the *simple factors* of G. Indeed, they are simple algebraic groups by definition. If G is semisimple, then the indexing set I is finite. Furthermore, there exists a surjective algebraic group homomorphism $\psi : \prod_{i \in I} G_i \to G$ with finite kernel.[10] The converse of this statement also holds. Another characterization of the semisimplicity is as follows: A connected algebraic group G is semisimple if and only if the commutator subgroup $(G, G) \subseteq G$ is a quotient of G by a finite central subgroup.

Let us assume for a moment that we already know the definition of a reductive group. Let G be a connected reductive group. It follows from [39, Sections 19.5 and 27.5] that G is isomorphic to a quotient of the form $(G_0 \times Z)/Z_0$, where

- $G_0 := (G, G)$;
- Z is a central torus;
- Z_0 is a finite normal central subgroup of $G_0 \times Z$.

This decomposition can be taken as a definition of a connected reductive group. Here is a proper way of introducing the reductive groups.

Definition 2.5. An algebraic group U is called a *unipotent group* if there exists an embedding of algebraic groups $\phi : U \hookrightarrow \mathbf{GL}_n(k)$ for some $n \in \mathbb{Z}^+$ such that, for every $u \in U$, the matrix $\phi(u)$ is unipotent.[11] A *reductive (algebraic) group* is an affine algebraic group that does not contain a nontrivial closed normal unipotent subgroup.

[10] Such homomorphisms are called *algebraic covering maps*, or *isogenies*.

[11] A linear operator is called *unipotent* if 1 is its unique eigenvalue.

Exercise:

(1) Show that every semisimple group is a reductive group.
(2) Show that $\mathbf{GL}_2(\mathbb{C})$ is a reductive group.
(3) Show that $\mathbf{GL}_2(\mathbb{C})$ is isomorphic to $(\mathbf{SL}_2(\mathbb{C}) \times Z)^{Z_0}$, where Z is the group of diagonal constant matrices in $\mathbf{GL}_2(\mathbb{C})$, and Z_0 is the finite subgroup,
$$Z_0 = \{(g,g) \in \mathbf{SL}_2(\mathbb{C}) \times Z : \ g \text{ is diagonal and } g^2 = e\}.$$
(4) Show that $\mathbf{GL}_2(\mathbb{C}) \times \mathbf{GL}_2(\mathbb{C})$ is a reductive group.
(5) Show that the additive one dimensional algebraic group $\mathbb{G}_a = (\mathbb{C}, +)$ is not a reductive group.
(6) Show that the subgroup of upper triangular matrices in $\mathbf{GL}_2(\mathbb{C})$ is not a reductive group.

Let G be an affine algebraic group. A *Borel subgroup* B of G is a maximal closed connected solvable subgroup of G. Let us denote the set of all Borel subgroups of G by $\mathscr{B}(G)$. Then $\cap_{B\in\mathscr{B}(G)} B$ is a normal solvable subgroup of G. The connected component of the identity element in $\cap_{B\in\mathscr{B}(G)} B$ is called the *(solvable) radical* of G; we denote it by $\mathcal{R}(G)$. The maximal closed normal unipotent subgroup of $\mathcal{R}(G)$, denoted by $\mathcal{R}_u(G)$, is called the *unipotent radical* of G. In this notation, we have

- G is semisimple if and only if $\mathcal{R}(G)$ is trivial;
- G is reductive if and only if $\mathcal{R}_u(G)$ is trivial.

2.4. The group of algebraic characters

In this section, we introduce an essential notion for the representation theory of algebraic groups. Let S be an affine algebraic group.

Definition 2.6. An *algebraic character of* S is an algebraic group homomorphism into $\mathbb{G}_m$.

In other words, an abstract group homomorphism $\chi : S \to k^*$ is a character of S if χ is a nowhere vanishing regular function on S. Equivalently, a group homomorphism $\chi : S \to k^*$ is an algebraic character of S if χ is an element of $\mathcal{O}(S)^*$. Clearly, the set of algebraic characters of S is a group under point-wise multiplication.

Let X be an algebraic variety. Recall that $E(X)$ denotes the quotient group $\mathcal{O}(X)^*/k^*$. We will show that if S is an algebraic group, then $E(S)$ is nothing but the group algebraic characters of S. This result was discovered by Rosenlicht ([76]) in the mid-1950s. Our exposition closely follows the paper [47, §1].

Proposition 2.7. *Let S be a connected algebraic group. Let f be an element of $\mathcal{O}(S)^*$. Then f is an algebraic character of S if and only if $f(e) = 1$. In particular, the group of algebraic characters of S is isomorphic to the quotient group $E(S)$.*

Proof. The essential ingredient of the proof is the following observation: If X and Y are two irreducible varieties, then the map $\mathcal{O}(X)^* \times \mathcal{O}(Y)^* \to \mathcal{O}(X \times Y)^*$ is surjective, [47, Proposition 1.1]. In particular, $\mathcal{O}(S)^* \times \mathcal{O}(S)^* \to \mathcal{O}(S \times S)^*$ is surjective. Let $f : S \to k^*$

be a regular function such that $f(e) = 1$. We view f as an invertible regular function on $S \times S$ via $(g_1, g_2) \mapsto f(g_1g_2)$ $(g_1, g_2 \in S)$. Then there exist two invertible regular functions $r_1, r_2 \in \mathcal{O}(S)^*$ such that $f(g_1g_2) = r_1(g_1)r_2(g_2)$ for every $g_1, g_2 \in S$. Notice that, since $f(e) = 1$, the equality $f(g_1g_2) = r_1(g_1)r_2(g_2)$ holds for every $g_1, g_2 \in S$ even if we replace r_i by $r_i/r_i(e)$ for $i \in \{1, 2\}$. But now we see that $f = r_i$ for $i \in \{1, 2\}$. □

The notion of an algebraic action will be introduced properly later in this section. It can be roughly understood as follows. Let S (resp. X) be an algebraic group (resp. an algebraic variety). Let $\varphi : S \times X \to X$ be an abstract group action. If the action map φ is a morphism of varieties, then it called an *algebraic action of S on X*. We are now ready to mention another important result of Rosenlicht.

Proposition 2.8. *Let X be an irreducible variety. Then*

(1) $E(X)$ is a finitely generated free abelian group.
(2) If a connected affine algebraic group S acts algebraically on X, then the corresponding canonical action of S on $E(X)$ is trivial.

In the second part of the proposition, the canonical action of S on $E(X)$ is the one that comes from the action of S on the ring of regular functions, $\mathcal{O}(X)$.

Proof. We will sketch the idea of the proof of (1). We may assume that X is a quasi-projective subvariety of $\mathbb{P}^n$ for some $n \in \mathbb{Z}^+$. Let $\overline{X}$ denote the normalization of the Zariski closure of X in $\mathbb{P}^n$. Let us associate a principal divisor to each invertible regular function on X by using the following assignment:

$$\begin{aligned} \mathrm{div} : \mathcal{O}(X)^* &\longrightarrow \bigoplus_{i=1}^{m} \mathbb{Z}D_i \\ f &\longmapsto \sum_{i=1}^{m} \mathrm{ord}_{D_i}(f)D_i. \end{aligned}$$

Here, $D_1, \dots, D_m$ are the irreducible components of the complement $\overline{X} \backslash X$. For $i \in \{1, \dots, m\}$ and a rational function $f \in k(X)$, the notation $\mathrm{ord}_{D_i}(f)$ stands for the order-of-vanishing of f on the divisor D_i. It is easy to check that div is homomorphism of abelian groups, and the constant functions are contained in its kernel. Thus, div descends to an injective homomorphism on $E(X) \to \bigoplus_{i=1}^m \mathbb{Z}D_i$. But $\bigoplus_{i=1}^m \mathbb{Z}D_i$ is a finitely generated free abelian group. Therefore, $E(X)$ is a finitely generated free abelian group. (2) We will see later that any algebraic action induces a locally finite rational representation on the ring of regular functions. It follows that, for any element f in $\mathcal{O}(X)^*$. the span of $S \cdot f$ in $\mathcal{O}(X)$ is contained in a finite dimensional vector subspace of $\mathcal{O}(X)$. Since S is a connected group, we see that the orbit $S \cdot f \subseteq \mathcal{O}(X)^*$ is an irreducible algebraic subset of $\mathcal{O}(X)$. Then the image of $S \cdot f$ under the quotient map $\pi : \mathcal{O}(X)^* \to \mathcal{O}(X)^*/k^*$ is an irreducible subset as well. Since $E(X)$ is a lattice, its irreducible subsets are its points. Therefore, we conclude that $S \cdot f$ is of the form $S \cdot f = k^*f$. In particular, the action of S on the quotient group $E(X) = \mathcal{O}(X)^*/k^*$ is trivial. □

We continue with an important observation about the characters of semisimple groups.

Lemma 2.9. *Let G be a connected semisimple algebraic group of dimension at least two (or three!). Then G does not possess any nontrivial algebraic character. Consequently, we have $E(G) = 1$, or $\mathcal{O}(G)^* = k^*$.*

Proof. Let $\chi : G \to k^*$ be an algebraic character. Then the restriction of χ to any normal subgroup $G' \subseteq G$ is an algebraic character of G'. Therefore, it suffices to prove our claim for the simple algebraic groups. The kernel of an algebraic group homomorphism is an algebraic subgroup. Therefore, if $\chi : G \to k^*$ is an algebraic character of a simple algebraic group G, then $\ker \chi$ is a normal subgroup of G. Let H denote the connected component of the identity in $\ker \chi$. We proceed with the assumption that χ is a nontrivial character; $\ker \chi$ is a proper subgroup of G. Then H is a normal subgroup of finite index in $\ker \chi$, [39, Proposition 7.3]. Since $\ker \chi$ is normal in G, so is H. But G is a simple algebraic group and H is connected. Hence, we see that $H = \{e\}$. Therefore, $\ker \chi$ is a finite group. Hence, we have $\dim G = \dim(G/\ker \chi)$. But since $\chi(G) \subseteq k^*$ and $\chi(G) \cong G/\ker \chi$, we see that G is at most one dimensional. This contradiction shows that χ must be trivial. □

We now prove an analogous statement for the unipotent groups.

Lemma 2.10. *Let U be a unipotent group. Then U does not possess any nontrivial algebraic characters. In particular, we have $E(U) = 1$.*

Proof. Since any linear representation of $\mathbb{G}_m$ is diagonalizable, $\mathbb{G}_m$ consists of semisimple elements. At the same time, by the Jordan Decomposition Theorem [39, Theorem 15.3], we know that the image of a unipotent element under any rational representation is unipotent. Now let $\chi : U \to \mathbb{G}_m$ be an algebraic character of U. The image of χ consists of unipotent elements in $\mathbb{G}_m$. But only unipotent element of $\mathbb{G}_m$ is the identity element. Therefore, χ is the constant (trivial) algebraic character. □

Although semisimple and unipotent groups do not have any nontrivial algebraic characters, reductive groups have algebraic characters. For example, tori are reductive groups; they possess plenty of algebraic characters.

Example 2.11. Let S be the following three dimensional complex torus,

$$S := \left\{ \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} : \ a_1, a_2, a_3 \in \mathbb{C}^* \right\}.$$

For every $i \in \{1, 2, 3\}$, the map $\lambda_i : S \to \mathbb{G}_m, \ \lambda_i((a_1, a_2, a_3)) = a_i$, is a character of S. The constant map $S \to \mathbb{G}_m$, $(a_1, a_2, a_3) \mapsto 2$ is not a character since it is not a homomorphism.

2.5. Abstract characters vs algebraic characters of algebraic groups

In the theory of finite group representations, the "trace" function on matrices is used for defining the "character" of a $\mathbb{C}$-representation. More precisely, if G is a finite group and $\rho : G \to GL(V)$ is a representation of G on a finite dimensional $\mathbb{C}$-vector space V, then the function $\chi : G \to \mathbb{C}$ defined by

$$\chi(g) = \text{trace}(\rho(g)) = \text{sum of eigenvalues of } \rho(g)$$

is called the *character of* ρ. We want to stress the fact that our definition of a character for affine algebraic groups is not exactly this one! Our definition requires that a character takes nonzero values only, but the trace of a linear operator might be zero. If for all $g \in G$, the value $\chi(g)$ is nonzero, then χ is an algebraic character as in Definition 2.6.

Example 2.12. The symmetric group S_n has only two algebraic characters; they are the trivial and the sign representations of S_n.

Next, we will discuss the algebraic characters of diagonalizable groups.

Lemma 2.13. *Let S be an algebraic group. The group of all algebraic characters of S is a linearly independent subset of the space of all k-valued functions on S.*

The proof of this lemma is not difficult but we will not present it here; see [39, §16.1]. We will demonstrate it by a simple example.

Example 2.14. We consider $S = \mathbb{G}_m$. The algebraic characters of S are the regular functions of the form $z \mapsto z^m$ ($z \in \mathbb{C}^*$) where $m \in \mathbb{Z}$. Clearly, each such function is an element of the ring $\mathbb{C}[z, 1/z] = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}z^m$. Then the linear independence of the algebraic characters is easily verified.

Let us call an affine group D a *diagonalizable group* if it is isomorphic to a closed subgroup of the group of diagonal $n \times n$ matrices with entries from k. An important consequence of the previous lemma is that if D is a diagonalizable group, then $E(D)$ is a vector space basis for $k[D]$. The proof of the next theorem about the structure of a diagonalizable group can be found in [39, §16.2].

Theorem 2.15. *Let D be a diagonalizable group. Then D is isomorphic to a direct product of the form $S \times H$, where S is a torus (always connected!) and H is a finite abelian group.*

Exercise. Prove that for every connected algebraic group G, the group of algebraic characters of G is torsion free.

2.6. Algebraic actions and rational representations

Let $f : S \times X \to X$ be an abstract group action of S on X. If, in addition, the map f is a morphism of varieties, then it is called an *algebraic S-action*. In this case, X is called an *S-variety*. If the map f is understandable from the context, then instead of writing $f : S \times X \to X$ we will write $S : X$.

Let X be an affine algebraic variety. Let S be an affine algebraic group. For each algebraic action $S : X$, there is a corresponding *left translation action* of S on the ring of regular functions of X [6, Ch I, §1.9]. It is defined as follows. Let s be an element of S. Then we define the linear automorphism $\lambda(s) : k[X] \to k[X]$ by

$$\lambda(s)(g(x)) := g(s^{-1} \cdot x) \qquad (g \in k[X]). \tag{2.16}$$

(The action (2.16) makes perfect sense if we define it as the action of the opposite group G^{op} on $k[X]$. Note that G^{op} is canonically isomorphic to G as an algebraic group.)

A simple but important fact regarding the representation defined by (2.16) is that it is a locally finite representation. More precisely, we have the following well-known fact.

Lemma 2.17. *Let X be an S-variety. Then the k-algebra $k[X]$ is a union of finite dimensional vector subspaces that are S-stable.*

Proof. Let $\alpha : S \times X \to X$ be the morphism that defines the action $S : X$. Then the comorphism $\alpha^* : k[X] \to k[S \times X]$ is defined by $\alpha^*(g)(s, x) = g(\alpha(s, x))$, where $g \in k[X], s \in S$, and $x \in X$. Now let f be a function from $k[X]$. We will construct an S-stable, finite dimensional vector subspace V of $k[X]$ such that $f \in V$. Since the coordinate ring of $S \times X$ is given by the tensor product, $k[S \times X] = k[S] \otimes k[X]$, we write $\alpha^*(f)$ in the form

$$\alpha^*(f) := \sum_{i=1}^{r} \phi_i \otimes \psi_i$$

for some $\phi_i \in k[S]$ ($i \in \{1, \ldots, r\}$) and $\psi_i \in k[X]$ ($i \in \{1, \ldots, r\}$). In other words, we have

$$f(s \cdot x) = f(\alpha(s, x)) = \alpha^*(f)(s, x) = \sum_{i=1}^{r} \phi_i(s)\psi_i(x). \tag{2.18}$$

Note that $f(s \cdot x) = (s^{-1} \cdot f)(x)$ for every $s \in S$ and $x \in X$. Thus, we conclude from (2.18) that the set $\{s \cdot f(x) : s \in S, x \in X\}$ is not only S-stable but also contained in the linear span of the functions $\psi_1, \ldots, \psi_r$ in $k[X]$. In particular, the linear span $V := \mathrm{span}_k\{s \cdot f(x) : s \in S, x \in X\}$ is a finite dimensional, S-stable subspace of $k[X]$. This finishes the proof. $\square$

Let S be an affine algebraic group. Let M be a k-module, that is, a k-vector space. Let $\sigma : S \to GL(M)$ be an abstract group homomorphism. We have two possibilities for the vector space dimension of M.

(1) First, we assume that $\dim M < \infty$. If, in addition, σ is an algebraic group homomorphism, then M is called a *rational S-module*, and σ is called a *(rational) representation of S*.
(2) Next, we assume that $\dim M = \infty$. If for each element $x \in M$ there exists a finite dimensional $\sigma(S)$-stable linear subspace $V \subset M$ such that $x \in V$, and the restrictions of the elements of $\sigma(S)$ to V defines a rational S-module as in (1),

then we say that M is a *rational S-module*. Also in this case, we say that φ is a *rational representation of S*.

Lemma 2.17 shows that if X is an affine S-variety, then the coordinate ring of X has the structure of a rational S-module. In our first Example 2.3, $\mathbb{C}^3$ is not a rational $H_3(\mathbb{C})$-module, since the action there is not an algebraic action. All of the other examples in this section are rational modules.

Let us agree on the following abbreviation. When we discuss rational S-modules, if there is no danger for confusion, then let us just write a S-module instead of writing a 'rational S-module'.

Now, a nonzero S-module M is called *simple* if M has no S-submodule other than 0 and itself. In this case, we will say that the representation $\sigma : S \to GL(M)$ is *irreducible*. There is no harm in mixing up these terminologies. For an S-module V, the sum of all simple S-submodules of M isomorphic to V is called the *V-isotypic component of M*.

2.7. Algebraic covering homomorphisms

Let G and $\tilde{G}$ be two connected affine algebraic groups. Let $f : \tilde{G} \to G$ be a surjective algebraic group homomorphism. If the kernel of f is a finite group, then f is called a *covering homomorphism* or an *isogeny*. The group G is called *(algebraically) simply connected* if every isogeny onto G is an isomorphism of algebraic groups. A convenient characterization of simply connected groups is that they are precisely the connected affine groups G such that every finite dimensional representation of the Lie algebra of G is the differential of a rational representation of G.

Let us note that if G is a connected affine complex algebraic group, then viewed as a real Lie group, if G is topologically simply connected, then G is algebraically simply connected. A proof of this useful result can be found in [33, Remark 6.3.10].

2.8. A word on our notation

Let G be an algebraic group. Some authors call an algebraic character of G a *linear character*. Hereafter, when we feel that it will not cause any confusion, we will drop the adjectives like 'linear' or 'algebraic' from our terminology.

In standard textbooks [6, 39, 80], the set of characters of G is usually denoted by one of the following notations: $X(G)$, $X_*(G)$, $X^*(G)$, $\mathcal{X}_G$, $\mathcal{X}(G)$, $\mathcal{X}_*(G)$, $\mathcal{X}^*(G)$.. Also, we just learned that the quotient group $\mathcal{O}(G)^*/k^*$, denoted by $E(G)$, is isomorphic to the group of characters of G as well. In this text, the character group of an algebraic group will be denoted by $\ddot{\mathbf{O}}(G)$[12]. In the sequel, for a G-variety X, where G is a reductive group and $B \subseteq G$ a Borel subgroup, we will use the notation $\ddot{\mathbf{O}}(X)$ to denote a certain sublattice of the character group of B. We note here that, in general, $E(X) = \mathcal{O}(X)^*/k^*$ is not the same thing as $\ddot{\mathbf{O}}(X)$. When we write $\ddot{\mathbf{O}}(X)_{\mathbb{Q}}$, we mean the $\mathbb{Q}$-vector space $\ddot{\mathbf{O}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

[12] We decided to denote the set of algebraic characters of G by $\ddot{\mathbf{O}}(G)$, since the letter O with umlaut is the initial of the Turkish word öz, which could be translated as "one's own"

3. Geometric Interpretations

In this section we will review some well-known results about the geometric aspects of characters. Our main references for this section are [63, 47, 46]. We proceed with the assumption that k is an algebraically closed field of characteristic 0.

Let X be an algebraic variety. The *Picard group* of X, denoted by $\mathrm{Pic}(X)$, is the group of isomorphism classes of line bundles on X. The multiplication in $\mathrm{Pic}(X)$ is given by the tensor products of line bundles.

Remark 3.1. We mentioned in Section 1 the following result of Serre: the restriction of the analytification functor $X \mapsto X^{\mathrm{an}}$ to the category of proper $\mathbb{C}$-schemes is an equivalence of categories onto the category of compact Hausdorff $\mathbb{C}$-analytic spaces. Along these lines, we know the following fact from [35, Expose XII, Theorem 4.4]: if X is a proper $\mathbb{C}$-scheme, then the functor $\mathscr{F} \mapsto \mathscr{F}^{\mathrm{an}}$, where $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module, is an equivalence of categories. It easily follows from [35, Expose XII, Theorem-Definition 4.1] that the inverse functor $\mathscr{F}^{\mathrm{an}} \to \mathscr{F}$ respects the tensor products of $\mathcal{O}_X$-modules. Consequently, the (algebraic) Picard group of X is isomorphic to the (analytic) Picard group of X^{an}.

For a homogeneous space G/H, there is a close relationship between $\mathrm{Pic}(G/H)$ and the group of characters of H. In the special case of a Borel subgroup of a reductive group, there is a well-known construction of a line bundle from a character of B; it is sometimes called the "Borel-Weil-Bott construction". We will present the general formulation.

Let G be an affine algebraic group, and H be a closed subgroup of G. Let $\chi : H \to \mathbb{G}_m$ be an element of $\ddot{\mathbf{O}}(H)$. We consider the following diagonal action of H on $G \times k$:

$$h \cdot (g, x) := (gh^{-1}, \chi(h)x) \qquad (h \in H, g \in G, x \in k).$$

Let us assume that the quotient $(G \times k)/H$, denoted by $L(\chi)$, exists as a variety. Then via the first projection map $p : L(\chi) \to G/H$, $L(\chi)$ becomes a line bundle on G/H. We notice that G acts on $L(\chi) = (G \times k)/H$ via its action on the first factor. Furthermore, the projection map is equivariant with respect to this action. Therefore, the space of global sections of $L(\chi)$ has the structure of a G-module. If H is a Borel subgroup of a connected reductive group G, then under some assumptions on the character χ, the vector space $H^0(G/B, L(\chi))$ turns out to be an irreducible representation of G. In fact, the Borel-Weil-Bott theorem states that all irreducible representations of G can be constructed this way, [40, Part II, Ch 5].

It is a natural question to ask if any line bundle on a G-variety has any representation theoretic meaning. The answer is 'yes' under suitable assumptions. To explain the scope of this question, we will introduce the notion of an 'equivariant Picard group.'

The equivariant Picard group Let G be an affine algebraic group, and let X be a G-variety. Let $\pi : L \to X$ be a line bundle on X. A *G-linearization of L* is a G-action

$$\varphi : G \times L \to L$$

such that

(1) the natural map $\pi : L \to X$ is G-equivariant,
(2) for every $x \in X, g \in G$, the map $\varphi_x : L_x \to L_{gx}$ is linear. Here, L_x is the fiber of L at $x \in X$.

Definition 3.2. Let X be a G-variety. The *equivariant Picard group of* X, denoted by $\mathrm{Pic}_G(X)$, is the group of isomorphism classes of the G-linearized line bundles on X.

The following proposition is useful for computing the (equivariant) Picard groups of homogeneous spaces; it is proved in [47, Proposition 5.1].

Proposition 3.3. *Let G be a reductive group, and let X be an irreducible G-variety which admits a geometric quotient $\pi : X \to X/G$. Then we have the following commutative diagram whose rows and columns are exact:*

$$\begin{tikzcd}
 & 1 \arrow[d] & & \\
 & E(X/G) \arrow[d] & & \\
 & E(X)^G \arrow[d] & 1 \arrow[d] & \\
 & \boldsymbol{\ddot{O}}(G) \arrow[d] & \mathrm{Pic}(X/G) \arrow[d, "\gamma"] & \\
1 \arrow[r] & \mathrm{H}^1_{alg}(G, \mathcal{O}(X)^*) \arrow[r] \arrow[d] & \mathrm{Pic}_G(X) \arrow[r] \arrow[d, "\delta"] & \mathrm{Pic}(X) \\
 & \mathrm{H}^1(G/G^0, E(X)) & \prod_{x \in C} E(G_x) &
\end{tikzcd}$$

In the diagram of the proposition, G^0 stands for the connected component containing the identity element $e \in G$. The indexing set C in the product $\prod_{x \in C} E(G_x)$ is a set of representatives of the closed G-orbits in X; G_x denotes the stabilizer of a point $x \in G$. The map δ is obtained by associating to a line bundle L the characters of the isotropy groups on the fibers L_x ($x \in C$). Finally, $\mathrm{H}^1_{alg}(G, \mathcal{O}(X)^*)$ denotes the *group of classes of algebraic cocycles*. (A morphism $\gamma : G \to \mathcal{O}(X)^*$ is said to be an algebraic cocycle if the equality $\gamma(gh) = (g \cdot \gamma(h))\gamma(h)$ holds for every $g, h \in G$.)

Remark 3.4. We have a note on the notation of Proposition 3.3. In the original form of this proposition in [47], what we called a geometric quotient is called a "quotient," and denoted by $X/\!/G$. The reader may see in the literature, such as [19], that the latter notation is more commonly used for a "categorical quotient". In our text, all quotient varieties are geometric quotients.

Remark 3.5. If G is connected, then we know that $E(X)^G = E(X)$ by Proposition 2.8. Of course, if G is connected, then we have $\mathrm{H}^1(G/G^0, E(X)) = 0$ as well. If X is normal, then the horizontal sequence in Proposition 3.3 has an extension by the homomorphism $\rho : \mathrm{Pic}(X) \to \mathrm{Pic}(G)$ defined by $\rho(L) = (\varphi^*(L) \otimes p_X^*(L)^{-1})|_{G\times\{x_0\}}$, where $\varphi : G \times X \to X$ is the action map, $p_X : G \times X \to X$ is the second projection, and $x_0 \in X$ is an arbitrary point. This is proved in [46, Lemma 2.2].

Proposition 3.3 is a culmination of several useful theorems which we want to state separately.

Theorem 3.6. *Let G be a connected affine group, and let H be a closed subgroup of G. Then there exists a canonical isomorphism,*

$$\delta : \mathrm{Pic}_G(G/H) \to \ddot{\boldsymbol{O}}(H).$$

Identify $\mathrm{Pic}_G(G/H)$ and $\ddot{\boldsymbol{O}}(H)$ via δ, and let ε be the morphism of "forgetting the G-action",

$$\varepsilon : \mathrm{Pic}_G(G/H) \to \mathrm{Pic}(G/H).$$

If $\mathrm{Pic}\, G = 0$, then ε is surjective and the kernel of ε is the subgroup $res(\ddot{\boldsymbol{O}}(G)) =: \ddot{\boldsymbol{O}}_G(H) \subseteq \ddot{\boldsymbol{O}}(H)$ of characters that are obtained from $\ddot{\boldsymbol{O}}(G)$ by restricting to $\ddot{\boldsymbol{O}}(H)$.

Remark 3.7. For any connected affine algebraic group G, and a closed subgroup $H \subseteq G$, the quotient group $\ddot{\mathbf{O}}(H)/\ddot{\mathbf{O}}_G(H)$ is isomorphic to the character group $\ddot{\mathbf{O}}(H \cap (G,G))$, [63, §4, Lemma 4]. Therefore, if $\mathrm{Pic}\, G = 0$, then by Theorem 3.6, we see that the Picard group of G/H is $\ddot{\mathbf{O}}(H \cap (G,G))$.

Theorem 3.8. *Let G be a connected affine group, and let H be a closed subgroup of G. Let ε be as in Theorem 3.6, and let $\pi : G \to G/H$ be the canonical projection. Then the following statements hold:*

(1) The sequence

$$\ddot{\boldsymbol{O}}(G) \xrightarrow{res} \ddot{\boldsymbol{O}}(H) \xrightarrow{\varepsilon} \mathrm{Pic}(G/H) \xrightarrow{\pi^*} \mathrm{Pic}(G) \tag{3.9}$$

is exact.

(2) If H is a connected solvable subgroup of G, then the natural morphism $\pi^* : \mathrm{Pic}(G/H) \to \mathrm{Pic}(G)$ is surjective.

(3) Let $\mathcal{R}$ denote the radical of G. We assume that $(G,G) \cap \mathcal{R} = 0$ holds. Then $\mathrm{Pic}(G)$ is isomorphic to the fundamental group of $G/\mathcal{R}$. (This statement is not correct if the assumption $(G,G) \cap \mathcal{R} = 0$ is missing.) It is also isomorphic to the kernel of the covering homomorphism $\widetilde{G} \to G$, where $\widetilde{G}$ is simply connected.

(4) If G is simply connected, then $\mathrm{Pic}(G) = 0$.

The proofs of (1) and (2) are given in [47, Proposition 3.2]. Notice that, in light of Remark 3.7, (1) can be seen as a restatement of Theorem 3.6. The proofs of (3) and (4) are given in [63]. For part (3) also see [90]. The statement in part (4) was first proven by Voskserenskii in characteristic 0, [87].

Corollary 3.10. *Let B be a Borel subgroup of a connected reductive group G. Let T be a maximal torus of B. Then we have*

- $\mathrm{Pic}(G/B) \cong \ddot{\mathbf{O}}(T_0)$*, where T_0 is the maximal torus in the semisimple part (G, G) of G.*
- $\mathrm{Pic}_G(G/B) \cong \ddot{\mathbf{O}}(T)$.

Proof. The statement in the second bullet follows directly from Theorem 3.6. We proceed with the proof of the claim in first bullet.

Recall that an isogeny is a surjective algebraic group homomorphism whose kernel is finite. It is easy to verify that the flag varieties are invariant under the morphisms induced by isogenies. Therefore, we can replace G/B by $\tilde{G}/\tilde{B}$, where $\tilde{G}$ is a simply connected cover of G, and $\tilde{B}$ is the preimage of B in $\tilde{G}$. Then by Part 2 of Theorem 3.8, we have $\mathrm{Pic}(G/B) \cong \ddot{\mathbf{O}}(\tilde{B})/res(\ddot{\mathbf{O}}(\tilde{G}))$. We notice that the quotient $\ddot{\mathbf{O}}(\tilde{B})/res(\ddot{\mathbf{O}}(\tilde{G}))$ is isomorphic to the quotient $\ddot{\mathbf{O}}(B)/res(\ddot{\mathbf{O}}(G))$. Finally, we notice that every central torus of G is contained in B; these facts follows from our remark in Subsection 2.3 about the definition of a connected reductive group. Hence, the quotient $\ddot{\mathbf{O}}(B)/res(\ddot{\mathbf{O}}(G))$ is isomorphic to the corresponding quotient $\ddot{\mathbf{O}}(B_0)/res(\ddot{\mathbf{O}}(G_0))$, where G_0 is the semisimple part $G_0 = (G, G)$ and B_0 is the Borel subgroup $B \cap G_0$. Since G_0 is a connected semisimple group, we know that $\ddot{\mathbf{O}}(G_0) = 1$. Therefore, $\mathrm{Pic}(G/B) = \ddot{\mathbf{O}}(B_0) = \ddot{\mathbf{O}}(T_0)$, where T_0 is the maximal torus of B_0. This finishes the proof of our assertion. □

Remark 3.11. Let G be a connected affine group, $\tilde{G} \to G$ be a simply connected cover. If H is a closed subgroup of G, then let $\tilde{H}$ denote preimage of H in $\tilde{G}$. The arguments that we used in the proof of Corollary 3.10 hold in general. First, we have the isomorphism

$$G/H \cong \tilde{G}/\tilde{H}.$$

It follows that G/H is a $\tilde{G}$-variety. Secondly, since $\tilde{G}$ is simply connected, $\mathrm{Pic}(\tilde{G}) = 0$, hence, every line bundle on G/H is $\tilde{G}$-linearizable. This is proved in [46, Proposition 2.4 and its Remark]. We now apply Theorem 3.6 (or part 1. of Theorem 3.8), which gives us

$$\mathrm{Pic}(G/H) = \mathrm{Pic}(\tilde{G}/\tilde{H}) = \ddot{\mathbf{O}}(\tilde{H})/res(\ddot{\mathbf{O}}(\tilde{G})). \tag{3.12}$$

3.1. The Picard group of $\mathbf{GL}_n(k)$

Next, we want to work out an instructive example. The statement we want to prove is the following: Let k be an algebraically closed field. Then we have

$$\mathrm{Pic}(\mathbf{GL}_n(k)) = 0. \tag{3.13}$$

We begin with an algebraic reinterpretation of Picard groups from [36, Ch II, §2].

Let X be a noetherian, integral, separated scheme which is regular in codimension one. The last condition means that every local ring $\mathcal{O}_x$ of X of dimension one is regular. For example, if X is a normal algebraic variety, then it has these properties.

Let Z be a prime divisor in X. The local ring $\mathcal{O}_{Z,X}$ is a discrete valuation ring in $k(X)$. In particular, the total quotient field of $\mathcal{O}_{Z,X}$ is equal to the function field of X. Let v_Z denote the valuation corresponding to $\mathcal{O}_{Z,X}$. For every element $f \in k(X)$, there are only finitely many prime divisors Z (depending on f) in X such that $v_Z(f) \neq 0$. Thus, we associate a divisor to f as follows:

$$\operatorname{div}(f) := \sum_Z v_Z(f)Z.$$

This is called the *principal divisor* defined by f. The free abelian group that is generated by all divisors is called the *group of Weil divisors*, denoted by $\operatorname{Div}(X)$.

Two divisors D and D' are said to be *linearly equivalent* if $D - D'$ is a principal divisor. Since $\operatorname{div}(f/g) = \operatorname{div}(f) - \operatorname{div}(g)$ for $f, g \in k(X)\backslash\{0\}$, the principal divisors form a subgroup of $\operatorname{Div}(X)$. The quotient of $\operatorname{Div}(X)$ by the subgroup of principal divisors is called the *(divisor) class group*, denoted by $\operatorname{Cl}(X)$.

Lemma 3.14. *([36, Proposition 6.2]) Let A be a noetherian domain. Then A is a UFD if and only if the affine scheme $X := \operatorname{Spec} A$ is normal and $\operatorname{Cl}(X) = 0$.*

Another well-known important property of the class groups is the following:

Lemma 3.15. *([36, Proposition 6.5 (c)] Let Z be a proper closed subset of X, and let U denote the open set $X\backslash Z$. If Z is an irreducible subset of codimension one in X, then the sequence*

$$\mathbb{Z} \to \operatorname{Cl}(X) \to \operatorname{Cl}(U) \to 0,$$

where the first map is given by $1 \mapsto 1 \cdot [Z]$, is exact.

A variety X is said to be *locally factorial* if for every $x \in X$ the local ring of x is a unique factorization domain. The relation of the class group to our discussion is the following result:

Lemma 3.16. *([36, Proposition 6.15]) Let X be a noetherian, integral, separated scheme. If X is locally factorial, then $\operatorname{Cl}(X)$ is isomorphic to $\operatorname{Pic}(X)$.*

Corollary 3.17. *Let n be a positive integer. Then the Picard group of $\mathbf{GL}_n(k)$ is trivial.*

Proof. Let U denote $\mathbf{GL}_n(k)$. Since U is smooth, it is a locally factorial variety. Thus we have $\operatorname{Pic}(U) \cong \operatorname{Cl}(U)$. We now view U as the complement of the hypersurface $\{a \in \mathbf{Mat}_n(k) : \det a = 0\}$ in $\mathbf{Mat}_n(k)$. Since $X := \mathbf{Mat}_n(k)$ is an affine space, we see from Lemma 3.14 that its class group is trivial. It follows from Lemma 3.16 that $\operatorname{Cl}(U) = 0$. This finishes the proof of our assertion. $\square$

3.2. The equivariant Picard groups of classical symmetric spaces

A *classical group* is one of the following simple groups: $\mathbf{SL}_n(k), \mathbf{SO}_n(k)$, and $\mathbf{Sp}_n(k)$. By a *classical symmetric space* we mean a pair (G, K), where G is a classical group, and $K = \{g \in G : \theta(g) = g\}$, where $\theta \in \mathrm{Aut}(G)$ such that $\theta^2 = id$. It turns out that there are 16 distinct classes of classical symmetrical spaces (G, K). Seven of these 16 classes have infinitely many members, and the remaining 9 of them are singletons. In our next table, we have listed the infinite families along with their titles in the "Cartan classification".

Non-exceptional Classical Symmetric Spaces	
AI	$\mathbf{SL}_n(k)/\,\mathbf{SO}_n(k)$
AII	$\mathbf{SL}_{2n}(k)/\,\mathbf{Sp}_n(k)$
AIII	$\mathbf{SL}_n(k)/\mathbf{S}(\mathbf{GL}_p(k) \times \mathbf{GL}_q(k))$ $\quad (p + q = n$ and $1 \leqslant p \leqslant q)$
CI	$\mathbf{Sp}_n(k)/\mathbf{GL}_n(k)$
DIII	$\mathbf{SO}_{2n}(k)/\mathbf{GL}_n(k)$
BDI	$\mathbf{SO}_n(k)/\mathbf{S}(\mathbf{O}_p(k) \times \mathbf{O}_q(k))$ $\quad (p + q = n$ and $1 \leqslant p \leqslant q)$
CII	$\mathbf{Sp}_n(k)/\,\mathbf{Sp}_p(k) \times \mathbf{Sp}_q(k)$ $\quad (p + q = n)$

Let us compute the Picard groups of these symmetric spaces.

Proposition 3.18. *Let (G, K) be a non-exceptional classical symmetric space, defined over an algebraically closed field. Then the (equivariant) Picard group of G/K is as follows:*

G/H	$\mathrm{Pic}(G/H)$	$\mathrm{Pic}_G(G/H)$
AI	*0*	*0*
AII	*0*	*0*
AIII	*0*	*0*
CI	$\mathbb{Z}$	$\mathbb{Z}$
DIII	$\mathbb{Z}$	$\mathbb{Z}$
BDI	*0*	*0*
CII	*0*	*0*

TABLE 1. The Picard groups of classical symmetric spaces.

Proof. The fact that the equivariant of Picard groups are as listed in the last column of Table 1 follows directly from Theorem 3.6, where it is shown that $\mathrm{Pic}_G(G/H) = \ddot{\mathbf{O}}(H)$. For the middle column, we will use an exact sequence, which is a direct consequence of Proposition 3.3. It is shown in [47, Example 5.2] that for every connected reductive group G and a reductive subgroup $H \subseteq G$, the following sequence is exact:

$$0 \to E(G/H) \to \ddot{\mathbf{O}}(G) \xrightarrow{res} \ddot{\mathbf{O}}(H) \xrightarrow{\varepsilon} \mathrm{Pic}(G/H) \xrightarrow{\pi^*} \mathrm{Pic}(G) \to \mathrm{Pic}(H). \tag{3.19}$$

In all of our cases, G is a connected, semisimple, and simply connected group, H is a reductive subgroup. Therefore we know that $\ddot{\mathbf{O}}(G) = 0$ (Lemma 2.9) and $\mathrm{Pic}(G) = 0$ (Theorem 3.8). Then by (3.19) we see that $\ddot{\mathbf{O}}(H)$ is isomorphic to $\mathrm{Pic}(G/H)$. We conclude that the middle column of our Table 1 is identical to the last column of the table. Alternatively, we can use Remark 3.7; since $\mathrm{Pic}(G) = 0$, we have $\mathrm{Pic}(G/H) \cong \ddot{\mathbf{O}}((G,G) \cap H)$. Since G is simple, $G/(G,G)$ is a finite group, hence, $H/(G,G) \cap H$ is a finite group. It follows that $\ddot{\mathbf{O}}((G,G) \cap H) = \ddot{\mathbf{O}}(H)$. □

4. Root Systems, Weyl Groups, Fundamental Weights

The simple algebraic groups defined over an algebraically closed field k are parametrized by the "indecomposable and reduced root systems" independently of the characteristic of the underlying field k. It turns out that there are four infinite families of such roots systems, labeled $\mathtt{A}_\ell$ ($\ell \geqslant 1$), $\mathtt{B}_\ell$ ($\ell \geqslant 2$), $\mathtt{C}_\ell$ ($\ell \geqslant 3$), and $\mathtt{D}_\ell$ ($\ell \geqslant 4$), and five exceptional types, labeled $\mathtt{E}_6$, $\mathtt{E}_7$, $\mathtt{E}_8$, $\mathtt{F}_4$, $\mathtt{G}_2$. Then the semisimple algebraic groups are classified by the possibly decomposable but still reduced root systems. In this section, we will briefly review the relevant notions. As a reference text, we recommend [7], which contains many historical remarks on the whole development of the theory.

Let E be a euclidean space. This means that E is a finite dimensional real vector space together with an inner product $(\cdot,\cdot) : \mathrm{E} \times \mathrm{E} \to \mathbb{R}$. For each vector $\alpha \in \mathrm{E}$, we have the corresponding *reflection operator*, denoted $s_\alpha : \mathrm{E} \to \mathrm{E}$, that reflects the vectors with respect to the hyperplane that is perpendicular to α. To formulate this more precisely, let us define another 'product' map:

$$\langle \beta, \alpha \rangle := \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \qquad (\alpha, \beta \in \mathrm{E},\ \alpha \neq 0).$$

Then for every $\beta \in \mathrm{E}$ we have

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha.$$

Definition 4.1. Let E be a finite dimensional vector space over $\mathbb{Q}$. A *root system in E*, denoted $\mathtt{R}$, is a finite set of nonzero vectors, called *roots*, in E such that

(1) $\mathtt{R}$ spans E;
(2) for every root $\alpha \in \mathtt{R}$, the corresponding reflection operator $s_\alpha : \mathrm{E} \to \mathrm{E}$ permutes $\mathtt{R}$;
(3) for every α and β from $\mathtt{R}$, we have $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

If, in addition, for each $\alpha \in \mathtt{R}$, the condition $c\alpha \in \mathtt{R}$, where $c \in \mathbb{Q}$, implies that $c \in \{-1, +1\}$, then $\mathtt{R}$ is said to be a *reduced root system in E*. In general, if α and $c\alpha$ ($c \in \mathbb{Q}$) are two proportional elements of a root system, then c can be any of the elements of $\{\pm\frac{1}{2}, \pm 1, \pm 2\}$. A root system is called *indecomposable* if it cannot be partitioned into a disjoint union of two mutually orthogonal proper subsets. The *rank* of $\mathtt{R}$ is the dimension of E.

In many texts, including [7], what we called here as an 'indecomposable root system' is called an *irreducible* root system. Later in this text we will encounter the root systems of

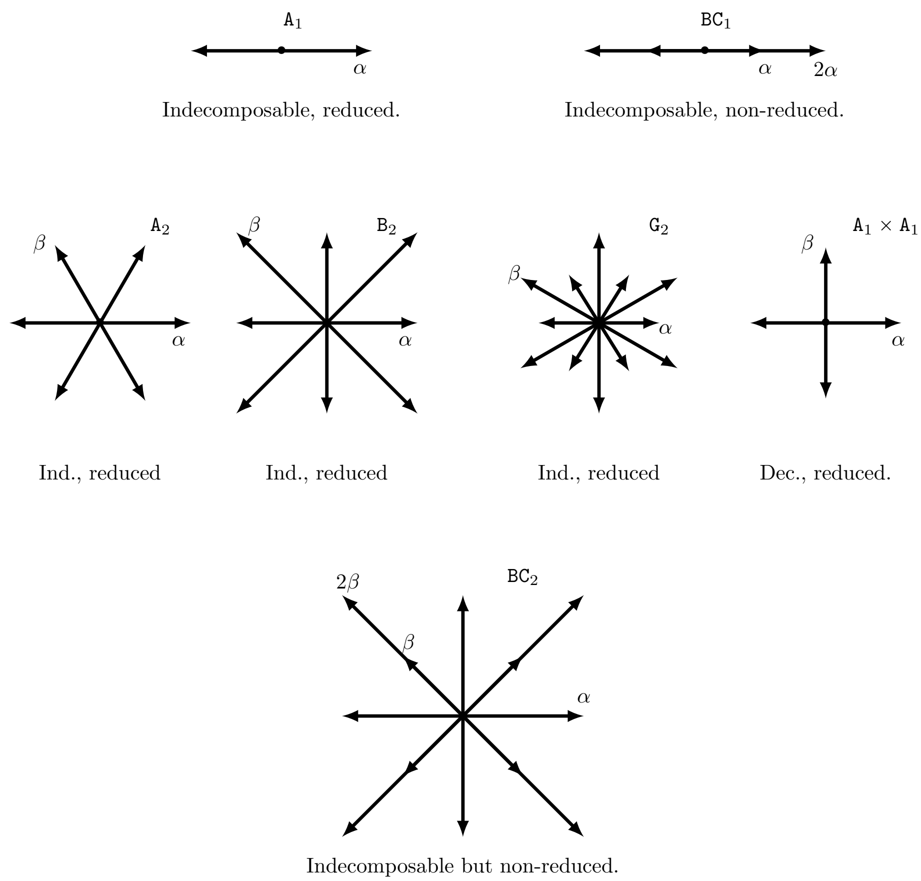

FIGURE 4.1. All rank one and rank two root systems. The abbreviations "Ind." and "Dec." stand for the words indecomposable and decomposable, respectively. The simple roots are indicated by α and β.

"spherical varieties". They can be non-reduced.[13] We mentioned at the beginning of this section that the classification of simple algebraic groups is achieved via the classification of indecomposable and reduced root systems, which are enumerated by the letters $\mathtt{A}_n$ to $\mathtt{G}_2$. It turns out that there is only one family of indecomposable and non-reduced root systems; it is labeled by $\mathtt{BC}_n$ ($n \geqslant 1$). For the details of the proof, see [7, Ch VI, §4, Subsection 14].

[13]We feel that the adjectives "non-reduced" and "irreducible" are too close to each other in their literal meanings. At the same time, the word "irreducible" is used in the names of many other geometric notions. For this reason we prefer to write indecomposable rather than irreducible.

Let us introduce the *root system of (the semisimple part of) a reductive group*. We fix a connected reductive group G, a Borel subgroup $B \subset G$, and a maximal torus $T \subset G$ contained in B. An algebraic group homomorphism $\varphi : \mathbb{G}_a \to G$ will be called a *one-parameter subgroup*, abbreviated to 1-psg. We will call a character $\alpha \in \ddot{\mathbf{O}}(T)$ a *root of G relative to T* if there exists an additive 1-psg $\varphi : \mathbb{G}_a \to G$ such that

$$t\varphi(c)t^{-1} = \varphi(\alpha(t)c) \qquad (t \in T, c \in \mathbb{G}_a). \tag{4.2}$$

We call the image of φ, denoted U_α, a *root subgroup* with respect to T. Although U_α is uniquely determined by α, the 1-psg φ is not unique; we can always modify the speed of φ by scaling its variable by a nonzero element of k^*. The set of all roots of G relative to T is denoted by $\mathtt{R}_{G,T}$. It turns out that G is generated by its maximal torus along with the root subgroups, see [39, Theorem 26.3 (d)].

A closely related construction is obtained via the "adjoint representation of G" on the tangent space at the identity element, T_eG. The Lie algebra structure on T_eG will be denoted by $\mathfrak{g}$. Since G is a linear algebraic group, it admits an embedding into a suitable general linear group as a closed subgroup. Then the *adjoint representation of G* is the representation of G on its Lie algebra that is given by the conjugation action:

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow GL(\mathfrak{g}) \\ g &\longmapsto \mathrm{Ad}(g), \end{aligned}$$

where $\mathrm{Ad}(g)(A) = gAg^{-1}$ for every $A \in \mathfrak{g}$. By differentiating this algebraic group homomorphism, we arrive at the *adjoint representation of the Lie algebra* $\mathfrak{g}$, defined by

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\longrightarrow \mathrm{End}(\mathfrak{g}) \\ x &\longmapsto \mathrm{ad}\ x, \end{aligned}$$

where ad x is the left translation map ad $x(y) = [x, y]$, $y \in \mathfrak{g}$. Let $\mathfrak{t}$ denote the Lie subalgebra in $\mathfrak{g}$ of the maximal torus $T \subset G$. Since T is a maximal abelian algebraic subgroup of G, $\mathfrak{t}$ is a maximal abelian Lie subalgebra of $\mathfrak{g}$. By restricting ad to $\mathfrak{t}$, we obtain the generalized eigenspace decomposition,

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \mathtt{R}_{G,T}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the Lie algebra of the root subgroup U_α. Let $\alpha : T \to \mathbb{G}_m$ be any root from $\mathtt{R}_{G,T}$. Then the action of $\mathfrak{t}$ on $\mathfrak{g}_\alpha$ is given by

$$[s, x] = (d\alpha|_e(s))x \qquad (s \in \mathfrak{t},\ x \in \mathfrak{g}_\alpha),$$

where $d\alpha|_e : \mathfrak{t} \to k = \mathrm{Lie}(\mathbb{G}_m)$ is the differential of α at $e \in T$. Below, we will discuss more general weight space decompositions from group theory viewpoint in more detail.

Homework: Find out about the Coxeter and Dynkin diagrams of a root system.

4.1. Weyl groups.

The relative positioning of the root subgroups in a reductive group is controlled by a finite group called the *Weyl group*. It is defined by $W := N_G(T)/Z_G(T)$, where $N_G(T)$ is the normalizer of T in G, and $Z_G(T)$ is the centralizer of T in G. If G is connected, then $Z_G(T) = T$.

The Weyl group acts as a group of automorphisms of the abelian group $\ddot{\mathbf{O}}(T)$. The action is given as follows. For $w \in W$ and $\alpha \in \ddot{\mathbf{O}}(T)$, the function $w \cdot \chi : T \to \mathbb{G}_m$ is defined by

$$(w \cdot \chi)(t) = \chi(w^{-1}tw) \qquad (t \in T). \tag{4.3}$$

Furthermore, we have $w \cdot (\chi\mu) = (w \cdot \chi)(w \cdot \mu)$ for every $\chi, \mu \in \ddot{\mathbf{O}}(T)$.

Let $\sigma : G \to GL(V)$ be a finite dimensional rational representation of G. We will verify that the action of W in (4.3) descends to a permutation action on a certain set of characters called the "weights of σ". To this end let us view σ (by restriction) as a rational representation of T. Then we have a decomposition $V = \bigoplus_{\chi \in \mathtt{R}(\sigma)} V_\chi$, where V_χ's are the joint eigenspaces for the elements of T:

$$V_\chi := \{v \in V : \ t \cdot v = \chi(t)v \text{ for all } t \in T\}. \tag{4.4}$$

It is easy to verify that the functions $\chi \in \mathtt{R}(\sigma)$ are actually algebraic characters of the maximal torus. We call them the *weights of the representation* σ. For $\chi \in \mathtt{R}(\sigma)$, the elements of V_χ are called the *weight vectors of weight* χ.

Now let χ be a weight of $\sigma : G \to GL(V)$. Let $v \in V_\chi$ be a weight vector of weight χ. Let w be an element of W. Let $\dot{w}$ be a representative of w in $N_G(T)$. We will check to see where v is sent to via the action of w. To this end, we will compute the action of T on $\sigma(\dot{w})(v)$:

$$\begin{aligned}
\sigma(t)(\sigma(\dot{w})(v)) = (\sigma(t)\sigma(\dot{w}))(v) = \sigma(\dot{w})(\sigma(\dot{w}^{-1}t\dot{w})(v)) &= \sigma(\dot{w})(\chi(\dot{w}^{-1}t\dot{w})\ v) \\
&= \sigma(\dot{w})((w \cdot \chi)(t)\ v) \\
&= (w \cdot \chi)(t)\sigma(\dot{w})(v).
\end{aligned}$$

Hence, we conclude that $w \cdot \chi$ is also a weight of σ. In other words, the weights of a rational representation are permuted by the Weyl group. In particular, if we use the adjoint representation in place of σ, then we find that $W \cdot \mathtt{R}_{G,T} = \mathtt{R}_{G,T}$.

It is a consequence of the general theory that $\mathtt{R}_{G,T}$ is indeed a reduced root system in the euclidean space $\ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, where the inner product is chosen so that it is W-invariant. We will not verify this assertion in these notes. The details of this development are clearly presented in [6, Ch IV, §14], or [39, Chapter X].

Homework:

(1) Show that W is isomorphic to the finite group generated by all the reflections s_β, where $\beta \in \mathtt{R}_{G,T}$.
(2) Determine the Weyl groups of the classical groups.
(3) Let $\tilde{G} \to G$ be a covering homomorphism. Show that the Weyl groups of $\tilde{G}$ and $\tilde{G}$ are isomorphic.
(4) Let Z be a central closed subgroup of G. Show that G/Z and G have isomorphic Weyl groups.

4.2. Terminology of weights

Let G be a connected reductive algebraic group. Let B and T denote a Borel subgroup of G, and a maximal torus contained in B, respectively. Then there is a unique Borel subgroup $B^- \subset G$ such that $B^- \cap B = T$. This Borel subgroup is called the *opposite Borel subgroup to* B. Let $\mathtt{R}_{G,T}$ denote the root system determined by (G,T). Recall that $\mathtt{R}_{G,T}$ is a finite subset of the character lattice $\ddot{\mathbf{O}}(T)$ of the torus T. The elements of the character lattice $\ddot{\mathbf{O}}(T)$ are sometimes called the *weights of* G *relative to* T. We set

$$\mathrm{E} := \ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

As usual, the Weyl group of (G,T) will be denoted by W.

A root $\alpha \in \mathtt{R}_{G,T}$ is called a *positive root* if the corresponding root subgroup U_α is contained in B. The set of positive roots will be denoted by $\mathtt{R}^+_{G,T}$. Note that there are several equivalent ways of defining the positive roots. We introduced them by using Borel subgroups to emphasize the combinatorial nature of such subgroups. Similarly to the definition a positive root, an element $\alpha \in \mathtt{R}_{G,T}$ is called *negative* if the corresponding root subgroup U_α is contained in the opposite Borel subgroup B^-. The set of negative roots will be denoted by $\mathtt{R}^-_{G,T}$. It is easy to check that $\mathtt{R}^-_{G,T} = -\mathtt{R}^+_{G,T}$. (In fact, we could introduce the negative roots this way!) A positive root $\alpha \in \mathtt{R}^+_{G,T}$ is called *simple* if it cannot be written as a sum of two positive roots. The set of all simple roots will be denoted by $\mathtt{S}$. It is easy to show that $\mathtt{S}$ is a basis for the vector space $\ddot{\mathbf{O}}(T)\otimes_{\mathbb{Z}}\mathbb{Q}$. Although in general $\mathtt{S}$ is not a basis for the lattice $\ddot{\mathbf{O}}(T)$, it will help us to define a distinguished basis for $\ddot{\mathbf{O}}(T)$. We proceed to explain.

The duals of the elements of $\mathtt{R}_{G,T}$ in E^* form the *dual root system*, denoted by $\mathtt{R}^\vee_{G,T}$. Furthermore, $\mathtt{S}^\vee := \{\alpha^\vee : \alpha \in \mathtt{S}\}$ is a set of simple roots for this root system. Let $(,)$ be any inner product on E. By using the averaging process, we can assume that $(,)$ is a W-invariant inner product. Using this invariant inner product, we identify the dual vector space E^* by E. Under this passage it is easy to check that the dual of 2α, where α is a root in $\mathtt{R}_{G,T}$ gives the following vector in E:

$$\alpha^\vee := 2\alpha/(\alpha,\alpha).$$

This vector is called the *coroot of* α. Now let us define the notion of an ‘abstract weight.’ Let $\varpi \in \mathrm{E} \otimes_{\mathbb{Z}} \mathbb{R}$. If for every $\alpha \in \mathtt{S}$, we have $\langle \varpi, \alpha^\vee \rangle \in \mathbb{Z}$, then ϖ is called an *abstract* *weight*. At the end of this section, we will assign the verification of the following fact as a homework: Every element of $\ddot{\mathbf{O}}(T)$ is an abstract weight.

We proceed by labeling the simple roots of $\mathtt{R}_{G,T}$ as in $\mathtt{S} = \{\alpha_1, \dots, \alpha_l\}$. For $i \in \{1, \dots, l\}$, the *fundamental dominant weight* is a character $\varpi_i \in \ddot{\mathbf{O}}(T)$ such that $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{i,j}$, where $\alpha_j^\vee$ is the coroot of α_j. A character $\omega \in \ddot{\mathbf{O}}(T)$ is said to be a *dominant weight* if it is can be written as a nonnegative integral combination of the fundamental dominant weights. Thus, set of all dominant weights becomes a subsemigroup of the free abelian group $\ddot{\mathbf{O}}(T)$.

The cone span of the semigroup of dominant weights in E will be called the *fundamental (or dominant) Weyl chamber*. The *anti-dominant Weyl chamber* is the cone spanned by $\{-\alpha_1^\vee, \dots, -\alpha_l^\vee\}$ in E.

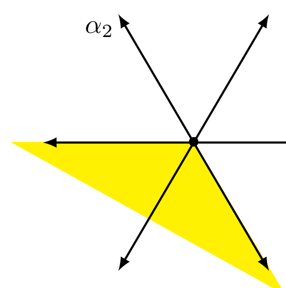

FIGURE 4.2. The anti-dominant Weyl chamber of $\mathtt{A}_2$.

Homework:

(1) Show that for every nontrivial element $w \in W$ and a vector v from the interior of the dominant Weyl chamber, $w \cdot v$ is not an element of the dominant chamber.
(2) Show that the set of abstract weights of (G, T) in $\mathrm{E} \otimes_{\mathbb{Z}} \mathbb{R}$ is a free abelian group of rank l (the number of simple roots of $\mathtt{R}_{G,T}$).
(3) Show that the elements of $\ddot{\mathbf{O}}(T)$ are abstract weights. Hint: Consider the W-action on the weights of a rational representation.
(4) Let χ and μ be two abstract weights in $\mathrm{E} \otimes_{\mathbb{Z}} \mathbb{R}$. We write $\mu \preceq \chi$ if and only if $\chi - \mu = \sum_{i=1}^{l} a_i \alpha_i$ for some nonnegative integers $a_1, \dots, a_l$.
    (a) Verify that $\preceq$ is a partial order on the set of all abstract weights. Draw the Hasse diagram of $\preceq$ for all rank 2 root systems.
    (b) Show that $\mathtt{R}_{G,T}^{+} = \{\beta \in \mathtt{R}_{G,T} :\ 0 \precneqq \beta\}$.
    (c) Show that for every $w \in W$ and v from the dominant Weyl chamber, we have $w \cdot v \preceq v$.

(5) Compare the root systems, weight lattices, as well as the lattices of abstract weights for the groups $\mathbf{SL}_2(\mathbb{C})$ and $\mathbf{PSL}_2(\mathbb{C})$.
(6) Show that every dominant Weyl chamber is a strictly convex rational polyhedral cone in $\ddot{\mathbf{O}}(T)\otimes_{\mathbb{Z}}\mathbb{R}$. (For the relevant convex geometry terminology, see Section 7).

4.3. The Bruhat-Chevalley decomposition and Schubert varieties

In this subsection we will introduce one of the most useful tool for studying the structure of a reductive group. We begin with setting up our conventions. As usual, G will denote a connected reductive group. We fix a Borel subgroup B and a maximal torus, $T \subset B$. Then B is of the form $B = T\mathcal{R}_u(B)$, where $\mathcal{R}_u(B)$ is the unipotent radical of B. It follows from our initial definition of the positive root system $\mathtt{R}^+_{G,T}$ that $\mathcal{R}_u(B)$ is directly spanned by the root subgroups that it contains,

$$\mathcal{R}_u(B) = \prod_{\alpha\in \mathtt{R}^+_{G,T}} U_\alpha,$$

where the product can be taken in any order [39, Proposition 28.1].

Let W denote the Weyl group. Every element $w \in W$ has a representative, denoted $\dot{w}$, in the subgroup $N_G(T) \subset G$. Hereafter, by abuse of notation, we will omit using the dot on the top of w. The *Bruhat-Chevalley decomposition* of G is the decomposition of G into two-sided orbits (= double-cosets) of B in G:

$$G = \bigsqcup_{w\in W} BwB. \tag{4.5}$$

In B, the maximal torus T normalizes the maximal unipotent subgroup, $U := \mathcal{R}_u(B)$. It follows that a double-coset BwB is canonically isomorphic to UwB. Therefore, we see that the coset space G/B is a union of finitely many U-orbits, indexed by the Weyl group of (G,T).

A closed subgroup $P \subseteq G$ is called a *standard parabolic subgroup with respect to B* if it contains B. We call a closed subgroup $Q \subseteq G$ a *parabolic subgroup* if Q is conjugate-isomorphic to a standard parabolic subgroup with respect to B. Another characterizing property of the parabolic subgroups is that the homogeneous space G/Q is a projective variety. We want to explain how the Bruhat-Chevalley decomposition (4.5) descends to a similar decomposition for each parabolic subgroup. For simplicity, we will work with the standard parabolic subgroups.

Let P be a standard parabolic subgroup. Then $P = \mathcal{R}_u(P) \rtimes L$, where $\mathcal{R}_u(P)$ is the unipotent radical of P and L is a (maximal) reductive subgroup of P that normalizes $\mathcal{R}_u(P)$. It is easy to see that the maximal torus T is entirely contained in the subgroup L. In other words, T is a maximal torus of the reductive group L. We will denote the Weyl group of the pair (L,T) by W_P. Then W_P is naturally isomorphic to the subgroup of W that is generated by the simple reflections s_α, where α is a simple root such that the corresponding root subgroup U_α is contained in L. Then we have analog of the Bruhat-Chevalley decomposition for P is given by

$$P = \bigsqcup_{w \in W_P} BwB. \tag{4.6}$$

Recall that G/P is a projective variety. Since both of G and P have related Bruhat-Chevalley decompositions, the structure of the variety G/P is closely related to the finite quotient set W/W_P.

The projective variety G/P is called a *partial flag variety* of G. If $P = B$, then it is called the *(full) flag variety of* G. By using (4.6), we see that G/P has a decomposition into finitely many B-orbits. More precisely, we have

$$G/P = \bigsqcup_{w \in W^P} BwP/P, \tag{4.7}$$

where $W^P \subseteq W$ is a certain transversal for W_P in W. There is a canonical choice for this transversal. To identify it, we will introduce the *rank function* of W:

$$\begin{aligned} \ell : W &\longrightarrow \mathbb{N} \\ w &\longmapsto \dim(BwB/B). \end{aligned} \tag{4.8}$$

Here, $\dim(BwB/B) = \dim(UwB/B)$ is the dimension of the B-orbit of w in G/B. Then, the *canonical transversal* for the subgroup $W_P \subseteq W$ is given by

$$w \in W^P \iff \ell(w) \leqslant \ell(v) \text{ for all } v \in wW_P. \tag{4.9}$$

Hereafter, we implicitly assume that the elements of W^P are the minimal length left coset representatives of W_P in W as in (4.9).

One of the most useful properties of the B-orbit closures in partial flag varieties is that they give a basis for the singular homology groups with integral coefficients. A proof of the following well-known fact is presented in [6, Ch IV, §14].

Lemma 4.10. *Let P be a parabolic subgroup of G. Then the decomposition of G/P in (4.7) is a cellular decomposition of G/P.*

We are now ready to introduce the Schubert varieties.

Definition 4.11. Let w be an element of W^P. The U-orbit of the coset $wP \in G/P$ is called the *Schubert cell determined by w in G/P*. Let us denote it by C_{wP}. The Zariski closure of C_{wP} in G/P is called the *Schubert variety determined by w in G/P*. It will be denoted by X_{wP}.

The Schubert varieties are partially ordered according to the gluing of their Schubert cells. We will explain this comment by using the Schubert varieties of the full flag variety.

Definition 4.12. The *Bruhat-Chevalley order*, denoted $\leqslant$, is the partial order defined by

$$w \leqslant v \iff C_{wB} \subseteq X_{vB} \qquad (w, v \in W).$$

The pair $(W, \leqslant)$ is a ranked poset with respect to the rank function defined in (4.8).

Example 4.13. Let us consider $G = \mathbf{GL}_3(k)$. We take T as the group of diagonal matrices in G, and we take B as the group of upper triangular matrices in G. The Weyl group W of (G,T) is the symmetric group S_3. The root system of (G,T) is given by $\{\varepsilon_i - \varepsilon_j : 1 \leqslant i \neq j \leqslant 3\}$, where $\varepsilon_i : T \to k^*$ is the character of T defined by $\varepsilon_i(\mathrm{diag}(t_1,t_2,t_3)) = t_i$ for $i \in \{1,2,3\}$. The set $\mathtt{S} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$ is a set of simple roots for $\mathtt{R}_{G,T}$. Let $s_i \in S_3$ $(i \in \{1,2\})$ denote the Coxeter generator corresponding to the simple root $\varepsilon_i - \varepsilon_{i+1}$. The Hasse diagram of the Bruhat-Chevalley order on S_3 is depicted in Figure 4.3.

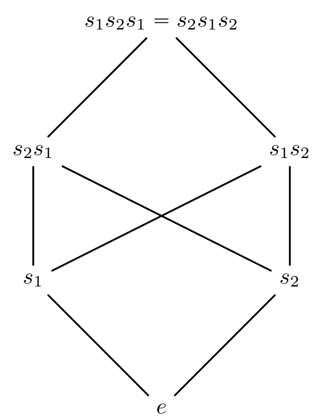

FIGURE 4.3. The Bruhat-Chevalley order on S_3.

Many geometric properties of a Schubert variety $X_{wB} \subseteq G/B$ can be extracted from the combinatorial properties of the lower interval $[e,w]$ in the Bruhat-Chevalley order on W. For example, it is easy to tell if a Schubert variety in G/B is rationally smooth or not just by checking if the rank generating function $p_w(t) := \sum_{0 \leqslant v \leqslant w} t^{\ell(v)}$ is palindromic[14] or not [24]. But understanding the exact nature of an interval in the Bruhat-Chevalley order is not only important for grasping the geometry of a Schubert variety but also for representation theory. Indeed, the multiplicities in the Jordan-Hölder series of Verma modules, or equivalently for the formal characters of irreducible highest weight modules (extending the celebrated Weyl character formula to "nondominant" weights), depend on these intervals [41]. It is natural to ask if there are similar combinatorial, geometric, or representation theoretic properties for the posets of B-orbit closures in other homogeneous spaces. However, in an arbitrary homogeneous space G/H, there might be infinitely many B-orbits.

[14]A polynomial in one variable $p(t)$ is called *palindromic* if $t^d p(t^{-1}) = p(t)$, where d is the degree of $p(t)$.

Definition 4.14. Let H be a closed subgroup of G. The homogeneous space G/H is called *spherical* if the number of B-orbits in G/H is finite.

It is known that the symmetric spaces are spherical homogeneous spaces. This fact was first shown for the symmetric spaces defined over $\mathbb{C}$ by Matsuki [53]. In [78], Springer showed that this fact holds true in arbitrary characteristic ($\neq 2$).

We close this section by explicitly describing the inclusion poset of the B-orbit closures in a very special symmetric space.

Example 4.15. Let G denote $\mathbf{SL}_{2n}(\mathbb{C})$, and let B denote a Borel subgroup of G. Let H denote the symmetric subgroup $\mathbf{Sp}_n(\mathbb{C})$. Then G/H can be identified with the space of all nondegenerate 2-forms on $\mathbb{C}^{2n}$.

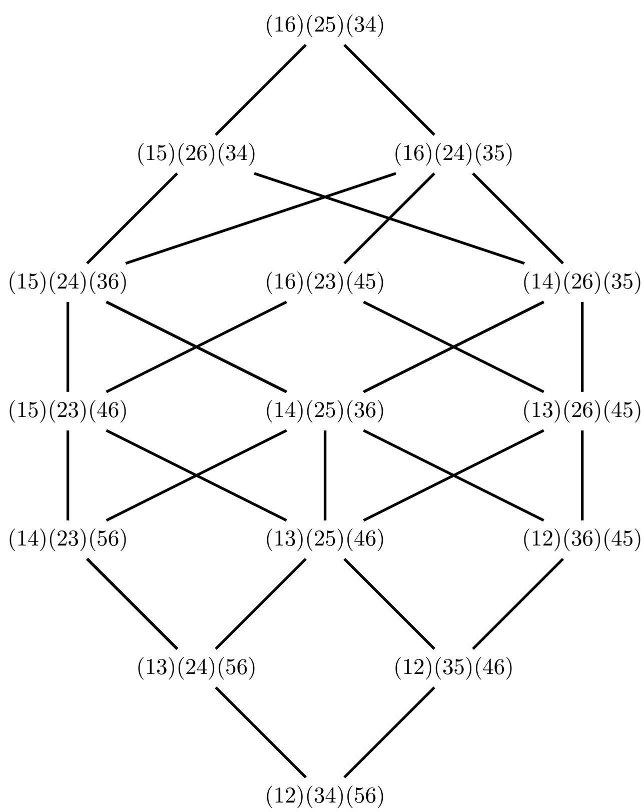

FIGURE 4.4. Bruhat-Chevalley order on $\mathbf{SL}_6/\mathbf{Sp}_6$.

Indeed, G acts transitively, by the "congruence action", on the space invertible, skew-symmetric, $2n \times 2n$ complex matrices. More precisely, let g be an element from G, and let A be a skew-symmetric $2n \times 2n$ complex matrix. Then the action of g on A is given by $g \cdot A := gAg^\top$. It is shown by Richardson and Springer [72, Example 10.4] that the set of B-orbits in G/H is in bijection with the set of all fixed-point-free involutions of the symmetric group S_{2n}. Richardson and Springer showed in [73] that the inclusion order on the B-orbit closures in G/H agrees with the restriction of the Bruhat-Chevalley order to the fixed-point-free involutions. Note that, conveniently, a fixed-point-free involution in S_{2n} can be represented by a product of transpositions of the form $(i_1\, j_1) \cdots (i_n\, j_n)$, where we have $1 \leqslant i_1 < \cdots < i_n \leqslant 2n-1$, $i_1 < j_1, \dots, i_n < j_n$, and $\{i_1, \dots, i_n, j_1, \dots, j_n\} = \{1, 2, \dots, 2n\}$. In Figure 4.4, we depict the Bruhat-Chevalley order on the fixed-point-free involutions of the symmetric group S_6.

5. Eigenvectors of Algebraic Actions

Let S be an affine algebraic group, and let V be an S-module. A nonzero vector $v \in V$ is said to be an S-*eigenvector*, or an S-*semiinvariant*, if there exists a character χ in $\ddot{\mathbf{O}}(S)$ such that $s \cdot v = \chi(s)v$ for all $s \in S$. We set

$$V^{(S)} := \{v \in V : \ v \text{ is an } S\text{-semiinvariant}\}.$$

If S is a torus T, and $\chi \in \ddot{\mathbf{O}}(T)$, then the T-semiinvariants in V relative to χ are precisely the weight vectors of weight χ as we defined in (4.4).

Example 5.1. Let T denote the maximal diagonal torus of $\mathbf{GL}_2(\mathbb{C})$. We consider the standard action of $\mathbf{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$. We will find all T-semiinvariants of the corresponding action on the coordinate ring $R := \mathcal{O}(\mathbb{C}^2) = \mathbb{C}[X, Y]$. The action of T on R is given by the restriction of the action in Example 2.4. Let $f(X, Y)$ be an element of $R^{(T)}$. Then there exists χ_f in $\ddot{\mathbf{O}}(T)$ such that, for every $t := \operatorname{diag}(a^{-1}, d^{-1}) \in T$,

$$t \cdot f(X, Y) = f(aX, dY) = \chi_f(t) f(X, Y).$$

First by setting $d = 1$ and varying $a \in \mathbb{C}^*$, and then by setting $a = 1$ and varying d in $\mathbb{C}^*$, we see that $f(X, Y)$ is a monomial of the form $\alpha X^s Y^r$ for some $\alpha \in \mathbb{C}^*$ and $r, s \in \mathbb{N}$.

The semiinvariants of a Borel subgroup are contained in the semiinvariants of its maximal torus. More generally, we have the following statement.

Lemma 5.2. *Let S be an affine algebraic group, and let $H \subseteq S$ be a closed subgroup. If V is an S-module, then $V^{(S)} \subseteq V^{(H)}$.*

Proof. Let v be an element of $V^{(S)}$, and let χ_v denote the corresponding S-semiinvariant. Then, the restriction of χ_v to H is a character of H. Therefore, we have $v \in V^{(H)}$. $\square$

Example 5.3. Let B denote the Borel subgroup of upper triangular matrices in $\mathbf{GL}_2(\mathbb{C})$. We will compute directly the B-semiinvariants in $R := \mathbb{C}[X, Y]$ for the action that we discussed in Example 2.4. Since $R^{(B)} \subseteq R^{(T)}$, it suffices to determine the T-semiinvariants that are B-semiinvariants. It follows from the argument of Example 5.1 that a B-semiinvariant is a monomial of the form $\alpha X^r Y^s$, where $\alpha \in \mathbb{C}^*$, $r, s \in \mathbb{N}$. Let g^{-1} be an element of B such that $g = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$. Then the action of g on $\alpha X^r Y^s$ is given by

$$g \cdot \alpha X^r Y^s = \alpha a^r X^r (bX + dY)^s. \tag{5.4}$$

Unless $b = 0$ or $s = 0$, the right hand side of (5.4) cannot be a scalar multiple of $\alpha X^r Y^s$. Since for an arbitrary $g \in B$, the entry b can be different than zero, we must have $s = 0$. In other words, $\alpha X^r Y^s$ is a B-semiinvariant if and only if $s = 0$. This argument shows that

$$R^{(B)} = \{\alpha X^r : \ \alpha \in \mathbb{C}^*, \ r \in \mathbb{N}\}.$$

On the other hand, as we observed in Example 5.1, the set of T-semiinvariants is given by

$$R^{(T)} = \{\alpha X^r Y^s : \ \alpha \in \mathbb{C}^*, \ r, s \in \mathbb{N}\}.$$

5.1. The weight monoid of an action

Let B be a Borel subgroup of a reductive group. In this section we will introduce some basic representation theoretic invariants of normal B-varieties. Hereafter, by writing $B = TU$, we implicitly assume that T is a maximal torus in B and U is the unipotent radical of B. Note that the groups of characters of B and T are naturally isomorphic. This follows from the decomposition $B = TU$ and the fact that the character group of U is trivial. Let r denote the dimension of T. Then $\ddot{\mathbf{O}}(B) \cong \ddot{\mathbf{O}}(T)$ is a free abelian group of rank r. The group operation on $\ddot{\mathbf{O}}(B)$ is given by the point-wise multiplication of the algebraic characters. However, it will be more convenient to switch to the additive notation.

Example 5.5. Let B denote the Borel subgroup of upper triangular matrices in $\mathbf{GL}_2(\mathbb{C})$. Then $\ddot{\mathbf{O}}(B)$ is freely generated by the following characters:

$$\lambda_i \left(\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} \right) = a_{ii} \qquad (i \in \{1, 2\}).$$

If we identify λ_1 (resp. λ_2) with the vector $(1, 0)$ (resp. by $(0, 1)$), then we see that the multiplication of the generators in $\ddot{\mathbf{O}}(B)$ corresponds to the addition of the corresponding integer vectors. Therefore, $\ddot{\mathbf{O}}(B)$ is isomorphic to $\mathbb{Z}^2$.

We proceed with the assumption that G is a connected reductive group containing B as a Borel subgroup. It turns out that the finite dimensional rational simple G-modules are in one-to-one correspondence with the elements of a certain submonoid of $\ddot{\mathbf{O}}(B)$. Since the main ideas behind this well-known result will be used several times in the sequel, we will review its proof.

Lemma 5.6. *([39, Theorem 31.3]) Let G be a connected reductive group having $B = TU$ as a Borel subgroup. If V is a simple G-module, then the subspace of U-invariants in V, that is $V^U = \{v \in V : \ u \cdot v = v$ for all $u \in U\}$ contains a one-dimensional subspace on* *which B acts by a character, $\chi \in \ddot{\mathbf{O}}(B)$. Furthermore, V is uniquely determined by χ up to G-isomorphism.*

Proof. By the Lie-Kolchin Theorem we know that V contains a B-stable one-dimensional weight subspace, $V_\chi \subseteq V$, such that B acts on V_χ by a character χ, see [39, Theorem 17.6]. Let v be a nonzero vector (weight vector) in V_χ. We claim that $U \cdot v = v$. Towards a contradiction, assume otherwise that $U \cdot v \neq v$. Since $b \cdot v = \chi(b)v$ for every $b \in B$, the restriction χ to U gives a nontrivial character for U, which is absurd. This contradiction shows that U point-wise fixes the whole line V_χ. This proves our first assertion.

For our second assertion, we fix some terminology; let us call the vector v of the previous paragraph a *maximal vector* for V. Let B^- denote the unique Borel subgroup opposite to B. Let U^- denote the maximal unipotent subgroup of B^-. Let $U^- = \prod_{\alpha \in \mathtt{R}^-_{G,T}} U_\alpha$ and $U = \prod_{\alpha \in \mathtt{R}_{G,T}} U_\alpha$ be the root subgroup decompositions of these unipotent subgroups. It is well-known that for all $\alpha \in \mathtt{R}_{G,T}$, the root subgroup U_α maps V_χ into $\sum_{k \in \mathbb{Z}^+} V_{\chi + k\alpha}$, see [39, Proposition 27.3]. Let V' denote the G-submodule of V that is generated by v. Since V is simple, we have $V' = V$. The weights of V' are of the form $\chi - \sum_{\alpha \in \mathtt{R}^+_{G,T}} c_\alpha \alpha$, where $c_\alpha \in \mathbb{Z}^+$, see [39, Proposition 31.2] (which essentially follows from [39, Proposition 27.3]). In other words, χ is the highest weight with respect to the partial order $\preceq$. Let us assume that there exists another 'maximal vector' $w \in V \backslash V_\chi$ so that $U \cdot w = w$ and on which B acts by a weight μ. Then by repeating the previous argument for w, we see that the span of $\chi \precnsim \mu$ which contradicts with the maximality of the weight χ. It follows from the uniqueness of the maximal vector in a simple module that if two simple G-modules V and W possess the same highest weight χ then they must be isomorphic. This finishes the proof of our second assertion. $\square$

Let us strengthen the part of Lemma 5.6 about the space of unipotent invariants.

Corollary 5.7. *Let G be a connected reductive group having $B = TU$ as a Borel subgroup. If V is a simple G-module, then V^U is one-dimensional.*

Proof. Let v and w be two non-proportional vectors in V such that $U \cdot w = w$ and $U \cdot v = v$. Without loss of generality we assume that v is a highest weight vector of weight χ. Then V decomposes as $V' \oplus V_\chi$, where w is an element of V'. Since V' is spanned by the weight spaces V_μ, where $\mu \precnsim \chi$, it is T-stable. In particular, we see that $B \cdot w \subseteq V'$. We now observe that the B^- orbit of w cannot be entirely contained in V'. Otherwise, since $B^- B$ is dense in G, we would have the span of $G \cdot w$ entirely contained in V', which is absurd. Now let $b' \in B^-$ be such that $b'w = v$. Then we have $w = u \cdot w = ub'^{-1}v$ for every $u \in U$. Since both of the spans of $B^- v$ and $B^- w$ are dense in V, the equalities $ub'^{-1}v = b'^{-1}v$ (for all $u \in U$) imply that $b' \in T$. But then the span of w in V is a line on which B acts by the character χ. This contradicts with our assumption that w is a weight vector of weight μ such that $\mu \neq \chi$. $\square$

Notation 5.8. Let G, B, T, U be as in Lemma 5.6. The group $\ddot{\mathbf{O}}(B)$ is called the *weight lattice* of (G, B, T). The elements of $\ddot{\mathbf{O}}(B)$ are called the *weights*. The highest weight of a simple G-module as in the proof of Lemma 5.6 is called a *dominant weight* of (G, B, T). We set

$$\ddot{\mathbf{O}}(B)^+ := \{\chi \in \ddot{\mathbf{O}}(B) : \ \chi \text{ is the highest weight of a simple } G\text{-module}\}.$$

This set is a submonoid of $\ddot{\mathbf{O}}(B)$. The identity element of this monoid is given by the trivial character defined by $\chi_0(b) := 1$ $(b \in B)$. This element corresponds to 0 in additive notation.

Now let X be an irreducible affine G-variety. Then $k(X)$ is the total quotient field of the ring of regular functions $k[X]$. Clearly, $k(X)$ has the structure of a G-module. For the analysis of this representation, the following fact, which was originally proved separately by Hadziev and Grosshans, is essential; it will enable us to define some important combinatorial invariants for the action $G : X$.

Lemma 5.9. *(Hadziev-Grosshans Theorem) The algebra of U-invariant regular functions $k[X]^U$ is a finitely generated k-algebra.*

Remark 5.10. Hadziev-Grosshans Theorem allows us to reduce the analyses of the coordinate rings of many affine varieties to the coordinate rings of toric varieties. We will make this comment more precise in Theorem 7.34.

One proof of Hadziev-Grosshans theorem relies on a useful method that is known as the "transfer principle". In our case this principle says that

$$k[X]^U \cong k[X \times G/U]^G \cong (k[X] \otimes k[G]^U)^G.$$

Since G is reductive, the k-algebra of G-invariants of a finitely generated algebra is finitely generated. Note that $k[X]$ is already a finitely generated k-algebra. Therefore, to prove the Hadziev-Grosshans theorem, it suffices to prove the finite generation of $k[G]^U$ only. In the literature, this invariant ring is sometimes called the *flag algebra of* G. It is naturally identified with the coordinate ring of G/U. We note here without proof that for arbitrary affine algebraic group G and a maximal unipotent subgroup $U \subset G$, the ring of invariants, $k[G/U] = k[G]^U$, is finitely generated; see the paragraph after [34, Theorem 5.6].

Lemma 5.11. *Let X be an irreducible quasi-affine S-variety, where S is a connected solvable affine algebraic group. Then any S-invariant in $k(X)$ is the quotient of two S-semiinvariants in $k[X]$ with the same weight. Furthermore, if S is a unipotent group, then $k(X)^S$ is isomorphic to the total quotient field of $k[X]^S$.*

Proof. Let $f = p/q$ be a S-invariant in $k(X)$, where p and q are regular functions. Since any element of the coordinate ring $k[X]$ is contained in a finite dimensional S-stable subspace, we see that the subspace $V := \text{span}\{s \cdot q : \ s \in S\} \subset k[X]$ is finite dimensional. Then by the Lie-Kolchin Theorem, there is a nonzero semiinvariant $h := \sum_{i \in I} c_i s_i q \in V$, where I is a finite set of indices. Then for every $i \in I$ we have $f = s_i \cdot f = \frac{s_i \cdot p}{s_i \cdot q}$. Equivalently, we have the following equations: $(c_i s_i \cdot q)f = (c_i s_i \cdot p)$ for $i \in \{1, \dots, |I|\}$. It follows that f is also equal to $f = \frac{\sum_{i \in I} c_i s_i \cdot p}{\sum_{i \in I} c_i s_i \cdot q}$. But since the denominator is a semiinvariant, and f is invariant, the numerator must also be a semiinvariant with the same weight as the denominator. This finishes the proof of our first assertion. For the proof of our second assertion, it suffices to mention that every semiinvariant of a unipotent group action is an invariant since such a group does not possess any nontrivial characters. $\square$

Remark 5.12. (1) The arguments that we used in the proof of Lemma 5.11 can be used for proving the following statement: any S-semiinvariant in $k(X)$ is the quotient of two S-semiinvariants in $k[X]$.

(2) The first assertion of Lemma 5.11 holds true even without the assumption on solvability as long as X is a "factorial" quasi-affine variety. Here, X is *factorial* means that the coordinate ring $k[X]$ is a unique factorization domain.

(3) Let U be a unipotent group. Let M be a rational U-module. Then we have $M^{(U)} = M^U$.

We will now focus on the B-semiinvariants of a quasi-affine irreducible B-variety X, where B is a Borel subgroup of some connected reductive group G. Although initially we do not need to require that X admits an algebraic G-action, let us first assume that this is the case. Recall that the coordinate ring of X is a locally finite representation of G which decomposes as a sum of finite dimensional irreducible G-modules. We further note that the B-highest weights of this representation is closed under addition. In other words, the B-semiinvariants that appear in $k[X]$ all belong to the monoid of dominant weights in the character group $\ddot{\mathbf{O}}(B)$. Since the dominant Weyl chamber is a strictly convex rational polyhedral cone in $\ddot{\mathbf{O}}(B) \otimes_{\mathbb{Z}} \mathbb{Q}$, the inverse weight $-\chi$ (in additive notation) does not appear as the weight of a regular function in $k[X]$. However, if a nonzero element $f \in k[X]$ is a B-semiinvariant of weight χ, then certainly the rational function $1/f \in k(X)$ is a B-semiinvariant of weight $-\chi$ in $k(X)$. At the same time, the pointwise multiplication of two B-semiinvariants in $k(X)^{(B)}$ is another B-semiinvariant in $k(X)$. Therefore, $k(X)^{(B)}$ is a subgroup of the multiplicative group of the field $k(X)$. Furthermore, the weights of $k(X)^{(B)}$ form a subgroup of $\ddot{\mathbf{O}}(B)$. In other words, the monoid homomorphism $k[X]^{(B)} \to \ddot{\mathbf{O}}(B)^+$, $f \mapsto \chi_f$ extends to a group homomorphism $k(X)^{(B)} \to \ddot{\mathbf{O}}(B)$. We are now ready to ramp up our Notation 5.8.

Definition 5.13. Let X be an irreducible affine (or, quasi-affine) G-variety. The map

$$\begin{aligned} \mathtt{r} : k(X)^{(B)} &\longrightarrow \ddot{\mathbf{O}}(B) \\ f &\longmapsto \chi_f \end{aligned}$$

is a $\mathbb{Z}$-module homomorphism. We call the image of $\mathtt{r}$, that is,

$$\ddot{\mathbf{O}}(X) := \mathtt{r}(k(X)^{(B)}),$$

the *weight lattice of* X. (Some authors call $\ddot{\mathbf{O}}(X)$ the *rank group of* X.) Since $\ddot{\mathbf{O}}(B)$ is a free abelian group, and $\ddot{\mathbf{O}}(X)$ is a subgroup of $\ddot{\mathbf{O}}(B)$, $\ddot{\mathbf{O}}(X)$ is a free abelian group as well. Its $\mathbb{Z}$-module rank, which we denote by $rk_G(X)$, will be called the *rank of the action* $G : X$.

We defined the weight lattice of an affine variety by using its coordinate ring. The ring of regular functions of a projective variety X consists of constant functions only. So it is not big enough to define any meaningful invariant of an action $G \times X \to X$. We can overcome this difficulty by using a B-stable affine open subset of X. We proceed with an example.

Example 5.14. Let P be a parabolic subgroup. We fix a Borel subgroup B in P, and let T be a maximal torus contained in B. By the Bruhat-Chevalley decomposition, we know that B has an open orbit in G/P. Let W denote the Weyl group $N_G(T)/T$, and let W_P denote its subgroup corresponding to the parabolic subgroup P. Let O denote the open orbit of B in G/P. It is well-known that O contains a unique T-fixed point, that is, $z_0 := w_0^P B$, where w_0^P is the longest element of W/W_P. Then we have $O = B \cdot z_0 \cong B/B_{z_0}$, where B_{z_0} is the stabilizer of z_0 in B. Now let $f(x)$ be a B-semiinvariant from $k(O)$, and let χ_f denote the corresponding character. For $b \in B$, let $t \in T$ and $u \in U$ be such that $b = tu$. Then we have $b^{-1} \cdot z_0 = u^{-1} \cdot z_0$. It follows that $b \cdot f(z_0) = f(b^{-1} \cdot z_0) = f(u^{-1} \cdot z_0) = \chi_f(u) \cdot f(z_0) = f(z_0)$. At the same time, as b varies in B, $b^{-1}z_0$ varies over all points of O. In other words, we have $f(O) = f(z_0)$. Since O is open in G/P, we see that $f(x)$ is a constant rational function, hence, $\chi_f = 1$. In conclusion, we see that the weight lattice of O, hence the weight lattice of G/P, is trivial.

In our next example, we will show an alternative approach towards the definition of the weight lattice of a projective variety.

Example 5.15. Let G denote $\mathbf{SL}_2(\mathbb{C})$, and let B be a Borel subgroup of G. Let X denote G/B. Then X is isomorphic to $\mathbb{P}^1$. We denote by $\pi : L \to X$ the hyperplane bundle $\mathcal{O}_X(1)$ on X. Clearly, G acts on the total space of L, and π is a G-equivariant morphism. In other words, L is a G-linearized line bundle for the simply connected semisimple group G. Let us define the *section ring of* L by

$$R(X, L) := \bigoplus_{n \geqslant 0} H^0(X, L^n). \tag{5.16}$$

Notice that this section ring is nothing but the (graded) coordinate ring of the total space of the dual line bundle $L^\vee$. The fiber of $L^\vee$ at a point $p \in X$ is the line in k^2 passing through the origin and the point p. By definition, the *affine cone associated with the line bundle* $L \to X$ is the affine variety whose coordinate ring is $R(X, L)$. Let $\tilde{X}$ denote this cone. Clearly, in our case, we have $\tilde{X} \cong k^2$. Let us have a closer look at the $G \times \mathbb{G}_m$-module structure on the coordinate ring $R(X, L) \cong k[x, y]$. The $\mathbb{G}_m$-action gives the usual grading on the polynomial ring $k[x, y]$. In particular, the n-th graded part, that is $H^0(X, L^n)$, is isomorphic to the space of homogeneous polynomials of degree n in $k[x, y]$. Furthermore, for every $n \in \mathbb{N}$, the subspace $H^0(X, L^n)$ is a G-module. It is well-known that this is an irreducible G-module of highest weight $\chi_n := n + 1$, $n \in \mathbb{N}$. Since the semisimple part of $G \times \mathbb{G}_m$ is G, the weight χ_n is also the highest weight of $H^0(X, L^n)$ as an irreducible rational representation of $G \times \mathbb{G}_m$. In conclusion, the weight lattice of the line bundle $L^\vee \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ as an $G \times \mathbb{G}_m$-variety is given by

$$\ddot{\mathbf{O}}(\mathcal{O}_{\mathbb{P}^1}(-1)) \cong \mathbb{Z}.$$

Remark 5.17. We maintain the notation of Example 5.15. It is not difficult to see that the homogeneous space G/U, where U is the unipotent radical of B, is isomorphic to $k^2\backslash\{0\}$. Thus, the weight lattice of G/U as a G-variety is identical to the weight lattice of k^2 as a $G \times \mathbb{G}_m$-variety.

5.2. Summary

Let X be an irreducible quasi-affine G-variety, where G is a connected reductive group. Then coordinate ring $k[X]$ has a natural decomposition into G-submodules as follows:

$$k[X] \cong \bigoplus_{\lambda \in \ddot{\mathbf{O}}(B)^+} (k[X]^{(B)}_\lambda) \otimes V(\lambda),$$

where $V(\lambda)$ is the simple G-module with highest weight λ, and $k[X]^{(B)}_\lambda$ is the G-module

$$k[X]^{(B)}_\lambda := \mathrm{Hom}_G(V(\lambda), k[X]),$$

that is the space of all G-equivariant morphisms from $V(\lambda)$ into $k[X]$. In characteristic 0, $\dim k[X]^{(B)}_\lambda$ is equal to the number of occurrence of $V(\lambda)$ in $k[X]$. To justify the notation, let us note the well-known fact that, for every λ and μ from $\ddot{\mathbf{O}}(B)^+$, the simple module $V(\lambda+\mu)$ occurs exactly once in the tensor product $V(\lambda) \otimes V(\mu)$. Therefore, highest weights of the simple G-submodules of $k[X]$ form a subsemigroup $\ddot{\mathbf{O}}(B)^+ \subseteq \ddot{\mathbf{O}}(B)$.

Definition 5.18. Let X be an irreducible affine (or, quasi-affine) G-variety. The *weight monoid of* $G : X$, denoted $\ddot{\mathbf{O}}(X)^+$, is the following submonoid of $\ddot{\mathbf{O}}(X)$:

$$\ddot{\mathbf{O}}(X)^+ := \{\lambda \in \ddot{\mathbf{O}}(X) :\ V(\lambda) \text{ occurs in } k[X]\}.$$

We have learned from the discussion preceding Remark 5.12 that the weight monoid $\ddot{\mathbf{O}}(X)^+$ generates the weight lattice $\ddot{\mathbf{O}}(X)$. Furthermore, we observed that $\ddot{\mathbf{O}}(X)^+$ spans a rational polyhedral cone in $\ddot{\mathbf{O}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Notation 5.19. We denote the $\mathbb{Q}$-vector space defined by the dual of $\ddot{\mathbf{O}}(X)$ as follows:

$$\mathbf{H}\ddot{\mathbf{O}}(X) := \mathrm{Hom}_{\mathbb{Z}}(\ddot{\mathbf{O}}(X), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Hom}_{\mathbb{Z}}(\ddot{\mathbf{O}}(X), \mathbb{Q}).$$

5.3. Modality

If an algebraic group action has an open orbit, then usually there is a combinatorially rich algebraic structure in force in the background. This is most apparent for the action of a Borel subgroup that is obtained by restricting a reductive group action.

Definition 5.20. Let G be a reductive group. Let X be a normal G-variety. If a Borel subgroup B of G has an open orbit in X, then X is said to be a *spherical G-variety*.

There are several different characterizations of the spherical varieties. We begin with interpreting the original definition in terms of invariants. For this purpose, the notion of a "modality" will be useful. Before introducing that notion, we want to mention an important result of Rosenlicht.

Lemma 5.21. *Let X be an irreducible H-variety, where H is an algebraic group. The transcendence degree over k of the field $k(X)^H$, denoted tr.deg$_k$ $k(X)^H$, is equal to* $\dim X - m$*, where* $m := \max\{\dim(H \cdot x) : x \in X\}$.

Definition 5.22. Let H be an algebraic group, let X be an H-variety. The *generic modality* of the action $H : X$ is the integer $d_H(X)$ defined by

$$d_H(X) := \text{tr.deg}_k\ k(X)^H.$$

In other words, by Lemma 5.21, the generic modality $d_H(X)$ is the minimum codimension of an H-orbit in X.

Definition 5.23. We maintain the notation of Definition 5.22. The *modality* of the action $H : X$, denoted $\operatorname{mod}(H : X)$, is the integer defined by

$$\operatorname{mod}(H : X) = \max\{d_H(Y) : \ Y \text{ is an } H\text{-stable, irreducible subvariety of } X\}.$$

Let X be an irreducible H-variety. Clearly, if X is one dimensional and H has an open orbit in X, then the complement of the open H-orbit consists of finitely many H-fixed points. More generally we have the following statement.

Proposition 5.24. *Let X be an irreducible H-variety. If* $\operatorname{mod}(H : X) = 0$*, then X is comprised of finitely many H-orbits.*

Proof. We use induction on $\dim X$. We already observed that if $\dim X = 1$, then our claim holds true. We proceed with a well-known statement [29, Theorem 3.3]: For $m \in \mathbb{N}$, the set $\{x \in X : \ \dim(H \cdot x) \geqslant m\}$ is open in X. In particular the set of points whose orbits have maximal dimension is open. Now we assume that for every H-variety Y with $\dim Y \leqslant n - 1$, where $n \in \mathbb{Z}^+$, if $\operatorname{mod}(H : Y) = 0$, then the set $\{H \cdot y : y \in Y\}$ has finitely many elements. Let X be an n-dimensional irreducible H-variety such that $\operatorname{mod}(H : X) = 0$. The latter condition implies that there exists an open H-orbit O in X. Furthermore, since X is irreducible, O is the unique open H-orbit in X. The complement $X\backslash O$ is an H-stable subvariety of X. By our assumption, each irreducible component of $X\backslash O$ is comprised of only finitely many H-orbits. Since the number of irreducible components of $X\backslash O$ is finite, we conclude that the total number of H-orbits in X is finite. □

Definition 5.25. Let G be a reductive group with a Borel subgroup B. Let X be a G-variety. The *complexity* of the action $G : X$, denoted by $c_G(X)$, is the generic modality $d_B(X)$.

We omit the proof of the following theorem.

Theorem 5.26. *([83, Theorems 2 and 3]) Let X be an irreducible G-variety, where G is a reductive group. Let B be a Borel subgroup. Then the following equalities hold:*

(1) $d_B(X) = mod(B : X)$,
(2) $d_U(X) = mod(U : X)$.

The following simple consequence of the previous theorem of Vinberg gives us another characterization of the spherical varieties. The same corollary was obtained independently by Brion by different methods.

Corollary 5.27. *([83, 14]) Let X be an irreducible variety as in Theorem 5.26. If B has an open orbit in X, then X is comprised of finitely many B-orbits.*

Proof. If B has an open orbit in X, then $d_B(X) = 0$. The proof follows from Theorem 5.26. ☐

Let H be a closed subgroup of a connected reductive group. Clearly, G/H is a G-variety. If G/H is a spherical variety, then H is called a *spherical subgroup*.

Lemma 5.28. *Let H be a closed subgroup of G. If H is a spherical subgroup of G, then we have the following short exact sequence of $\mathbb{Z}$-modules:*

$$\{1\} \to k^* \to k(G/H)^{(B)} \xrightarrow{r} \ddot{\mathbf{O}}(G/H) \to \{0\}.$$

Proof. Let $\frac{f}{g}$ be a rational function contained in the kernel of $r : k(G/H)^{(B)} \to \ddot{\mathbf{O}}(B)$. This means that the character of B that acts on $\frac{f}{g}$ is the character $\mathbf{1} : B \to \mathbb{G}_m$, $b \mapsto 1$. In other words, $\frac{f}{g}$ is a B-invariant rational function. We proceed to calculate some values of $\frac{f}{g}$. Let x_0 be a point from the open B-orbit in G/H. Since $\frac{f}{g}(b^{-1} \cdot x_0) = b \cdot \frac{f}{g}(x_0) = \frac{f}{g}(x_0)$ for all $b \in B$, we see that $\frac{f}{g}$ is constant along the open orbit. But G/H is a normal variety, hence, $\frac{f}{g}$ is constant on all points of X. In other words, $\frac{f}{g}$ is contained in the image of k^* in $k(G/H)$. This finishes the proof. ☐

There are many interesting non-spherical B-varieties with open B-orbits. We give two examples here.

(1) Let B denote the Borel subgroup of $G := GL(V)$, where V is a finite dimensional k-vector space. Then the Zariski closure $\overline{B}$ of B in $End(V)$ has an open $B \times B$-orbit. It is easy to verify that $\overline{B}$ is not a $G \times G$-variety.
(2) Let X_{wB} be a Schubert variety in G/B. Then the B-orbit of $wB \in G/B$ is open in X_{wB}. Unless w is the longest element $w_0 \in W$, the Schubert variety X_{wB} is not G-stable.

We finally want to point out a simple but useful relationship between the notions of complexity, rank, and the generic U-modality of a G-variety.

Lemma 5.29. *([44, Lemma 2.1], [58, Theorem 1.2.9 (2)]) Let X be an irreducible G-variety. Then we have the following identity:*

$$c_G(X) + rk_G(X) = d_U(X).$$

Proof. By replacing X with an open subset (using Rosenlicht's theorem on quotients) we may assume that the orbit spaces X/U and $X/B = (X/U)/T$ exist. Then the following equations hold: $\dim(X/B) = c_G(X)$ and $\dim X = \dim(X/U) + \max\{\dim Ux : x \in X\}$. The latter equation is equivalent to $d_U(X) = \dim(X/U)$. Moreover, it is easy to see that the image of T in the automorphism group $\mathrm{Aut}(X/U)$ is of dimension $rk_G(X)$. It follows from these facts that $\dim(X/U) = \dim(X/B) + rk_G(X)$. This finishes the proof of our assertion. □

Corollary 5.30. *Let P be a parabolic subgroup of G. If $X = G/P$, then we have $c_G(X) = rk_G(X) = 0$ and $\ddot{\mathbf{O}}(X) = \{0\}$.*

Proof. By Bruhat-Chevalley decomposition we know that U has an open orbit in G/P. Therefore, $d_U(X) = 0$. Since both of the integers $c_G(X)$ and $rk_G(X)$ are nonnegative, they are equal to 0. In particular, $\ddot{\mathbf{O}}(X)$ is trivial. □

6. Valuations

From a purely combinatorial viewpoint, a *valuation ring*[15] is nothing but a maximal element of a certain partial order on the set of all local rings of a field. Let K be a field and let **Loc** denote the set of all local subrings of K. We order **Loc** with respect to the *domination order* which is defined as follows. Let A and B be two local subrings of K. We say that B *dominates* A if A is a subring of B and the maximal ideal of A is the intersection of the maximal ring of B with A. Then the maximal elements of **Loc** exist; they are precisely the valuation rings whose quotient field is K. Since we are interested in the geometric applications of valuation rings, we proceed to identify the most useful ones for our purposes.

Let X be a variety. The multiplicative group of field of rational functions on X will be denoted by $k(X)^*$. A *valuation on X*[16] (or, more precisely, a *valuation on $k(X)$*) is a map

$$v : k(X) \to \mathbb{Q} \cup \{\infty\}$$

such that

[15] Let B be an integral domain, and let K denote the quotient field of B. If for every $x \in K$ either $x \in B$ or $x^{-1} \in B$, then B is called a *valuation ring*. Let U denote the group of units of B. Clearly, U is a subgroup of $K^* := K\backslash\{0\}$. Let Γ denote the quotient group K^*/U. It is easy to check that Γ is a totally ordered abelian group with respect to the partial order defined by $\bar{x} \geqslant \bar{y}$ if and only if $xy^{-1} \in B$, where $\bar{x}$ (resp. $\bar{y}$) is the image of x (resp. y) in Γ. Then the canonical projection $v : K^* \to \Gamma$ is a group homomorphism such that $v(x+y) \geqslant \min\{v(x), v(y)\}$ for every $x, y \in K^*$. The map v is called a *valuation of K with values in* Γ. A good reference for studying valuation rings is [54].

[16] Strictly speaking, this should be called a *discrete valuation on* $k(X)$.

(1) $v(f_1+f_2) \geqslant \min\{v(f_1), v(f_2)\}$ whenever we have $\{f_1, f_2, f_1+f_2\} \subset k(X)^*$,
(2) $v(f_1 f_2) = v(f_1) + v(f_2)$ for all $\{f_1, f_2\} \subset k(X)^*$,
(3) $v(k^*) = 0$ and $v(0) = \infty$.

We note that, according to this definition, the map defined by $w(k(X)^*) = 0$ and $w(0) = \infty$ is a valuation on X. We will call this valuation the *trivial valuation* on $k(X)$.

It is easy to verify that, for every nontrivial valuation v on $k(X)$, the image $v(k(X)^*) \subset \mathbb{Q}$ is a discrete subgroup of $\mathbb{Q}$. Although it is isomorphic to $\mathbb{Z}$, $v(k(X)^*)$, it may not be identically equal to $\mathbb{Z}$ in $\mathbb{Q}$. If the equality $v(k(X)^*) = \mathbb{Z}$ holds, then we say that "v is *normalized*".

Example 6.1. Let $v_{\deg}$ denote the degree-difference function on the field of rational functions $k(x)$ on $\mathbb{A}^1$. More precisely, for $f(x) = h(x)/g(x) \in k(x)$, we define

$$v_{\deg}(f(x)) := \begin{cases} \deg h(x) - \deg g(x) & \text{if } h(x) \neq 0; \\ \infty & \text{if } h(x) = 0. \end{cases}$$

Then it is easy to check that $v_{\deg}$ is a normalized valuation. We note that "deg" in $v_{\deg}$ should not be taken as 'degree' but rather for 'degenerate.' As we will see in later in the text, the most interesting valuations on $k(x)$ are the ones that correspond to the geometric points of $\mathbb{P}^1$. The valuation $v_{\deg}$ is not one of them; this is the reason for why we called it the degenerate valuation.

The *valuation ring of* v[17] is the local ring $\mathcal{O}_v$ defined by

$$\mathcal{O}_v := \{f \in k(X)^* :\ v(f) \geqslant 0\} \cup \{0\}.$$

Note that the quotient field of $\mathcal{O}_v$ is exactly the full function field $k(X)$. The unique maximal of $\mathcal{O}_v$ is given by

$$\mathfrak{m}_v := \{f \in k(X)^* :\ v(f) > 0\} \cup \{0\}.$$

The *residue field* of v is the field defined by

$$k(v) := \mathcal{O}_v/\mathfrak{m}_v.$$

Remark 6.2. (1) $\mathcal{O}_v$ is a maximal subring of $k(X)$.
(2) $\mathcal{O}_v$ determines v up to a positive scalar multiple.

We now want to explain why valuations are actually at the heart of algebraic geometry. Let K/k be a field extension with transcendence degree one. Let C_K denote the set of valuation rings $\mathcal{O}_v$ such that

(1) $k \subseteq \mathcal{O}_v \subseteq K$,
(2) v is a normalized valuation of K.

[17] More generally, let B be a (not necessarily discrete) valuation ring. We assume that B is not a field. Then B is the local ring of a discrete valuation if and only if B is noetherian [54, Theorem 11.2].

The set C_K possesses the "Zariski topology" whose closed sets are finite subsets of C_K or C_K itself. Then *an abstract nonsingular curve* is an open subset $U \subset C_K$ with the induced topology. It is well-known that every nonsingular quasi-projective curve Y is isomorphic to an abstract nonsingular curve [36, Ch I, Proposition 6.7].

6.1. Valuations associated with subvarieties

Hereafter, we assume that X is a normal variety. Let Z be an irreducible divisor in X. Since X is normal, the local ring of Z, denoted by $\mathcal{O}_{Z,X}$, is integrally closed. Furthermore, it is noetherian and it has dimension one. In other words, it is a Dedekind domain[18] The fact that $\mathcal{O}_{Z,X}$ is a local ring implies that it is a discrete valuation ring. Let f be a nonzero element from $\mathcal{O}_{Z,X}$. Then, as an $\mathcal{O}_{Z,X}$-module, the quotient $\mathcal{O}_{Z,X}/(f)$ has finite length. This observation allows us to introduce the *order-of-vanishing function*:

$$\begin{aligned} \mathrm{ord}_Z : \mathcal{O}_{Z,X} &\longrightarrow \mathbb{Z} \\ f &\longmapsto \text{the length of the module } \mathcal{O}_{Z,X}/(f). \end{aligned}$$

This function has the property that $\mathrm{ord}_Z(fg) = \mathrm{ord}_Z(f) + \mathrm{ord}_Z(g)$. At the same time, we can view $\mathcal{O}_{Z,X}$ as a subring of the function field $k(X)$. In fact, we know the fact that the quotient field of $\mathcal{O}_{Z,X}$ is $k(X)$ (see, for example, [36, pg. 130]). In particular, any element of $k(X)$ is of the form f/g with $f, g \in \mathcal{O}_{Z,X}$. Hence, the function ord_Z extends to a function $v_Z : k(X)^* \to \mathbb{Z}$ defined by

$$v_Z(f/g) = \mathrm{ord}_Z(f) - \mathrm{ord}_Z(g). \tag{6.3}$$

It is not hard to see that v_Z is indeed a valuation of $k(X)$. Thus, we associated to the divisor Z a valuation v_Z on $k(X)$.

Definition 6.4. Let D be a prime divisor in a normal variety X. The valuation v_D defined as in (6.3) will be called the *order-of-vanishing valuation of D*.

Example 6.5. This example is the most elementary example of a valuation. Let p be a prime number. Every nonzero rational number $\alpha \in \mathbb{Q}$ has a unique expression of the form

$$\alpha = p^n \frac{a}{b},$$

where $n \in \mathbb{Z}$, $a \in \mathbb{Z}\backslash\{0\}$, $b \in \mathbb{Z}^+$, and $gcd(a, b) = gcd(a, p) = gcd(b, p) = 1$. In this notation, we define a function $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ as follows:

$$v_p(\alpha) = \begin{cases} n & \text{if } \alpha \neq 0 \text{ and } \alpha = p^n \frac{a}{b}; \\ \infty & \text{if } \alpha = 0. \end{cases}$$

It is easy to verify that v_p is a valuation on $\mathbb{Q}$; it will be called the *p-adic valuation*. There is a useful geometric way of viewing v_p. We will explain this in very rough terms. Since $\mathbb{Z}$ is a principal ideal domain, its prime ideals are all maximal ideals. Let $\operatorname{Spec} \mathbb{Z}$ denote

[18] Let A be a noetherian domain of dimension one. If A is integrally closed, then it is called a *Dedekind domain*. Equivalently, if every local ring A_P, where P is a nonzero prime ideal in A, is a DVR, then A is a Dedekind domain.

the affine scheme of prime ideals of $\mathbb{Z}$. Since it is one-dimensional as a scheme, $\operatorname{Spec}\mathbb{Z}$ can be viewed as a curve. Moreover, since $\mathbb{Z}$ is integrally closed in its fraction field, we can view $\operatorname{Spec}\mathbb{Z}$ as a normal curve. From this viewpoint, a prime ideal of $\mathbb{Z}$ represents an irreducible divisor in $\operatorname{Spec}\mathbb{Z}$. Hence, each p-adic valuation can be regarded as the order-of-vanishing valuation on the curve $\operatorname{Spec}\mathbb{Z}$.

Next, we will discuss the order-of-vanishing valuations of the affine line, $\mathbb{A}^1(k) := \operatorname{Spec} k[x]$.

Example 6.6. Let x be a transcendental element over k, and let f be an irreducible polynomial in the polynomial ring $k[x]$. Clearly every nonzero element of the field $k(x)$ is of the form $f^n g/h$, where $n \in \mathbb{Z}$, and both of g and h are nonzero polynomials in $k[x]$ such that $gcd(f,h) = gcd(f,g) = gcd(g,h) = 1$. Let $v : k(x) \to \mathbb{Z} \cup \{\infty\}$ denote the function defined by

$$v(f^n g/h) = \begin{cases} n & \text{if } g \neq 0; \\ \infty & \text{if } g = 0. \end{cases}$$

Then it is easy to verify that v is a normalized valuation. Since k is algebraically closed, we have $f = x-\alpha$ for some $\alpha \in k$. Thus by applying v to a rational function in $k(x)$ we find out whether that rational function has a pole, or zero of certain order at the point $\alpha \in \mathbb{A}^1$. In summary we see that v is the discrete valuation of the divisor α in $\mathbb{A}^1$. Thus, it makes sense to write $v = v_\alpha$. Note also that the local ring of v is precisely the localization of $k[x]$ at the ideal (f). In other words, we have $\mathcal{O}_v = k[x]_{(f)} = \{a/b \in k(x) : \ gcd(f,b) = 1\}$.

We now introduce the most important valuations for our purposes.

Definition 6.7. Let X be a variety. Let v be a valuation on $k(X)$. Then v is said to be a *geometric valuation on* $k(X)$ if there exists an irreducible divisor $D \subset X$ and $c \in \mathbb{Q}^+$ such that $v(f) = c \cdot v_D(f)$ for every $f \in k(X)$. In this case, the DVRs of the valuations v and v_D are equal. A closed subvariety Z of X is called the *center of* v if the local ring of Z in X is dominated by the local ring of v. (Recall that this means that $\mathcal{O}_{Z,X} \subseteq \mathcal{O}_v$ and $\mathfrak{m}_{Z,X} = \mathcal{O}_{Z,X} \cap \mathfrak{m}_v$.)

The center of a valuation may not exist. Nevertheless, if it exists, then it is unique.

Example 6.8. Let us consider the affine variety, $X := \mathbb{C}^*$. The function field of X is given by $\mathbb{C}(x)$. As in Example 6.6, we will define the valuation $v_0 : \mathbb{C}(x) \to \mathbb{Q} \cup \{\infty\}$ as follows. Every nonzero element $f \in \mathbb{C}(x)$ can be written uniquely in the form $x^n g/h$ for some $n \in \mathbb{Z}$, $g, h \in \mathbb{C}[x]$ such that $gcd(x,g) = gcd(x,h) = gcd(g,h) = 1$. We then set $v_0(f) := n$, and $v_0(0) = \infty$. It is easy to check that v_0 is a valuation of $\mathbb{C}(x)$. Furthermore, the valuation ring of v_0 is $\mathcal{O}_v = \mathbb{C}[x]_{(x)} = \{g/h : \ g, h \in \mathbb{C}[x], \ gcd(h,x) = 1\}$, and the maximal ideal of $\mathcal{O}_v$ is generated by x in $\mathcal{O}_v$. Since $\mathbb{C}^*$ is affine, if the center of v were existent, then the center would be given by the vanishing of x in $\mathbb{C}^*$. Clearly, there is no such point in $\mathbb{C}^*$. Indeed, v_0 is the valuation associated with the origin $0 \in \mathbb{C}$, which is missing from $\mathbb{C}^*$.

For each closed subvariety Z in X, there is a corresponding geometric valuation on $k(X)$. The construction of this valuation goes as follows. Let $\pi : \widetilde{X} \to Bl_Z(X) \to X$ denote the normalization of the blow-up of X along Z. Let $\widetilde{Z}$ denote an irreducible component of the exceptional divisor, $\pi^{-1}(Z)$. Then we get a geometric valuation $v_{\widetilde{Z}}$ on $k(\widetilde{X})$ corresponding to $\widetilde{Z}$. Since π is a dominant map, the induced map on the function fields, $\pi^* : k(X) \to k(\widetilde{X})$ is injective. The valuation v_Z that we seek is then the restriction of $v_{\widetilde{Z}}$ to the subfield $k(X)$ in $k(\widetilde{X})$.

Let v_Z be the valuation associated with a closed subvariety $Z \subset X$ as defined in the previous paragraph. Then the proofs of the following assertions follow from the definition of a valuation associated to a divisor.

- Let f be an element of $\mathcal{O}_{Z,X}$. Then $v_Z(f) \geqslant 0$.
- Let f be an element of $k(X)$. If $f|_Z = 0$, then $v_Z(f) > 0$.
- Since $\mathcal{O}_v$ dominates $\mathcal{O}_{Z,X}$, the induced map $k(Z) = \mathcal{O}_{Z,X}/\mathfrak{m}_{Z,X} \to k(v) = \mathcal{O}_v/\mathfrak{m}_v$ is injective.

Remark 6.9. (1) If X is an affine variety, then a valuation v on $k(X)$ has a center $Z := Z_v \subset X$ if and only if $v(f) \geqslant 0$ for every $f \in k[X]$. In this case, Z is defined by the ideal $k[X] \cap \mathfrak{m}_v$.

(2) Let $f : X \to X'$ be a dominant morphism of varieties. Then the induced map $f^* : k(X') \to k(X)$ of function fields is injective. We identify $k(X')$ with its image in $k(X)$. Now, the map $f : X \to X'$ is proper if and only if any geometric valuation of $k(X)$ has a center in X provided that its restriction to $k(X')$ has a center in X'.

(3) A variety X is complete if and only if every geometric valuation has a center in X.

Example 6.10. Let X denote $\mathbb{P}^1$. We decompose $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$, where ∞ is the point-at-infinity. Then the field of rational function on X is the total quotient field of the coordinate ring $k[x]$ of $\mathbb{A}^1$. Since X is complete, every valuation of $k(x)$ has a unique center on $\mathbb{P}^1$. Since X is one dimensional, the proper irreducible subvarieties of X are points; they are precisely the prime divisors of $\mathbb{P}^1$. In other words, the geometric valuations of X are the valuations of $k(X)$ associated with the points of X. Let α be a point in X. Let us first assume that $\alpha \neq \infty$. Then we have, as in Example 6.6, v_α as the corresponding valuation. Now let us define the valuation corresponding to the divisor $\alpha = \infty$. In this case the local defining equation of our divisor is given by $f := \frac{1}{x} = 0$. Clearly, any nonzero element of $k(x)$ has a unique expression of the form $x^n g/h$, where $n \in \mathbb{Z}$ and g and h are polynomials without constant terms. Once again as in Example 6.6 we see that the valuation corresponding to ∞ is given by

$$v_\infty(x^n g/h) = \begin{cases} -n & \text{if } g \neq 0; \\ 0 & \text{if } g = 0. \end{cases}$$

It turns out that the geometric valuations of $k(X)$ are characterized by the transcendence degrees of their residue fields.

Proposition 6.11. *Let K' be an intermediate field between k and $k(X)$. Let v be a nonzero valuation on $k(X)$. Then we have,*

1. *v is a geometric valuation if and only if $tr.deg_k k(v) = tr.deg_k k(X) - 1$.*
2. *If v is a geometric valuation of $k(X)$, then $v|_{K'}$ is a geometric valuation of K'.*
3. *Any geometric valuation of K' extends to a geometric valuation of $k(X)$.*

The proof of this proposition is not difficult. We refer the reader to [82, Propositions B.7 and B.8].

6.2. Invariant valuations

In this subsection, as usual, X denotes an irreducible G-variety, where G is a connected affine algebraic group.

Definition 6.12. A valuation v on $k(X)$ is said to be a *G-invariant valuation* if $v(g \cdot f) = v(f)$ for every $g \in G$ and $f \in k(X)^*$. A G-invariant valuation v is called a *G-valuation* if it is a geometric valuation. The set of all G-valuations on X is denoted by $\mathcal{V}(X)$. Although the trivial valuation is not a geometric valuation, we include it in the set $\mathcal{V}(X)$ as a convention. Therefore, we always have $\mathcal{V}(X) \neq \varnothing$.

We note that a G-valuation does not need to be a normalized valuation.

Example 6.13. In this example, we will determine all normalized $\mathbb{G}_m$-valuations on $\mathbb{P}^1$. Let s_1, s_2 be two distinct positive integers. We consider the algebraic action $\mathbb{G}_m \times \mathbb{P}^1 \to \mathbb{P}^1$ defined by $t \cdot [(a,b)] = [(t^{s_1}a, t^{s_2}b)]$, where $[(a,b)] \in \mathbb{P}^1$ and $t \in \mathbb{G}_m$. Here, we use the identification of $\mathbb{P}^1$ with the quotient $(k^2 \backslash \{0\})/\mathbb{G}_m$, where $\mathbb{G}_m$ acts on the right on $k^2 \backslash \{0\}$ by the coordinate-wise multiplication action. Now let v_α ($\alpha \in \mathbb{P}^1$) be a geometric valuation on $\mathbb{P}^1$. Let us also assume that v_α is a $\mathbb{G}_m$-valuation. As before, we decompose $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$. Let us first assume that $\alpha \in \mathbb{A}^1 \backslash \{0\}$. Let $f(x) \in k(x)$ be a rational function defined on an open subset contained in $\mathbb{A}^1$. Then $f(x) = c\prod_{i=1}^r (x - a_i)^{n_i}$, where $c \in k$, $n_i \in \mathbb{Z}$, and $a_i \in k$ for all i. Since v_α is a $\mathbb{G}_m$-valuation, for every $t \in \mathbb{G}_m$, the following equalities hold:

$$\begin{aligned} v_\alpha(f(x)) &= v_\alpha(t \cdot f(x)) \\ &= v_\alpha(f(t^{-1} \cdot x)) \\ &= v_\alpha(ct^{s_1 \sum n_i} \prod_{i=1}^r (x - t^{s_1}a_i)^{n_i}) \\ &= v_\alpha(\prod_{i=1}^r (x - t^{s_1}a_i)^{n_i}). \end{aligned} \tag{6.14}$$

But since $v_\alpha(f(x))$ is the order of vanishing of $f(x)$ at α, and since t varies over all elements of $\mathbb{G}_m$, we see that the equality (6.14) cannot hold unless the valuation v_α takes the constant value 0 on $k(x)^*$. Therefore, v_α cannot be a $\mathbb{G}_m$-valuation. Next, we assume that $\alpha = 0$ in $\mathbb{A}^1$. We notice that a similar calculation to (6.14) shows that v_α is a valid

$\mathbb{G}_m$-valuation. Adopting these arguments to the valuation of the divisor (∞), we see that v_∞ is also a $\mathbb{G}_m$-valuation. (Indeed, we have the equality $v_\infty = -v_0$.) In conclusion, we see that on $\mathbb{P}^1$ there are only two nontrivial $\mathbb{G}_m$-valuations, that are, v_0 and v_∞. Therefore, on $k(\mathbb{P}^1)$, there are exactly three normalized $\mathbb{G}_m$-valuations; one of them is the trivial valuation, and the other two are the normalized valuations v_0 and v_∞.

Example 6.15. In this example, we view $\mathbb{P}^1$ as the flag variety $\mathbf{GL}_2/\mathbf{B}$. Then we obtain a natural $\mathbf{GL}_2$ action on $\mathbb{P}^1$. Since any $\mathbf{GL}_2$-valuation is also a $\mathbb{G}_m$-valuation, where $\mathbb{G}_m$ is viewed as the center of $\mathbf{GL}_2$, we see that up to normalization, there are at most three normalized $\mathbf{GL}_2$-valuation on $\mathbb{P}^1$. But $\mathbf{GL}_2$ acts transitively on $\mathbb{P}^1$; this was not possible by the $\mathbb{G}_m$-action. It follows from a calculation similarly to (6.14) that the normalized valuation v_0 is not a $\mathbf{GL}_2$-valuation. Likewise, v_∞ is not a $\mathbf{GL}_2$-valuation. In fact, as we will see later, on any homogeneous space of the form G/P, where G is a connected reductive group and $P \subseteq G$ a parabolic subgroup, there is a unique G-valuation, which is the trivial one.

Remark 6.16. Let v be a G-valuation in $\mathcal{V}(X)$. We assume that v has the center on a subvariety $Z \subseteq X$. By intersecting it with an affine open G-stable subset in X, we may assume that Z is affine. Now it is easy to check that Z is G-stable. Conversely, if Z is an irreducible G-stable closed subvariety of X, then let v_Z denote the valuation in $k(X)$ that is obtained by the blowing up construction as in the previous subsection (in the paragraph after Example 6.8). Since it comes from the restriction of a G-valuation onto the G-stable field $k(X)$, the valuation v_Z is a G-valuation as well.

One of the earliest and the most useful results about geometric valuations is the following lemma of Sumihiro [81, Theorem 1]. Here we follow Timashev's proof [82, Chapter 19].

Lemma 6.17 (Sumihiro's Lemma)**.** *Let X be an irreducible G-variety. Let v be a geometric valuation on $k(X)$. Then there exists a G-valuation $\bar{v} \in \mathcal{V}(X)$ such that if f is a rational function on X, then there is an open subset $U_f \subseteq G$ having the following property,*

$$\bar{v}(f) = v(g \cdot f) \quad \text{for all } g \in U_f.$$

Furthermore, if A is a G-stable subalgebra of $k(X)$, then for every $f \in A$, the value $\bar{v}(f)$ is found by

$$\bar{v}(f) = \min\{v(g \cdot f) : \ g \in G\}.$$

Proof. Thanks to the algebraic action $G : X$, for each $f \in k(X)$, the G-translates of f, that are $g \cdot f$ where $g \in G$, glue together to give a rational function $f' \in k(G \times X)$ defined by $f'(g, x) = f(g^{-1} \cdot x)$, $(g, x) \in G \times U$. Here, U is the domain of definition of f in X. We now let D be a prime divisor of X such that $v = v_D$. Then $G \times D$ is a prime divisor of $G \times X$. Let $v' := v_{G\times D}$ denote the corresponding geometric valuation. Let $f' \in k(G \times X)$ be a rational function with domain of definition $A \subseteq G \times X$. Of course, f' might be an element of the subfield $k(X) \hookrightarrow k(G \times X)$. In any case, since v' is the valuation of the divisor $G \times D$, $v'(f')$ is the order zeros or poles of f' on $G \times D$. In particular, there is an open subset $U_{f'}$ of G such that for every $g \in U_{f'}$, the order of zeros or poles of the rational function $f'(g,\cdot) : A \to k$, $(h,x) \mapsto f'(g,x)$ is given by $v'(f')$. In other words, we have $v'(f') = v(f(g,\cdot))$. Therefore, by setting $\bar{v} := v'|_{k(X)}$ we obtain a G-valuation that extends v.

For the second assertion, we first notice that, for $d \in \mathbb{Q}$, the linear spaces $A^{(d)} := \{f \in A : \ v(f) \geqslant d\}$ form a filtration of the (infinite dimensional) affine space A. The orbit $G \cdot f$ in A is a finite dimensional variety. Then the intersections $(G \cdot f) \cap A^{(d)}$ are closed subvarieties of $G \cdot f$. Therefore our claim follows from the first assertion combined with the definition of $A^{(d)}$. □

As a consequence of Sumihiro's lemma and Proposition 6.11, we know the following useful statement.

Lemma 6.18. *Let $k \subseteq K' \subseteq k(X)$ be an intermediate G-stable field. Let v' be a G-valuation on K'. Then there exists a G-valuation on $k(X)$ such that $v|_{K'} = v'$. Conversely, the restriction of a G-valuation $v \in \mathcal{V}(X)$ to K' is a G-valuation v' on K'.*

Convention 6.19. For the rest of this subsection we assume that G denotes a connected reductive group. We assume that X is a normal G-variety.

Let H be a closed subgroup of G. As we pointed out before, the canonical map $G \to G/H$ is dominant. Then $k(G/H)$ is a G-stable subfield of $k(G)$. Therefore, by Lemma 6.18, any G-valuation on $k(G/H)$ extends to a G-valuation on $k(G)$. More generally, if the modality of $G : X$ is zero, then there is an open G-orbit $O \subseteq X$. Let H denote the stabilizer of a point in general position in O. Then $k(X) = k(O) = k(G/H)$. It follows that any G-valuation of $k(X)$ extends to a G-valuation on G/H.

We are ready to discuss a very useful but somewhat technical result of Knop. Let char-exp(k) denote the *characteristic exponent of* k, that is,

$$\text{char-exp}(k) := \begin{cases} 1 & \text{if } \text{char}(k) = 0; \\ \text{char}(k) & \text{otherwise.} \end{cases}$$

Let M be a subset of a G-field K. For $n \in \mathbb{Z}^+$, we denote by M^n the subspace of K that is spanned by all n-fold products of the form $f_1 \cdots f_n$ for $f_i \in M$.

Theorem 6.20 (Knop's Approximation Theorem)**.** *Let K be a G-field such that $k \subseteq K \subseteq k(G/H)$. Let A be a rational G-subalgebra of K. Let v be an element of $\mathcal{V}(G/H)$. Then for every $f \in A$ we have*

$$v(f) = \min \left\{ \frac{v(\tilde{f})}{n} : \ \tilde{f} \in (M^n)^{(B)} \right\}, \tag{6.21}$$

where M is the G-submodule generated by f in A, and $n = n(f)$ is a sufficiently big power of $p :=$ char-exp(k).

Notice that, in characteristic zero, the formula in (6.21) simplifies to

$$v(f) = \min \left\{ v(\tilde{f}) : \ \tilde{f} \in (M^n)^{(B)} \right\}.$$

A proof of this theorem can be found in [82, Section 19]. We will mention some of its important consequences.

Corollary 6.22. *Let H be a spherical subgroup of G, and let v be a G-valuation from $\mathcal{V}(G/H)$. Let f be a regular function on the open B-orbit $Bx_0 \subseteq G/H$. Then there exists a sufficiently big power $n := p^N$ (depending on f), and a B-semiinvariant $f' \in k(G/H)^{(B)}$ such that*

(1) $v(f') = v(f^n)$,
(2) $w(f') \geqslant w(f^n)$ for all $w \in \mathcal{V}(G/H)$,
(3) $v_D(f') \geqslant v_D(f^n)$ for every B-invariant divisor D of G/H.

Proof of this corollary follows immediately from a more general statement (Lemma 6.24) that is applicable to any G-variety X such that $\mathrm{mod}(G : X) = 0$. In characteristic 0, Corollary 6.22 implies that, on a spherical homogeneous space G/H, a G-valuation is completely determined by its values on the B-semiinvariants $k(G/H)^{(B)}$. To state the more general version of Corollary 6.22, we consider a special subalgebra of the field rational functions.

Notation 6.23. Let X be a G-variety. We denote by K_B the subalgebra of $k(X)$ that consists of rational functions with B-stable divisor of poles on X. In other words, we set

$$K_B := \{f/g \in k(X) : \quad \text{the divisor of } g \text{ in } X \text{ is } B\text{-stable}\}.$$

Recall that a nonzero rational function $f \in k(X)$ is called a B-semiinvariant if there exists a character $\chi \in \ddot{\mathbf{O}}(B)$ such that $b \cdot f = \chi(b) f$ for every $b \in B$. Recall also that the set of all B-semiinvariant rational functions in $k(X)$ is denoted by $k(X)^{(B)}$.

Lemma 6.24. *Let X be a normal G-variety with an open orbit. Let K denote the field of rational functions on X. Let v be a G-valuation on G. Then for any rational function $f \in K_B$, there exists an element $\tilde{f} \in K^{(B)}$ and a sufficiently big power $n := p^N$ (depending on f) such that*

(1) $v(\tilde{f}) = v(f^n)$,
(2) $w(\tilde{f}) \geqslant w(f^n)$ for every $w \in \mathcal{V}$,
(3) $v_D(\tilde{f}) \geqslant v_D(f^n)$ for every B-stable, but not G-stable divisor D of K.

Proof. Let $\delta := \mathrm{div}_\infty f$ denote the divisor of poles of f in X. Let σ denote the element $f\mu_\delta$ in $H^0(X, \mathcal{O}(\delta))$, where μ_δ is the canonical section of the locally free sheaf $\mathcal{O}(\delta)$. After extending all G-valuations to $\bigoplus_{n \geqslant 0} H^0(X, \mathcal{O}(n\delta))$ by using Sumihiro's lemma, we consider the G-submodule $M \subseteq H^0(X, \mathcal{O}(\delta))$ generated by σ. For any B-eigensection $\tilde{\sigma} \in (M^n)^{(B)}$, we define $\tilde{f} := \tilde{\sigma}/\mu_\delta^n$. It follows from Theorem 6.20 that $v_D(\tilde{f}) \geqslant nv_D(f)$, $w(\tilde{f}) \geqslant nw(f)$, and $v(\tilde{f}) \geqslant nv(f)$ for some $\tilde{\sigma}$. □

Lemma 6.24 implies the following important fact that is noticed by Luna and Vust [52].

Proposition 6.25. *Let X be a G-variety with an open orbit. Then the elements of $\mathcal{V}(X)$ are distinguished by their restrictions to $k(X)^{(B)}$.*

Proof. Since G has an open orbit in X, the field K of rational functions on X is a G-stable subfield of $k(G)$. Let v' and w' be two distinct valuations on K. Since the quotient field of K_B is K, there exists $f \in K$ such that $v'(f) \neq w'(f)$. Without loss of generality we may assume that $v'(f) < w'(f)$. By virtue of Sumihiro's lemma, there are distinct valuations v and w of G extending v' and w', respectively. Then by Lemma 6.24 (1), there exists a B-semiinvariant $\tilde{f}$ such that $v(\tilde{f}) = v(f^n)$ for some n. Once again by applying Lemma 6.24 (part (2)), we see that $v(\tilde{f}) < w(\tilde{f})$ for some $K^{(B)}$. Therefore, the restrictions of v and w to $K^{(B)}$ give distinct functions. □

6.3. Recollections

In this subsection, we will reorganize some facts that we already mentioned for pedagogical reasons. Along the way, we will introduce several new concepts.

Let X be an affine G-variety, where G is a connected reductive group. Let K denote field of rational functions on X. Then K is the quotient field of $\mathcal{O}(X)$. Recall that the weight lattice of X is the subgroup $\ddot{\mathbf{O}}(X) \subseteq \ddot{\mathbf{O}}(B)$ generated by the highest weights of the irreducible representations of G that appear in $k[X]$. The weight lattice of X fits into a short exact sequence,

$$1 \to (K^B)^\times \to K^{(B)} \xrightarrow{\mathtt{r}} \ddot{\mathbf{O}}(X) \to 0, \tag{6.26}$$

where $\mathtt{r}(f) = \chi_f$. Here, χ_f is the character that determines the action of B on f. We now assume that X is an affine spherical G-variety. Then, as we explained earlier (Lemma 5.28), the short exact sequence (6.26) becomes

$$1 \to k^* \to k(X)^{(B)} \xrightarrow{\mathtt{r}} \ddot{\mathbf{O}}(X) \to 0.$$

Hence, we have the identification, $k(X)^{(B)}/k^* \cong \ddot{\mathbf{O}}(X)$.

Remark 6.27. The requirement that X be an affine variety is not very critical. We know that every normal G-variety X is locally linearizable (thanks to Sumihiro's Equivariant Embedding Theorem, which will be reviewed later). Thanks to the local structure theorem [82, Theorems 4.6 and 4.7] we can always find a B-stable affine open subset X_0 in X. Then we use X_0 to define the set of G-valuations as well as the weight lattice of X.

We continue with the assumption that X is a not necessarily affine spherical G-variety. For any valuation v on $k(X)$, there is an associated homomorphism $k(X)^{(B)} \to \mathbb{Q}$ defined by $f \mapsto v(f)$. If v is a constant function on the group of nonzero B-invariants, then, thanks to (6.26), v descends to a homomorphism on $\ddot{\mathbf{O}}(X)$. Let us denote this homomorphism by ϱ_v. Since X is a spherical variety, the assignment $v \mapsto \varrho_v \in \mathbf{H}\ddot{\mathbf{O}}(X)$ is well-defined. In other words, we have the following function:

$$\begin{aligned} \varrho : \{\text{all valuations on } X\} &\longrightarrow \mathbf{H}\ddot{\mathbf{O}}(X) \\ v &\longmapsto \varrho_v. \end{aligned}$$

There is a good choice of notation for the evaluations of ϱ_v's on the elements of $\mathbf{H\ddot{O}}(X)$. It is given by

$$\langle v, \chi \rangle := \varrho_v(\chi) \qquad (\chi \in \mathbf{H\ddot{O}}(X)).$$

This notation is indicative of a duality pairing between a certain set of valuations and the weights. When there is no danger for confusion, we will use this notation.

The map ϱ has utmost importance in the classification of spherical varieties. The analysis of this function becomes more useful once its domain is carefully restricted.

Notation 6.28. Let v be a G-valuation on $k(X)$. We denote the evaluation map ϱ_v by $\mathtt{e}_v$. Similarly, the restriction of ϱ to $\mathcal{V}(X)$ will be denoted by $\mathtt{e}$. In other words, we set

$$\begin{aligned} \mathtt{e} : \mathcal{V}(X) &\longrightarrow \mathbf{H\ddot{O}}(X) \\ v &\longmapsto \mathtt{e}_v. \end{aligned} \tag{6.29}$$

We now paraphrase Proposition 6.25 in the notation of (6.29).

Proposition 6.30. *Let X be a spherical G-variety. Then the evaluation map on the G-valuations, that is $\mathtt{e} : \mathcal{V}(X) \to \boldsymbol{H\ddot{O}}(X)$, is injective.*

Proof. Let G/H be the homogeneous space that is isomorphic to the open B-orbit in X. Since all of the ingredients are birational invariants, without loss of generality we may assume that $X = G/H$. Let v_0 and v_1 be two distinct G-valuations on G/H. Then there exists a rational function f defined on $B \cdot x_0$ such that $v_0(f) < v_1(f)$. By Corollary 6.22, we have a B-semiinvariant rational function $f' \in k(G/H)$ such that $v_0(f') = v_0(f^n) < v_1(f^n) \leqslant v_1(f')$. This shows that $val(v_0)$ and $val(v_1)$ are distinct. □

Example 6.31. Let X denote a partial flag variety G/P, where P is a parabolic subgroup of G. Recall from Example 5.14 that $\mathbf{\ddot{O}}(G/P) = \{0\}$. Therefore, by Proposition 6.30, we see that the set of G-valuations on G/P has only one element.

6.4. Central valuations

Let X be an irreducible G-variety, where G is a connected reductive group.

Definition 6.32. A G-valuation v on $k(X)$ is called a *central valuation* if v vanishes on the nonzero B-invariant rational functions. In other words, v is a central valuation if and only if $v((k(X)^B)^\times) = 0$.

Notice that if X is a spherical variety, then every G-valuation on $k(X)$ is a central valuation. We proceed with the more general assumption that $d_G(X) = 0$. In other words, G has an open orbit in X. Thanks to (6.26), a central G-valuation on $k(X)$ factors through a linear functional on $\mathbf{\ddot{O}}(X)$. We will denote by $\mathcal{Z}(X)$ the set of linear functionals in $\mathbf{\ddot{O}}(X)$ that come from central valuations on $k(X)$. By Proposition 6.25, we know that for each central valuation on $k(X)$ there exists a unique element of $\mathcal{Z}(X)$. Of course, for a spherical variety X, the set $\mathcal{Z}(X)$ is none other than the cone of invariant valuations $\mathcal{V}(X)$. The following theorem is very important for our purposes. In this generality it is due to Knop.

Theorem 6.33. *Let X be an irreducible G-variety, where G is a connected reductive group. If $d_G(X) = 0$, then $\mathcal{Z}(X)$ is a finitely generated solid convex polyhedral cone in $\boldsymbol{H\ddot{O}}(X)$. Furthermore, $\mathcal{Z}(X)$ contains the image of the anti-dominant Weyl chamber of (G, B, T).*

For spherical varieties, Pauer realized that the set $\mathcal{V}(X)$ $(= \mathcal{Z}(X))$ is a cone. In the same setup, the fact that $\mathcal{V}(X)$ is a finitely generated cone[19] was first proved by Brion and Pauer in [20]. Actually, in [20], Brion and Pauer identify the inequalities that define $\mathcal{V}(X)$.

Theorem 6.34. *Let X be a spherical variety. Then there exists a finite linearly independent set of weights $\{\chi_1, \dots, \chi_n\} \subset \boldsymbol{\ddot{O}}(X)$ such that*

$$\mathcal{V}(X) = \{v \in \boldsymbol{H\ddot{O}}(X) : \langle v, -\chi_i \rangle \geqslant 0 \ \ \text{for all } i \in \{1, \dots, n\}\}.$$

Theorem 6.34 implies immediately that the dual of the valuation cone in $\mathbf{\ddot{O}}(X)_{\mathbb{Q}}$ is a rational polyhedral cone generated by the linearly independent set $\{-\chi_1, \dots, -\chi_n\} \subset \mathbf{\ddot{O}}(X)_{\mathbb{Q}}$. In particular, the valuation cone itself is a rational polyhedral cone since these properties are preserved under passing to the duals. For relatively easy proofs of these extremely important facts, in the setup of spherical varieties, we recommend Knop's "expository" article [43, Section 5]. We will sketch the proof of Theorem 6.34 in Subsection 8.3.

It turns out that the valuation cone $\mathcal{V}(X)$ of a spherical variety is a fundamental domain for a finite reflection group which naturally acts on $\mathbf{H\ddot{O}}(X)$. This reflection group is called the *little Weyl group of X*.

Definition 6.35. Let X be a spherical homogeneous space G/H. The set of linearly independent and primitive weights $\mathtt{S} := \{-\chi_1, \dots, -\chi_n\}$ that are introduced in Theorem 6.34 is a basis for a crystallographic root system. The corresponding reflection group is called the *little Weyl group* of X. The elements of $\mathtt{S}$ are called the *spherical roots of X* relative to (G, B, T).

These facts were proved by Brion [16, 18] after [20]. Later, Knop extended the notion of a little Weyl group of a spherical variety to arbitrary G-varieties. In particular, Knop identifies the little Weyl group of a G-variety as the Galois group of a certain covering of the cotangent bundle of X. A complete account of these results are nicely presented by Timashev in his memoir [82, Chapter 4].

[19] In the language of convex geometry, which we will review soon, a finitely generated convex cone is called a *polyhedral cone*. A polyhedral cone σ in $N \otimes_{\mathbb{Z}} \mathbb{Q}$, where N is a lattice, is called *rational* if a generating set for σ is contained in N. Let us mention in passing that the valuation cone of a spherical variety is not necessarily "strictly convex" although it is a rational polyhedral cone.

7. Toric Varieties

Definition 7.1. Let T be a torus. A normal T-variety X is called a *toric variety* if it is a spherical T-variety.

It is safe to say that the theory of spherical varieties sprung from the toric varieties and reductive groups. The purpose of this section is to review their construction in terms of combinatorial fans. The following resources are helpful: [30, 57, 26, 28]. Throughout this section we will follow the standard notation related to the toric varieties.

N	:	a free abelian group of rank n
M	:	the dual lattice of N, that is, $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$
$N_{\mathbb{Q}}$	:	$N \otimes_{\mathbb{Q}} \mathbb{Z}$
$M_{\mathbb{Q}}$	:	$M \otimes_{\mathbb{Q}} \mathbb{Z} \cong \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q})$

In our previous notation, if T is an n-dimensional torus defined over k, then M will be the character group $\ddot{\mathbf{O}}(T)$, and N will denote the dual of $\ddot{\mathbf{O}}(T)$. Then we have $N_{\mathbb{Q}} = \mathbf{H}\ddot{\mathbf{O}}(T)$. Conversely, if M and N are free abelian groups as above, then we obtain an n-dimensional torus T_N and the character lattice of T_N by setting

$$T_N := N \otimes_{\mathbb{Z}} k^* \quad \text{and} \quad \ddot{\mathbf{O}}(T_N) = M.$$

Later, when we want to emphasize the fact that an element $m \in M$ is a character of the torus T_N, we will write χ^m.

Soon we will construct the affine toric varieties; they will be obtained from certain cones in $\mathbf{H}\ddot{\mathbf{O}}(T)$ via the spectra of finitely generated semigroup algebras. To this end, let us first introduce the necessary combinatorial gadgets.

A *(convex) polyhedral cone in* $N_{\mathbb{Q}}$ is a finitely generated convex subset of $N_{\mathbb{Q}}$. If $\sigma \subseteq N_{\mathbb{Q}}$ is a polyhedral cone of the form

$$\sigma := \langle v_1, \dots, v_j \rangle := \left\{ \sum_{i=1}^{j} r_i v_i : \ r_i \in \mathbb{Q}_{\geqslant 0} \right\}, \tag{7.2}$$

for some vectors $v_1, \dots, v_j$ in $N_{\mathbb{Q}}$, then the v_i's are called the *generators of* σ. We say that $\{v_1, \dots, v_j\}$ is a minimal generating set if σ cannot be generated by less than j vectors. If a generator v_i ($i \in \{1, \dots, j\}$) is the first nonzero lattice point along its *ray*, that is $\mathbb{Q}_{\geqslant 0} \cdot v_i$, then it is called the *primitive vector for* $\mathbb{Q}^+ v_i$. We now introduce several related notions.

- A *rational polyhedral cone in* $N_{\mathbb{Q}}$ is a polyhedral cone whose generators are from the lattice N.
- A polyhedral cone is called a *strictly convex polyhedral cone* if it does not contain any vector subspace of $N_{\mathbb{Q}}$.
- A strictly convex rational polyhedral cone σ in $N_{\mathbb{Q}}$ is said to be a *simplicial cone* if a set of generators $\{v_1, \dots, v_j\}$ for σ is a linearly independent set of vectors from N.

- A strictly convex rational polyhedral cone σ is called *smooth* if its minimal generating set $\langle v_1, \ldots, v_j \rangle$ is part of a $\mathbb{Z}$-basis for N.
- The *dual cone of* σ is the cone defined by $\sigma^\vee = \{m \in M_\mathbb{Q} : m(u) \geqslant 0 \text{ for all } u \in \sigma\}$.
- A *face of* σ is a subcone of the form $H_m \cap \sigma$ for some $m \in \sigma^\vee$, where $H_m \subset N_\mathbb{Q}$ is the kernel hyperplane of m, that is, $H_m := \{u \in N_\mathbb{Q} : m(u) = 0\}$.

Remark 7.3. (1) The set of all faces of a polyhedral cone σ forms a finite poset with respect to inclusion, called the *face lattice of* σ. This naming is justified by the fact that a face lattice is a lower meet semilattice in a combinatorial sense. Furthermore, the face lattices are always coatomic posets.

(2) Let τ be a face of a σ be a polyhedral cone in $N_\mathbb{Q}$, and let τ be a face of σ. Let $\tau^\perp$ denote the orthogonal complement of τ in $M_\mathbb{Q}$[20]. Then there is a unique orthogonal face, denoted τ^*, of the dual cone of σ. It is defined by $\tau^* := \sigma^\vee \cap \tau^\perp$. The assignment $\tau \mapsto \tau^*$ is an isomorphism between the face lattice of σ and the opposite poset of the face lattice of $\sigma^\vee$. Furthermore, we have $\dim \tau + \dim \tau^* = n$, where $\dim \tau$ is the dimensional of the smallest subspace of $N_\mathbb{Q}$ that contains τ.

(3) The notion of a *relative interior of a face* can be defined by the following implication:
$m \in \mathrm{Relint}(\tau^*) \iff \tau = H_m \cap \sigma$.

(4) The strictly convex polyhedral cones can be characterized in several different ways:
   (a) σ is strictly convex;
   (b) $\{0\}$ is a face of σ;
   (c) $\sigma \cap -\sigma = \{0\}$;
   (d) $\dim \sigma^\vee = \dim M_\mathbb{Q}$.

(5) Let σ be a full dimensional cone in $N_\mathbb{Q}$. Then σ is smooth (resp. simplicial) if and only if $\sigma^\vee$ is smooth (resp. simplicial).

The monoid of integral points in a rational convex polyhedral cone is always finitely generated. This fact is known as the *Gordan's lemma* in literature. We will rewrite it in terms of the coordinate rings.

Lemma 7.4 (Gordan's lemma)**.** *If σ is a rational convex polyhedral cone in $N_\mathbb{Q}$, then $\sigma^\vee \cap M$ is a finitely generated subsemigroup (submonoid) of M. In particular, the semigroup algebra $k[\sigma^\vee \cap M]$ is a finitely generated k-algebra.*

Let σ be a rational polyhedral cone in $M_\mathbb{Q}$. By Gordan's lemma $\sigma^\vee \cap M$ is a finitely generated affine semigroup in M. We fix a minimal generating set $\{m_1, \ldots, m_j\}$ for $\sigma^\vee \cap M$. As before, let T_N denote the torus defined by N, so that $M = \ddot{\mathbf{O}}(T_N)$.

[20]The orthogonal complement of a cone $\tau \subset N_\mathbb{Q}$ is defined by $\tau^\perp := \{m \in M_\mathbb{Q} : m(u) = 0 \text{ for all } u \in \tau\}$.

A minimal generating set of $\sigma^\vee \cap M$.

$$\begin{aligned} \mathtt{t}_\sigma : T_N &\longrightarrow k^j \\ t &\longmapsto (\chi^{m_1}(t), \dots, \chi^{m_j}(t)), \end{aligned} \tag{7.5}$$

where χ^m ($m \in M$) stands for the character of T_N corresponding to the vector m. Then the image of $\mathtt{t}_\sigma$, denoted by T_σ, is canonically isomorphic to the quotient torus,

$$T_\sigma \cong T_N / \cap_{i=1}^{j} \ker \chi^{m_i}. \tag{7.6}$$

Theorem 7.7. *Let σ be a rational polyhedral cone in $N_\mathbb{Q}$. Then the affine scheme*

$$U_\sigma := \operatorname{Spec}(k[\sigma^\vee \cap M])$$

is an irreducible affine variety; it admits a natural algebraic action of the torus T_σ defined as in (7.6) with an open orbit. Furthermore, the following statements are equivalent:

(1) Then $\dim U_\sigma = n$. (Recall that we set $n := \dim N_\mathbb{Q}$.)
(2) The torus T_N acts on U_σ with an open orbit. In other words, we have $T_N = T_\sigma$.
(3) σ is strictly convex.
(4) $\sigma^\vee \cap M$ is a saturated lattice cone.
(5) U_σ is normal.

Proof. Let $\{m_1, \dots, m_j\}$ be a minimal generating set for $\sigma^\vee \cap M$. Then the characters $\chi^{m_1}, \dots, \chi^{m_j}$ minimally generate the coordinate ring $k[\sigma^\vee \cap M]$. Let T_σ denote the torus defined by $\{m_1, \dots, m_j\}$ as in (7.6). To show that U_σ is an affine variety we will show that the k-algebra $k[\sigma^\vee \cap M]$ is a finitely generated integral domain. The finite generation follows from the Gordan's lemma. To show that $k[\sigma^\vee \cap M]$ is an integral domain (that is to say it has no zero divisors), we simply notice that $k[\sigma^\vee \cap M]$ is a subalgebra of the polynomial ring $k[M]$. Let us discuss the open orbit question.

Recall that the k-valued points of an affine algebraic variety $\operatorname{Spec}(A)$ correspond to the maximal ideals of its coordinate ring A. Since a maximal ideal of A is the kernel of a k-algebra homomorphism from A to the field k (viewed as a k-algebra), we will further identify the k-valued points of $\operatorname{Spec}(A)$ by the k-algebra homomorphisms $A \to k$.

Returning to our setup, we see that the k-valued points of U_σ are given by the k-algebra homomorphism $k[\sigma^\vee \cap M] \to k$. But we know from its definition that the k-algebra $k[\sigma^\vee \cap M]$ is minimally generated by the functions $\chi^{m_1}, \dots, \chi^{m_j}$. The image of the map (7.5) defined by $t \mapsto (\chi^{m_1}(t), \dots, \chi^{m_j}(t))$ ($t \in T_N$) is exactly our torus T_σ. Therefore, it gives a dominant morphism $\mathtt{t}_\sigma : T_N \to \overline{T_\sigma}$, where $\overline{T_\sigma}$ is the Zariski closure of T_σ in k^j. Then the induced map of k-algebras $\mathtt{t}_\sigma^* : k[\overline{T_\sigma}] \to k[T_N]$ is injective. Since the coordinate functions of $\mathtt{t}_\sigma = (f_1(t), \dots, f_j(t))$ are given by $f_1(t) = \chi^{m_1}(t), \dots, f_j(t) = \chi^{m_j}(t)$, we see that any k-algebra homomorphism $k[\overline{T_\sigma}] \to k$ is contained in the k-algebra generated by $\chi^{m_1}, \dots, \chi^{m_j}$. In other words, these characters generate the coordinate ring of $\overline{T_\sigma}$. At the same time, we know that they generate $k[\sigma^\vee \cap M]$ as a k-algebra. Therefore, the affine variety U_σ is isomorphic to $\overline{T_\sigma}$. It follows that T_σ acts algebraically on U_σ with an open orbit.

We are now ready to prove the second assertion. The equivalencies of (1), (2), and (3) follow from various characterizations of the strictly convex polyhedral cones. The equivalencies of (3), (4), and (5) is proved in [26, Theorem 1.3.5]. □

Theorem 7.7 shows that the affine toric varieties are classified by the strictly convex rational polyhedral cones in $N_{\mathbb{Q}}$. Arbitrary toric varieties are obtained from affine toric varieties by gluing. This idea is formalized by the combinatorial notion of a fan. Before we introduce the notion of a fan, we will explain how the morphisms of affine varieties arise from the lattice maps.

Let α and β be two strictly convex rational polyhedral cones contained in the lattices $N_{\mathbb{Q}}$ and $N'_{\mathbb{Q}}$ respectively. Let $\varphi : N \to N'$ be a homomorphism of lattices such that $\varphi(\alpha) \subseteq \beta$. Then we have the dual map $\varphi^\vee : \beta^\vee \to \alpha^\vee$. Hence, we have a k-algebra homomorphism $\varphi_* : k[\beta^\vee \cap M] \to k[\alpha^\vee \cap M]$. Thus, we obtain a map of affine toric varieties $\varphi : U_\alpha \to U_\beta$. Let us focus on a more specific situation, where $\varphi : N \to N$ is the identity map and $\tau \subset \sigma$ is a face. Recall that τ is a face of σ if there exists $m \in \sigma^\vee$ such that $\tau = H_m \cap \sigma$. It is not difficult to check that $\tau^\vee \cap M$ is generated by m and $\sigma^\vee \cap M$. It follows from that $k[\tau^\vee \cap M]$ is equal to the localization of $k[\sigma^\vee \cap M]$ at χ^m. In other words, U_τ is an open subset of U_σ. Hence we observed the following useful fact.

Lemma 7.8. *Let $(\mathbf{P}(\sigma), \preceq)$ denote the face lattice of σ. Then the inclusion order on the affine open subsets U_τ ($\tau \in \mathbf{P}(\sigma)$) is isomorphic to $(\mathbf{P}(\sigma), \preceq)$.*

Definition 7.9. A *fan* Σ in $N_{\mathbb{Q}}$ is a finite set of strictly convex rational polyhedral cones $\sigma \subset N_{\mathbb{Q}}$ such that

1. every face of σ is a cone in Σ;
2. the intersection of two cones from Σ is a face of both of the cones, hence, it is an element of Σ.

The dimension of a fan Σ is the dimension of the smallest vector subspace in $N_{\mathbb{Q}}$ that contains the *support of* Σ, that is, $\cup_{\sigma \in \Sigma} \sigma$. A fan Σ is said to be an *affine fan* if Σ consists of a cone σ and all of its faces. A *complete fan in* $N_{\mathbb{Q}}$ is a fan Σ in $N_{\mathbb{Q}}$ such that $\cup_{\sigma \in \Sigma} \sigma = N_{\mathbb{Q}}$.

The toric variety associated with Σ, denoted by $X(\Sigma)$, is obtained by gluing the affine toric varieties U_σ ($\sigma \in \Sigma$) along their common open affine toric subvarieties. An instructive example here is the construction of the n-dimensional projective space from its affine open subsets. Note that in the case of a projective space $\mathbb{P}^n$, the inclusion order on the fan of $\mathbb{P}^n$ is isomorphic to the face lattice of the regular n simplex. We leave the details to the reader. More formally stated, the the toric variety associated with a fan is obtained by taking the direct limits of the affine toric varieties associated with the cones of the fan:

Definition 7.10. The toric variety associated with a fan Σ in $N_{\mathbb{Q}}$ is the direct limit

$$X(\Sigma) := \varinjlim_{\sigma \in \Sigma} U_\sigma.$$

For a given fan Σ, let us agree on calling the open cover $\{U_\sigma\}_{\sigma\in\Sigma}$ the *distinguished open affine covering* of $X(\Sigma)$.

7.1. The T-orbits of a toric variety.

In the previous subsection we learned that every affine toric variety U_σ, where σ is a strictly convex rational polyhedral cone in $N_{\mathbb{Q}}$, is covered by the affine open toric subvarieties U_τ, where τ is a face of σ. In particular, for the unique smallest face ν of σ, that is $\nu := \{0\}$, we have $U_\nu \subseteq U_\tau$, where $\tau \in \mathbf{P}(\sigma)$. We notice that, since $\nu^\vee = M$, U_ν is the torus T_N,

$$U_{\{0\}} = U_\nu = (\operatorname{Spec} k[\nu^\vee \cap M]) = \operatorname{Spec}(k[M]) \cong T_N.$$

In this subsection we will discuss the structures of the T_N-orbits in U_σ. For this purpose, it will be useful to view N as the group of one-parameter subgroups of T_N. To see that this is a natural point of view let us describe the perfect pairing the group of 1-psgs and the group of characters of T_N. Let χ be a character of T, and let λ be a 1-psg of T_N. Then the composition $\chi\circ\lambda : \mathbb{G}_m \to \mathbb{G}_m$ is an algebraic group homomorphism. Hence, there exists $a \in \mathbb{Z}$ such that for every $s \in \mathbb{G}_m$, we have $\chi\circ\lambda(s) = s^a$. The pairing between the 1-psgs and the characters is then defined by $\langle \lambda, \chi\rangle := a$. Hereafter, whenever it is useful, we will view N as the lattice of 1-psgs of T_N.

In the proof of Theorem 7.7 we observed that the points of the affine toric variety U_σ corresponds to the semigroup homomorphisms $\sigma^\vee \cap M \to k$. It turns out that there is a distinguished point of U_σ which we denote by e_σ. It is defined by the semigroup homomorphism

$$v \mapsto \begin{cases} 0 & \text{if } v \notin \sigma^\perp, \\ 1 & \text{if } v \in \sigma^\perp. \end{cases}$$

It can be shown that $\sigma \subseteq N_{\mathbb{Q}}$ is full dimensional if and only if the point e_σ is the unique closed T_N-orbit in U_σ. In the standard literature on toric varieties, the point e_σ is either called the *distinguished point*, or the *special point* of U_σ. It turns out that the distinguished point of U_σ is nothing but an idempotent if we view U_σ as a commutative algebraic monoid. The distinguished point e_σ is characterized as the point-of-flow for all 1-psgs that are contained in the relative interior of σ. The proofs of the following facts are not difficult.

- e_σ is fixed under the T_N-action if and only if $\dim\sigma = n$, see [26, Corollary 1.3.3].
- If τ is a face of σ, then e_τ is in U_σ.
- Let u be a vector from $N_{\mathbb{Q}}$. Then $u \in \sigma$ if and only if $\lim_{s\to 0}\lambda^u(s)$ exists in U_σ. Furthermore, if u is contained in the relative interior of σ, then $\lim_{s\to 0}\lambda^u(s) = e_\sigma$, see [26, Proposition 3.2.2].

For $\sigma \in \Sigma$, let U_σ denote the affine toric variety $\operatorname{Spec} k[\sigma^\vee \cap M]$. We will denote by O_σ the T orbit determined by σ. Since T is a solvable group, the O_σ is an affine variety. We are going to describe the coordinate ring of O_σ.

Let σ be a saturated rational polyhedral cone in $N \otimes_{\mathbb{Z}} \mathbb{Q}$. Let N_σ denote the sublattice of N generated by $\sigma \cap N$. We do not assume that σ is full dimensional, so, N_σ is not necessarily equal to N. We set $N(\sigma) := N/N_\sigma$ and $T_{N(\sigma)} := N(\sigma) \otimes_{\mathbb{Z}} k^*$. Since the sequence of free $\mathbb{Z}$-modules $0 \to N_\sigma \to N \to N(\sigma) \to 0$ is exact, the torus $T_{N(\sigma)}$ is isomorphic to the quotient torus T_N/T_{N_σ}. Clearly, the dimension of $T_{N(\sigma)}$ is given by $\dim T_{N(\sigma)} = \mathrm{rank}(N(\sigma)) = \mathrm{rank}(N) - \dim \sigma$. It is easy to check that the duality pairing $\langle , \rangle : N \times M \to \mathbb{Z}$ restricts to give the following perfect pairing:

$$(\sigma^\perp \cap M) \times N(\sigma) \to \mathbb{Z}.$$

Hereafter, we write $M(\sigma)$ in place of $\sigma^\perp \cap M$. In this notation, we have $M(\sigma) = \mathbf{H\ddot{O}}(T_{N(\sigma)})$.

Let Σ be a fan in $N_{\mathbb{Q}}$, and let σ be an r-dimensional cone in Σ. We will define the notion of the *star of* σ. If τ is a cone such that $\sigma \preceq \tau$, then under the natural projection $N \to N(\sigma)$, τ maps onto a cone denoted by $\bar{\tau}$ in $N(\sigma)$.

Definition 7.11. The following collection of cones in $N(\sigma)$ is called the *star* of σ (with respect to Σ):

$$\mathrm{star}(\sigma) := \{\bar{\tau} :\ \tau \in \Sigma,\ \sigma \preceq \tau\}.$$

It is easy to check that the star of σ is a fan in $N(\sigma)_{\mathbb{Q}}$. We will denote the associated toric variety $X(\mathrm{star}(\sigma))$ by $V(\sigma)$. The dimension of this variety is equal to the dimension of the euclidean space $N(\sigma)_{\mathbb{Q}}$, hence, we have $\dim V(\sigma) = n - r$. Clearly, the image of σ in $\mathrm{star}(\sigma)$ gives the unique zero dimensional cone $\bar{\sigma}$. The corresponding affine toric variety $U_{\bar{\sigma}}$, that is, $\mathrm{Spec}\, k[\bar{\sigma}^\vee \cap M(\sigma)]$ is naturally isomorphic to the torus $T_{N(\sigma)}$.

Let σ be a cone in the fan Σ. We set

$$O(\sigma) := T_{N(\sigma)} \cdot e_\sigma \qquad \text{and} \qquad V(\sigma) := \overline{O(\sigma)}.$$

Hence, $V(\sigma)$ is a toric variety itself. Notice that T_N still acts on $O(\sigma)$ by its quotient group $T_{N(\sigma)}$. In other words, $O(\sigma)$ is a T_N-orbit in U_σ, hence, $V(\sigma)$ is a toric T_N-subvariety of U_σ. The inclusion orders on the sets $\{V(\sigma) :\ \sigma \in \Sigma\}$ and $\{U_\sigma :\ \sigma \in \Sigma\}$ are closely related to the inclusion order on the fan Σ. Here is the main theorem that summarizes these relationships [26, §3.2].

Theorem 7.12. *Let Σ be a fan in $N_{\mathbb{Q}}$. Then the following statements hold:*

1. *There is an order reversing isomorphism between the inclusion poset of cones $\sigma \in \Sigma$ and the inclusion poset of the Zariski closures of the corresponding orbits $O(\sigma)$. Under this isomorphism the dimension of a cone $\sigma \in \Sigma$ is equal to the codimension of $O(\sigma)$ in $X(\Sigma)$.*
2. *$U_\sigma = \bigsqcup_{\tau \preceq \sigma} O_\tau$,*
3. *$V_\sigma = \bigsqcup_{\sigma \preceq \tau} O_\tau$.*
4. *$V_\sigma = X(star(\sigma))$.*

One of the many useful facts that we immediately deduce from Theorem 7.12 is that each affine open toric variety U_σ contains a unique closed T_N-orbit, that is the orbit of the distinguished point e_σ. Therefore, e_σ is a T_N-fixed point if and only if σ is a maximal cone in Σ.

7.2. Σ-linear support functions and the equivariant Picard group

Let Σ be a fan in $N_{\mathbb{Q}}$, and let $|\Sigma|$ denote its support, $\cup_{\sigma\in\Sigma}\sigma$ in $N_{\mathbb{Q}}$. A $\mathbb{Q}$-valued function $h : |\Sigma| \to \mathbb{Q}$ is called an *integral Σ-linear support function* if the following condition hold:

- for each cone $\sigma \in \Sigma$, there exists a character $m_\sigma \in M$ such that $h(v) = m_\sigma(v)$ for all $v \in \sigma$.

Clearly, such a function respects the inclusion relations between the elements of Σ. Therefore, an integral Σ-linear support function is a piecewise linear function $h : |\Sigma| \to \mathbb{Q}$ such that the domains of the *pieces of* h are the maximal cones of Σ. The integral Σ-linear support functions forms an abelian group with respect to pointwise addition. We will denote this group by $SF(\Sigma, N)$, and we will call it the *group of integral Σ-linear support functions*. For each element m of M, we have a *principal integral Σ-linear support function*, denoted by h_m. It is defined in an obvious way,

$$h_m|_\sigma(v) = m(v) = \langle m, v\rangle \qquad \text{for every } v \in \sigma.$$

Thus we have a natural group homomorphism $M \to SF(\Sigma, N)$. It is easy to verify that if $|\Sigma|$ spans $N_{\mathbb{Q}}$, then this inclusion of abelian groups is injective. Less obvious is the fact that if Σ is a smooth fan, then $SF(\Sigma, N)$ is a free abelian group of rank r, where r is the number of rays in Σ. The integral Σ-linear support functions are interesting from geometry viewpoint also.

Theorem 7.13. *Let Σ be a fan in $N_{\mathbb{Q}}$. Then there is a right exact sequence,*

$$M \to SF(\Sigma, N) \to \mathrm{Pic}(X(\Sigma)) \to 0. \tag{7.14}$$

The sequence (7.14) is exact if and only if the rays of Σ spans $N_{\mathbb{Q}}$.

A proof of Theorem 7.13 can be pieced together from [26, Theorems 4.2.1 and 4.2.12]. ([26, Theorem 4.2.1] states that the exact sequence (7.14) holds true if $SF(\Sigma, N)$ is replaced by the group of T_N-stable Cartier divisors on $X(\Sigma)$; [26, Theorem 4.2.12] states that these two groups are isomorphic.) Note that the Picard groups of the toric varieties $X(\Sigma)$, where Σ contains a cone of dimension n are torsion free [26, Proposition 4.2.5]. If $X(\Sigma)$ is factorial (for example, if $X(\Sigma)$ is smooth), then every Weil divisor on $X(\Sigma)$ is Cartier. Hence, in this case, the class group of $X(\Sigma)$ is isomorphic to $\mathrm{Pic}(X(\Sigma))$. If Σ is simplicial only, then $\mathrm{Pic}(X(\Sigma))$ is isomorphic to a finite index subgroup of $\mathrm{Cl}(X(\Sigma))$, see [26, Propositions 4.2.6 and 4.2.7]. Below we will discuss the group of T_N-stable Cartier divisors in more detail. Here, we want to a reinterpretation of $SF(\Sigma, N)$.

Corollary 7.15. *If $X(\Sigma)$ is a complete fan, then $SF(\Sigma, N) \cong CDiv_T(X) \cong \mathrm{Pic}_T(X(\Sigma))$, where $CDiv_T(X)$ is the group of T_N-stable Cartier divisors.*

Proof. The first isomorphism follows from [26, Theorem 4.2.12]; it is independent of the completeness of Σ. For the second isomorphism, we observe that if Σ is complete, then its rays span $N_{\mathbb{Q}}$. Then (7.14) is a short exact sequence. But the image of M in $\mathrm{CDiv}_T(X)$ is exactly the group of T_N-stable principal divisors. Thus by [57, Propositions 2.4 and 2.1], we see that the natural homomorphism that associates a T-equivariant line bundle L_h to each integral Σ-linear support function h is an isomorphism. □

We want to point out another usefulness of the integral Σ-linear support functions.

Definition 7.16. An integral Σ-linear support function $h : |\Sigma| \to \mathbb{Q}$ is called *strictly convex* if for every cone $\sigma \in \Sigma$, the corresponding piece of h, that is, $h|_\sigma = m_\sigma : N_{\mathbb{Q}} \to \mathbb{Q}$, where $m_\sigma \in M$, satisfies the following property:

$$m_\sigma(v) \geqslant h(v) \qquad \text{for every } v \in N_{\mathbb{Q}}.$$

Not all complete toric varieties are projective (although the converse is always true). It is known that a complete toric variety $X(\Sigma)$ is projective if and only if Σ possesses a strictly convex Σ-linear support function. In fact, this fact is closely related to the existence of an ample line bundle on $X(\Sigma)$. The relevant notion here is the notion of a *lattice polytope* in $M_{\mathbb{Q}}$, which, by definition, is a polytope[21] whose vertices are contained in M.

7.3. Polytopes and the projective toric varieties

Let P be a (not necessarily lattice) polytope in $M_{\mathbb{Q}}$ such that $n := \dim P = \dim M_{\mathbb{Q}}$. Then each facet of P is $n-1$ dimensional, hence, each facet F of P is uniquely determined by the hyperplane that contains F. Let us denote the hyperplane corresponding to the facet F by H_F. Then H_F splits $M_{\mathbb{Q}}$ into two disjoint path connected halves. We will denote the half-space containing P by H_F^+. By translating H_F to the origin, we obtain a codimension one subspace of $M_{\mathbb{Q}}$. Consequently, we get a linear functional on $N_{\mathbb{Q}}$ that annihilates this translated copy of H_F in $M_{\mathbb{Q}}$. Equivalently, we keep H_F in its original position and take a vector $u \in N_{\mathbb{Q}}$ such that some translate of u annihilates H_F. Any such u is uniquely defined up to a scalar multiple, so, we fix one of them and denote it by u_F. In this notation, the hyperplane corresponding to F is given by

$$H_F := \{v \in M_{\mathbb{Q}} : \langle v, u_F \rangle = -a_F\},$$

where a_F is a fixed scalar that measures the distance of the translation of u_F to the origin. This u_F is sometimes called the *inward-pointing normal vector for H_F*. The polytope P now has a handy description:

$$\begin{aligned} P = \bigcap_{F\text{: facet of } P} H_F^+ &= \bigcap_{F\text{: facet of } P} \{v \in M_{\mathbb{Q}} : \langle v, u_F \rangle \geqslant -a_F\} \\ &= \{v \in M_{\mathbb{Q}} : \langle v, u_F \rangle \geqslant -a_F \ \text{ for every facet } F \subseteq P\}. \end{aligned}$$

[21] A *polytope in $M_{\mathbb{Q}}$* is a compact convex subset that is obtained by intersecting finitely many hyperplanes in $M_{\mathbb{Q}}$.

Next, we will describe the fan that is canonically associated to a full dimensional lattice polytope.

Let $P \subset M_{\mathbb{Q}}$ be a polytope such that $\dim P = \dim M_{\mathbb{Q}}$. Let Q be a face of P. Let $\sigma_Q \subset N_{\mathbb{Q}}$ be the cone generated by the vectors u_F, where F is a facet that contains Q,

$$\sigma_Q := \langle u_F \in N_{\mathbb{Q}} : \ F \text{ is a facet containing } Q \rangle. \tag{7.17}$$

In particular, if Q is a facet, then σ_Q is a ray in $N_{\mathbb{Q}}$. It is not difficult to see that σ_Q is a strictly convex polyhedral cone.

Definition 7.18. Let P be a full dimensional lattice polytope in $M_{\mathbb{Q}}$. Then as Q varies over all faces of P, the cones σ_Q defined in (7.17) form a fan,

$$\Sigma_P := \{\sigma_Q : \ Q \text{ is a face of } P\}.$$

The fan Σ_P is called the *normal fan of* P.

It is not difficult to show that the normal fans are complete fans. Therefore, the toric variety defined by a normal fan Σ_P, where P is a full dimensional lattice polytope in $N_{\mathbb{Q}}$, is a normal projective variety. We should also point out the obvious fact that, for every full dimensional lattice polytope P, the normal fan of any dilation aP $(a \in \mathbb{Z}^+)$ of P and the normal fan of any translation $m + P$ $(m \in M)$ of P are isomorphic.

7.4. Reminders about divisors

The purpose of this section is to re-introduce, for pedagogical reasons, many geometrical objects that we have been using. Earlier we defined most of these objects but sometimes they were defined by only via a reference to them. We hope that this subsection will be helpful to the reader for tying some of the loose ends.

Let X be a variety (or a scheme). A *Weil divisor of* X is a finite formal sum of the form $D = \sum a_i V_i$, where $a_i \in \mathbb{Z}$ and V_i's are irreducible subvarieties of codimension one in X. Clearly, the Weil divisors is an abelian group with respect to formal addition. A Weil divisor is called *effective* if the coefficients are nonnegative integers.

A *Cartier divisor*, which is also called a *locally principal Weil divisor*, is a Weil divisor D that is defined by the following data: First of all, there is covering $X = \bigcup_{\alpha \in I} U_\alpha$, where U_α's are some affine open subsets of X, On each U_α, there is a nonzero rational function f_α called the *local equation of* D *on* U_α; these rational functions are put together by the requirements that the ratios f_α / f_β are nowhere zero regular functions on $U_\alpha \cap U_\beta$. Furthermore, we ask that the restriction[22] of D to each U_α $(\alpha \in I)$ is given by the divisor of f_α, that is $D|_{U_\alpha} = \mathrm{div}(f_\alpha)|_{U_\alpha}$.

Let us briefly expand on this point. Recall that if Z is a prime divisor of a normal variety X, then the local ring $\mathcal{O}_{X,Z}$ is a DVR and the valuation of this local ring is given by the order of vanishing function, that is, $\mathrm{ord}_Z(h/g) := \mathrm{ord}_Z(h) - \mathrm{ord}_Z(g)$ for every

[22] The restriction of a Weil divisor $D = \sum a_i D_i$ to an open subset $U \subseteq X$ is the Weil divisor of U defined by $D|_U := \sum a_i U \cap D_i$.

nonzero element $h/g \in \mathcal{O}_{X,Z}$. For every nonzero rational function f on X, we have a corresponding Weil divisor, called the *principal Weil divisor of f*, defined by

$$\mathrm{div}(f) := \sum_{\mathrm{codim}(Z,X)=1} \mathrm{ord}_Z(f)Z.$$

It is easy to see that the principal Weil divisors is a subgroup of the group of Weil divisors. Indeed, the additivity of the order of vanishing function implies that

$$\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g).$$

Two Weil divisors D and D' are said to be *linearly equivalent* if there exists a principal divisor $\mathrm{div}(f)$ such that

$$D - D' = \mathrm{div}(f).$$

Definition 7.19. Let X be an n-dimensional variety. The *class group of X*, denoted by $A_{n-1}(X)$, is the group of Weil divisors modulo the linear equivalence. In other words, if $\mathrm{Div}(X)$ (resp. $\mathrm{PDiv}(X)$) denotes the group of all Weil divisors (resp. the group of all principal Weil divisors) on X, then

$$A_{n-1}(X) := \mathrm{Div}(X)/\mathrm{PDiv}(X).$$

We use the notation $A_{n-1}(X)$ instead of $\mathrm{Cl}(X)$ since the former notation suggests a deeper connection to the Chow groups; the class group of X is the $(n-1)$-th Chow groupof X.

A related group is determined by the Cartier divisors. First, let us note that every principal Weil divisor is a Cartier divisor.

Definition 7.20. The *Picard group of X* is the group of Cartier divisors modulo the subgroup of principal (Cartier) divisors. In other words, the class group of Cartier divisors is the Picard group of the variety.

An equivalent description of the Picard group is obtained by associating to each Cartier divisor a line bundle. More precisely, on a variety X, every Cartier divisor D determines a line bundle[23] $\mathcal{O}(D)$ on X. Conversely, for each line bundle on X, there is a unique Cartier divisor up to linear equivalence on X. Under this bijection, two Cartier divisors D_1 and D_2 are linearly equivalent if and only if the corresponding line bundles $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ are isomorphic [36, Proposition 6.13]. This isomorphism is obtained as follows. Let $D = \{(U_\alpha, f_\alpha)\}_{\alpha\in I}$ be the local data of D. Then we interpret the inverse of the local equation of D on U_α, that is f_α^{-1}, as the generating section on U_α of a line bundle on X. Thus the Picard group of X is given by the group of isomorphism classes of line bundles on X with multiplication given by the tensor product.

If for every point p on a variety X, the local ring $\mathcal{O}_{X,p}$ is a UFD, then X is said to be a *factorial variety*. In this case, every Weil divisor is a Cartier divisor, hence, we have $A_{n-1}(X) = \mathrm{Pic}(X)$. There is an interesting characterization of the unique factorization

[23] A line bundle on a variety is a locally free sheaf of rank 1.

domains via Chow groups: Let A be a noetherian integral domain. Then $X := \mathrm{Spec}(A)$ is normal and $A_{n-1}(X) = 0$ if and only if A is a UFD. A proof of this fact can be found in [36, Ch II, Proposition 6.2]. An important example for our purposes is the n-dimensional torus T. Since the coordinate ring of T is $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, which is a UFD, $\mathrm{Pic}(T) = 0$. It turns out that the Picard group of every normal affine toric variety is trivial, but this is less obvious compared to the fact that $\mathrm{Pic}(T) = 0$. We will explain this in next subsections.

7.5. The localization sequence for the class groups

Let X be a variety, let Z be a closed subset of X, and let U denote $X \backslash Z$. Recall that by intersecting the irreducible divisors of X with U, we get a natural map

$$\begin{aligned} \rho : \mathrm{Div}(X) &\longrightarrow \mathrm{Div}(U) \\ D &\longmapsto D \cap U \end{aligned}$$

It is easy to see that ρ is a surjective homomorphism. Moreover, it maps the divisor of a rational function (f) to $(f|_U)$. In other words, ρ descends to a homomorphism $\tilde{\rho} : A_{n-1}(X) \to A_{n-1}(U)$. To compute the kernel of $\tilde{\rho}$, we decompose Z as a union $Z_1 \cup \cdots \cup Z_r \cup Z'$, where Z_i's are prime divisors of X contained in Z, and $Z' := Z \backslash \cup_{i=1}^r Z_i$. Clearly, we have $\mathrm{codim} Z' \geqslant 2$. Thus, the classes of Z_i's are mapped to 0 in $A_{n-1}(U)$. In other words, $\ker \tilde{\rho}$ is generated by the subset $\{[Z_1], \dots, [Z_r]\}$ in $A_{n-1}(X)$.

Lemma 7.21. *Let X be a normal variety, let Z be a closed subset of X, and let U denote $X \backslash Z$. Then there exists an exact sequence*

$$\bigoplus_{i=1}^{r} \mathbb{Z} Z_i \to A_{n-1}(X) \to A_{n-1}(U) \to 0, \tag{7.22}$$

where $Z_1, \dots, Z_r$'s are the prime divisors of X that are contained in Z.

7.6. Divisors on toric varieties

Let T be an algebraic torus, and let X be a toric variety with respect to T. We do not assume that X is complete, however, we will assume that the fan Σ that defines X is full dimensional. (This assumption is needed for having enough number of divisors to generate the kernel of a homomorphism below.) Let U denote the open T-orbit in X. Then $X \backslash U$, which we denote by Z, is T stable, therefore, it is a union of finitely many T-divisors, denoted by $D_1, \dots, D_r$. Thus, we have the short exact sequence $\mathbb{Z}^r \to A_{n-1}(X) \to A_{n-1}(U) \to 0$ as in Lemma 7.21. But since $U \cong T$ and T is a factorial variety, $A_{n-1}(U) = 0$. In other words, the exact sequence in (7.22) is a short exact sequence of the form

$$0 \to R \to \bigoplus_{i=1}^{r} \mathbb{Z} D_i \to A_{n-1}(X) \to 0. \tag{7.23}$$

It follows from the definition of $A_{n-1}(X)$ that R consists of the classes of the divisors of rational functions that are supported on $\cup D_i$. Hence, if $\mathrm{div} f \in R$, then f does not have any zeros or poles on U; this means that f is a nowhere vanishing regular map on T. But the ring of regular functions on T is precisely the group algebra of M, that is, $k[M]$. In other words, $R = M$.

Next, we will compute the principal divisors of the regular functions of the form χ^m ($m \in M$). Hereafter, we are going to fix and use the following notation.

Notation 7.24. The set of all rays of the fan Σ will be denoted by $\Sigma(1)$. If ρ is a ray in $\Sigma(1)$, then u_ρ will denote the lattice vector of minimal length in $\rho \cap N$. Finally, D_ρ will denote the T stable divisor corresponding to ρ.

Lemma 7.25. *If m is an element of M, then the divisor of the character $\chi^m \in k[M]$ is given by*

$$div\,(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

Proof. Since χ^m is nowhere vanishing regular function on X it is supported on the union of the (T stable) prime divisors $\cup_{\rho\in\Sigma(1)} D_\rho$. The order of vanishing of χ^m on D_ρ is computed by the valuation of the local ring of D_ρ in X. The rest of the proof follows from a local computation. □

Remark 7.26. Let $f = \sum a_m \chi^m$ be an element from $k[M]$. Then by Lemma 7.25, $\mathrm{div}(f) = \sum_{\rho\in\Sigma(1)} a_i \langle m, u_\rho \rangle D_\rho$.

Next, we will focus on Cartier divisors on toric varieties.

Notation 7.27. By $\mathrm{CDiv}_T(X)$ we will denote the group of T-stable Cartier divisors on X. The elements of $\mathrm{CDiv}_T(X)$ will be called the *T-Cartier divisors.*

We will show that $\mathrm{CDiv}_T(X)$ is a subgroup of $\bigoplus_{\rho\in\Sigma(1)} \mathbb{Z} D_\rho$.

Theorem 7.28. *Let X be a normal toric variety defined by the fan Σ. Then the diagram in (7.1) is commutative.*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z} D_\rho \arrow[r] & A_{n-1}(X) \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, equal] & CDiv_T(X) \arrow[r] \arrow[u, hook] & \mathrm{Pic}(X) \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

FIGURE 7.1. The relationship between the class group and the Picard group of a toric variety.

Proof. The exactness of the first row is already proved in (7.23). It remains to show that the bottom row is exact, and that the vertical arrows are inclusions. We begin with the injectivity of the middle map. The proof here is split into two parts: 1) X is an affine toric variety. 2) X is a general toric variety. We proceed with 1). Let σ be the cone that defines the affine toric variety $X = U_\sigma$. Without loss of generality we assume that σ is full dimensional. Let $\mathcal{K}$ denote the sheaf of total quotient ring of $\mathcal{O}_X$. Let D be a Cartier divisor from $\mathrm{CDiv}_T(X)$, and let $\mathcal{O}_X(D)$ denote the corresponding sheaf. Then $I := H^0(X, \mathcal{O}_X(D))$ is a fractional ideal in $H^0(X, \mathcal{K})$. Since D is T stable, the vector space I is graded by M; we can decompose I into a sum of eigenspaces. Since D is Cartier, the sheaf of fractional ideals $\mathcal{O}_X(D)$ is principal at the distinguished point of U_σ, so, $I/\mathfrak{m}I$ is one dimensional. Here $\mathfrak{m}$ is the maximal ideal $\mathfrak{m} = \sum k \cdot \chi^u$. Let m be an element of M such that the image of χ^m in $I/\mathfrak{m}I$ is nonzero. Then, by the Nakayama's lemma, we have $I = k\chi^m$. Therefore, I, hence, D is determined by the character χ^m. More precisely, we proved that $D = \mathrm{div}(\chi^m)$ for $X = U_\sigma$. 2) We now proceed with a general toric variety. Since X is obtained by gluing together the affine toric varieties U_σ, where $\sigma \in \Sigma$, we need to understand the corresponding local picture. The tricky part of this analysis is that σ does not need to be a full dimensional cone in $N_{\mathbb{Q}}$. The following simple trick takes us to the right setup: Let N_σ denote the subgroup of N that is generated by $\sigma' := \sigma \cap N$ in N. Let $N(\sigma)$ denote the quotient group N/N_σ, and let $M(\sigma)$ denote the dual lattice, which can be computed as $\sigma^\perp \cap M$. Let $T_{N(\sigma)}$ denote the torus corresponding to $N(\sigma)$. Then we have the decomposition,

$$U_\sigma \cong U_{\sigma'} \times T_{N(\sigma)},$$

where we view $U_{\sigma'}$ as an affine toric variety of $T' := T/T_{N(\sigma)}$. Since the rays of the cones σ and σ' are (combinatorially) the same, the groups of Weil divisors on U_σ and $U_{\sigma'}$ are isomorphic. Furthermore, for every m and m' from M, we have

$$\mathrm{div}(\chi^m) = \mathrm{div}(\chi^{m'}) \iff m - m' \in \sigma^\perp \cap M = M(\sigma). \tag{7.29}$$

It follows that the T-Cartier divisors on U_σ are determined by the elements of the lattice $M/M(\sigma)$. In summary, on a general normal toric variety $X = X(\Sigma)$, each T-Cartier divisor is determined by an element $\oplus_{\sigma\in\Sigma} m_\sigma \in \bigoplus_{\sigma\in\Sigma} M/M(\sigma)$. Of course, there are some compatibility restrictions between the summands in $\oplus_{\sigma\in\Sigma} m_\sigma$. More precisely, we insist that m_σ restricts onto m_τ under the canonical map $M/M(\sigma) \to M/M(\tau)$. After agreeing on this compatibility requirement, we can now focus solely on the maximal cones of Σ. Then a T-Cartier divisor on a general normal toric variety $X(\Sigma)$ can be expressed as a sum of the T-stable Weil divisors. In turn, this observation makes it apparent that there is an injective group homomorphism $\mathrm{CDiv}_T(X) \hookrightarrow \bigoplus_{\rho\in\Sigma(1)} \mathbb{Z}D_\rho$. This takes care of the injectivity of the middle map in (7.1).

The right most vertical arrow in Figure 7.1 is obtained by assigning a Cartier divisor to its Weil divisor. Since X is normal, this assignment is well defined and injective. Furthermore, the map we obtained by this way is natural in the sense that it makes the right most square in Figure 7.1 commutative. Therefore, by using the exactness of the first row we get the short exactness in the bottom row. This finishes the proof of our assertion. □

Corollary 7.30. *If X is an affine toric variety, then* $\mathrm{Pic}(X) = 0$.

Proof. We identify X with U_σ, where σ is a full dimensional cone in $N \otimes_{\mathbb{Z}} \mathbb{Q}$. The first part of the proof of Theorem 7.28 shows that on every affine open toric variety U_σ, the T-Cartier divisors are uniquely determined by the characters of T. In other words, for an affine toric variety, the first map in the bottom row of Figure 7.1 is injective. It follows that $\mathrm{Pic}(X) = 0$. □

Remark 7.31. The conclusion of Corollary 7.30 holds true more generally for the affine spherical varieties whose unique closed orbit is factorial.

In light of (the proof of) Theorem 7.28, we see several different characterizations of the T-Cartier divisors on a toric variety $X = X(\Sigma)$. Let D be a T-stable Weil divisor of the form $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$. Then the following assertions are equivalent:

(1) D is a T-Cartier divisor.
(2) D is principal on every affine open U_σ, where $\sigma \in \Sigma$.
(3) For every cone σ in Σ, there exists $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho \rangle = -a_\rho$ for every $\rho \in \Sigma(1)$.
(4) For every maximal cone σ in Σ, there exists $m_\sigma \in M$ such that $\langle m_\sigma, u_\rho \rangle = -a_\rho$.

Furthermore, if one of these equivalent conditions hold for D, and $\{m_\sigma\}_{\sigma \in \Sigma}$ is the data determined as in part 3., then we see that

(1) for every $\sigma \in \Sigma$, the image of m_σ in $M/M(\sigma)$, where $M(\sigma) = \sigma^\perp \cap M$, is uniquely determined by D.
(2) If τ is a face of σ, then $m_\sigma = m_\tau \mod M(\tau)$.

7.7. Additional remarks about distinguished coverings of toric varieties

We learned in the previous sections that every toric variety $X(\Sigma)$ has a distinguished open affine covering by affine toric subvarieties, that is, $X(\Sigma) = \bigcup_{\sigma \in \Sigma} U_\sigma$, where $U_\sigma = \mathrm{Spec}[\sigma^\vee \cap M]$. We saw also that the combinatorial structure (the face lattice) of σ explains exactly how U_σ's as well as their T_N-orbits fit together. We observed that the T_N-orbit of the distinguished point $e_\sigma \in U_\sigma$ is the unique closed orbit of U_σ. These facts are the pillars of the whole theory of toric varieties.

We already mentioned in a remark a related fact for the actions of a connected affine group action on a normal variety.

Theorem 7.32. *(Sumihiro's Embedding Theorem, [81]) Let G be a connected affine group and let X be a G-variety. Let X be a G-orbit in X. If X is normal, then there exists a G-stable open subset $U \subseteq X$ such that $Y \subseteq U$. Furthermore, Y is G-equivariantly isomorphic to a G-stable subvariety of $\mathbb{P}(V)$ for some rational G-module V.*

For a modern proof of the Sumihiro's Embedding Theorem, we recommend [46, Theorem 1.1].

Definition 7.33. Let X be a G-variety such that $d_G(X) = 0$. If X has a unique closed G-orbit, then X is called a *simple embedding*.

Sumihiro's Theorem amplifies the importance of understanding the geometry of the G-stable subvarieties of rational G-modules. In particular, we see from this viewpoint that the affine spherical embeddings play a special role for the whole theory of spherical embeddings. At this junction, we want to emphasize the close relationship between the affine toric varieties and the affine spherical varieties. Recall that the weight cone of a spherical variety X is the full dimensional rational polyhedral cone spanned by $\ddot{\mathbf{O}}(X)^+$ in $\ddot{\mathbf{O}}(X)_{\mathbb{Q}}$.

Theorem 7.34. *Let G be a connected reductive group, and let $B = UT$ be a Borel subgroup. If X is an irreducible affine G-variety, then the following conditions are equivalent:*

(1) X is a spherical G-variety.

(2) The G-module $k[X]$ is multiplicity-free, and the weight monoid $\ddot{\mathbf{O}}(X)^+$ is saturated.

(3) The affine T-variety $X//U := \mathrm{Spec}(k[X]^U)$ is an affine toric variety.

Furthermore, in this case, the finite inclusion poset of G-orbit closures in X is isomorphic to the face lattice of the weight cone of X.

A proof of this theorem can be found in Brion's Luminy lecture notes on algebraic group actions (see Theorem 2.14 therein). We want to point out a subtlety in the last assertion of Theorem 7.34. It says that the inclusion poset of G-orbit closures in X is isomorphic to the face lattice of the weight cone. Here, the weight cone lives in the character lattice of T, which we denoted by M in our discussion of the toric varieties. But in our construction of the toric varieties, we began with a strictly convex rational polyhedral cone σ in N. Nevertheless this is just a nuisance since there is an order reversing bijection between the faces of the cones σ and $\sigma^\vee$. Another consequence of this observation is that the dual of the weight cone of X is strictly convex, hence, the associated toric variety has a unique closed orbit of T, where $T = T_N$-orbit. Here, N stands for the dual of $\ddot{\mathbf{O}}(T)$. Therefore, Theorem 7.34 implies the following useful fact.

Corollary 7.35. *Every affine spherical variety is a simple embedding.*

The converse of this observation holds for toric varieties.

Corollary 7.36. *Let X be a toric variety with open orbit T. Then X is simple as a T-variety if and only if X is affine.*

Proof. It suffices to show "⇒" direction. Let Σ be the fan of X. For every $\sigma \in X = X(\Sigma)$, the unique closed T-orbit of the affine subvariety U_σ is given by $T \cdot e_\sigma$, where e_σ is the distinguished point of U_σ. Now let σ be a maximal cone in Σ. Then $T \cdot e_\sigma$ is a closed orbit in X. Since X is simple, $T \cdot e_\sigma$ is the unique closed orbit. Then every T-orbit closure $\overline{T \cdot e_\tau}$ $(\tau \in \Sigma)$ contains $T \cdot e_\sigma$. Equivalently, for every $\tau \in \Sigma$ we have $\tau \preceq \sigma$. This means that every element of Σ is a subset of the face lattice of σ. Hence, there exists a unique affine open toric variety U_σ in X. □

At this point it is a natural question to ask about the description of the cone of T-invariant valuations of a toric variety. We will answer this question at the end of the next section. We want to finish this section by some pointers to the literature.

Recall that for a toric variety $X(\Sigma)$ there is a close relationship between its Picard group and the $n-1$-th Chow group $A_{n-1}(X)$ as we saw in Theorem 7.28. Recall also the relationships $SF(\Sigma, N) \cong \mathrm{CDiv}_T(X) \cong \mathrm{Pic}_T(X(\Sigma))$, where $X(\Sigma)$ is a complete toric variety. There is a natural and important generalization of the group $SF(\Sigma, N)$. The *ring of integral piecewise polynomial functions on Σ* is defined by

$$PP^*(\Sigma, N) = \{f : \mathrm{supp}(\Sigma) \to \mathbb{Q} : \ f|_\sigma \in \mathrm{Sym}(M_\sigma) \text{ for each } \sigma \in \Sigma\},$$

where $\mathrm{Sym}(M_\sigma)$ is the ring of integral polynomial functions on the cone σ, and $\mathrm{supp}(\Sigma)$ is the support of Σ in $N_{\mathbb{Q}}$. Clearly, $SF(\Sigma, N)$ is a subgroup $PP^*(\Sigma, N)$. It is shown by Brion [17] (for simplicial fans) and Payne [59] (for a general fan) that the T-equivariant Chow ring with rational coefficients is isomorphic to $PP^*(\Sigma, N)$.

There are interesting generalizations of these results in the setting of spherical varieties. For example, the Picard groups of spherical varieties have been computed by Brion in [15]. Brion also computed the equivariant Chow groups of projective nonsingular spherical varieties in [17].

8. Colored Fans

In previous sections we reviewed the T-stable divisors on a normal toric variety. We learned from the top row of diagram in Figure 7.1 that the linear equivalence class of a Weil divisor on a normal toric variety is given by a T-stable Weil divisor modulo a T-stable principal divisor, the latter being the divisor of a character of T. There is a related, more general statement for normal varieties that admit solvable group actions. If a connected solvable group B acts on a normal variety X, then any Weil divisor on X is rationally equivalent to a B-stable Weil divisor, see [82, Proposition 17.1]. In this section, more specifically, we will consider the theory of the B-stable divisors on the spherical varieties.

Throughout this section, we will denote by G a connected reductive group. We fix a Borel subgroup $B = UT$ in G. Finally, we fix a spherical subgroup $H \subseteq G$. Let X be a spherical embedding of G/H. This means that X is normal and the stabilizer in G of a point x from the open G-orbit of X is the subgroup H. We set

$$\mathcal{D}(X) := \{\text{the set of } B\text{-stable prime divisors of } X\},$$

and

$$\mathcal{F}(X) := \{\text{the set of } B\text{-stable but not } G\text{-stable prime divisors of } X\} \subseteq \mathcal{D}(X).$$

Remark 8.1. Let Z be a B-stable prime divisor in X. Since there are only finitely many B-orbits, and since the closure of every B-orbit is irreducible, we see that Z is the closure of a B-orbit. Let us denote the open B-orbit of Z by Z_0. Clearly, if Z_0 is not contained in

G/H, then it is contained in one of the G-stable components of the complement $X - G/H$. It is also obvious that such a G-stable connected component is a G-stable prime divisor of X. By comparing their dimensions, we see that Z_0 is the open B-orbit of that G-stable prime divisor of X. In other words, Z is a G-stable prime divisor. This argument shows that, if X is a spherical variety, then $\mathcal{F}(X)$ is nothing but the set of Zariski closures in X of the codimension one B-orbits of the open G-orbit, $G/H \subseteq X$.

Definition 8.2. The elements of $\mathcal{F}(X)$ are called the *colors of* X. It is easy to verify that

$$\mathcal{F}(X) = \{D \in \mathcal{D}(X) : \ D \cap G/H \neq \varnothing\}.$$

If Y is a G-orbit in X, then we set

$$\mathcal{D}_Y(X) := \{D \in \mathcal{D}(X) : \ Y \subseteq D\} \tag{8.3}$$

and

$$\mathcal{F}_Y(X) := \mathcal{D}_Y(X) \cap \mathcal{F}(X). \tag{8.4}$$

Definition 8.5. A normal G-equivariant embedding X of G/H is called a *toroidal embedding* if there is no B-stable prime divisor of G/H whose closure in X contains a G-orbit. Equivalently, for every G-orbit $Y \subseteq X$, we have $\mathcal{F}_Y(X) = \varnothing$.

Example 8.6. Let X be a toric variety with an open orbit isomorphic to the torus T_N, where N is a lattice of rank of n. Then X is a spherical embedding of T_N. Notice that the open orbit of X cannot contain any T_N-stable divisor since T_N acts on itself transitively. In other words, we have $\mathcal{F}(X) = \varnothing$. Thus, according to Definition 8.5 every toric variety is a toroidal embedding of its open orbit.

The notion of a "regular G-variety" is introduced by Bifet, De Concini, and Procesi in [5] as a generalization of the smooth toric varieties.

Definition 8.7. Let G be a connected affine variety and let X be a smooth, irreducible G-variety such that $d_G(X) = 0$. Let X_0 denote the open G-orbit. Under these assumptions the variety X is called *regular* if the following conditions are satisfied:

(1) The *boundary of* X_0, that is $\partial X := X \backslash X_0$, is a union of smooth prime divisors with normal crossings.
(2) Every G-orbit closure in X is the transversal intersection of the boundary divisors.
(3) For every $x \in X$, the normal space $T_x X / T_x(G \cdot x)$ contains a dense orbit of $Stab_G(x)$.

These conditions that are used for defining the regular G-varieties are satisfied by every smooth toric variety. It turns out that every complete regular variety is spherical. Conversely, every homogeneous spherical variety admits a completion that is regular. The following result is due to Bien and Brion [4].

Theorem 8.8. *Let X be a smooth and complete G-variety. Then X is a toroidal embedding if and only if X is a regular G-variety.*

Remark 8.9. As we will see later, regular G-varieties having unique closed G-orbit are rather wonderful varieties.

It is natural to wonder how toroidal embeddings can be distinguished from each other in combinatorial terms since they are very close to being a toric variety. In the next several subsections we will summarize how such a classification is achieved.

8.1. Colored cones

We begin with setting our conventions which will be in force throughout this subsection.

As usual, G denotes a connected reductive group, H denotes a closed spherical subgroup of G. And B denotes a Borel subgroup; T denotes a maximal torus of B. We will use X to denote a spherical G-variety with open orbit G/H. We will refer to it as a *spherical embedding of* G/H. A B-stable divisor of G/H is called a *color* of G/H[24]. The set of colors of X is denoted by $\mathcal{F}(X)$. The colors of X that contain a G-orbit Y are denoted by $\mathcal{F}_Y(X)$.

Since X is normal, every prime divisor $D \subset G/H$ defines a DVR whose valuation is given by the order-of-vanishing valuation $\mathrm{ord}_D : k(G/H) = k(X) \to \mathbb{Q} \cup \{\infty\}$. Recall that we introduced earlier the map $\varrho : \{\text{all valuations on } X\} \to \mathbf{H\ddot{O}}(X)$ and denoted its restriction to G-valuations by $\mathtt{e}$. Here we will consider the restriction of ϱ to the set of valuations of the form ord_D. But we do not want to double-use the notation $\mathtt{e}$. In fact, we want to emphasize the difference between the valuations of the prime G-stable divisors and the valuations of colors. So, we will use a different notation for the restriction of ϱ on the set of the latter kind of valuations. We set

$$\begin{aligned} \mathtt{f} : \mathcal{F}(X) &\longrightarrow \mathbf{H\ddot{O}}(G/H) = \mathbf{H\ddot{O}}(X) \\ D &\longmapsto \mathtt{f}_D, \end{aligned}$$

where $\mathtt{f}_D : \mathbf{\ddot{O}}(G/H) \to \mathbb{Q}$ is the homomorphism ϱ_{ord_D}, that is,

$$\mathtt{f}_D(\mathtt{r}(f)) = \mathtt{f}_D(\chi_f) := \mathrm{ord}_D(f) \qquad (f \in k(G/H)^{(B)}).$$

Remember that $\mathtt{e} : \mathcal{V}(X) \to \mathbf{H\ddot{O}}(G/H)$ is injective. This property is in general not true for $\mathtt{f} : \mathcal{F}(X) \to \mathbf{H\ddot{O}}(G/H)$.

Definition 8.10. Let X be a simple embedding of G/H, let Y denote the closed orbit of X. The *strictly convex colored cone of* X, denoted $\mathscr{C}_{G/H}(X)$, is the pair $(\mathcal{C}(X), \mathcal{F}_Y(X))$, where $\mathcal{C}(X)$ is the (strictly convex) polyhedral cone in $\mathbf{H\ddot{O}}(G/H)$ defined by

$$\mathcal{C}(X) := \langle \mathtt{f}(D), \mathtt{e}(v_E) : \; D \in \mathcal{F}_Y(X), \quad E \text{ is a } G\text{-stable prime divisor of } X \rangle.$$

[24]More generally, for a G-variety X, a *color* is a B-stable but not G-stable divisor in X. If X is a homogeneous space of the form G/H, then X has no G-stable divisors.

We now reformulate this definition without giving reference to any simple embedding X. A *strictly convex colored cone of* G/H is a pair $(\mathcal{C}, \mathcal{F})$, where $\mathcal{F} \subseteq \mathcal{D}$ such that

CC1. $\mathcal{C}$ is generated by $\mathtt{f}(\mathcal{F})$ and a finite number of elements from $\mathcal{V}(G/H)$;
CC2. the relative interior of $\mathcal{C}$ intersects $\mathcal{V}(G/H)$;
CC3. 0 is not an element of $\mathtt{f}(\mathcal{F})$.

It is easy to verify that the strictly convex colored cone of a simple embedding $G/H \hookrightarrow X$ satisfies the conditions CC1, CC2, and CC3.

The following result was first proved by Luna and Vust in [52, Proposition 8.10].

Theorem 8.11. *(Classification of simple spherical embeddings* [43, Theorem 3.1]*.) The assignment*

$$X \rightsquigarrow \mathscr{C}_{G/H}(X)$$

is a bijection between the set of all simple embeddings of G/H (up to G-isomorphisms) and the set of all strictly convex colored cones of G/H.

Let $(\mathcal{C}, \mathcal{F})$ be a strictly convex colored cone of G/H. The faces of $(\mathcal{C}, \mathcal{F})$ are defined in a natural way as follows: A strictly convex colored cone $(\mathcal{C}', \mathcal{F}')$ (for some spherical homogeneous space G/K) is called a *face* of $(\mathcal{C}, \mathcal{F})$ if the cone $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{F}' = \mathcal{F} \cap \mathtt{f}^{-1}(\mathcal{C}')$. The faces of a strictly convex colored cone corresponds to the G-orbit of the simple embedding.

Proposition 8.12. *(*[43, Lemma 4.2]*) Let X be a simple embedding of G/H, and let $Y \subseteq X$ denote the unique closed G-orbit. Let $(\mathcal{C}(X), \mathcal{F}_Y(X))$ denote the corresponding strictly convex colored cone. Then there is an inclusion order reversing bijection between the set of faces of $(\mathcal{C}(X), \mathcal{F}_Y(X))$ and the set of simple embeddings obtained by taking the closures of the G-orbits in X.*

We refer the reader to Knop's article for the proof of the above proposition.

Lemma 8.13. *(Brion-Pauer's Standard Embedding Theorem* [20, §3.1]*) For every spherical homogeneous space G/H, there exists a complete simple embedding of G/H without colors.*

Proof. Let $\pi : G \to G/H$ denote the canonical projection. Let $f_0 \in k[G]^{(B\times H)}$ be a regular function such that f_0 vanishes on any divisor $\pi^{-1}(D)$ where $D \in \mathcal{D}(G/H)$. Let $V \subseteq k[G]$ be the simple G-submodule generated by f_0. By choosing a basis for V, and by using the fact that f_0 is a $B \times H$-semiinvariant function we obtain a G-equivariant map $\psi : G/H \to \mathbb{P}(V^\vee)$[25]. Let X' denote $\overline{\psi(G/H)}$. Since we can view f_0 as a linear functional

[25]Let $f_1, \dots, f_m$ be a basis for V. Then we define $\psi' : G \to V^\vee,\ g \mapsto (f_1(g^{-1}), \dots, f_m(g^{-1}))$. Clearly, this map is a morphism. We claim that the image of G does not contain 0. Otherwise, there exists $g_0 \in G$ such that $f_i(g_0^{-1}) = 0$ for every $i \in \{1, \dots, m\}$. This means that $f'(g_0^{-1}) = 0$ for every $f' \in V^\vee$. Then for every $g \in G$ the equality $(g_0 g) \cdot f'(g_0^{-1}) = 0$ holds. This implies that $g \cdot f' = 0$ for every $g \in G$, hence, $f' = 0$. Now since the image of ψ' does not contain 0 and since f is $B \times H$-semiinvariant, ψ' descends to a G-equivariant map $\psi : G/H \to \mathbb{P}(V^\vee)$.

on $V^\vee$, we can consider the set of zeros $L := \{f_0 = 0\}$ in $\mathbb{P}(V^\vee)$. It is easy to see, by using the fact that V is generated by f_0, that L does not contain any G-orbits. Clearly, we have $\psi(D) \subseteq L$ for every $D \in \mathcal{D}(G/H)$. Therefore, X does not have any "color" that contains a G-orbit. Now let X'' be any complete embedding of G/H and consider the diagonal embedding of G/H in $X' \times X''$. Then the normalization of the closure of the image of G/H is a colorless complete simple embedding of G/H. □

8.2. Colored fans

Sumihiro's Embedding Theorem indicates that a spherical variety can be built from certain simple embeddings that it contains. This is formalized combinatorially by the notion of a "colored fan". Before we introduce colored fans, we will describe the 'simple pieces' of a spherical variety. Let X be an equivariant embedding of the spherical homogeneous space G/H. For each G-orbit $Y \subseteq X$ we have a special open subset defined by

$$X_{Y,B} := X \backslash \bigcup_{D \in \mathcal{D} \backslash \mathcal{D}_X(Y)} D.$$

In other words, $X_{Y,B}$ is the complement of the B-stable divisor whose irreducible components are the prime divisors that do not contain Y. It turns out that $X_{Y,B}$ is the minimal B-stable affine open subset of X such that $Y \cap X_{Y,B} \neq \varnothing$. We now set

$$X_{Y,G} := G \cdot X_{Y,B}.$$

Then $X_{Y,G}$ is a G-stable open subset of X. It is not difficult to verify that the open subvariety $X_{Y,G} \hookrightarrow X$ is a simple embedding G/H. Furthermore, the unique closed orbit of $X_{Y,G}$ is Y, and the intersection $Y \cap X_{Y,B}$ is a single B-orbit. The proofs of these facts, which are due to Luna and Vust can be found in [43, Theorem 2.1]. Since the union $\bigcup_{Y:\ G\text{-orbit}} X_{Y,G}$ is equal to X, we found an open covering of X by a finite number of simple embeddings. This is a precise generalization of the fact that for every fan Σ, the toric variety $X(\Sigma)$ is covered by the open toric subvarieties U_σ, where σ runs over all cones of the fan Σ.

Definition 8.14. Let G/H be a spherical homogeneous space. A finite set $\mathscr{F}$ of strictly convex colored cones of G/H is called a *strictly convex colored fan of* G/H if the following conditions hold:

CF1. If $(\mathcal{C}', \mathcal{D}')$ is a face of a strictly convex colored cone from $\mathscr{F}$, then $(\mathcal{C}', \mathcal{D}') \in \mathscr{F}$.
CF2. The relative interiors of the cones in $\mathscr{F}$ do not intersect.

We are now ready to define the strictly convex colored fan of a spherical variety X whose open G-orbit is G/H. Let Y be a G-orbit in X. Since $X_{Y,G}$ is a simple embedding, it comes with a strictly convex colored cone, $(\mathcal{C}(X_{Y,G}), \mathcal{F}_{X_{Y,G}}(Y))$. The *strictly convex colored fan of* X, denoted $\mathscr{F}_{G/H}(X)$, is the set

$$\mathscr{F}_{G/H}(X) := \{(\mathcal{C}(X_{Y,G}), \mathcal{F}_{X_{Y,G}}(Y)) : Y \text{ is a } G\text{-orbit}\}.$$

It is easy to check that $\mathscr{F}_{G/H}(X)$ satisfies the two conditions CF1 and CF2.

The classification theorem of simple embeddings has a natural extension in terms of colored fans.

Theorem 8.15. *(Classification of spherical embeddings,* [43, Theorem 3.3]*) Let G/H be a spherical homogeneous space. The assignment*

$$X \rightsquigarrow \mathscr{F}_{G/H}(X)$$

is a bijection between the set of all spherical embeddings of G/H (up to G-isomorphisms) and the set of all strictly convex colored fans of G/H.

One of the many merits of the Knop's "expository" article [43] is its clarification of the functorial properties of the classification. In particular, Knop characterizes the proper morphisms of spherical varieties in terms of the strictly convex colored fans.

Definition 8.16. Let X be a spherical embedding of G/H. The *support* of the strictly convex colored fan of X is defined by

$$\operatorname{supp}(\mathscr{F}(X)) := \bigcup_{(\mathcal{C}(X_{Y,G}),\mathcal{F}_Y(X))\in\mathscr{F}(X)} \mathcal{C}(X_{Y,G}) \cap \mathcal{V}(G/H).$$

Theorem 8.17. *(*[43, Theorem 4.2]*) Let $\varphi : X \to X'$ be a dominant G-morphism between a G/H- and a G/H'-embedding. Then φ is proper if and only if the support of the strictly convex colored fan of X is equal to $\varphi_*^{-1}(supp(\mathscr{F}(X')))$. In particular, X is a complete spherical variety if and only if the support of $\mathscr{F}(X)$ is equal to $\mathcal{V}(G/H)$.*

Here is another useful statement from Knop's exposition.

Theorem 8.18. *(*[43, Theorem 6.7]*) Let X be a spherical embedding of G/H. Then X is affine if and only if the following two conditions hold:*

(1) X is a simple embedding.
(2) There exists a weight $\chi \in \ddot{\mathbf{O}}(X) \subseteq \ddot{\mathbf{O}}(B)$ such that
  (a) $\chi(\mathsf{e}(\mathcal{V}(X))) \leqslant 0$,
  (b) $\chi(\mathcal{C}(X)) = 0$, and
  (c) $\chi(\mathcal{F}(X)\backslash\mathcal{F}_Y(X)) > 0$.

Corollary 8.19. *Let X be a spherical embedding of G/H. Then the following statements hold:*

(1) If $\mathcal{F}(X) = \mathcal{D}(X)$, then X is affine.
(2) G/H is affine if and only if $\mathcal{V}(G/H)$ and $\mathtt{f}(\mathcal{F}(G/H))$ are separated by a hyperplane.
(3) G/H is quasi-affine if and only if $\mathtt{f}(\mathcal{F}(G/H))$ does not contain 0, and it generates a strictly convex rational polyhedral cone.

8.3. Valuation cone revisited

Let $k[G]^{(H)}$ denote the set of right H-semiinvariants in $k[G]$, that is,

$$k[G]^{(H)} = \{f \in k[G] : \ f(xh) = \chi(h)f(x) \ \text{ for some } \chi \in \ddot{\mathbf{O}}(H) \text{ and for every } x \in G,\ h \in H\}.$$

It is easy to check that G acts (on the left) on $k[G]^{(H)}$, hence it is a rational G-submodule of $k[G]$. Let $f_1,\dots,f_s$ be a list of elements from $k[G]^{(H)}$. Let M_i denote the G-submodule of $k[G]^{(H)}$ generated by f_i ($i\in\{1,\dots,s\}$). Finally let $M_1\cdots M_s$ denote the G-submodule of $k[G]$ generated by the products $g_1\cdots g_s$, where $g_i\in M_i$ for $i\in\{1,\dots,s\}$. Notice that every element $g_1\cdots g_s\in M_1\cdots M_s$ has the same H-weight as the element $f_1\cdots f_s$. Therefore, for every $f\in M_1\cdots M_s$, the element $f_1\cdots f_s/f$ is an H-invariant rational function in $k(G)$. Equivalently, we have $f_1\cdots f_s/f\in k(G)^H = k(G/H)$. Now since f is from $M_1\cdots M_s$, it is of the form $\prod_{i=1}^s\sum_{j\in I_i} g_j\cdot f_i$, where I_i's are some finite sets of indices. Thus if $\bar v$ is any G-invariant valuation on $k(G)$, then we have $\bar v(f)\geqslant \bar v(f_1)+\cdots+\bar v(f_i)$. This inequality holds also for $v\in\mathcal{V}(G/H)$ since it extends to a G-invariant valuation on G. It follows that $(\sum_{i=1}^s v(f_i)) - v(f)\geqslant 0$, or that $v\left(\frac{f_1\cdots f_s}{f}\right)\leqslant 0$. This observation is very helpful for analyzing the B-semiinvariants of G/H. Let us assume that $f_1,\dots,f_s$ are B-semiinvariants in $k[G]^{(H)}$ with respect to the left B-action on $k[G]$. Likewise, let f be a (left) B-semiinvariant in $M_1\cdots M_s$. Let $\chi_{f_1},\dots,\chi_{f_s},\chi_f$ denote the corresponding weights. Then the rational function $h:=\frac{f_1\cdots f_s}{f}$ is a B-semiinvariant in ${}^{(B)}k(G/H)$[26]. The B-weight of h is given by

$$\chi_h := \chi_{f_1}+\cdots+\chi_{f_s}-\chi_f. \tag{8.20}$$

Since $v(h)\leqslant 0$, we see that

$$\varrho_v(h)=\langle v,\chi_h\rangle\leqslant 0.$$

Let us denote the set of all weights of the form χ_h as in (8.20) by Δ. The sublattice generated by Δ in $\ddot{\mathbf{O}}(G/H)$ plays the role of the root lattice of a semisimple group. We denote the minimal generating set of primitive vectors of the lattice generated by Δ in $\ddot{\mathbf{O}}(G/H)$ by $\mathtt{S}_{G/H}$, or, simply by $\mathtt{S}$ if there is danger for confusion. This is the set of spherical roots of G/H.

We are now ready to describe the linear conditions that define the valuation cone.

Theorem 8.21. *Let G/H be a spherical homogeneous space. Then the set of G-valuations on G/H, that is $\mathcal{V}(G/H)$, is the dual cone of the cone generated by $-\mathtt{S}$ in $\ddot{\boldsymbol{O}}(G/H)_{\mathbb{Q}}$. In other words, we have*

$$\mathcal{V}(G/H)=\{v\in \boldsymbol{H}\ddot{\boldsymbol{O}}(G/H):\ \langle v,-\alpha\rangle\geqslant 0 \text{ for every } \alpha\in\mathtt{S}\}. \tag{8.22}$$

Furthermore, $\mathcal{V}(G/H)$ is a rational polyhedral cone.

Proof. We will explain the idea behind the proof; the details can be found in [18, Section 4] or [43, Section 5]. The inclusion "$\subseteq$" follows from the previous paragraph. For "$\supseteq$", let v be a vector from the right hand side of (8.22). The main idea is to identify an "elementary embedding" X' of G/H whose strictly convex colored cone is of the form $(\mathbb{Q}^+v',\varnothing)$ and such that $v\in\mathbb{Q}^+v'$. It then follows that v is also a G-valuation of G/H. This proves the

[26] Even though B acts on the left, by abuse of notation, the group of B-semiinvariants ${}^{(B)}k(G/H)$ is usually denoted by $k(G/H)^{(B)}$.

equality in (8.22) as well as its rationality. It remains to show that $\mathcal{V}(G/H)$ is a finitely generated convex cone. A quick proof of this fact follows from another result of Brion and Pauer (Lemma 8.13): every spherical variety G/H has a simple complete toroidal embedding X. Then the strictly convex colored cone $\mathscr{C}_{G/H}(X)$ has the property that the supports of its faces give a subdivision of $\mathcal{V}(G/H)$. (This will be explained in more detail in Proposition 8.25.) In particular, we see that 1) $\mathcal{V}(G/H)$ is covered by finitely many strictly convex colored cones, hence, it is finitely generated. 2) $\mathcal{V}(G/H)$ is convex since its elements are contained in the cone of $\mathscr{C}_{G/H}(X)$. □

Remark 8.23. Recall the definition of the anti-dominant Weyl chamber $\c{C}(G)$ of (G, B, T): Let $\{\alpha_1, \dots, \alpha_l\}$ be the set of simple roots $\mathtt{S}_{G,B,T}$ determined by (G, B, T). Then $\c{C}(G)$ is the cone spanned by the elements of $\{-\alpha_1^\vee, \dots, -\alpha_l^\vee\}$ in $\mathbf{\ddot{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. We can write it as in

$$\c{C}(G) := \{f \in \mathbf{H\ddot{O}}(T) : \langle f, \alpha \rangle \leqslant 0 \text{ for every } \alpha \in \mathtt{S}_{G,B,T}\}. \tag{8.24}$$

Since $\mathbf{\ddot{O}}(G/H)$ is a sublattice of $\mathbf{\ddot{O}}(B)$, dually, we have the surjection $\mathbf{H\ddot{O}}(B) \to \mathbf{H\ddot{O}}(G/B)$. The *anti-dominant Weyl chamber of* G/H, denoted by $\c{C}(G/H)$, is the image of $\c{C}(G)$ in $\mathbf{H\ddot{O}}(G/B)$. Theorem 8.21 implies that the anti-dominant Weyl chamber of G/H is contained in $\mathcal{V}(G/H)$. In turn, this fact implies that the linear span of $\mathcal{V}(G/H)$ is equal to $\mathbf{H\ddot{O}}(G/H)$. In other words, $\mathcal{V}(G/H)$ is a full dimensional cone in $\mathbf{H\ddot{O}}(G/H)$.

8.4. The linear part of the valuation cone

Recall that a spherical embedding X of G/H is a toroidal embedding if and only if its strictly convex colored fan is of the form

$$\mathscr{F}(X) = \{(\mathcal{C}(X_{Y,G}), \varnothing) : Y \text{ is a } G\text{-orbit}\}.$$

In other words, for any G-orbit $Y \subseteq X$, the simple embedding $X_{Y,G}$ has no colors.

Proposition 8.25. *Let G/H be a spherical homogeneous space. Then every complete toroidal embedding of G/H is obtained from a subdivision of $\mathcal{V}(G/H)$ into strictly convex rational polyhedral cones. Furthermore, a simple toroidal embedding of G/H exists if and only if $\mathcal{V}(G/H)$ is a strictly convex rational polyhedral cone.*

Proof. Let X be a simple toroidal embedding of G/H and let Y be the closed orbit in X. By Proposition 8.12, we know that the G-orbit closures containing Y correspond to the faces of the cone $\mathcal{C}(X)$. Since $\mathcal{F}_Y(X) = \varnothing$, the cone $\mathcal{C}(X)$ is generated by a finitely many elements from $\mathcal{V}(G/H)$, hence, the following inclusion holds:

$$\mathcal{C}(X) \subseteq \mathcal{V}(G/H). \tag{8.26}$$

Thus, for any toroidal embedding $G/H \hookrightarrow X$, the inclusion $\bigcup_{Y:\ G\text{-orbit}} \mathcal{C}(X_{Y,G}) \subseteq \mathcal{V}(G/H)$ holds. Recall that, by Theorem 8.17, X is complete if and only if the support of the strictly convex colored fan of X is equal to the valuation cone. It follows that if X is a complete toroidal embedding of G/H, then we have the equality,

$$\bigcup_{Y\colon G\text{-orbit}} \mathcal{C}(X_{Y,G}) = \mathcal{V}(G/H). \tag{8.27}$$

Therefore, every complete toroidal embedding is given by a subdivision of $\mathcal{V}(G/H)$ into strictly convex rational polyhedral cones.

For our second assertion, we observe once again by (8.27) that a simple toroidal embedding X is complete if and only if $\mathcal{V}(G/H) = \mathcal{C}(X)$ holds. In other words, a simple complete toroidal embedding of X exists if and only if $\mathcal{V}(G/H)$ is a strictly convex rational polyhedral cone. $\square$

There are several useful consequences of Proposition 8.25. First of all, it implies that if $\mathcal{V}(G/H)$ is a strictly convex rational polyhedral cone, then there exists a unique maximally dominating complete simple toroidal embedding of G/H; it is the one with the strictly convex colored cone $(\mathcal{C}(X), \varnothing) = (\mathcal{V}(X), \varnothing)$. Assuming that it exists, let us call this complete simple toroidal embedding the *standard embedding of* G/H. All other complete toroidal simple embeddings are obtained from the standard embedding by subdividing $\mathcal{V}(X)$. Now it is natural to wonder the following questions: 1) Is there a practical method for deciding when $\mathcal{V}(G/H)$ is strictly convex? 2) What are the geometric properties of the standard embedding of G/H? The first question has a nice answer in terms of the normalizer of H. The second question has a broad answer.

Recall the useful description of $\mathcal{V}(G/H)$ from Theorem 8.21:

$$\mathcal{V}(G/H) = \{v \in \mathbf{H\ddot{O}}(G/H) : \ v = -\sum_{\alpha_i \in \mathtt{S}} a_i \alpha_i^\vee \text{ for some } a_i \in \mathbb{Q}^+\},$$

where $\mathtt{S}$ is the set of simple roots of the little root system of G/H. The following important theorem of Brion and Pauer [20] immediately answers the first question in characteristic 0.

Theorem 8.28. *(*[18, Theorem 4.3]*) Let $N_G(H)$ denote the normalizer of H in G. Let $\langle \mathtt{S} \rangle$ denote the lattice generated by $\mathtt{S}$ in $\boldsymbol{H\ddot{O}}(G/H)$. Then the following statements hold:*

(1) The quotient $N_G(H)/H$ is isomorphic to $(\boldsymbol{\ddot{O}}(G/H)/\langle \mathtt{S} \rangle) \otimes_{\mathbb{Z}} k^*$. Consequently, $N_G(H)/H$ is a diagonalizable algebraic group, and its dimension is equal to the dimension of the linear part of $\mathcal{V}(G/H)$.

(2) There is a short exact sequence

$$0 \to \mathit{linear}(\mathcal{V}(G/H)) \to \boldsymbol{\ddot{O}}(G/H) \to \boldsymbol{\ddot{O}}(G/N_G(H)) \to 0,$$

where $\mathit{linear}(\mathcal{V}(G/H))$ is the linear part of $\mathcal{V}(G/H)$ in $\boldsymbol{\ddot{O}}(G/H)$.

Let us list some immediate consequences of Theorem 8.28.

- $\mathcal{V}(G/H)$ is strictly convex if and only if $N_G(H)/H$ is a finite group.
- If $\mathbf{\ddot{O}}(G/N_G(H)) = 0$, then $\text{linear}(\mathcal{V}(G/H)) = \mathbf{\ddot{O}}(G/H)$. This happens, for example, when $G = B = T$, and $H = \{0\}$. It follows that the cone of T-valuation on any spherical embedding of T (a toric variety) is the whole space $\mathbf{H\ddot{O}}(T)$.
- If $N_G(H) = H$, then $\text{linear}(\mathcal{V}(G/H)) = 0$. This happens, for example, when $H = N_G(G^\theta)$, where $\theta : G \to G$ is an involutory automorphism of G.

9. The Classification of Spherical Homogeneous Spaces and Wonderful Varieties

Notice that we only discussed the classification of the spherical embeddings of a fixed spherical homogeneous space G/H. However, we did not mention a single theorem about how one can classify all spherical homogeneous spaces. In this section we will briefly mention how this classification is achieved. We begin with introducing a very special spherical variety whose roots go back to the work of Demazure. The definition here is due to Luna. It is a natural generalization of the "De Concini-Procesi wonderful compactification" of a symmetric space.

Definition 9.1. Let X be an irreducible G-variety. We call X a *wonderful variety of rank* r if

(1) X is smooth and complete;
(2) X has an open G-orbit, denoted X_0, such that $X\backslash X_0$ is a union of r smooth G-stable prime divisors $X_1, \dots, X_r$ that intersect transversally[27] and such that $X_1 \cap \cdots \cap X_r$ is nonempty. In other words, the *boundary*, that is, $\partial X := X\backslash X_0$, is a normal crossings divisor.[28]
(3) For two points $x, y \in X$, the G-orbits $G \cdot x$ and $G \cdot y$ coincide if and only if $\{i : \ x \in X_i\} = \{j : \ y \in X_j\}$.

Conditions (2) and (3) show that X has a unique closed orbit that is given by $X_1 \cap \cdots \cap X_r$. Clearly, any G-orbit O such that $O \neq X_0$ is contained in the boundary. Therefore, once again by (2) and (3) the closure of O is equal to $\cap_{i\in I} X_i$, where I is the maximal subset of $\{1, \dots, r\}$ such that $O \subset X_i$ for every $i \in I$. It follows that the inclusion poset of G-orbit closures in X is isomorphic to the boolean lattice on $\{1, \dots, r\}$.

It is shown by Luna in [49] that every wonderful variety is spherical. After knowing this fact, it is not difficult to show that X is a toroidal embedding, [61, Proposition 3.3.1]. Therefore, the open orbit X_0 has no B-stable prime divisors that contain $X_1 \cap \cdots \cap X_r$.

[27] Loosely speaking, two subvarieties Y and Z in X are said to intersect *transversally* if for every $x \in Y \cap Z$, the tangent subspace $T_x(Y \cap Z)$ is equal to $T_x Y \cap T_x Z$ in $T_x X$.

[28] An effective Cartier divisor D is said to be a *strict normal crossings divisor* if for every $p \in D$ the local ring $\mathcal{O}_{p,X}$ is regular and there exists a regular system of parameters $x_1, \dots, x_d$ for $\mathfrak{m}_p$ and $s \in \{1, \dots, d\}$ such that D is defined by the rational function $\prod_{i=1}^s x_i$ in $\mathcal{O}_{X,p}$.

We mentioned before that $\mathcal{V}(X)^\vee \subseteq \ddot{\mathbf{O}}(X)_\mathbb{Q}$ is a simplicial cone spanned by the spherical roots $\mathtt{S} \subset \ddot{\mathbf{O}}(X)$. In the case of a wonderful variety of rank r, since $\mathcal{V}(X)$ is strictly convex, we have $\mathcal{V}(X)^\vee = \ddot{\mathbf{O}}(X)_\mathbb{Q}$. It follows that the rank of the weight lattice $\ddot{\mathbf{O}}(X)$ is r. In other words, $\mathtt{S}$ is a basis for $\ddot{\mathbf{O}}(X)$.

Let us call a spherical subgroup $H \subseteq G$ *sober* if $N_G(H)/H$ is finite (or, equivalently, by Theorem 8.28, $\mathcal{V}(G/H)$ is strictly convex). Since wonderful varieties are smooth simple toroidal embeddings, the stabilizer of a point in general position in a wonderful variety is a sober subgroup. A sober group is said to be *very sober* if the finite group $N_G(H)/H$ acts effectively on the set of B-stable divisors of G/H.

Theorem 9.2. *Let H be a sober spherical subgroup of G, and let X denote G/H. Then we have*

(1) If H is very sober, then the standard embedding of X is a wonderful variety.
(2) If $H = N_G(H)$ (hence H is very sober), then the wonderful embedding of X is the Zariski closure of the orbit $Ad(G) \cdot \mathfrak{h}$ in $Gr(k, \mathfrak{g})$, where $k := \dim \mathfrak{h}$, and $\mathfrak{g}$ (resp. $\mathfrak{h}$) is the Lie algebra of G (resp. of H).

The first assertion of Theorem 9.2 was originally discovered by Luna. In fact, Luna showed in type $\mathtt{A}$ that (1) is an if and only if statement. However, in general, the converse is not correct; see [82, Remark 30.6] for a counterexample. The general proof of (1) is given by Knop in [45]. The second part of Theorem 9.2, which was a conjecture of Brion, is proved in the special cases by Luna [51] (in type $\mathtt{A}$), Bravi and Pezzini [12] (in types $\mathtt{A}$ and $\mathtt{D}$), and by Bravi [9] (in $\mathtt{E}$ types). The general case is due to [48].

Let G/H be a spherical homogeneous space. As before, let $\mathcal{F}(G/H)$ denote the set of all colors of G/H. The normalizer $N_G(H)$ acts on G/H via $n \cdot gH = gn^{-1}H$, where $n \in N_G(H)$ and $gH \in G/H$. Clearly, the restriction of this action to H is trivial. It is also evident that the action $N_G(H) : G/H$ descends to an action on the set of (B-stable) divisors of G/H. The *very sober hull of H in G*, denoted $\overline{H}$, is the kernel of the action $N_G(H) : \mathcal{F}(G/H)$. Note that $\overline{H}$ contains the connected component of the identity element of $N_G(H)$. It follows from [11, Lemma 2.4.2] that the very sober hull $\overline{H}$ is a very sober spherical subgroup of G. Therefore, thanks to Theorem 9.2, for every spherical subgroup $H \subseteq G$, the wonderful embedding of $G/\overline{H}$ exists. Now the strategy for classifying all spherical homogeneous spaces becomes apparent: First classify all spherical subgroups H with a fixed very sober hull $\overline{H}$. Then classify all wonderful varieties. This strategy was proposed by Luna, who completed the first step in [50]. For the second step Luna conjectured that the wonderful G-varieties can be classified by certain combinatorial objects that he called the "spherical systems". This part of the classification has recently been completed by the joint efforts of many mathematicians in a series of papers [50, 12, 9, 10, 27, 13].

10. Some Interesting Examples

As usual, we denote by G a connected reductive group; $B = UT$ denotes a Borel subgroup of G.

10.1. Horospherical varieties

A closed subgroup $H \subseteq G$ is called a *horospherical subgroup* if a maximal unipotent subgroup of G is contained in H. Examples of horospherical subgroups include all parabolic subgroups. By using the Bruhat-Chevalley decomposition it is easy to see that a horospherical subgroup is indeed a spherical subgroup. Naturally, a homogeneous space G/H is called a *horospherical homogeneous space* if H is a horospherical subgroup. By definition, a horospherical homogeneous space is the one that admits a surjective G-equivariant map from G/U. It is also characterized by the property that the evaluation map (6.29) is surjective [43, Corollary 6.2].

A G-variety X is called a *horospherical variety* if the stabilizer in G of any point $x \in X$ is a horospherical subgroup. Let us assume, for the purpose of giving an example, that X is a quasi-affine G-variety. For a dominant weight $\lambda \in \ddot{\mathbf{O}}(T)^+$, let $k[X]^{(B)}_\chi$ denote, as before, the $V(\lambda)$-isotypic component of $k[X]$ as a G-module. It is well-known that [64] X is a horospherical G-variety if and only if $k[X]^{(B)}_\chi \cdot k[X]^{(B)}_\lambda \subseteq k[X]^{(B)}_{\chi+\lambda}$ for every $\chi, \lambda \in \ddot{\mathbf{O}}(T)^+$.

Horospherical spaces are very useful for the theory of deformations of algebraic group actions [82, Section 7.3]. We will encounter an important example of a 'horospherical contraction' in the subsection below about the reductive monoids.

10.2. Strongly solvable spherical subgroups

A closed subgroup of G is called *strongly solvable* if it is contained in a Borel subgroup of G. An explicit classification (up to conjugacy) of strongly solvable spherical subgroups of connected reductive groups is due to Avdeev [3, 2]. The explicit parameterization of the B-orbits in a strongly solvable spherical homogeneous space G/H is due to Gandini and Pezzini [32]. In this article, the authors determine the action of W on the set of Borel orbits in G/H as well. One of the most important examples of spherical subgroups is the unipotent radical $U := \mathcal{R}_u(B)$ of B. Clearly, U is a strongly solvable spherical subgroup of G. Let U' denote the derived subgroup of U. Then we have $U' = \prod_{\mathtt{R}^+_{G,T}\setminus\Delta} U_\alpha$. It is known that TU' is a strongly solvable spherical subgroup. It is conjectured by Knop that the spherical variety G/TU' has the largest number of B-orbits among all spherical homogeneous spaces G/H. A theorem of Gandini and Pezzini partially resolves this conjecture: as H varies in all strongly solvable spherical subgroups, the largest number of B-orbits in G/H is attained for $H = TU'$ [32, Theorem 6.14].

10.3. Reductive monoids

Clearly, every group is a *monoid*, that is, a set that is endowed with an associative multiplication and a unit. Every monoid is a *semigroup*, that is, a set that is endowed with an associative multiplication. An *algebraic semigroup* is an algebraic variety S endowed with an associative multiplication $S \times S \to S$ which is a morphism. If there is a unit element in an algebraic semigroup, then it is called an algebraic monoid. These objects have been becoming increasingly important in various parts of algebraic group theory. We already mentioned in the introduction that they appear naturally in the study of homogeneous bundles on abelian varieties. But certain algebraic monoids are even closer to the underlying affine algebraic group structure. This is the way of an "asymptotic semigroup". Let G_0 be a connected semisimple group, which acts on itself on the right by multiplication. Let T_0 be a maximal torus of G_0. For a dominant weight $\lambda \in \ddot{\mathbf{O}}(T_0)^+$, let $k[G_0]_\chi$ denote the $V(\lambda)$-isotypic component of $k[G_0]$ as a G_0-module. Let f (resp. g) be an element from $k[G_0]_\chi$ (resp. from $k[G_0]_\lambda$) for some dominant weight χ (λ). Then their product fg in $k[G_0]$ can be written as $fg = \sum_{\nu \in \ddot{\mathbf{O}}(T_0)^+} p_\nu(f,g)$, where $p_\nu(f,g) \in k[G_0]_\nu$. By truncating it at the highest weight $\chi + \lambda$, we obtain a new multiplication,

$$f * g := p_\nu(f,g) \qquad (\nu = \chi + \lambda).$$

It is easy to check that this new multiplication is compatible with addition of functions, so, we get a new ring structure, denoted gr $k[G_0]$, on the coordinate ring of G_0. It turns out that, as shown by Vinberg in [84], the affine variety

$$\mathrm{As}(G_0) := \mathrm{Spec}(\mathrm{gr}\ k[G_0])$$

is normal and irreducible. Moreover, it has the structure of an algebraic semigroup. What is amazing about this algebraic semigroup is that, any algebraic action $G_0 : X$ can be deformed to an algebraic action of $G_0 \cup \mathrm{As}(G_0)$ on $X \cup \mathrm{As}(X)$, where $\mathrm{As}(X)$ is the contraction of X, [64]. Furthermore, we can view $\mathrm{As}(G_0)$ as the boundary $M \backslash G$ of a certain reductive group G in an algebraic monoid M with unit group G, [70, §6.3].

A *reductive monoid* is an algebraic monoid M whose group of invertible elements $G := G(M)$ is a reductive group. It turns out that the theory is irreducible reductive monoids is directly related to the theory of spherical embeddings of its unit group. Indeed, by a deep result of Putcha and Renner it is known that every normal reductive monoid is a spherical $G \times G$-equivariant embedding of G. For more in this fascinating subject, we recommend the two monographs [65, 70]. The classification of reductive monoids is due to Renner [66] (in the semisimple case), Vinberg [85] (in characteristic 0), and Rittatore [74] (in the most general situation). In the same paper, Rittatore showed that every reductive monoid with unit group G is an affine, spherical, $G \times G$-equivariant embedding of G [74, Theorem 2]. Conversely, if G is a connected reductive group, then every simple $G \times G$-equivariant embedding of G is a reductive monoid [74, Theorem 3]. Note that every affine toric variety X whose open orbit is isomorphic to a torus T is at the same time a $T \times T$-variety. Indeed, since T is commutative, $(T \times T)/\mathrm{diag}(T)$ is isomorphic, as an algebraic group, to T. It follows that every T-orbit in X is a $T \times T$-orbit in a natural way. In particular, we see that every affine toric variety is a reductive monoids.

Also, the equivariant completions of semisimple groups are directly related to the quotients of certain reductive monoids. Let G_0 be a semisimple group. Let X be a projective variety such that $G_0 \times G_0$ acts algebraically on X with an open orbit that is isomorphic to G_0. Loosely following the terminology from [68], we call X a *semisimple variety for* G_0 if there exists a $G_0 \times G_0$-equivariant embedding of G_0 into X. Notice that in this case G_0 can be assumed to be centerless.

Theorem 10.1. *(*[68, Corollary 3.4]*) Let X be a semisimple variety for a semisimple adjoint group G_0. Then there exists a normal reductive monoid M with zero such that*

(1) $(G, G) \cong G_0$, where G is the unit group of M,
(2) $\dim Z(G) = 1$,
(3) $X \cong (M \backslash \{0\})/k^*$.

Here, k^* is the image of a regular 1-psg of a maximal torus of G.

Of course, the above theorem shows that the wonderful completion of G_0 can be obtained from a reductive monoid. In fact, this monoid captures all essential information about the wonderful completion of G_0 [69]. It is quite remarkable that the asymptotic semigroup is natural part of this development [75].

In the rest of this example, we will explain Rittatore's classification of the normal reductive monoids whose unit group G is a connected reductive group. Since we already know that the the affine toric varieties are classified by the strictly convex rational polyhedral cones, we will proceed with the assumption that the semisimple rank of G is positive. This means that the commutator subgroup of G is nontrivial, hence, G is not a torus. In fact, we already know that a connected reductive group G is isomorphic to $(G_0 \times Z)/Z_0$, where Z is a central torus of G, $Z_0 \subset Z$ is a finite subgroup, and G_0 is the commutator subgroup, $G_0 = (G, G)$. Therefore, the coordinate ring of G is given by

$$k[G] \cong k[(G_0 \times Z)/Z_0] \cong (k[G_0] \otimes_k k[Z])^{Z_0}.$$

We fix a maximal torus T_0 and a Borel subgroup B_0 in G_0 such that $T_0 \subset B_0$. Let $\alpha_1, \dots, \alpha_l$ denote the simple roots determined by B_0, and let $\varpi_1, \dots, \varpi_l$ denote the corresponding fundamental weights. Then $T = T_0 Z$ is a maximal torus of G. The character group of T is given by $\ddot{\mathbf{O}}(T) = (\ddot{\mathbf{O}}(T_0) \times \ddot{\mathbf{O}}(Z))^{Z_0}$, which is a sublattice of finite index in $\ddot{\mathbf{O}}(T_0) \times \ddot{\mathbf{O}}(Z)$. It follows that

$$\mathbf{H}\ddot{\mathbf{O}}(T) \cong \mathbf{H}\ddot{\mathbf{O}}(T_0) \times \mathbf{H}\ddot{\mathbf{O}}(Z). \tag{10.2}$$

Thanks to (10.2), we have a splitting of the cone of invariant valuations also:

$$\mathcal{V}(G) \cong \mathcal{V}(G_0) \times \mathcal{V}(Z) \cong \mathcal{V}(G_0) \times \mathbf{H}\ddot{\mathbf{O}}(Z).$$

There is a convenient characterization of the elements of $\mathcal{V}(G)$ by one parameter subgroups. Let $\lambda : \mathbb{G}_m \to T$ be a 1-psg (induced from the $G \times G$-action on G). It induces a grading on $k[G]$:

$$k[G] = \bigoplus_{n \in \mathbb{Z}} k[G]_n, \quad \text{where} \quad k[G]_n = \{f \in k[G] : t \cdot f = t^n f \text{ for every } t \in \mathbb{G}_m\}.$$

Let v_λ be the valuation whose restriction to $k[G]^{(B\times B^-)}$ is defined by $v_\lambda(f) = \langle \lambda, \chi_f \rangle$ where $\chi_f \in \ddot{\mathbf{O}}(B \times B^-)$ is the weight of f. On $k[G] = \bigoplus_{n\in\mathbb{Z}} k[G]_n$, v_λ is calculated by $v_\lambda(f) = \min\{n : f_n \neq 0\}$. We mentioned before that every element $v \in \mathcal{V}(G)$ gives an elementary embedding whose strictly convex colored cone is given by $(\mathbb{Q}^+ v, \varnothing)$. Since this cone has only two faces, such an embedding has only two orbits; one of them is the open orbit and the other is closed. The following observation of Rittatore shows that the convergent 1-psgs of T give invariant valuations on G.

Lemma 10.3. *Let v be an element from $\mathcal{V}(G)$, and let X be an elementary embedding associated with v. Let $Y \subseteq X$ be the closed orbit of X. Let $\lambda : \mathbb{G}_m \to T$ be a 1-psg. If $\lim_{t\to 0} \lambda(t)$ exists in X and belongs to the open $B \times B^-$-orbit of Y, then v_λ is equivalent to v.*

The colors of G as a $G \times G$-variety are easy to calculate from the Bruhat-Chevalley decomposition of G. Recall that G_0 denotes the semisimple part of G. We set $B_0 := B \cap G_0$ and $T_0 := T \cap G_0$. Then the Weyl group of (G, T) is equal to the Weyl group of (G_0, T_0). The colors of G are the divisors of the form $\overline{B s_\alpha B^-}$, where α is a simple root for the triplet (G_0, B_0, T_0). Here, s_α is the Coxeter generator of W corresponding to α. If the semisimple rank of G is 1, then G has a unique $B \times B^-$-stable divisor, which is given by $\overline{B s_\alpha}$, where α is the unique root of (G_0, T_0).

Proposition 10.4. *The valuation cone $\mathcal{V}(G)$ is equal to $\c{C}(G) = \c{C}(G_0) \times \boldsymbol{H\ddot{O}}(Z)$. Furthermore, the valuation associated with the divisor $\overline{B s_\alpha B^-}$ is given by $(\alpha^\vee, 0) \in \boldsymbol{H\ddot{O}}(T_0) \times \boldsymbol{H\ddot{O}}(Z)$.*

This proposition is closely related to a result of Vust [89] and Knop [43], who proved that the cone G-valuations of a symmetric space is the anti-dominant chamber of the "restricted root system" of the symmetric space. We will review the notion of a restricted root system in our next example.

Proof of Proposition 10.4. We will sketch the idea of the proof of Rittatore [74, Proposition 9] which might be viewed as an adaptation of the proof [89, Proposition 2] to the monoid setting. The second assertion is actually a consequence of the first one since the anti-dominant chamber of the restricted root system is generated by the coroots. We proceed with the proof of the first assertion. We already know that $\mathcal{V}(G)$ contains the anti-dominant chamber, so it suffices to show the containment in the other direction. Let v be a G-invariant valuation. Then there exists an elementary embedding $X_{Z,G}$, and a 1-psg λ whose limit exists in $X_{Z,G}$. Recall that any valuation of a spherical variety is uniquely determined by its values on the semiinvariant rational functions. It follows that our 1-psg λ defines a valuation, denoted v_λ, by the formula,

$$v_\lambda(f) := \langle \lambda, \chi_f \rangle \qquad (f \in k(G)^{(B\times B^-)}).$$

Here, $\langle , \rangle$ is the non-degenerate duality pairing between the group of 1-psg's of T and $\ddot{\mathbf{O}}(T)$. Then by Lemma 10.3, the valuation v_λ is equal to v. The rest of the proof follows from checking that v_λ is a G-valuation. $\square$

By Proposition 10.4 we see that, for any normal reductive monoid M with the unit group G, the set of colors $\mathcal{F}(M) = \{\overline{Bs_\alpha B^-} : \ \alpha \text{ is a simple root of } (G_0, B_0, T_0)\}$ is in bijection with the set of simple coroots. From this we conclude that the evaluation-of-valuation map on the colors, that is $\mathtt{f} : \mathcal{F}(M) \to \mathbf{H}\ddot{\mathbf{O}}(M)$, is an injective map. By abuse of notation, we identify $\mathcal{F}(M)$ with its image,

$$\mathcal{F}(M) = \{(\alpha^\vee, 0) \in \mathbf{H}\ddot{\mathbf{O}}(T_0) \times \mathbf{H}\ddot{\mathbf{O}}(Z) : \ \alpha \text{ is a simple root of } (G, B, T)\}.$$

In summary, Rittatore proved the following result:

Theorem 10.5. *Let G be a connected reductive group. Then the normal reductive monoids with 0 are in 1-1 correspondence with the strictly convex polyhedral cones in $\boldsymbol{H}\ddot{\boldsymbol{O}}(T)$ generated by all of the simple coroots and a finite set of elements from $\c{C}(G)$.*

In [67], Renner extended the Bruhat-Chevalley decomposition of reductive groups to the setting of reductive monoids as follows. Let M be a reductive monoid. Let G, B, and T denote respectively the unit group of M, a Borel subgroup of G, and a maximal torus of B. Let R denote the quotient $\overline{N_G(T)}/T$, where $\overline{N_G(T)}$ is the Zariski closure in M of the normalizer of T in G. Then the *Bruhat-Chevalley-Renner decomposition* of M is given by $M = \cup_{r\in R} BrB$. where the closure is taken in M. The indexing set R is a finite monoid, called the *Renner monoid of M*. The Weyl group of G is the group of invertible elements of R. The *Bruhat-Chevalley-Renner order* on R is defined by

$$r \leqslant s \iff BrB \subseteq \overline{BsB} \qquad (r, s \in R),$$

Notice that the above descriptions of the cone of invariant valuations as well as colors are built on the $B \times B^-$-stable divisors not $B \times B$-stable divisors. This is a nuisance since we can transform every $B \times B$-orbit into a $B \times B^-$-orbit as follows. Let w_0 denote the longest element of W. Then the opposite Borel subgroup is given by $B^- = w_0 B w_0$. In particular for every $r \in R_n$ we have $BrB = Brw_0w_0Bw_0w_0 = Brw_0B^-w_0$. Note that $R_n w_0 = R_n$ and $Mw_0 = M$. Since the multiplication by w_0 preserves the inclusions of orbit closures, the Bruhat-Chevalley-Renner order can be used for analyzing the inclusion order on the closures of the $B \times B^-$-orbits as well.

The monoid of $n \times n$ matrices: Among all reductive monoids, the monoid of $n \times n$ matrices, denoted $\mathbf{Mat}_n$, has a special place. The reason is that every reductive monoid with 0 has a closed embedding into $\mathbf{Mat}_n$ for some $n \in \mathbb{Z}^+$. We will finish our example by explaining the place of $\mathbf{Mat}_n$ in the classification schematics but first we want to introduce some useful combinatorial ingredients.

The unit group of $\mathbf{Mat}_n$ is the general linear group $\mathbf{GL}_n$. We fix the following notation: $\mathbf{B}_n$ denotes the Borel subgroup of upper triangular matrices, and $\mathbf{T}_n$ denotes the maximal torus of diagonal matrices in $\mathbf{B}_n$. We will denote by $\mathbf{B}_n^-$ the Borel subgroup opposite to $\mathbf{B}_n$. Recall that the Weyl group $\mathbf{GL}_n$ is the symmetric group S_n. We represent the elements of S_n by the $n \times n$ permutation matrices; they are the $n \times n$, $0/1$ matrices with exactly one 1 in each row and each column. Likewise, the Renner monoid of $\mathbf{Mat}_n$, denoted R_n, consists of $n \times n$, $0/1$ matrices with at most one 1 in each row and each column. The inclusion order on the $\mathbf{B}_n \times \mathbf{B}_n$-orbit closures, that is the Bruhat-Chevalley-Renner order on R_n, is described in [23]. To demonstrate how one can use this partial order on R_n for the purposes of identifying the colored equipment of $\mathbf{Mat}_n$ as a $\mathbf{GL}_n \times \mathbf{GL}_n$-equivariant embedding of $\mathbf{GL}_n$, we will focus on the smallest nontrivial example, that is, $\mathbf{Mat}_2$.

The Renner monoid of $\mathbf{Mat}_2$ is given by

$$R_2 = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

The elements of R_2 are the natural representatives of the closures of the $\mathbf{B}_2 \times \mathbf{B}_2$-orbits in $\mathbf{Mat}_2$. The Hasse diagram of the inclusion order (the Bruhat-Chevalley-Renner order) on these orbit closures are depicted in Figure 10.1.

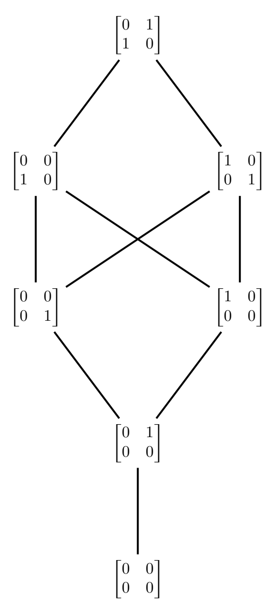

Figure 10.1. Bruhat-Chevalley-Renner order on R_2.

Every line in that figure corresponds to a unique inclusion of the form $\mathbf{B}_2 s \mathbf{B}_2 \subsetneq \overline{\mathbf{B}_2 r \mathbf{B}_2}$, where $r, s \in R_2$ such that $\dim \mathbf{B}_2 r \mathbf{B}_2 = \dim \mathbf{B}_2 s \mathbf{B}_2 + 1$. It follows that we have two $\mathbf{B}_2 \times \mathbf{B}_2$-stable divisors in $\mathbf{Mat}_2$; they are given by the closures of the orbits $O_1 := \mathbf{B}_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{B}_2$ and $O_2 := \mathbf{B}_2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{B}_2$. Let D_i' ($i \in \{1,2\}$) denote the Zariski closure $\overline{O_i}$ ($i \in \{1,2\}$). Then D_i' ($i \in \{1,2\}$) is a $\mathbf{B}_2 \times \mathbf{B}_2$-stable divisor. It is easy to check that D_2' is $\mathbf{GL}_2 \times \mathbf{GL}_2$-stable. However, D_1' is not $\mathbf{GL}_2 \times \mathbf{GL}_2$-stable.

Recall that the colored equipments were introduced by using the $\mathbf{B}_n \times \mathbf{B}_n^-$-orbits. Since the $\mathbf{B}_2 \times \mathbf{B}_2^-$-orbit closures are obtained from the $\mathbf{B}_2 \times \mathbf{B}_2$-orbits by replacing r by rw_0 and then shifting the orbit by w_0, we see that in $\mathbf{Mat}_2$ there are two $\mathbf{B}_2 \times \mathbf{B}_2^-$-stable divisors; one of them is $\mathbf{GL}_2 \times \mathbf{GL}_2$-stable but the other is not. Let us denote the $\mathbf{GL}_2 \times \mathbf{GL}_2$-stable divisor by D_2 and the other one by D_1. Clearly, we have $D_2 = D_2'$ but $D_1 \neq D_1'$. Note that D_1 is the unique color of $\mathbf{Mat}_2$. In summary, the orbit structure of $\mathbf{Mat}_2$ is as follows: there are exactly three $\mathbf{GL}_2 \times \mathbf{GL}_2$-orbits; their closures are linearly ordered. The open $\mathbf{GL}_2 \times \mathbf{GL}_2$-orbit, that is $\mathbf{GL}_2$, is equal to $\mathbf{B}_2\mathbf{B}_2^- \sqcup \mathbf{B}_2 w_0 \mathbf{B}_2^-$, where $w_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The codimension one $\mathbf{GL}_2 \times \mathbf{GL}_2$-orbit contains four $\mathbf{B}_2 \times \mathbf{B}_2^-$-orbits. The 0-matrix is the unique closed $\mathbf{B}_2 \times \mathbf{B}_2^-$-orbit.

Next, we will describe the valuation cone of $\mathbf{Mat}_2$. Let $\mathbf{Z}$ denote the center of $\mathbf{GL}_2$. Let $\mathbf{B}_2'$ denote the Borel subgroup of $\mathbf{SL}_2$ defined by $\mathbf{B}_2' := \mathbf{B}_2 \cap \mathbf{SL}_2$. Finally, let $\mathbf{T}_2'$ denote the maximal diagonal torus of $\mathbf{SL}_2$. In light of Proposition 10.4, we know that the cone of $\mathbf{GL}_2 \times \mathbf{GL}_2$-valuations on $k(\mathbf{GL}_2)$ is given by

$$\mathcal{V}(\mathbf{GL}_2) = \mathcal{V}(\mathbf{SL}_2) \times \mathbf{H\ddot{O}}(\mathbf{Z}) \subset \mathbf{H\ddot{O}}(\mathbf{T}_2') \times \mathbf{H\ddot{O}}(\mathbf{Z}).$$

Let α denote the unique simple root of the triplet $(\mathbf{SL}_2, \mathbf{B}_2', \mathbf{T}_2')$. The antidominant Weyl chamber of $\mathbf{SL}_2$ is the one dimensional strongly convex polyhedral cone spanned by the vector $-\alpha^\vee$. Therefore, the valuation cone $\mathcal{V}(\mathbf{GL}_2)$ is given by the half space $\mathbb{Q}^+(-\alpha^\vee, 0) + \mathbb{Q}(0,1)$ in $\mathbf{H\ddot{O}}(\mathbf{T}_2') \times \mathbf{H\ddot{O}}(\mathbf{Z}) \cong \mathbb{Q}^2$.

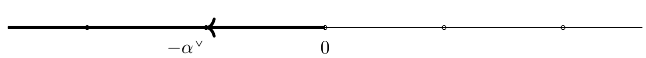

Figure 10.2. The cone of invariant valuations of $\mathbf{SL}_2$.

We go back to the divisors in $\mathbf{Mat}_2$. We notice that D_2 is a principal divisor defined by the polynomial equation $\det x = 0$ for $x \in \mathbf{Mat}_2$. This is the unique $\mathbf{GL}_2 \times \mathbf{GL}_2$-stable divisor in $\mathbf{Mat}_2$. Note that the determinant function is not only a regular function on $\mathbf{GL}_2$ but also the unique positive generator of the character group of the center of $\mathbf{GL}_2$. We will compute $\mathsf{e}(v_{D_2})$ in $\mathbf{H\ddot{O}}(\mathbf{GL}_2) \cong \mathbf{H\ddot{O}}(\mathbf{T}_2)$.

Recall that we have a short exact sequence

$$1 \to k^* \to k(\mathbf{Mat}_2)^{(\mathbf{B}_2 \times \mathbf{B}_2^-)} \to \ddot{\mathbf{O}}(\mathbf{T}_2) \to 0.$$

A straightforward computation shows that the group of $\mathbf{B}_2 \times \mathbf{B}_2^-$-semiinvariants of $\mathbf{Mat}_2$ are generated by $1, x_{2,2}, \det \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}$, where $x_{i,j}$ $(1 \leqslant i, j \leqslant 2)$ is the (i,j)-th coordinate function on $\mathbf{Mat}_2$. Therefore, the vector $\mathtt{e}(v_{D_2})$ is determined by how v_{D_2} evaluates on the semiinvariants $x_{2,2}$ and $\det X$, where $X := \begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{bmatrix}$. Since $x_{2,2}$ is a unit in the localization $k[\mathbf{Mat}_2]_{(\det X)}$, we see that $v_{D_2}(x_{2,2}) = 0$. At the same time, it is obvious that $v_{D_2}(\det X) = 1$. We now look at the splitting $\varphi : \mathbf{H}\ddot{\mathbf{O}}(\mathbf{T}_2) \to \mathbf{H}\ddot{\mathbf{O}}(\mathbf{T}_2') \times \mathbf{H}\ddot{\mathbf{O}}(\mathbf{Z}) \cong \mathbb{Q}^2$ more carefully. On one hand, $\mathbf{H}\ddot{\mathbf{O}}(\mathbf{Z})$ is generated by the dual of central character, $\det X$. On the other hand, $\mathbf{H}\ddot{\mathbf{O}}(\mathbf{T}_2')$ is generated by the dual of the character, $x_{1,1}x_{2,2}^{-1} \in \ddot{\mathbf{O}}(\mathbf{T}_2')$. Clearly, this character is equal to $x_{2,2}^{-2} \det X$. Therefore, in terms of the lattice generators of $\mathbf{H}\ddot{\mathbf{O}}(\mathbf{T}_2)$, the map φ is determined by $(x_{2,2})^\vee \mapsto (-2, 1)$ and $(\det X)^\vee \mapsto (0, 1)$. Since $v_{D_2}(x_{2,2}^{-2} \det X) = v_{D_2}(\det X) = 1$, we now see that $\mathtt{e}(v_{D_2})$ corresponds to the vector $(-2, 2) \in \mathbf{H}\ddot{\mathbf{O}}(\mathbf{T}_2') \times \mathbf{H}\ddot{\mathbf{O}}(\mathbf{Z})$. We are now ready to describe the colored cone of $\mathbf{Mat}_2$:

$$\begin{aligned} \mathcal{C}(\mathbf{Mat}_2) &= \mathbb{Q}^+ \mathtt{e}(v_{D_2}) + \mathbb{Q}^+ \mathtt{f}(v_{D_1}) \\ &= \mathbb{Q}^+(-2, 2) + \mathbb{Q}^+(\alpha^\vee, 0). \end{aligned}$$

In Figure 10.3 we depicted the valuation cone and the colored cone of $\mathbf{Mat}_2$.

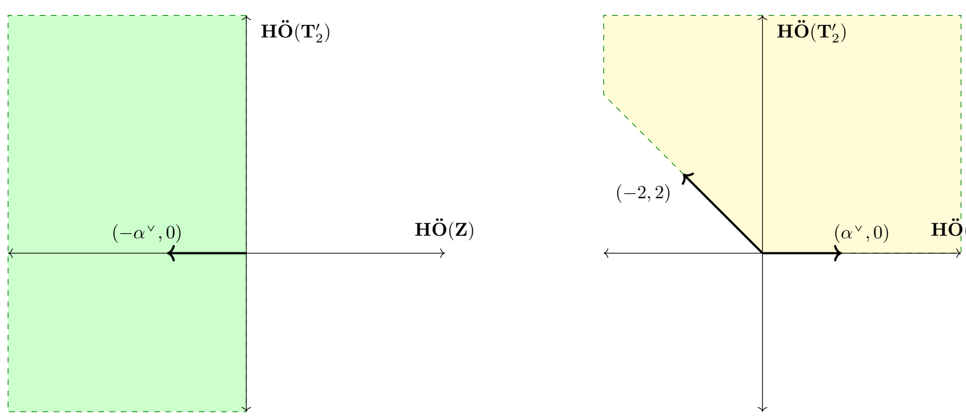

FIGURE 10.3. The valuation cone (on the left) and the strictly convex colored cone (on the right).

The calculation we performed for $\mathbf{Mat}_2$ easily extends to all $\mathbf{Mat}_n$ for $n \in \mathbb{Z}^+$. The strictly convex colored cone of $\mathbf{Mat}_n$ is given by

$$(\mathcal{C}(\mathbf{Mat}_n), \mathcal{F}(\mathbf{Mat}_n)) = (\mathbb{Q}^+\{(\alpha_1^\vee, 0), \ldots, (\alpha_{n-1}^\vee, 0), v\}, \{\overline{\mathbf{B}_n s_{\alpha_i} \mathbf{B}_n^-}\}_{i=1}^{n-1}),$$

where v is a vector that is contained in the relative interior of $\mathcal{V}(\mathbf{Mat}_n)$.

10.4. Symmetric spaces

Recall that a *symmetric space* is a homogeneous space of the form G/K, where K is the fixed subgroup of automorphism $\theta : G \to G$ such that $\theta^2 = id$. The classification of automorphisms of order on algebraic groups has a long history. For G_0 defined over $\mathbb{C}$, this problem is equivalent to classifying the real forms of simple complex Lie algebras and it was solved by É. Cartan. Subsequently, Araki [1] simplified this classification for simple algebraic groups defined over $\mathbb{C}$, and Helminck [37] extended Araki's result over arbitrary algebraically closed fields of characteristic $\neq 2$. In [79], Springer describes an alternative and simple approach of this classification in a way that is suitable for investigating the equivariant embeddings of symmetric spaces.

We note that our previous example can be thought of as a symmetric space. Indeed, the algebraic group G is isomorphic to the homogeneous space $(G \times G)/\mathrm{diag}(G)$, where $\mathrm{diag}(G)$ is the diagonal embedding of G into $G \times G$. The subgroup $\mathrm{diag}(G)$ is precisely the group of θ-fixed elements, where θ is the involution defined by $\theta(x, y) = (y, x)$ for every $(x, y) \in G \times G$. From this viewpoint, the classification of normal reductive monoids is part of the classification of the affine spherical embeddings of symmetric spaces.

Hereafter, for simplicity, here we will focus on the symmetric spaces of a connected complex semisimple algebraic group G. In fact, passing to a simply connected cover does not change the symmetric space. Therefore, we will assume that G is a connected, simply connected, complex semisimple algebraic group.

Let B be a Borel subgroup of G. Then there exists a maximal torus T in B such that $\theta(T) = T$. Indeed, we may take T as a maximal torus of the intersection $B \cap \theta(B)$. We set $T_0 := \{t \in T : \ \theta(t) = t\}$ and $T_1 := \{t \in T : \ \theta(t) = t^{-1}\}$. Then the map $m : T_0 \times T_1 \to T$, $(a, b) \mapsto ab$ is an isogeny. The subtorus T_1 is called a θ-split torus. We now fix a maximal θ-split torus $S \subseteq G$. Then any maximal torus that contains S is θ-stable [88, §1]. We fix one such a maximal torus T. Then the quotient $\overline{S} := S/S \cap G^\theta$ for G/G^θ is what is the maximal torus T for G. Observe that $S \cap G^\theta = \{s \in S : \ s^2 = e\}$. Every element of finite order is semisimple, hence, a central element in characteristic 0. Since G is semisimple, we see that $\overline{S}$ is a finite quotient of S. The character lattice of the torus $\overline{S}$ is given by

$$\ddot{\mathbf{O}}(\overline{S}) = \{\chi - \theta(\chi) : \ \chi \in \ddot{\mathbf{O}}(T)\}.$$

We set

$$\overline{\mathtt{R}} := \{\beta - \theta(\beta) \in \ddot{\mathbf{O}}(\overline{S}) : \ \beta \in \mathtt{R}_{(G,T)}\}. \tag{10.6}$$

It turns out that $\overline{\mathtt{R}}$ is a root system, which is naturally associated with the pair $(G/G^\theta, S/S\cap G^\theta)$. We will explain this in more detail.

Since $\overline{S}$ is a quotient of S, the character group of $\overline{S}$ is a sublattice of the character group of S. In other words, we have $\ddot{\mathbf{O}}(\overline{S}) \subseteq \ddot{\mathbf{O}}(S)$. Let $res : \ddot{\mathbf{O}}(T) \to \ddot{\mathbf{O}}(S)$ denote the homomorphism that is defined by restricting the characters of T onto S. Let $\mathtt{R}_0$ denote the roots of (G,T) that are fixed by the induced involution $\tilde{\theta} : \mathtt{R} \to \mathtt{R}$. Let $\mathtt{R}_1 := \mathtt{R}\backslash\mathtt{R}_0$. The image $res(\mathtt{R}_1) \subseteq \ddot{\mathbf{O}}(S)$ turns out to be a not necessarily a reduced root system; this is shown by Richardson in [71, §4].

Definition 10.7. The root system $res(\mathtt{R}_1)$ is called the *restricted root system* of G/K relative to (T,S).

We will see below that $res(\mathtt{R}_1) = \overline{\mathtt{R}}$. Let $\mathtt{S}$ denote the set of simple roots that is defined by $\mathtt{R}^+$, and let $\mathtt{S}_1$ denote the intersection $\mathtt{S}\cap\mathtt{R}_1$. Finally, the set $\overline{\mathtt{S}} := res(\mathtt{S}_1)$ is a set of simple roots for the restricted root system $\overline{\mathtt{R}} = res(\mathtt{R}_1)$. The elements of $\overline{\mathtt{S}}$ are called the *restricted simple roots*. Furthermore, it turns out that [71, §4] the Weyl group of $\overline{\mathtt{R}}$, denoted W_0, is given by

$$W_0 = N_G(S)/Z_G(S),$$

where $N_G(S)$ denotes the normalizer of S in G, and $Z_G(S)$ is the centralizer of S in G. This Weyl group is called the *little Weyl group* of the symmetric space G/K. It naturally acts on $\mathbf{H}\ddot{\mathbf{O}}(G/K)$. We will show below that the valuation cone $\mathcal{V}(G/K)$ is the domain of transitivity for the action of W_0.

Let us explicitly identify the restricted simple roots in terms of the simple roots of (G,B,T), where B is a θ-stable Borel subgroup. To this end, let us set

$$\mathtt{S}_0 := \{\beta_1,\dots,\beta_k\} \ \text{ and } \ \mathtt{S}_1 := \{\alpha_1,\dots,\alpha_j\}.$$

We will denote the corresponding fundamental dominant weights of $\mathtt{R}$ by the sets

$$\{\zeta_1,\dots,\zeta_k\} \ \text{ and } \ \{\varpi_1,\dots,\varpi_j\}.$$

Note that $\tilde{\theta}$ is an involution on the set of indices $\{1,\dots,j\}$. Then, under the induced involution $\tilde{\theta}$, we have $\tilde{\theta}(\varpi_i) = -\varpi_{\tilde{\theta}(i)}$ for $i\in\{1,\dots,j\}$. Let l denote the number of orbits of $\tilde{\theta}$ on $\{1,\dots,j\}$. Then the restricted simple roots are given by

$$res(\alpha_i) = \frac{1}{2}(\alpha_i - \tilde{\theta}(\alpha_i)) \qquad (i\in\{1,\dots,l\}). \tag{10.8}$$

Note that (10.8) proves the equality $res(\mathtt{R}_1) = \overline{\mathtt{R}}$. From now on, we will use the notation $\overline{\alpha_i}$ to denote the restricted simple root $res(\alpha_i)$ for $i\in\{1,\dots,l\}$.

For the sake of giving an 'explicit' example, let us describe the restricted root system of the symmetric space $(G\times G)/\mathrm{diag}(G)$. We set $\mathbf{G} := G\times G$. The involution on $\mathbf{G}$ is given by $\theta : (x,y)\mapsto (y,x)$, where $(x,y)\in G\times G$. The maximal torus $\mathbf{T} := T\times T \subset G\times G$ is θ-stable. The maximal split torus is given by $\mathbf{S} := \{(s,s^{-1}) : \ s\in T\}$. The character group of $\mathbf{T}$ is $\ddot{\mathbf{O}}(T)\times\ddot{\mathbf{O}}(T)$, and the character group of $\mathbf{S}$ is $\{(\chi,-\chi) : \ \chi\in\ddot{\mathbf{O}}(T)\}$. Let

$\{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots of G with respect to (G, B, T). It is now easy to check that the restricted simple roots of $(G \times G)/\mathrm{diag}(G)$ are of the form

$$\overline{(\alpha_i, 0)} = \overline{(0, -\alpha_i)} = \frac{1}{2}(\alpha_i, -\alpha_i) \qquad (i \in \{1, \ldots, l\}). \tag{10.9}$$

We conclude from (10.9) that the restricted root system of $(G \times G)/\mathrm{diag}(G)$ relative to $(T \times T, \mathrm{diag}(T))$ and the ordinary root system of G relative to T are isomorphic root systems.

The anti-dominant Weyl chamber of (G, B, T) is the cone spanned by the negatives of the simple coroots. Then in the notation of the previous paragraph, it is the cone spanned by $\{-\beta_1^\vee, \ldots, -\beta_k^\vee, -\alpha_1^\vee, \ldots, -\alpha_j^\vee\}$ in $\mathbf{H\ddot{O}}(T)$. The anti-dominant Weyl chamber of $(G/G^\theta, S/S \cap G^\theta)$ is given by the image of this cone in $\mathbf{H\ddot{O}}(\overline{S})$. In [89] by Vust and in [43] by Knop, it is shown that this image is equal to the cone spanned by the negatives of the restricted simple coroots.

Proposition 10.10. *Let* $\mathtt{S} := \{\overline{\alpha_1}, \ldots, \overline{\alpha_l}\}$ *denote the set of restricted simple roots of the symmetric space* G/G^θ*. Then the anti-dominant Weyl chamber of* G/G^θ *is given by*

$$\mathcal{C}(G/G^\theta) = \{f \in \boldsymbol{H\ddot{O}}(\overline{S}) : \langle f, \overline{\alpha_i}\rangle \leqslant 0 \textit{ for all } i \in \{1, \ldots, l\}\}.$$

Proof. We will sketch the idea of the proof due to Knop [43]. Let us denote by W_0 the Weyl group of $\overline{\mathtt{R}}$. Then the dominant Weyl chamber as well as the anti-dominant Weyl chamber of $\overline{\mathtt{R}}$ are fundamental domains for the action of W_0 on $\mathbf{\ddot{O}}(\overline{S})$. Now let us denote by $N(\mathbf{\ddot{O}}(\overline{S}))$ the stabilizer in W of the sublattice $\mathbf{\ddot{O}}(\overline{S}) \subset \mathbf{\ddot{O}}(T)$, and let us denote by $W(\mathbf{\ddot{O}}(\overline{S}))$ the group of automorphisms of $\mathbf{\ddot{O}}(\overline{S})$ induced by $N(\mathbf{\ddot{O}}(\overline{S}))$. Then we see that $W_0 \cong W(\mathbf{\ddot{O}}(\overline{S}))$. In [16], Brion shows that there is a subgroup $W_{G/H} \subseteq W(\mathbf{\ddot{O}}(\overline{S}))$ which is generated by reflections and such that $\mathcal{V}(G/K)$ is a domain of transitivity for its action. Let M denote the $\mathbb{C}$-algebra of K-invariant polynomial functions on $\mathrm{Lie}(G)/\mathrm{Lie}(K)$. By [42, Korollar 7.2], we know that M is isomorphic to $W_{G/H}$-invariants in the coordinate ring $\mathbb{C}[S]$. But M is canonically isomorphic to the $\mathbb{C}$-algebra of W_0-invariant polynomial functions on $\mathrm{Lie}(S)$. It follows that $\mathbb{C}[S]^{W_{G/H}} \cong \mathbb{C}[S]^{W_0}$. This argument shows that the groups $W_{G/H}$ and W_0 are isomorphic. In particular, their domain of transitivity are equal. Therefore, $\mathcal{V}(G/K)$ is equal to the anti-dominant Weyl chamber of the restricted root system $\overline{\mathtt{R}}$. $\square$

Finally, we will mention another result of Vust which describes the evaluation-of-valuation map on the colors of a symmetric space. A symmetric space G/K is said to *have a hermitian type factor* if the center of the normalizer of K in G has positive dimension.

Proposition 10.11. *(*[89, Proposition 1]*) Let* $\mathtt{S} := \{\overline{\alpha_1}, \ldots, \overline{\alpha_l}\}$ *denote the set of restricted simple roots of the symmetric space* G/G^θ*. Then under the map* $\mathtt{f} : \mathcal{F}(G/K) \to \boldsymbol{H\ddot{O}}(G/K)$ *the colors are mapped onto the negatives of the simple coroots of* $\overline{\mathtt{R}}$*. If* G/K *does not have a hermitian type factor, then* $\mathtt{f}$ *is injective. Otherwise, the fibers of* $\mathtt{f}$ *may contain two colors.*

Finally, we want to mention that it is possible to construct some interesting affine embeddings of symmetric spaces by using reductive monoids as in [22]. There are many open questions in that direction.

Acknowledgments This work is partially supported by the Louisiana Board of Regents grant no. 090ENH-21. We thank the organizers of the webinar series, Selman Akbulut, Eylem Zeliha Yıldız, Üstün Yıldırım, and Emily Windes. These notes were prepared with the help of the following expository resources: [19, 31, 60, 61, 62, 65, 70, 82]. We thank the authors of these texts. Finally, we thank the referees. Their comments and suggestions greatly improved the quality of our article.

11. Problems and Solutions

11.1. The first exercise set and hints for their solutions

(1) Let $\mathbb{G}_m$ denote the multiplicative group of nonzero complex numbers, $\mathbb{C}^*$.
  (a) Show that $\mathbb{G}_m$ is a complex algebraic group.
  (b) Show directly by using the definition of a complete variety that G is not a complete variety.

(2) The *n-th complex unitary group*, denoted by $\mathbf{U}_n(\mathbb{C})$, is defined by

$$\mathbf{U}_n(\mathbb{C}) = \{g \in \mathbf{GL}_n(\mathbb{C}) : \; g^{-1} = \bar{g}^{\top}\},$$

where the bar on a matrix $g = (g_{i,j})$ means that we are taking the complex-conjugates of the entries of that matrix. (The "$\top$" stands for the transpose.)
  (a) Show that $\mathbf{U}_1(\mathbb{C})$ is a real Lie group.
  (b) Show that $\mathbf{U}_1(\mathbb{C})$ is not a complex Lie group.
  (c) Show that for every $n \in \mathbb{Z}^+$ the n-th complex unitary group is not a complex Lie group but a compact real Lie group.
  (d) Show that the complexification of $\mathbf{U}_n(\mathbb{C})$ is $\mathbf{GL}_n(\mathbb{C})$.

(3) The *n-th complex special unitary group*, denoted by $\mathbf{SU}_n(\mathbb{C})$, is defined by

$$\mathbf{SU}_n(\mathbb{C}) = \{g \in \mathbf{U}_n(\mathbb{C}) : \; \det g = 1\}.$$

  (a) For $n \in \mathbb{Z}^+$, determine the anti-holomorphic involution $\tau : \mathbf{SL}_n(\mathbb{C}) \to \mathbf{SL}_n(\mathbb{C})$ for which the fixed subgroup is $\mathbf{SU}_n(\mathbb{C})$, In other words, show that

$$\mathbf{SL}_n(\mathbb{C})^{\tau} := \{g \in \mathbf{SL}_n(\mathbb{C}) : \; \tau(g) = g\} = \mathbf{SU}_n(\mathbb{C}).$$

  Conclude that the complexification of $\mathbf{SU}_n(\mathbb{C})$ is $\mathbf{SL}_n(\mathbb{C})$.
  (b) For $n = 2$, determine the equations of the six dimensional differentiable manifold $\mathbf{SL}_2(\mathbb{C})$ viewed as a submanifold of $\mathbf{Mat}_2(\mathbb{C}) \cong \mathbb{C}^4$.
  (c) Determine the cotangent bundle of $\mathbf{SU}_2(\mathbb{C})$. Compare it with $\mathbf{SL}_2(\mathbb{C})$.

(4) Let A be denote the image in $\mathbb{C}^2$ of the holomorphic map $s \mapsto (e^s, e^{is})$, $s \in \mathbb{C}$.
  (a) Show that A is a one-dimensional complex Lie group.
  (b) Show that A is not a complex algebraic variety.

(5) Recall that an abelian variety is a complete algebraic group. Recall also that the complexification of every compact real matrix Lie group has the structure of a complex reductive algebraic group. Explain why we cannot readily apply this method to conclude that every compact complex Lie group has the structure of an abelian variety.

Solutions/Hints:

(1) (a) The set $\mathbb{C}\backslash\{0\}$ is open in $\mathbb{C}$. Its coordinate ring is obtained from the coordinate ring $\mathbb{C}[x]$ of $\mathbb{C}$ by the localization at $x = 0$. In other words, $\mathbb{G}_m$ is a principal open subset of $\mathbb{C}$, therefore, it is an irreducible algebraic subvariety of $\mathbb{C}$. To show that $\mathbb{G}_m$ is an algebraic group, consider the ring the quotient ring $\mathbb{C}[x, y]/(xy - 1)$, where x and y are two algebraically independent variables over $\mathbb{C}$. Check that R is a finitely generated integral domain. Thus, the prime spectrum $X := \mathrm{Spec}(R)$ is a reduced, integral, and separated scheme of finite type over $\mathbb{C}$. In other words, X is an affine variety. Note that the geometric points of X, that is the spectrum of maximal ideals of R, can be identified with the set $\{(x, y) \in \mathbb{C}^2 : \ xy = 1\}$. We can identify $\mathbb{G}_m = \{s \in \mathbb{C} : \ s \neq 0\}$ as an algebraic variety with $\{(x, y) \in \mathbb{C}^2 : \ xy = 1\}$ via the map $\varphi : \mathbb{G}_m \to \{(x, y) \in \mathbb{C}^2 : \ xy = 1\}, \ x \mapsto (x, y)$, where $y \in \mathbb{C}$ is such that $xy = 1$. (The map φ is an invertible polynomial map, where the inverse map is given by the first projection.) Note that X has a natural algebraic group structure, where the group operations on $\{(x, y) \in \mathbb{C}^2 : \ xy = 1\}$ are given by

$$\begin{aligned} m : X \times X &\to X \\ ((x, y), (x', y')) &\mapsto (xx', yy') \end{aligned}$$

and

$$\begin{aligned} i : X &\to X \\ (x, y) &\mapsto (y, x). \end{aligned}$$

Clearly m and i are polynomial maps on X. Since φ is an abstract group isomorphism and an invertible polynomial map, the algebraic group structure on X gives the algebraic group structure on $\mathbb{G}_m$. (b) Let X denote, as in the previous part, the curve $\{(x, y) \in \mathbb{C}^2 : \ xy = 1\} \subset \mathbb{C} \times \mathbb{C}$. Then $X \cap (\mathbb{C}^* \times \mathbb{C})$ is a closed subset of $\mathbb{C}^* \times \mathbb{C}$. Show that the second projection $\pi : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}$ maps X to a non-closed subset of $\mathbb{C}$.

(2) (a) We leave this to the reader.
(b) The unit circle is not a complex manifold.
(c) The determinant function $\det : \mathbf{Mat}_n(\mathbb{C}) \to \mathbb{C}$ is a holomorphic map. Assume that $\mathbf{U}(n)$ is a complex manifold. Then we find $\det(\mathbf{U}(n)) = \mathbf{U}(1)$ is a complex manifold, which is absurd. We leave checking that $\mathbf{U}(n)$ is a Lie group to the reader. To see that $\mathbf{U}(n)$ is compact, use the standard metric Check that $\det(\mathbf{U}(n)) = \mathbf{U}(1)$. By the previous part we obtain a contradiction. To show that $\mathbf{U}(n)$ is compact, first, observe that the map $g \mapsto \sqrt{\bar{g}^\top g}$ defines a norm function on $\mathbf{Mat}_n(\mathbb{C})$. Verify that $\mathbf{U}(n)$ is bounded under this norm. Use the same norm function to conclude that $\mathbf{U}(n)$ is a closed subset as well. Finally use the Heine-Borel Theorem.
(d) The map $\sigma : \mathbf{GL}_n(\mathbb{C}) \to \mathbf{GL}_n(\mathbb{C})$, $\sigma(g) := (\bar{g}^\top)^{-1}$ is an anti-holomorphic involutory automorphism of $\mathbf{GL}_n(\mathbb{C})$. The fixed point set of σ is exactly $\mathbf{U}_n(\mathbb{C})$.

(3) (a) Use the solution of part (d) of the previous problem.
(b) Left to the reader.
(c) Left to the reader.

(4) (a) The map $s \mapsto (e^s, e^{is})$ is an injective holomorphic map; its image is a complex manifold. It is easy to check that the image of this map is a group with respect to entrywise multiplication.
(b) Verify that there is no polynomial function in two variables $f(x, y) \in \mathbb{C}[x, y]$ such that $f(e^s, e^{is}) = 0$ for every $s \in \mathbb{C}$.

(5) A compact complex Lie group of dimension > 1 does not admit a holomorphic embedding into matrices. Therefore, we cannot readily apply the technique of polynomial invariants to separate orbits to conclude that a compact complex Lie group is an algebraic group. As we mentioned in the notes, a compact complex Lie group has the structure of an algebraic variety if and only if it admits an embedding into a projective space via its meromorphic functions.

11.2. The second exercise set and hints for the solutions

(1) Show that the dimension of a complex semisimple algebraic group is at least 3.

(2) Explain why a finite group is an algebraic group. Show that a finite group consists of semisimple elements only.

(3) Calculate the group of algebraic characters of the following groups:
(a) The cyclic group of order n.
(b) The symmetric group S_n.
(c) The alternating subgroup $A_n \subseteq S_n$.
(d) The n-dimensional complex torus.
(e) The general linear group $\mathbf{GL}_2(\mathbb{C})$.
(f) The special linear group $\mathbf{SL}_2(\mathbb{C})$.
(g) $P = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & f & g \end{bmatrix} \in \mathbf{GL}_3(\mathbb{C}) \right\}.$
(h) $Q = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \in \mathbf{GL}_4(\mathbb{C}) \right\}.$
(i) $H = \left\{ \begin{bmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g \end{bmatrix} \in \mathbf{GL}_3(\mathbb{C}) \right\}.$

(4) Which of the groups in the previous exercise are simple, semisimple, or reductive?

(5) Determine the Picard group of $\mathbb{P}^n$.

(6) Determine the Picard group of the Grassmann variety of 2 planes in $\mathbb{C}^4$.

(7) Show that $\mathbf{GL}_n(\mathbb{C})$ is not simply connected in the Hausdorff topology. Show directly that $\mathbf{GL}_n(\mathbb{C})$ is not algebraically simply connected.

(8) Let G be a complex semisimple algebraic group. Determine the equivariant Picard group $\mathrm{Pic}_{G\times G}(G)$, where $G \times G$ acts on G as follows:

$$(g,h)\cdot x = gxh^{-1} \qquad (g,h,x \in G).$$

Solutions/Hints:

(1) The complex semisimple algebraic groups are parametrized by their root systems. The number of simple roots in the root system of a complex semisimple algebraic group is equal to the dimension of the maximal torus. The root system with the smallest number of roots is $\mathtt{A}_1$; the corresponding group is either $\mathbf{SL}_2(\mathbb{C})$ (the simply connected one) or $\mathbf{PGL}_2(\mathbb{C})$ (the adjoint one). Both of these groups are three dimensional.

(2) For the first part show that a finite set of points is an algebraic set. For the second part of the question, a semisimple element means a diagonalizable element. Let m be the order of an element $g \in G$ in a finite group. Then the characteristic polynomial is given by $x^m - 1$.

(3) The image of an algebraic character is a subgroup of $\mathbb{C}^*$, hence, it is an abelian group. If G is a finite group, then show that the image of an algebraic character of G is contained in the unit circle in $\mathbb{C}^*$. In particular, the image of an algebraic character is an abelian group.
(a) If G is a cyclic group, show that the group of algebraic characters of G is isomorphic to G itself.
(b) (Hint for $n \geqslant 5$.) Let $\varphi : S_n \to \mathbb{C}^*$ be a homomorphism. The restriction of φ to any (normal) subgroup $H \subseteq S_n$ is a homomorphism of that subgroup. Since A_n is a normal subgroup, and since $\varphi|_{A_n}$ is trivial by part (c), we see that φ factors through S_n/A_n which is a cyclic group of order two. Conclude from part (a) that S_n ($n \geqslant 5$) has only two algebraic characters. Can you identify the nontrivial homomorphism?
(c) (Hint for $n \geqslant 5$.) The alternating group A_n, where $n \geqslant 5$, has no nontrivial normal subgroups. This means any homomorphism from A_n is either an isomorphism or it is trivial. Since the image of an algebraic group homomorphism is abelian, the only algebraic character of A_n ($n \geqslant 5$) is the trivial character.
(d) (Hint for $n = 1$. The general case is obtained from this case by direct products.) Let $\varphi : \mathbb{C}^* \to \mathbb{C}^*$ be an algebraic group homomorphism. Show that there exists $m \in \mathbb{Z}$ such that $\varphi(z) = z^m$ for every $z \in \mathbb{C}^*$. Therefore, φ is uniquely determined by the integer m. Check that the assignment $\varphi \mapsto m$ is an isomorphism between the group of algebraic characters of $\mathbb{C}^*$ and the additive group of integers.
(f) The special linear group $\mathbf{SL}_2(\mathbb{C})$ is a semisimple group. We showed in the text that such a group does not have any nontrivial characters.

(e) Let $\mathbf{Z}$ denote the center of $\mathbf{GL}_2(\mathbb{C})$. Observe that the center of $\mathbf{SL}_2(\mathbb{C})$ is an order two cyclic subgroup of $\mathbf{Z}$. Convince yourself $\mathbf{GL}_2(\mathbb{C})$ is isomorphic to $(\mathbf{SL}_2(\mathbb{C}) \times \mathbf{Z})/\mathbf{Z}_0$, where $\mathbf{Z}_0$ is the diagonal copy the center of $\mathbf{SL}_2(\mathbb{C})$ in $\mathbf{SL}_2(\mathbb{C}) \times \mathbf{Z}$. By part (f) we know that $\mathbf{SL}_2(\mathbb{C})$ has no nontrivial algebraic characters. Therefore, an algebraic character of $\mathbf{GL}_2(\mathbb{C})$ is completely determined by its center, which is isomorphic to $\mathbb{C}^*$. It follows from part (d) that the group of algebraic characters of $\mathbf{GL}_2(\mathbb{C})$ is isomorphic to $(\mathbb{Z}, +)$.
(g) Since P contains the Borel subgroup of upper triangular matrices, it is a parabolic subgroup. Therefore, it has a decomposition of the form $L\mathcal{R}_u(P)$, where L is a maximal reductive group, and $\mathcal{R}_u(P)$ is its unipotent radical. Since unipotent subgroups have no nontrivial algebraic characters, we see that the group of algebraic characters of P is isomorphic to the group of algebraic characters of L. In our case, L is of the form $\mathbf{GL}_1(\mathbb{C}) \times \mathbf{GL}_2(\mathbb{C})$; the factors are diagonal square matrices that we can fit into P. Recall that an algebraic group G can be written as a semidirect product of By using the previous parts, we see that the group of algebraic characters of P is isomorphic to $\mathbb{Z}^2$.
(h) Similarly to the previous part, in this part also, we observe that the group of algebraic characters of Q is isomorphic to the group of algebraic characters of $\mathbf{GL}_1(\mathbb{C}) \times \mathbf{GL}_2(\mathbb{C}) \times \mathbf{GL}_1(\mathbb{C})$, which is isomorphic to $\mathbb{Z}^3$.
(i) Show that H is isomorphic to the direct product $\mathbf{G}_a \times \mathbf{G}_m$. Therefore, its group of algebraic characters is $\mathbb{Z}$.

(4) Left to the reader.

(5) Projective spaces are Grassmannians. The solution of this exercise follows from the solution of the next one.

(6) The Grassmann variety of k-dimensional vector subspaces of $\mathbb{C}^n$ is isomorphic to $\mathbf{SL}_n(\mathbb{C})/\mathbf{P}$, where $\mathbf{P}$ is the maximal parabolic subgroup,

$$\mathbf{P} = \left\{ \begin{bmatrix} A & * \\ 0 & B \end{bmatrix} : \ A \in \mathbf{GL}_k(\mathbb{C}),\ B \in \mathbf{GL}_{n-k}(\mathbb{C}),\ \det(A)\det(B) = 1 \right\}.$$

Use Theorem 3.8 to show that the Picard group of $\mathbf{SL}_n(\mathbb{C})/\mathbf{P}$ is isomorphic to the group of algebraic characters of $\mathbf{P}$. Show that this character group is isomorphic to $\mathbb{Z}$.

(7) Assume $\mathbf{GL}_n(\mathbb{C})$ is simply connected. Then the homomorphism $\det : \mathbf{GL}_n(\mathbb{C}) \to \mathbb{G}_m$ lifts to a surjective homomorphism to the universal cover of $\mathbb{G}_m$ which is $\mathbb{G}_a$. This would mean that $\mathbf{GL}_n(\mathbb{C})$ has a nontrivial normal unipotent subgroup which is absurd.

(8) Since G is isomorphic to $(G \times G)/\mathrm{diag}(G)$, by Theorem 3.6 we see that $\mathrm{Pic}_{G\times G}(G) \cong \ddot{\mathbf{O}}(G)$. Since G is semisimple, $\ddot{\mathbf{O}}(G)$ is trivial.

11.3. The third exercise set and hints for the solutions

For the next four problems the following notation is fixed: let G denote $\mathbf{SL}_3(\mathbb{C})$, let B denote the Borel subgroup of upper triangular matrices in G, and let T denote the maximal diagonal torus in B.

(1) Compute the weight lattice of (G, B, T).

(2) Compute the Weyl group W of (G, T).

(3) Let $\{\varpi_1, \varpi_2\}$ be the set of fundamental dominant weights of (G, B, T). Compute the W-orbits of ϖ_1 and $\varpi_1 + \varpi_2$.

(4) Let W denote the vector space of 3×3 skew-symmetric matrices on which G acts by the congruence action:
$$g \cdot A = gAg^{\top} \qquad (g \in G, A \in W).$$
Show that W is an irreducible representation of G. Find the highest weight of W.

For the next three problems the following notation is fixed: let G denote $\mathbf{SL}_2(\mathbb{C})$, let B denote the Borel subgroup of upper triangular matrices in G, and let T denote the maximal diagonal torus in B.

(5) Find the character group $\ddot{\mathbf{O}}(B)$ of B.

(6) Let V denote the two dimensional complex vector space of 2×1 column matrices on which G acts by matrix multiplication. Compute the weight lattice of V in $\ddot{\mathbf{O}}(B)$.

(7) For each $n \in \mathbb{Z}^+$ find an irreducible rational representation V of G such that $\dim V = n$. Show that in each dimension, there is exactly one irreducible representations of G.

Solutions/Hints:

(1) Let (G, B, T) denote $(\mathbf{SL}_3(\mathbb{C}), \mathbf{B}_3(\mathbb{C}), \mathbf{T}_3(\mathbb{C}))$, where $\mathbf{B}_3(\mathbb{C})$ is the Borel subgroup of upper triangular matrices in $\mathbf{SL}_3(\mathbb{C})$ and $\mathbf{T}_3(\mathbb{C})$ is the maximal diagonal torus in $\mathbf{B}_3(\mathbb{C})$.

The root system $\mathtt{R}$ of (G, B, T) is the following set of algebraic characters:
$$\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\} \subset \ddot{\mathbf{O}}(T) \cong \mathbb{Z}^2,$$
where α_1 and α_2 are the characters
$$\alpha_1(\mathrm{diag}(t_1, t_2, t_3)) = t_1/t_2 \quad \text{and} \quad \alpha_1(\mathrm{diag}(t_1, t_2, t_3)) = t_2/t_3,$$
where $\mathrm{diag}(t_1, t_2, t_3) \in T$.

We will view $\mathtt{R}$ inside the euclidean space $\ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, and we will use the additive notation. The latter convention conforms with the standard practice of constructing a root system by using the differentials of algebraic characters.

Thus, if $\varepsilon_1, \varepsilon_2$, and ε_3 denote the standard coordinate functionals on $\mathrm{Lie}(T)$, then we can identify $d\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $d\alpha_2 = \varepsilon_2 - \varepsilon_3$. Although this transition is not necessary, it makes the computations more visual; we can view $\mathtt{R}$ as a set of vectors on the hyperplane $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ in $\ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. In other words, $\mathtt{R}$ can be identified with the following configuration of six vectors in the plane:

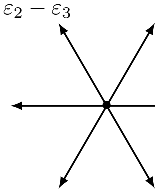

FIGURE 11.1. The root system of $\mathbf{SL}_3$.

Since $\mathbf{SL}_3(\mathbb{C})$ is a simply connected algebraic group, its weight lattice is identical with the following (abstractly defined) sublattice of $\ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$\{\chi \in \ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q} : \langle \chi, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for } i \in \{1,2\}\} \cong \ddot{\mathbf{O}}(T). \tag{11.1}$$

Here $\langle , \rangle$ is the pairing defined by $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\beta, \beta)$ for every $\alpha, \beta \in \ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $(,)$ is an inner product that is invariant under all reflection operators defined by the elements of $\mathtt{R}$. (In fact, if we identify the plane in Figure 11.1 by $\mathbb{Q}^2$, where α_1 lies on the x-axis, then the usual dot product can be used in place of $(,)$). The weight lattice (11.1) is generated by the fundamental dominant weights ϖ_1 and ϖ_2 which are defined by

$$2\frac{(\varpi_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have $\varpi_i \perp \alpha_j$ for $1 \leqslant i \neq j \leqslant 2$. This means that, the fundamental dominant weight ϖ_1 (resp. ϖ_2) is contained in the hyperplane H_2 (resp. in H_1), which is drawn by the dashed line in Figure 11.2 .

Since $(\varpi_1, \alpha_1) = \frac{1}{2}(\alpha_1, \alpha_1) > 1$, the angle between ϖ_1 and α_1 is acute. This implies that the angle between ϖ_1 and α_1 is $\pi/6$. Likewise, the angle between ϖ_2 and α_2 is $\pi/6$. It remains to calculate the length of ϖ_i ($i \in \{1,2\}$). The equation $(\varpi_1, \alpha_1) = \frac{1}{2}(\alpha_1, \alpha_1)$ is equivalent to $|\varpi_1||\alpha_1| \cos(\pi/6) = \frac{1}{2}|\alpha_1|^2$, or,

$$\sqrt{3}|\varpi_1| = |\alpha_1|.$$

Of course, in $\mathtt{A}_2$, all roots have equal length. Therefore, we find

$$\sqrt{3}|\varpi_1| = |\alpha_1| = |\alpha_2| = \sqrt{3}|\varpi_2|.$$

We place these vectors in Figure 11.3.

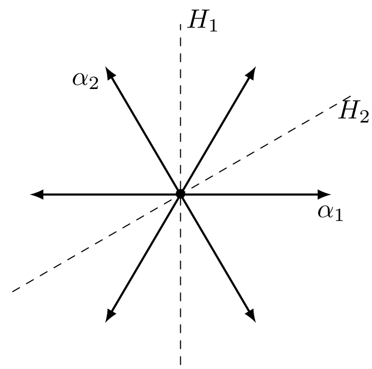

FIGURE 11.2. The fundamental hyperplanes of $\mathbf{SL}_3$.

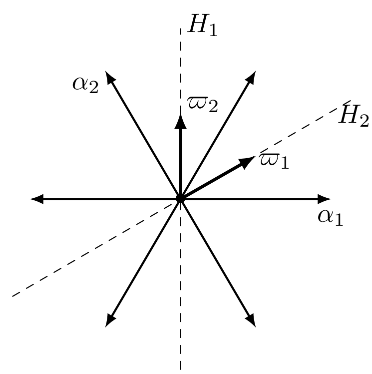

FIGURE 11.3. The fundamental weights of $\mathbf{SL}_3$. The ratio of lengths $|\alpha_1|/|\varpi_1|$ is $\sqrt{3}$.

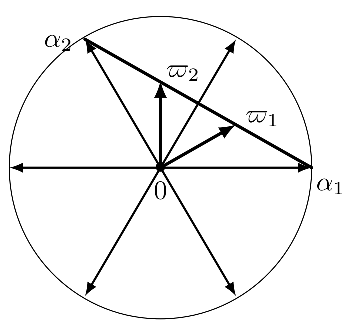

FIGURE 11.4. The fundamental circle of $\mathbf{SL}_3$.

At this point, we gathered sufficient information to draw the lattice spanned by ϖ_1 and ϖ_2 in $\ddot{\mathbf{O}}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. But we want to further analyze the elementary geometry of these vectors in relation with the root vectors. So, we draw a circle of radius $|\alpha_1|$ centered at the origin in Figure 11.4, and then we connect the roots α_1 and α_2 by a line segment.

(2) The answer is S_3. The generators are the reflection operators with respect hyperplanes H_1 and H_2 that we draw in Figure 11.2.

(3) Answers:
  (a) $S_3 \cdot \varpi_1 = \{\varpi_1, -\varpi_2, \frac{1}{2}(-\varpi_1 + \varpi_2)\}$.
  (b) $S_3 \cdot (\varpi_1 + \varpi_2) = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\}$.

(4) Assume towards a contradiction that W is not irreducible. Since it is a three dimensional vector space, there exists a one dimensional subrepresentation of W. Now show that $\mathbf{SL}_3(\mathbb{C})$-action on skew-symmetric 3×3 matrices does not fix a line in W. This contradiction shows that W is an irreducible $\mathbf{SL}_3(\mathbb{C})$-module. Alternatively, first show that W is isomorphic to the second exterior power of $\mathbb{C}^3$, and the action on W is equivalent to the induced action of $\mathbf{SL}_3(\mathbb{C})$ on $\bigwedge^2 \mathbb{C}^3$. Notice also that $\bigwedge^2 \mathbb{C}^3$ is isomorphic ($\mathbf{SL}_3(\mathbb{C})$-equivariantly) to the dual of the standard representation $\mathbb{C}^3$ of $\mathbf{SL}_3(\mathbb{C})$. In other words, the contragradient representation of $\mathbf{SL}_3(\mathbb{C})$ on $\mathbb{C}^3$ is given by $\bigwedge^2(\mathbb{C}^3)$. Since the highest weight of the former representation is ϖ_1, the highest weight of its dual is given by ϖ_2.

Let G denote $\mathbf{SL}_2(\mathbb{C})$. Let B denote the Borel subgroup of upper triangular matrices in G and let T denote the maximal diagonal torus in B.

(5) The group of algebraic characters of B is equal to the group of algebraic characters of its maximal torus, which is $T \cong \mathbb{G}_m$. Therefore, $\ddot{\mathbf{O}}(B) = \mathbb{Z}$.

(6) The coordinate ring of $\mathbb{C}^2$ is $R := \mathbb{C}[x, y]$. It decomposes as $\bigoplus_{n \geqslant 0} R_n$, where R_n is the subspace that consists of homogeneous degree n polynomials in the variables x and y. Check that R_n is an irreducible representation of highest weight n and $\dim R_n = n + 1$. Therefore, the weight lattice of $\mathbb{C}^2$ in $\ddot{\mathbf{O}}(B)$, which is generated by all of the highest weights that appear in its coordinate ring, is equal to $\ddot{\mathbf{O}}(B)$.

(7) The solution of this problem follows from the solution of (2).

11.4. The fourth exercise set and hints for the solutions

(1) Consider the natural action of $\mathbf{GL}_n$ on the $n - 1$ dimensional projective space $\mathbb{P}^{n-1}$. Let $\lambda : \mathbb{G}_m \to \mathbf{T}$ be a 1-psg of the n-dimensional diagonal torus $\mathbf{T} \subseteq \mathbf{GL}_n$. What can you say about the fixed points of the action of $\lambda(\mathbb{G}_m)$ on $\mathbb{P}^{n-1}$? If $n = 2$, then how many $\lambda(\mathbb{G}_m)$-orbits are there in $\mathbb{P}^1$?

(2) Prove that $\mathbb{P}^1$ is isomorphic to $\mathbf{GL}_2/\mathbf{B}$, where $\mathbf{B}$ is the Borel subgroup of upper triangular matrices in $\mathbf{GL}_2$.

(3) Prove that $\mathbf{GL}_2/\mathbf{B} \cong \mathbf{SL}_2/\mathbf{B}_0$, where $\mathbf{B}_0 = \mathbf{B} \cap \mathbf{SL}_2$.

(4) Let $\mathbf{T}_0$ be a maximal torus in $\mathbf{B}_0$, where $\mathbf{B}_0 \subset \mathbf{SL}_2$ is as in the previous example.
   (a) Show that $\mathbf{SL}_2/\mathbf{T}_0$ is an affine variety.
   (b) Show that $\mathbf{SL}_2/\mathbf{T}_0$ is a spherical variety.

(5) Let $\mathbf{B}_0$ be as in the previous example.
   (a) Compute the unipotent radical $\mathbf{U}_0$ of $\mathbf{B}_0$, and show that $\mathbf{SL}_2/\mathbf{U}_0$ is a spherical variety.
   (b) Show that $\mathbf{SL}_2/\mathbf{U}_0$ is a quasi-affine variety.
   (c) Explain why $\mathbf{SL}_2/\mathbf{U}_0$ is not an affine variety.

(6) Show that $\mathbf{SL}_n/\mathbf{U}$, where $\mathbf{U}$ is the unipotent radical of a Borel subgroup of $\mathbf{SL}_n$ is a spherical variety.

(7) Let $\mathbf{T}_0$ denote the maximal diagonal torus of $\mathbf{SL}_3$.
   (a) Prove that $\mathbf{SL}_3/\mathbf{T}_0$ is an affine variety.
   (b) Prove that $\mathbf{SL}_3/\mathbf{T}_0$ is not a spherical variety.

Solutions/Hints:

(1) If λ is a *regular 1-psg*, that is to say λ is of the form $\lambda(t) = (t^{a_1}, \dots, t^{a_n})$, for some integers $a_1, \dots, a_n$ such that $a_1 < a_2 < \cdots < a_n$, then the fixed point set of $\lambda(\mathbb{G}_m)$ has exactly $n+1$-fixed points. Check that, for $n = 2$, a regular 1-psg $\lambda : \mathbb{G}_m \to T$ has only two fixed points and an open orbit in $\mathbb{P}^1$.

(2) Let x denote the point of $\mathbb{P}^1$ that is represented by the 2×1 matrix $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Compute the stabilizer of x in $\mathbf{GL}_2$.

(3) $\mathbf{SL}_2$ is the semisimple part of $\mathbf{GL}_2$. The difference between these two groups is the central torus of $\mathbf{GL}_2$. At the same time, the center of a reductive group G is contained in every maximal torus T in G. Therefore, the difference between the Borel subgroups $\mathbf{B} \subset \mathbf{GL}_2$ and $\mathbf{B}_0 \subset \mathbf{SL}_2$ is that central torus. It follows that the quotients $\mathbf{GL}_2/\mathbf{B}$ and $\mathbf{SL}_2/\mathbf{B}_0$ are equal.

(4) Let $\mathbf{T}_0$ denote the maximal diagonal torus in the Borel subgroup $\mathbf{B}_0$ of upper triangular matrices in $\mathbf{SL}_2$.
   (a) We begin by noticing that $\dim \mathbf{SL}_2/\mathbf{T}_0 = 3 - 1 = 2$. The coordinate ring R of $\mathbf{SL}_2$ is given by $k[x_1, x_2, x_3, x_4]/(\det X - 1)$, where $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. The right action of $\mathbf{T}_0$ on $\mathbf{SL}_2$ induces the action $\mathbf{T}_0 : R$ that is given by $t \cdot x_i = tx_i$ for $i \in \{1, 2\}$ and $t \cdot x_j = t^{-1}x_j$ for $j \in \{3, 4\}$, where we view t as the diagonal matrix $\operatorname{diag}(t, t^{-1})$. Then the $\mathbf{T}_0$-invariants of R is a finitely generated k-algebra, denoted by $R^{\mathbf{T}_0}$. The k-algebra generators of $R^{\mathbf{T}_0}$ are given by x_1x_4 and x_2x_3. Let Y denote the affine variety $Y := \operatorname{Spec} k[x_1x_4, x_2x_3] = \operatorname{Spec}(R^{\mathbf{T}_0})$. Clearly, we have $\dim Y = 2$. Now since $k[G/H] = k[G]^H$, $R^{\mathbf{T}_0}$ is the coordinate ring of $\mathbf{SL}_2/\mathbf{T}_0$. At the same time,

since $\dim Y = 2 = \dim \mathbf{SL}_2/\mathbf{T}_0$, we see that the spectrum of the ring of invariants is exactly the quotient $\mathbf{SL}_2/\mathbf{T}_0$.

(b) An affine G-variety X is spherical if and only if X/U is a toric variety [19, Theorem 2.14] In our case, $X = \mathbf{SL}_2/\mathbf{T}_0$. Since X/U is isomorphic to $\mathbf{SL}_2/\mathbf{B}_0$, and since $\mathbb{P}^1 \cong \mathbf{SL}_2/\mathbf{B}_0$ is a toric variety for the action of $\mathbf{T}_0$, $\mathbf{SL}_2/\mathbf{T}_0$ is a spherical variety.

(5) We leave part (a) to the reader. For part (b), observe that $\mathbf{SL}_2$ acts on $\mathbb{C}^2\backslash\{0\}$. The stabilizer of the point $(1,0)$ is exactly U. Verify that $\mathbf{SL}_2 \cdot (1,0) = \mathbb{C}^2\backslash\{0\}$. Finally show that $\mathbb{C}^2\backslash\{0\}$ is not an affine variety but a quasi-affine variety.

(6) Hint: The Bruhat-Chevalley decomposition.

(7) The solution of the problem uses arguments that are similar to those used in the solution of Problem 4.

12. Appendix: k-structures

Many of the concepts that we discussed in these lectures are sensitive to the choice of the underlying field of definitions. In this section we will briefly review the concepts that make it possible to transfer (mostly by restring or extending) geometric structures defined over different fields. Our basic reference for this section is Springer's book [80], especially its Chapter 11.

For now, let k be an arbitrary field. Let K be a field extension of k. Let V be a vector space over K. Then V becomes a vector space over k as well. A subspace V_0 of V is called a *k-form* of V if there is a k-basis of V_0 that is also a K-basis of V.[29] For example, a polynomial ring $k[x_1,\dots,x_n]$, viewed as a k-vector space, is a k-form of $K[x_1,\dots,x_n]$.

Now let V_0 be a k-form of V, and let U be a K-vector subspace of V. Then U is said to be *defined over k with respect to* V_0 if it is spanned by some vectors from V_0. In this case, $U_0 := V_0 \cap U$ becomes a k-form of U.

Example 12.1. Let M_0 be a subset of k^n defined by a set of polynomials S from $k[x_1,\dots,x_n]$ as follows:

$$M_0 := \{\mathbf{a} \in k^n : \ f(\mathbf{a}) = 0 \ \text{ for every } f \in S\}. \tag{12.2}$$

Since S is also a subset of $K[x_1,\dots,x_n]$, we can consider the set

$$M := \{\mathbf{b} \in K^n : \ f(\mathbf{b}) = 0 \ \text{ for every } f \in S\}.$$

Let I_0 (resp. I) denote the ideal of all polynomials $f \in k[x_1,\dots,x_n]$ (resp. $\tilde{f} \in K[x_1,\dots,x_n]$) such that $f(\mathbf{a}) = 0$ for every $\mathbf{a} \in M_0$ (resp. $\tilde{f}(\mathbf{b}) = 0$ for every $\mathbf{b} \in M$). Clearly, I_0 is a k-vector space, and I is a K-vector space. It is easy to verify that $I = K \otimes_k I_0$. In other words, I_0 is a k-form of I. For this reason, M is said to be *defined over k*. Also, M_0 is called a *k-form of M*.

[29] Equivalently, the inclusion map $V_0 \hookrightarrow V$ extends to an isomorphism $V_0 \otimes_k K \to V$.

Next, we will briefly discuss the role of the Galois group of the field extension K/k. For simplicity, let us assume that K is a finite extension of k. Let us denote by $\mathrm{Aut}(K/k)$ the group of all field automorphisms $\sigma : K \to K$ such that $\sigma(x) = x$ for every $x \in k$. If the k-vector space dimension of K is equal to the order of $\mathrm{Aut}(K/k)$, then the field extension K/k is called a *Galois extension*. In this case, $\mathrm{Aut}(K/k)$ is called the *Galois group of the extension, K/k*. For example, the field $\mathbb{C}$ is a Galois extension of $\mathbb{R}$ with $\mathrm{Aut}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, where the nontrivial automorphism is given by the map $z \mapsto \overline{z}$, $z \in \mathbb{C}$.

We proceed with the assumption that K/k is a Galois extension. Let Γ denote $\mathrm{Aut}(K/k)$. Let V_0 be a k-vector space. We set $V := K \otimes_k V_0$. Then there is a natural action of $\Gamma \times V \to V$, $(\sigma, a \otimes v_0) \mapsto \sigma(a \otimes v_0)$, where

$$\sigma(a \otimes v_0) := \sigma(a) \otimes v_0 \qquad (a \in K, v_0 \in V_0,\ \sigma \in \Gamma).$$

Example 12.3. We continue the previous example. Let K/k be a Galois extension with Galois group Γ. Then Γ operates on $K[x_1, \dots, x_n]$ in an obvious manner by acting on the coefficients of polynomials. Let $S' \subset K[x_1, \dots, x_n]$ be a set of polynomials. Then the algebraic set $M = \{\mathbf{b} \in K^n : \ f(\mathbf{b}) = 0 \ \text{ for every } f \in S'\}$ is defined over k if and only if the ideal I generated by S' in $K[x_1, \dots, x_n]$ is stable under the action of Γ. Let I_0 denote the ideal $I \cap k[x_1, \dots, x_n]$. Then $M_0 = \{\mathbf{a} \in k^n : \ f(\mathbf{a}) = 0 \ \text{ for every } f \in I_0\}$ is a k-form of M.

Let $\varphi : M \to Y$ be an algebraic map between two algebraic sets $M \subset K^n$ and $Y \subset K^m$. We say that φ is *defined over k*, or is a *k-morphism* if $\varphi^* k[Y] \subset k[M]$. Here, $k[Y]$ (resp. $k[M]$) is the algebra of k-valued functions on Y (resp. on M).

We now define the forms of an extension. Let K be a Galois extension of k such that $K \subseteq \bar{k}$, where $\bar{k}$ is an algebraic closure of k. Let M_0 and M be as in Example 12.2. A *K-form of M_0* is an algebraic set $Y \subset K^n$ such that

(1) Y is defined over k,
(2) Y is isomorphic to M via a K-morphism.

It is natural to wonder how many such Y's are there? Equivalently, is there a good description of the set of k-isomorphism classes of K-forms of M_0? Let $\varPhi(K/k, M_0)$ denote this set.

Proposition 12.4. *Let $Aut_K(M)$ denote the group of K-algebra automorphisms of $K[M]$. Then Γ naturally acts on $Aut_K(M)$. Furthermore, there is a bijection between $\varPhi(K/k, M_0)$ and the group cohomology set, $H^1(\Gamma, Aut_K(M))$.*

A proof of the previous proposition can be found in [80, Proposition 11.3.3].

Example 12.5. Let us determine how many complex forms of the affine real space $\mathbb{R}^n$ there are. Let Γ denote $\mathbb{Z}/2\mathbb{Z}$, the Galois group of $\mathbb{C}/\mathbb{R}$. Clearly, a $\mathbb{C}$-form of $\mathbb{R}^n$ is given by $\mathbb{C}^n$. Let A denote the coordinate ring of $\mathbb{C}^n$. It is naturally isomorphic to the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$. The $\mathbb{C}$-algebra automorphism group of A is the complex general linear group $\mathbf{GL}_n(\mathbb{C})$. It is well-known that $H^1(\Gamma, \mathbf{GL}_n(\mathbb{C})) = 1$. Therefore, there is only one complex form of $\mathbb{R}^n$; it is given by $\mathbb{C}^n$.

References

[1] S. Araki, *On root systems and an infinitesimal classification of irreducible symmetric spaces*, J. Math. Osaka City Univ., **13**, (1962), 1–34.

[2] R. Avdeev, *Strongly solvable spherical subgroups and their combinatorial invariants*, Selecta Math. (N.S.), **21(3)**, (2015), 931–993.

[3] R. S. Avdeev, *On solvable spherical subgroups of semisimple algebraic groups,* Trans. Moscow Math. Soc., (2011), 1–44.

[4] F. Bien and M. Brion, *Automorphisms and local rigidity of regular varieties*, Compositio Math., **104(1)**, (1996), 1–26.

[5] E. Bifet, C. De Concini, and C. Procesi, *Cohomology of regular embeddings*, Adv. Math., **82(1)**, (1990), 1–34.

[6] A. Borel, *Linear algebraic groups*, Springer-Verlag, New York, (1991).

[7] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Springer-Verlag, Berlin, (2002).

[8] L. Brambila-Paz and A. Rittatore, *The endomorphisms monoid of a homogeneous vector bundle*, In Algebraic monoids, group embeddings, and algebraic combinatorics, Springer, New York, (2014).

[9] P. Bravi, *Wonderful varieties of type E*, Represent. Theory, **11**, (2007), 174–191.

[10] P. Bravi and S. Cupit-Foutou, *Classification of strict wonderful varieties,* Ann. Inst. Fourier (Grenoble), **60(2)**, (2010), 641–681.

[11] P. Bravi and D. Luna, *An introduction to wonderful varieties with many examples of type* F_4, J. Algebra, **329**, (2011), 4–51.

[12] P. Bravi and G. Pezzini, *Wonderful varieties of type D*, Represent. Theory, **9**, (2005), 578–637.

[13] P. Bravi and G. Pezzini, *Wonderful subgroups of reductive groups and spherical systems,* J. Algebra, **409**, (2014), 101–147.

[14] M. Brion, *Quelques propriétés des espaces homogènes sphériques,* Manuscripta Math., **55(2)**, (1986), 191–198.

[15] M. Brion, *Groupe de Picard et nombres caractéristiques des variétés sphériques,* Duke Math. J., **58(2)**, (1989), 397–424.

[16] M. Brion, *Vers une généralisation des espaces symétriques,* J. Algebra, **134(1)**, (1990), 115–143.

[17] M. Brion, *Equivariant Chow groups for torus actions,* Transform. Groups, **2(3)**, (1997), 225–267.

[18] M. Brion, *Variétiés sphériques,* http://www-fourier.univ-grenoble-alpes.fr/~mbrion/spheriques.pdf, (1997).

[19] M. Brion, *Introduction to actions of algebraic groups,* Les cours du CIRM, **1(1)**, (2010), 1–22.

[20] M. Brion and F. Pauer, *Valuations des espaces homogènes sphériques,* Comment. Math. Helv., **62(2)**, (1987), 265–285.

[21] M. Brion and A. Rittatore, *The structure of normal algebraic monoids,* Semigroup Forum, **74(3)**, (2007), 410–422.

[22] M. B. Can, R. Howe, and L. Renner, *Monoid embeddings of symmetric varieties,* Colloq. Math., **157(1)**, (2019), 17–33.

[23] M. B. Can and L. E. Renner, *Bruhat-Chevalley order on the rook monoid,* Turkish J. Math., **36(4)**, (2012), 499–519.

[24] J. B. Carrell, *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties,* In Algebraic groups and their generalizations, American Mathematical Society, Providence, RI, (1994).

[25] C. Chevalley, *Theory of Lie groups. I*, Princeton University Press, Princeton, NJ, (1999).

[26] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, American Mathematical Society, Providence, RI, (2011)

[27] S. Cupit-Foutou, *Wonderful varieties A geometrical realization*, https://arxiv.org/abs/0907.2852.

[28] V. I. Danilov, *The geometry of toric varieties,* Uspekhi Mat. Nauk, **33**, 247, (1978), 85–134.

[29] W.R. Ferrer Santos and A. Rittatore, *Actions and invariants of algebraic groups*, CRC Press (2017).

[30] W. Fulton, *Introduction to toric varieties*, Princeton University Press, (1993).

[31] J. Gandini, *Embeddings of spherical homogeneous spaces,* Acta Math. Sin. (Engl. Ser.), **34(3)**, (2018), 299–340.
[32] J. Gandini and G. Pezzini, *Orbits of strongly solvable spherical subgroups on the flag variety,* J. Algebraic Combin., **47(3)**, (2018), 357–401.
[33] R. Goodman and N. R. Wallach, *Symmetry, representations, and invariants*, Springer, (2009).
[34] F. D. Grosshans, *Algebraic homogeneous spaces and invariant theory*, Springer-Verlag, (1997).
[35] A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondamental*, Lecture Notes in Mathematics, Vol. 224. Springer-Verlag, (1971).
[36] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, (1977).
[37] A. G. Helminck, *Algebraic groups with a commuting pair of involutions and semisimple symmetric spaces,* Adv. in Math., **71(1)**, (1988), 21–91.
[38] G. Hochschild, *The structure of Lie groups*, Holden-Day, Inc., (1965).
[39] J. E. Humphreys, *Linear algebraic groups*, Springer-Verlag, New York-Heidelberg, (1975).
[40] J. C. Jantzen, *Representations of algebraic groups*, American Mathematical Society, Providence, RI, (2003).
[41] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras,* Invent. Math., **53(2)**, (1979), 165–184.
[42] F. Knop, *Weylgruppe und Momentabbildung,* Invent. Math., **99(1)**, (1990), 1–23.
[43] F. Knop, *The Luna-Vust theory of spherical embeddings,* In Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), Manoj Prakashan, Madras, (1991).
[44] F. Knop, *On the set of orbits for a Borel subgroup,* Comment. Math. Helv., **70(2)**, (1995), 285–309.
[45] F. Knop, *Automorphisms, root systems, and compactifications of homogeneous varieties,* J. Amer. Math. Soc., **9(1)**, (1996),153–174.
[46] F. Knop, H. Kraft, D. Luna, and T. Vust, *Local properties of algebraic group actions,* In Algebraische Transformationsgruppen und Invariantentheorie, Birkhäuser, Basel, (1989).
[47] F. Knop, H. Kraft, and T. Vust, *The Picard group of a G-variety,* In Algebraische Transformationsgruppen und Invariantentheorie, Birkhäuser, Basel, (1989).
[48] I. Losev, *Demazure embeddings are smooth,* Int. Math. Res. Not. IMRN, **(14)**, (2009), 2588–2596.
[49] D. Luna, *Toute variété magnifique est sphérique,* Transform. Groups, **1(3)**, (1996), 249–258.
[50] D. Luna, *Variétés sphériques de type A,* Publ. Math. Inst. Hautes Études Sci., **(94)**, (2001), 161–226.
[51] D. Luna, *Sur les plongements de Demazure,* J. Algebra, **258(1)**, (2002), 205–215.
[52] D. Luna and Th. Vust, *Plongements d'espaces homogènes,* Comment. Math. Helv., **58(2)**, (1983), 186–245.
[53] T. Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups,* J. Math. Soc. Japan, **31(2)**, (1979), 331–357.
[54] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, (1989).
[55] D. Montgomery and L. Zippin, *Topological transformation groups*, Robert E. Krieger Publishing Co., Huntington, N.Y., (1974).
[56] D. Mumford, *Abelian varieties*, Hindustan Book Agency, New Delhi, (2008).
[57] T. Oda, *Convex bodies and algebraic geometry*, Springer-Verlag, Berlin, (1988).
[58] D. I. Panyushev, *Complexity and rank of actions in invariant theory,* J. Math. Sci. (New York), **95(1)**, (1999), 1925–1985.
[59] S. Payne, *Equivariant Chow cohomology of toric varieties,* Math. Res. Lett., **13(1)**, 2006, 29–41.
[60] N. Perrin, *Sanya lectures, geometry of spherical varieties,* Acta Math. Sin. (Engl. Ser.), **34(3)**, (2018), 371–416.
[61] G. Pezzini, *Lectures on spherical and wonderful varieties,* Les cours du CIRM, **1(1)**, (2010), 33–53.
[62] G. Pezzini, *Lectures on wonderful varieties,* Acta Math. Sin. (Engl. Ser.), **34(3)**, (2018), 417–438.
[63] V. L. Popov, *Picard groups of homogeneous spaces of linear algebraic groups and one-dimensional homogeneous vector fiberings,* Izv. Akad. Nauk SSSR Ser. Mat., **38**, (1974), 294–322.

[64] V. L. Popov, *Contractions of actions of reductive algebraic groups,* Mat. Sb. (N.S.), **130**, (1986), 310–334.

[65] M. S. Putcha, *Linear algebraic monoids*, Cambridge University Press, (1988).

[66] L. E. Renner, *Classification of semisimple algebraic monoids*, Trans. Amer. Math. Soc., **292(1)**, (1985), 193–223.

[67] L. E. Renner, *Analogue of the Bruhat decomposition for algebraic monoids,* J. Algebra, 101(2), (1986), 303–338.

[68] L. E. Renner, *Classification of semisimple varieties,* J. Algebra, **122(2)**, (1989), 275–287.

[69] L. E. Renner, *An explicit cell decomposition of the wonderful compactification of a semisimple algebraic group,* Canad. Math. Bull., **46(1)**, (2003), 140–148.

[70] L. E. Renner, *Linear algebraic monoids*, Springer-Verlag, Berlin, (2005).

[71] R. W. Richardson, *Orbits, invariants, and representations associated to involutions of reductive groups,* Invent. Math., **66(2)**, (1982), 287–312.

[72] R. W. Richardson and T. A. Springer, *The Bruhat order on symmetric varieties,* Geom. Dedicata, **35(1-3)**, (1990), 389–436.

[73] R. W. Richardson and T. A. Springer, *Complements to, "The Bruhat order on symmetric varieties"* Geom. Dedicata, **49(2)**, (1994), 231–238.

[74] A. Rittatore, *Algebraic monoids and group embeddings,* Transform. Groups, **3(4)**, (1998), 375–396.

[75] A. Rittatore, *Very flat reductive monoids,* Publ. Mat. Urug., **9**, (2001), 93–121.

[76] M. Rosenlicht, *Some rationality questions on algebraic groups,* Ann. Mat. Pura App., **43(1)**, (1957), 25–50.

[77] J. P. Serre, *Géométrie algébrique et géométrie analytique,* Ann. Inst. Fourier (Grenoble), **6**, (1955/56), 1–42.

[78] T. A. Springer, *Some results on algebraic groups with involutions,* In Algebraic groups and related topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math. (1985).

[79] T. A. Springer, *The classification of involutions of simple algebraic groups,* J. Fac. Sci. Univ. Tokyo Sect. IA Math., **34(3)**, (1987), 655–670.

[80] T. A. Springer, *Linear algebraic groups*, Birkhäuser Boston, (2009).

[81] H. Sumihiro, *Equivariant completion,* J. Math. Kyoto Univ., **14**, (1974), 1–28.

[82] D. A. Timashev, *Homogeneous spaces and equivariant embeddings*, Springer, Heidelberg, (2011).

[83] E. B. Vinberg, *Complexity of actions of reductive groups,* Funktsional. Anal. i Prilozhen., **20(1)**, 96, (1986), 1–13.

[84] E. B. Vinberg, *The asymptotic semigroup of a semisimple Lie group,* In Semigroups in algebra, geometry and analysis (Oberwolfach, 1993), de Gruyter, Berlin, (1995).

[85] E. B. Vinberg, *On reductive algebraic semigroups,* In Lie groups and Lie algebras, E. B. Dynkin's Seminar, Amer. Math. Soc., Providence, RI, (1995).

[86] J. von Neumann, *Die Einführung analytischer Parameter in topologischen Gruppen,* Ann. of Math. (2), **34(1)**, (1933),, 170–190.

[87] V. E. Voskresenskiĭ, *Picard groups of linear algebraic groups*, In Studies in Number Theory, No. 3 (Russian) 7–16 Izdat. Saratov. Univ., Saratov, (1969).

[88] T. Vust, *Opération de groupes réductifs dans un type de cônes presque homogènes,* Bull. Soc. Math. France, **102**, (1974), 317–333.

[89] T. Vust, *Plongements d'espaces symétriques algébriques, une classification,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **17(2)**, (1990), 165–195.

[90] *Picard groups and fundamental groups of connected algebraic groups*, Mathematics Stack Exchange, https://math.stackexchange.com/questions/1999823/picard-groups-and-fundamental-groups-of-connected-algebraic-groups, (2016).

TULANE UNIVERSITY, NEW ORLEANS, LOUISIANA
Email address: mahirbilencan@gmail.com