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An introduction to the Chekanov torus

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Abstract. This is an introductory article which arose from expanded notes for a set of talks given in the GGTI Online Seminars organized by the Gökova Geometry and Topology Institute.

We give a mostly self-contained and elementary introduction to the Chekanov torus in $\mathbb{C}^2$, i.e. the first example of a Lagrangian torus in $(\mathbb{R}^4, \omega_0)$ which cannot be mapped to a product torus by a symplectomorphism. Our exposition emphasizes the relationship with Hamiltonian torus actions and symplectic reduction, and our goal is to explain how the Chekanov torus can be constructed in more general toric symplectic manifolds. Furthermore, we give a detailed introduction to the technique of versal deformations, which we use, together with the displacement energy of product tori, to prove that the Chekanov torus is not symplectomorphic to a product torus.

1. Introduction

1.1. A classification question in dimension two

Let $\mathcal{C}$ be a closed embedded smooth curve in the plane $\mathbb{R}^2$, i.e. a submanifold diffeomorphic to the circle. We are interested in classifying such curves up to applying area-preserving diffeomorphisms of $\mathbb{R}^2$. One obvious invariant of this classification question is the area contained by $\mathcal{C}$. More precisely, we know that the complement of $\mathcal{C}$ has one bounded component, say $U \subset \mathbb{R}^2$, and we can measure its Euclidean area,

$$\operatorname{area} \mathcal{C} = \operatorname{area} U = \int_U dx \wedge dy = \frac{1}{2} \int_{\mathcal{C}} x dy - y dx. \tag{1}$$

In fact, two curves $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{R}^2$ are equivalent up to area-preserving diffeomorphism if and only if they contain the same area. Thus our first classification question is settled.

Let us formalize this question in terms of symplectic geometry. We replace area-preserving diffeomorphisms of $\mathbb{R}^2$ by diffeomorphisms preserving the two-form $\omega_0 = dx \wedge dy$. These are the diffeomorphisms which preserve area *and orientation*. They are the *symplectomorphisms* of $(\mathbb{R}^2, \omega_0)$,

$$\operatorname{Symp}(\mathbb{R}^2, \omega_0) = \{\phi \in \operatorname{Diff}(\mathbb{R}^2) \mid \phi^* \omega_0 = \omega_0\}. \tag{2}$$

In terms of symplectic geometry, curves in the plane are *Lagrangian submanifolds*. This means that they have half the dimension of the ambient manifold and that the symplectic

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form vanishes on their tangent space. By the same token as above, *Lagrangian circles in the plane are classified by the area they contain.* In other words, we can take the round circles in the plane as models, meaning that every Lagrangian circle $\mathcal{C}$ in the plane is of the form

$$S^1(a) = \{\pi(x^2 + y^2) = a\}, \quad a = \operatorname{area} \mathcal{C}, \tag{3}$$

after applying a symplectomorphism.

1.2. A classification question in dimension four

Let us now ask the same question in dimension four. The symplectic manifold we consider is

$$\mathbb{R}^4 = \{(x_1, y_1, x_2, y_2)\}, \quad \omega_0 = dx_1 \wedge dy_1 + dx_2 \wedge dy_2. \tag{4}$$

The symplectomorphisms are again defined as the group of diffeomorphisms which preserve ω_0. The symplectic manifold $(\mathbb{R}^4, \omega_0)$ can be seen as the product of $(\mathbb{R}^2, \omega_0)$ with itself, where the symplectic form on the product of two symplectic manifolds is the sum of symplectic forms. This means that we obtain a natural family of Lagrangian submanifolds by taking the product $\mathcal{C}_1 \times \mathcal{C}_2$ of smooth embedded closed curves $\mathcal{C}_1 \subset \{(x_1, y_1)\}$, $\mathcal{C}_2 \subset \{(x_2, y_2)\}$ in the factors. These submanifolds are diffeomorphic to tori, and hence we will refer to them as *Lagrangian tori* from now on. For convenience, we can again put these Lagrangian submanifolds into the normal form given by products of round circles,

$$T(a_1, a_2) = \{\pi(x_1^2 + y_1^2) = a_1,\, \pi(x_2^2 + y_2^2) = a_2\}, \quad a_i = \operatorname{area} \mathcal{C}_i. \tag{5}$$

Indeed, we can achieve this normal form by applying symplectomorphisms in the factors of $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$. Members of the two-parameter family of tori $T(a_1, a_2)$ are called *product tori.* In light of the classification in dimension two, it is natural to ask

Question 1.1. *Is every Lagrangian torus $L \subset \mathbb{R}^4$ a product torus, up to applying a symplectomorphism?*

This question was answered in the negative by Chekanov, who constructed the first counter-example, now called *Chekanov torus*, in [8]. See also Eliashberg–Polterovich [14]. Our goal is to give a detailed description of the Chekanov torus and discuss a symplectic invariant of Lagrangian submanifolds one can use to distinguish it from the product tori.

1.3. Hamiltonian torus actions

Note that the circles (3) are the orbits of the *harmonic oscillator.* By this, we mean the orbits of the vector field X_H defined by *Hamilton's equation*

$$dH = -\omega_0(X_H, \cdot) \tag{6}$$

for the Hamiltonian $H(x, y) = \pi(x^2 + y^2)$. Indeed, this implicit equation defines the rotational vector field $X_H = 2\pi(x\partial_y - y\partial_x)$ and hence its orbits are concentric circles. In terms of classical mechanics, this vector field defines the flow in the phase space of a point particle in one dimension submitted to a (suitably normalized) square potential. Hence the name *harmonic oscillator.* Note that the flow of X_H is periodic of unit period and hence it induces a so-called *Hamiltonian group action* by the circle $S^1 = \mathbb{R}/\mathbb{Z}$. In higher dimensions, one can make a similar construction. Indeed, let us define the pair of Hamiltonians

$$H_1(x,y) = \pi(x_1^2 + y_1^2), \quad H_2(x,y) = \pi(x_2^2 + y_2^2) \tag{7}$$

on $(\mathbb{R}^4, \omega_0)$. The Hamiltonian circle actions associated to H_1, H_2 are given by rotation in the coordinate planes $\{(x_1, y_1)\}$ and $\{(x_2, y_2)\}$, respectively. Furthermore, the Hamiltonians H_1, H_2 *(Poisson-)commute*, meaning their respective Hamiltonian vector fields X_{H_1}, X_{H_2} satisfy $\omega(X_{H_1}, X_{H_2}) = 0$. Since they commute, the two circle actions can be combined to a Hamiltonian action of the two-torus on $\mathbb{R}^4$, see also Definition 2.27. The point here is that the orbits of the group action are precisely the product tori $T(a_1, a_2)$. Thinking slightly more geometrically, we obtain a singular fibration $(H_1, H_2)\colon \mathbb{R}^4 \to \mathbb{R}^2_{\geqslant 0}$ over the non-negative quadrant with product tori as fibres. Since the product tori are Lagrangian submanifolds, this is an example of a (singular) *Lagrangian torus fibration.* Readers who are more familiar with the language of integrable systems may think of (H_1, H_2) as an integrable system and of the torus orbits as Liouville tori. This kinship between torus actions and Lagrangian tori will play a central role in our exposition of the Chekanov torus. In fact, we will view the Chekanov torus as fibre of an *exotic* Lagrangian torus fibration on the complement $\mathbb{R}^4 \setminus P$ of the Lagrangian plane

$$P = \{(x, y, x, -y) \in \mathbb{R}^4 \,|\, (x, y) \in \mathbb{R}^2\}. \tag{8}$$

We construct this exotic Lagrangian torus fibration by an explicit deformation of the standard fibration in certain symplectic quotients. Since we carry out this construction in $\mathbb{R}^4$, we do everything by hand and refrain from making reference to the general theory of toric manifolds except in Section 6.

1.4. Outline

The main goal of this expository paper is to give an elementary description of the Chekanov torus, highlighting its relationship with symplectic reduction and Hamiltonian torus actions. Very little a priori knowledge of symplectic geometry is required since we discuss all relevant concepts before using them. Our exposition is largely self-contained except for

(1) Some references in §2.3 to concepts from Morse theory which we do not introduce;
(2) The discussion of relations to almost toric geometry in §3.4. See §1.5 for a discussion of references on this topic;
(3) The computation of the displacement energy of product tori in $\mathbb{R}^{2n}$, see Example 5.5;
(4) Section 6 on toric symplectic manifolds, where we only give an overview.

This paper roughly splits into two parts.

The first part consists of Section 2 giving an introduction to Hamiltonian circle actions and symplectic reduction, and of Section 3, where we define the Chekanov torus using the techniques from Section 2. The main theme of the first part is to exhibit the relationship between Lagrangian tori on the one hand and Hamiltonian torus actions on the other.

In fact, we will modify the standard toric structure (7) on $\mathbb{R}^4$ to get a Lagrangian torus fibration on the complement of a Lagrangian plane $P \subset \mathbb{R}^4$ (see (8)) having Chekanov tori as some of its fibres. For one, this approach is naturally compatible with the use of versal deformations. Secondly, the same method readily generalizes to toric symplectic manifolds of any dimension and yields an exotic copy of the Chekanov torus in many cases. The latter generalization will be outlined in Section 6.

The second part of the paper consists of an introduction to versal deformations in Section 4 and a short introduction to displacement energy in Section 5. Combining these two ingredients yields a proof of the fact that the Chekanov torus is exotic. The last section is an outlook on the case of general toric symplectic manifolds and as such is less detailed than the rest of the paper. However, we will see that the construction of the Chekanov torus, as well as the invariant used to prove its exoticity, can be carried over almost verbatim to toric symplectic manifolds.

As an expository text, this paper contains little to no original research. The only thing that is somewhat new is our approach to the construction of the exotic Lagrangian torus fibration on $\mathbb{R}^4 \setminus P$, although very similar ideas have appeared in a recent preprint by Groman–Varolgunes [17, Chapter 7]. What is interesting here is that the construction we give is related (an probably equivalent) to certain tools from *almost toric geometry*. As we have tried to outline in §3.4, the construction of the exotic Lagrangian torus fibration we give in §3.3 should be the same as performing a *nodal trade* on $\mathbb{C}^2$ followed by a *nodal slide* which sends the node to infinity.

1.5. Further references

For a general introduction to all topics related to Lagrangian torus fibrations and integrable systems we heartily recommend the recent book by Evans [15]. The classical references for Hamiltonian torus actions and toric structures are too many to name. Among them are Audin [3], Canas da Silva [7], Guillemin [18] and the original article by Delzant [12]. For an introduction to integrable systems, offering a complementary point of view on the material we discuss, we recommend the foundational article by Duistermaat [13]. For the construction of the Chekanov torus, see Eliashberg–Polterovich [14] and the articles [8],[10] and [11]. The latter three texts also use and discuss versal deformations. For a slightly different point of view, see Auroux [4]. The construction of the Chekanov torus in toric manifolds given in Section 6 is based on [5]. For more details on almost toric geometry, we again recommend the excellent exposition in [15] as well as the original articles by Symington [24] and Vianna [25], [26].

2. Hamiltonian circle actions

We introduce basic notions from symplectic geometry, Hamiltonian circle actions and symplectic reduction. We especially focus on Hamiltonian circle actions on two-dimensional symplectic manifolds, since this is a special case of toric manifolds. In particular, as a toy case of the Delzant theorem, we classify all Hamiltonian circle actions on surfaces.

2.1. Background: Symplectic manifolds and Hamilton's equation

In §1.1 and §1.2, we have seen the standard symplectic structures on $\mathbb{R}^2$ and $\mathbb{R}^4$. In general, a *symplectic manifold* is a smooth manifold M equipped with a closed differential two-form ω which is *non-degenerate*. Non-degenerate here means that the map

$$TM \to T^*M, \quad X \mapsto \iota(X)\omega = \omega(X, \cdot) \tag{9}$$

is a bundle isomorphism, i.e. an isomorphism of vector spaces $T_pM \to T_p^*M$ in every point $p \in M$. Note that we have already made use of the non-degeneracy in equation (6). If ω_0 was degenerate, the vector field X_H would not be well-defined. In general, a vector field X_H defined by *Hamilton's equation*

$$dH = -\iota(X_H)\omega \tag{10}$$

is called *Hamiltonian vector field* and the smooth function H is called the *Hamiltonian (function)*. By integrating a Hamiltonian vector field, we obtain a *Hamiltonian flow*, denoted by ϕ_t^H. We say that it is *generated by the Hamiltonian H*. In general, we allow for time-dependent Hamiltonians in the definition of Hamiltonian flows, see Definition 5.1. For now, however, it is enough to restrict our attention to *autonomous* Hamiltonian flows, i.e. flows coming from a time-independent Hamiltonian function.

Proposition 2.1. *Hamiltonian flows preserve the symplectic form. Furthermore, autonomous Hamiltonian flows preserve the Hamiltonian function,*

$$(\phi_t^H)^*\omega = \omega, \quad H \circ \phi_t^H = H. \tag{11}$$

The second statement follows directly from Hamilton's equation (10) and the anti-symmetry of the symplectic form, $dH(X_H) = \omega(X_H, X_H) = 0$. The first statement follows from *Cartan's magic formula*, which expresses the Lie derivative of a differential form $\alpha \in \Omega^k(M)$ in terms of the exterior derivative and contraction with the vector field,

$$\mathcal{L}_X\alpha = d\iota(X)\alpha + \iota(X)d\alpha, \tag{12}$$

by which

$$\frac{d}{dt}(\phi_t^H)^*\omega = (\phi_t^H)^*\mathcal{L}_{X_H}\omega = (\phi_t^H)^*(d\iota(X_H)\omega + 0) = (\phi_t^H)^*ddH = 0. \tag{13}$$

2.2. Hamiltonian circle actions

Definition 2.2. *A Hamiltonian system (M, ω, H) is called* Hamiltonian circle action *if its flow is periodic of unit period, i.e. $\phi_1^H = \mathrm{id}$.*

The name comes from the fact that such a Hamiltonian system induces a smooth action of $S^1 = \mathbb{R}/\mathbb{Z}$ which preserves the symplectic form,

$$S^1 \times M \to M, \quad (\theta, p) \mapsto \phi_\theta^H(p). \tag{14}$$

Note that we are a bit sloppy with terminology here, since *Hamiltonian circle action* can mean two things: The Hamiltonian system (M, ω, H) or the circle action itself.

Remark 2.3. (Periods of circle actions) Requiring Hamiltonian circle actions to have unit period is a choice which fixes the normalization of the Hamiltonian function. Indeed, if the Hamiltonian H has period τ, then we can replace it by τH, which has unit period. This explains why we have chosen $H = \pi(x^2 + y^2)$ in §1.1.

Let us now turn to some examples of symplectic manifolds and Hamiltonian circle actions on them.

Example 2.4. (The harmonic oscillator) Let $\mathbb{R}^2$ be equipped with the coordinates $(x, y) \in \mathbb{R}^2$. Throughout the text, we identify $\mathbb{C} = \mathbb{R}^2$ via the coordinate $z = x + iy$. The two form $\omega_0 = dx \wedge dy$ is symplectic. It is closed and, in fact, even exact. For example, one can pick any of the one-forms xdy, $-ydx$ or $\frac{1}{2}(xdy - ydx)$ as primitive. The latter primitive has the distinction of being invariant under the circle action $z \mapsto e^{2\pi i\theta}z$. The form ω_0 is non-degenerate. In fact, it induces the map $\partial_x \mapsto dy, \partial_y \mapsto -dx$ on $T\mathbb{C} \to T^*\mathbb{C}$ in the standard trivializations. The Hamiltonian $F = \pi(x^2 + y^2) = \pi|z|^2$ defines the Hamiltonian circle action $\phi_\theta^F(z) = e^{2\pi i\theta}z$ on $\mathbb{C}$. The image of the Hamiltonian is the non-negative reals, $F(\mathbb{C}) = \mathbb{R}_{\geqslant 0}$. See also Figure 1.

Example 2.5. (Rotation of the sphere) Let $S^2 \subset \mathbb{R}^3$ be the unit two-sphere equipped with the symplectic form ω_{S^2}, which measure Euclidean area. Let $G(x, y, z) = 2\pi z$ be the Hamiltonian given by a rescaling of the projection to the z-axis. The corresponding Hamiltonian flow ϕ_t^G is rotation of unit period around the z-axis. Indeed, one can write the symplectic form out in coordinates,

$$\omega_{S^2} = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy, \tag{15}$$

and check that the Hamiltonian vector field is $X_G = 2\pi(y\partial_x - x\partial_y)$. The image of the Hamiltonian is the closed segment $G(S^2) = [-2\pi, 2\pi]$. See also Figure 1.

Example 2.6. (Rotation of the cylinder) Let $Z = T^*S^1 \cong \mathbb{R} \times S^1$ be the cylinder equipped with the symplectic form $\omega_Z = dx \wedge d\theta$, where $x \in \mathbb{R}$ and $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$. Then the projection $K(\theta, x) = x$ generates the rotation $\phi_t^K(\theta, x) = (\theta + t, x)$ of unit period. The image of the Hamiltonian is the real line $K(Z) = \mathbb{R}$. See also Figure 1.

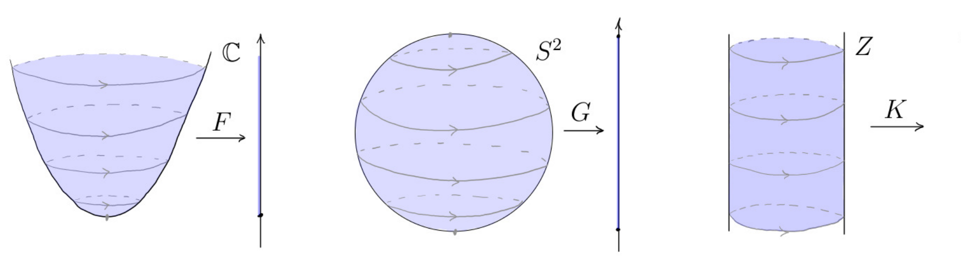

FIGURE 1. The circle actions from Examples 2.4, 2.5 and 2.6

Remark 2.7. Note that in the Examples 2.4, 2.5 and 2.6, the traces of the orbits of the Hamiltonian flow can be read off from the Hamiltonian function without any computation since the orbits are contained in level sets of the Hamiltonian and the ambient space has dimension two. However, this does not fix their parametrization. Indeed, take the example of $(\mathbb{C}, \omega_0)$ and note that all Hamiltonian functions of the form $H_f = f(\pi|z|^2)$ for a smooth function f have orbits contained in the circles $\{|z| = c\}$. Since the Hamiltonian vector field is given by

$$X_{H_f} = f' X_F, \tag{16}$$

for $F = \pi|z|^2$, the only H_f which yield circle actions are given by $H_f = kF + c$, where $k \in \mathbb{Z}$ and $c \in \mathbb{R}$ is a constant. See also Remark 2.15.

Example 2.8. (Linear combinations of harmonic oscillators) Note that all of the above circle actions are defined on two-dimensional symplectic manifolds. Let us now consider $\mathbb{C}^n$ equipped with the standard symplectic form $\omega_0 = \sum_i dx_i \wedge dy_i$. We view this as the product of n copies of $(\mathbb{C}, \omega_0 = dx \wedge dy)$ and thus there are circle actions $F_i = \pi|z_i|^2$ which rotate the i-th coordinate plane and leave the others invariant. We can construct more complicated Hamiltonian systems from the F_i by taking linear combinations

$$F_\xi = \xi_1 F_1 + \ldots + \xi_n F_n, \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n. \tag{17}$$

Note that Hamilton's equation (10) defines an $\mathbb{R}$-linear mapping $H \mapsto X_H$ and hence the corresponding Hamiltonian vector field is $X_{F_\xi} = \sum \xi_i X_{F_i}$. The corresponding Hamiltonian flow is given by

$$\phi_t^{F_\xi}(z_1, \ldots, z_n) = (e^{2\pi i \xi_1 t} z_1, \ldots, e^{2\pi i \xi_n t} z_n). \tag{18}$$

This yields a circle action of unit period if and only if $\xi \in \mathbb{Z}^n$. In §2.6, we will further generalize and geometrize this class of examples. Indeed, we will see in Example 2.28 that there is a *Hamiltonian group action* of the n-torus on $\mathbb{C}^n$. Hamiltonian flows of the type (18) arise by picking one-parameter subgroups of the torus.

2.3. Some Morse theory

We will see that Hamiltonian circle actions are in some sense very rigid. For example, the only compact surface admitting a Hamiltonian circle action is the sphere. In order to prove this, we will make use of Morse theory. Here, we assume that the reader is familiar with the basic ideas of Morse theory. In particular, we will make reference to the *Morse lemma* and *Morse indices* without introducing these concepts. The link between Hamiltonian circle actions and Morse theory comes from a local model of fixed points of the action. Let us first recall Darboux's theorem.

Proposition 2.9. *Let $p \in M$ be a point in a symplectic manifold (M^{2n}, ω). Then there is a chart $\varphi\colon \mathbb{C}^n \supset U \to V$ around p such that $\varphi^*\omega = \omega_0$.*

Sketch of proof. The proof has two steps. First one proves that the symplectic vector space $(T_pM, \Omega = \omega_p)$ obtained by restricting the symplectic form to the tangent space is equivalent to $(\mathbb{C}^n, \omega_0)$ by a symplectic isomorphism of vector spaces. In the second step, one picks a local diffeomorphism $\psi\colon T_pM \supset U' \to V$ for a contractible neighbourhood V of $p \in M$, for example by the exponential map associated to a Riemannian metric. Now there are two different symplectic forms on V, namely $\omega|_V$ and $\psi_*\Omega$. The discrepancy between these two symplectic forms can be corrected by applying the so-called *Moser trick*. The latter can be applied because V is contractible and hence the difference between the symplectic forms is exact, $\omega|_V - \psi_*\Omega = d\alpha$. □

As it turns out, Darboux's theorem can be made equivariant in some cases. Let us return to Hamiltonian circle actions. Let $H\colon M \to \mathbb{R}$ be a Hamiltonian generating a circle action $\theta \mapsto \psi_\theta$ on (M,ω). Furthermore, let $p \in M$ be a fixed point of ψ_θ. By Hamilton's equation (10) this is equivalent to $p \in M$ being a critical point of H.

Proposition 2.10. *Let $p \in M$ be a fixed point of a Hamiltonian circle action generated by $H\colon M \to \mathbb{R}$. Then there is an S^1-invariant neighbourhood V of p and a chart $\varphi\colon \mathbb{C}^n \supset U \to V$ such that*

$$H_\xi = H \circ \varphi = \sum_{i=1}^{n} \xi_i \pi |z_i|^2, \quad \varphi^*\omega = \omega_0, \tag{19}$$

for some $\xi_i \in \mathbb{Z}^n$.

Sketch of proof. The strategy of proof is similar to that used for the Darboux theorem. The circle action can be linearized at the fixed point to yield a circle action on $(T_pM, \Omega = \omega_p)$ and the local diffeomorphism $\psi\colon T_pM \supset U' \to V$ can be chosen to be equivariant, by taking the exponential map of an averaged Riemannian metric for example. As it turns out, the Moser trick can be carried out equivariantly and hence one obtains an S^1-equivariant symplectomorphism $\varphi\colon \mathbb{C}^n \supset U \to V$. In order to see that $H \circ \varphi$ has the desired form (19) all that is left to do is to classify linear Hamiltonian circle actions on symplectic vector spaces. These are always of the form (18) for some $\xi_i \in \mathbb{Z}$ and since φ is a symplectomorphism, we obtain $H \circ \varphi = H_\xi$. □

Let us make two comments about this local normal form. First note that we have encountered Hamiltonians of the type (19) in Example 2.8. Proposition 2.10 thus shows that every Hamiltonian circle action is locally a linear combination of harmonic oscillators around its fixed points. Secondly, the proposition can be interpreted in terms of Morse theory. Indeed note that (19) is a variation on the *Morse Lemma*. However, note that in general H is not Morse properly speaking, since some ξ_i may be zero. However it is always *Morse-Bott*, meaning that it is non-degenerate on the directions transverse to the submanifold of fixed points.

Corollary 2.11. *Hamiltonians generating Hamiltonian circle actions are Morse-Bott and their Morse-Bott indices are even.*

Evenness of the Morse-Bott index follows immediately from the formula (19), since the coordinates $x_i = \mathrm{Re}(z_i)$ and $y_i = \mathrm{Im}(z_i)$ come in pairs which both have the same contribution to the Morse index. Thus Hamiltonians which generate circle actions are *perfect*, meaning that the Morse inequalities are equalities. In terms of Morse homology, the boundary operator vanishes and the Morse chain complex is isomorphic to Morse homology. In particular, every perfect Morse function on a compact connected orientable manifold has a unique local maximum and minimum, since the top and bottom homologies of such spaces have exactly one generator.

Corollary 2.12. *The only compact symplectic surface admitting a non-trivial Hamiltonian circle action is the sphere.*

Proof. First note that a surface admitting a symplectic structure is orientable. Let (Σ, ω_Σ) be a compact symplectic surface with a Hamiltonian circle action generated by H. We may assume that H is Morse, since otherwise it vanishes on a neighbourhood of its degenerate fixed points and hence vanishes on all of Σ. If Σ has genus > 0, then it has non-vanishing first homology. This is in contradiction with the fact that H is Morse and has only even indices. □

2.4. Hamiltonian circle actions on surfaces

Let (Σ, ω) be a connected symplectic surface equipped with a Hamiltonian circle action generated by H. Let us assume that the circle action is *effective*. Recall that a group action is called *effective* if there is no element of the group which acts trivially, except for the neutral element. For us, this just means that we normalize H such that $\phi_1^H = \mathrm{id}$, but $\phi_t^H \neq \mathrm{id}$ for all $0 < t < 1$. Hence assuming effectiveness is not really a restriction. The goal of this subsection is to give a classification of effective Hamiltonian circle actions on connected symplectic surfaces, see Theorem 2.17. This is interesting for us in two respects. First, it serves as a toy case for the general classification of toric symplectic manifolds by Delzant, see Theorem 6.3. Secondly, we heavily rely on this classification for the construction of the Chekanov torus in Section 3. Before going into details about this, let us gain some intuition for Hamiltonian circle actions on surfaces by looking at the symplectic area contained between two orbits. This is closely related to a very old theorem by Archimedes.

Remark 2.13. (Archimedes' theorem) Let $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be the round sphere of unit radius and $Z = \{x^2 + y^2 = 1\} \subset \mathbb{R}^3$ the cylinder obtained as product of the unit circle and the real line. Archimedes observed that the area of slices of equal height

$$S^2 \cap \{a \leqslant z \leqslant b\}, \quad Z \cap \{a \leqslant z \leqslant b\} \tag{20}$$

is equal for all $-1 \leqslant a \leqslant b \leqslant 1$. We will see that this is equivalent to the fact that the Hamiltonian $H(x, y, z) = 2\pi z$ defines a Hamiltonian S^1-action on both surfaces.

Proposition 2.14. *Let (Σ, ω) be a connected symplectic surface and $H \colon \Sigma \to \mathbb{R}$ a proper Hamiltonian. Then (Σ, ω, H) is an effective Hamiltonian circle action if and only if H*

has only critical points of even index and for all $a, b \in H(\Sigma)$ we have

$$\int_{H^{-1}(a,b)} \omega = b - a. \tag{21}$$

Proof. Let us first assume that (Σ, ω, H) is an effective circle action. By Corollary 2.11, the function H is Morse and all its critical points have even index. In particular, since Σ is connected, there are no critical values in (a, b). Thus the level set $H^{-1}(x)$ for $x \in (a, b)$ is the circle run through by an orbit of the Hamiltonian system. In particular, the set $H^{-1}(a, b)$ is a cylinder and our goal is to prove that this cylinder has area $b - a$. We start by picking a smooth curve $\gamma\colon (a, b) \to \Sigma$ such that

$$H(\gamma(x)) = x. \tag{22}$$

We can think of γ as a local section of the system. By (22), γ intersects the orbits of the S^1-action ϕ_t^H transversely. Together with the circle action, the curve γ allows us to define a parametrization of the cylinder $H^{-1}(a, b)$,

$$h\colon (a, b) \times S^1 \to H^{-1}(a, b) \subset \Sigma, \quad (x, t) \mapsto \phi_t^H(\gamma(x)). \tag{23}$$

Note that we have used the fact that the action is effective here. Otherwise, h may not be a diffeomorphism. We claim that h is a symplectomorphism when we equip $(a, b) \times S^1$ with the symplectic form $dx \wedge dt$. The claim then follows, since $(a, b) \times S^1$ has area $b - a$. To show that h is a symplectomorphism, it suffices to compute

$$h^*\omega(\partial_t, \partial_t) = h^*\omega(\partial_x, \partial_x) = 0, \quad h^*\omega(\partial_x, \partial_t) = 1, \tag{24}$$

since ∂_x, ∂_t form a basis at each tangent space of the cylinder $(a, b) \times S^1$. The first two equalities in (24) follow directly from anti-symmetry of ω. For the third, compute

$$h_*\partial_x = \frac{\partial h}{\partial x} = (\phi_t^H)_*\dot{\gamma}, \quad h_*\partial_t = \frac{\partial h}{\partial t} = X_H \circ h. \tag{25}$$

From this, we deduce

$$h^*\omega(\partial_x, \partial_t) = \omega((\phi_t^H)_*\dot{\gamma}, X_H \circ h) = dH((\phi_t^H)_*\dot{\gamma}) = dH(\dot{\gamma}) = 1 \tag{26}$$

For the second identity, we have used Hamilton's equation, for the third the invariance of H under its Hamiltonian flow and the fourth follows from the choice of γ by differentiating (22) with respect to x.

Let us now prove the converse. The condition that H only has critical points of even Morse index (here this means that the indices are equal to 0 or 2) implies that the level sets $H^{-1}(x)$ are connected. This follows from the connectedness of Σ and basic Morse theory, see for example [3, Theorem IV.3.1]. In particular, every level $H^{-1}(x)$ is diffeomorphic to a circle by the properness of H. To prove that H induces a circle action, let $a, b \in H(\Sigma)$ such that $a < x < b$. By the above, we know that $H^{-1}(a, b)$ is an open cylinder. Again, we pick a section $\gamma\colon (a, b) \to \Sigma$ with $H(\gamma(x)) = x$ and define the corresponding map

$$h\colon (a, b) \times \mathbb{R} \to H^{-1}(a, b) \subset \Sigma, \quad (x, t) \mapsto \phi_t^H(\gamma(x)). \tag{27}$$

By the same arguments as above, this map satisfies $h^*\omega = dx \wedge dt$, but as opposed to (23), this is not a diffeomorphism but rather a covering. Indeed, since H is proper and has no critical values in (a, b), there is a well-defined function $P\colon (a, b) \to \mathbb{R}$ such that $P(x)$ is the period of the Hamiltonian system at the level $x \in (a, b)$. This function is smooth since it is defined by the implicit equation $\phi^H_{P(x)}(\gamma(x)) = \gamma(x)$. Our goal is to show that $P(x) = 1$ for all $x \in (a, b)$. Note that h induces a symplectomorphism from $((a, b) \times \mathbb{R})/(x, t) \sim (x, t + P(x))$ to $H^{-1}(a, b)$. But this implies that the symplectic area of slices $H^{-1}(a, y)$ for $a < y < b$ can be measured by integrating the period,

$$\int_a^y P(x)dx = \int_{H^{-1}(a,y)} \omega. \tag{28}$$

Differentiation with respect to y together with (21) yields $P(x) = 1$. □

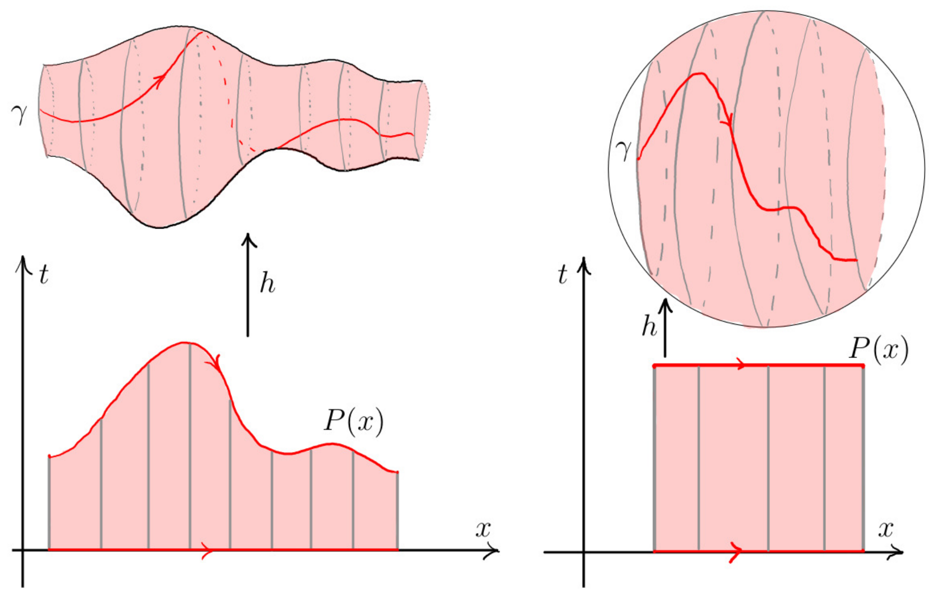

FIGURE 2. The map $h(x, t) = \phi^H_t(\gamma(x))$ induced by a Hamiltonian H as in (23) and (27). The Hamiltonian system is a circle action if and only if the period function $x \mapsto P(x)$ is constant.

Note that the symplectomorphism (23) appears in the proof of both directions of the equivalence. To recapitulate, the map h defined in this way satisfies

$$h^*\omega = dx \wedge dt, \quad H \circ h = K, \tag{29}$$

where $K(x,t) = x$ is the Hamiltonian which induces the rotation on the cylinder. See also Example 2.6. In particular, h is equivariant, $h \circ \phi_t^K = \phi_t^H \circ h$. This means that it yields a *local equivalence* of the systems (Σ, ω, H) and the model space $(Z, dx \wedge dt, K)$. Local models of this kind are ubiquitous in the study of Hamiltonian group actions and integrable systems. In favourable situations, the local models can be pieced together to yield a global classification of the spaces at hand. This happens for example for toric symplectic manifolds, see Theorem 6.3 or the original paper [12]. The goal of this section is to illustrate this in the most basic case, namely the one of surfaces equipped with a Hamiltonian circle action. Before moving to this classification, let us make some more remarks concerning Proposition 2.14.

Remark 2.15. (Periods and area) Let $H\colon \Sigma \to \mathbb{R}$ be a proper Hamiltonian with only critical points of even Morse index. Even if H itself does not generate a Hamiltonian S^1-action, we can come up with a Hamiltonian $\tilde{H}$ which generates a circle action and has the same orbits as H. This follows directly from Proposition 2.14 and one can given an explicit formula for $\tilde{H}$ by setting

$$\tilde{H}\colon \Sigma \to \mathbb{R}, \quad \tilde{H}(p) = \int_{H^{-1}(a,H(p))} \omega, \tag{30}$$

where $a \in \mathbb{R}$ is any real number. The function $\tilde{H}$ is well defined and smooth since all the level sets of H are smooth by the hypothesis that it has only critical points of even index. Note that changing a only changes $\tilde{H}$ by an additive constant. Furthermore, the function $\tilde{H}$ is the unique Hamiltonian generating a circle action with the same orbits as H. This sheds some more light on Remark 2.7. Using (68), it is easy to define a Hamiltonian circle action that has a given smooth family of circles in $\mathbb{C}$ as orbits, see Figure 3. Indeed, we can assign to any point the symplectic area bounded by the curve the point lies on.

Remark 2.16. (Hypotheses of Proposition 2.14) The left side of Figure 4 illustrates why the properness of H is important. One can define a Hamiltonian which has only critical points of even index and has the set of curves given in this figure as orbits, however one cannot find a corresponding circle action. One of the orbits is non-compact. In terms of Remark 2.15, this is reflected in the fact that the area contained by the orbits diverges as one approaches the non-compact orbit. After removal of the non-compact orbit, one can however define two distinct circle actions on the remaining halves. The right side of Figure 4 illustrates why one needs to exclude critical points of odd index. Again, one of the supposed orbits of the circle action is not a circle. This time the function assigning to each point the symplectic area of the sublevel set is well-defined but there is a jump in the period when one crosses the figure eight. One can define circle actions in the three regions arising after the removal of the figure eight.

The following theorem illustrates, both in the statement and in the strategy of proof, the classical Delzant theorem 6.3 on the classification of toric symplectic manifolds. We will heavily rely on the classification of Hamiltonian circle actions on surfaces in Section 3.

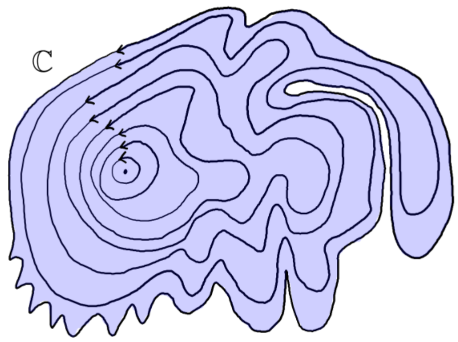

FIGURE 3. Any smooth family of circles in the plane is the orbit set of a Hamiltonian circle action.

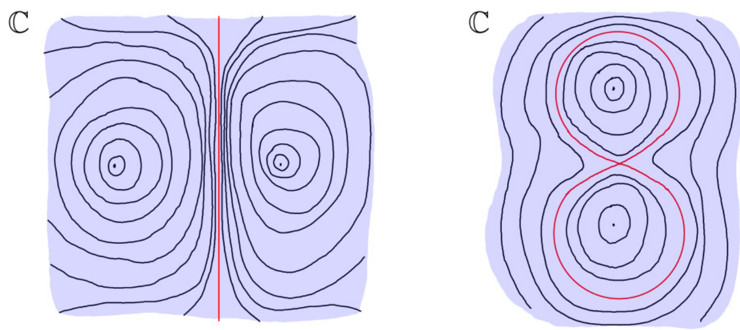

FIGURE 4. On the left: level sets of a non-proper function. On the right: level sets of a function with a critical point of odd Morse index.

Theorem 2.17. (toy Delzant theorem) *Connected symplectic surfaces carrying an effective Hamiltonian circle action are classified by the image of their Hamiltonian function. More precisely, two Hamiltonian circle actions* (Σ, ω, H) *and* (Σ', ω', H') *are* S^1*-equivariantly symplectomorphic if and only if the images* $H(\Sigma), H'(\Sigma') \subset \mathbb{R}$ *agree up to translation.*

Proof. The *only if* direction is obvious. For the other direction, we will see that most of the work is already contained in Proposition 2.14 and the normal form at fixed points which we have discussed in Proposition 2.10. Let (Σ, ω, H) be a Hamiltonian circle action on a connected surfaces with image $I = H(\Sigma)$. We will show that (Σ, ω, H) is equivariantly symplectomorphic to a model space determined by I up to translation. By connectedness, the images I is a (possibly infinite) segment. The set $H^{-1}(\operatorname{int} I)$ fibering over the interior of the image is equivariantly symplectomorphic to the standard Hamiltonian circle action on the open cylinder $\operatorname{int} I \times S^1$. This follows from the same argument as in the first half of the proof of Proposition 2.14. We denote this symplectomorphism by $\phi\colon \operatorname{int} I \times S^1 \to H^{-1}(\operatorname{int} I)$. If $I = \operatorname{int} I$, then we are done. If not, note that the set $I \setminus \operatorname{int} I$ consists of at most two boundary points, which are critical values of H and correspond to fixed points on Σ. This means that we need to extend the symplectomorphism ϕ over at most two fixed points. Assume that I has a boundary point which is a minimum of H (maxima can be treated similarly) and, up to translation, we can furthermore assume that this minimum is $0 \in \mathbb{R}$. Denote the corresponding fixed point on the surface by p. The normal form from Proposition 2.10 yields

$$\psi\colon \mathbb{C} \supset D(a) \to V, \quad H \circ \psi = |z|^2, \quad \psi^*\omega = \omega_0, \quad \varphi(0) = p, \quad i \in \{1, 2\}. \tag{31}$$

Here $D(a)$ denotes a disk of area a and V is an S^1-invariant neighbourhood of $p \in \Sigma$. In the normal form $H \circ \psi = \xi |z|^2$ given by Proposition 2.10 we have $\xi = 1$, since the action would not be effective for $|\xi| \neq 1$ and since we are dealing with a minimum (for a maximum, we have $\xi = -1$). Note that the normal form ψ is S^1-equivariant with respect to the harmonic oscillator system restricted to $D(a)$. Thus we now have a suitable model for a neighbourhood of p and this can be glued S^1-equivariantly to the existing model ϕ to obtain an extension over the fixed point p. This can be repeated in case there is another fixed point. ☐

Note that the classification of connected symplectic surfaces carrying an effective Hamiltonian circle action boils down to classifying segments in $\mathbb{R}$ up to translation. Thus the list given by Examples 2.4, 2.5 and 2.6 is complete up to changing the total area for S^2, removing some of the fixed points (to get the open segments) and flipping the orientation of the circle action by passing from H to $-H$.

2.5. Background: Lagrangian submanifolds

Before continuing our discussion, let us introduce some more basic notions from symplectic geometry. Note that a submanifold $N \subset M$ of a symplectic manifold (M, ω) is not necessarily symplectic itself. Indeed, the restriction of the symplectic form $\omega|_{TN}$ to the subbundle TN may be degenerate.

Definition 2.18. *A submanifold $L^n \subset (M^{2n}, \omega)$ of half the dimension of M such that $\omega|_{TL} = 0$ is called* Lagrangian*.*

More generally, a submanifold N on which ω vanishes is called *isotropic*. An elementary argument shows that isotropic submanifolds have at most half the dimension of the ambient space. This means that *Lagrangians are maximal among the submanifolds on which ω is maximally degenerate.*

Definition 2.19. *Two Lagrangian submanifolds $L, L' \subset (M, \omega)$ are called* symplectomorphic *if there is a symplectomorphism $\phi \in \mathrm{Symp}(M, \omega)$ mapping L to L'. Similarly, they are called* Hamiltonian isotopic *if there is a Hamiltonian diffeomorphism mapping L to L'.*

The second equivalence relation is stronger than the first one, since Hamiltonian diffeomorphisms are in particular symplectomorphisms. However, in all of the examples we discuss in detail, there will be no distinction between these two equivalence relations. The Chekanov torus for example is exotic with respect to both relations. There is another, considerably weaker, equivalence relation on Lagrangian submanifolds. Two Lagrangian submanifolds are called *Lagrangian isotopic* if there is a smooth isotopy of submanifolds *all of which are Lagrangian* having as endpoints the two given Lagrangians. We will see that the Chekanov torus *is not* exotic with respect to this equivalence relation. There are too many interesting examples of Lagrangians in symplectic geometry to even start mentioning them all and so we will restrict our attention to the strict minimum of examples that we will need to proceed in our discussion. In fact, we have already encountered an important set of Lagrangians in the Introduction, see §1.2.

Example 2.20. (Product tori) Let $(\mathbb{C}^n, \omega_0)$ be the standard symplectic vector space and let $a_1, \dots, a_n \in \mathbb{R}_{>0}$ be positive numbers. Then the *product tori*,

$$T(a_1, \dots, a_n) = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \pi |z_i|^2 = a_i\} \tag{32}$$

are Lagrangian submanifolds. This follows from the fact that the product of Lagrangian submanifolds is again Lagrangian and curves in symplectic surfaces are automatically Lagrangian. Note the the parameters a_i are the area of the disk bounded by the respective circles (not their radius). We will sometimes denote the product tori by $T(a)$ for $a \in \mathbb{R}^n_{>0}$ for short.

Definition 2.21. *A Lagrangian torus $L \subset \mathbb{C}^n$ is called* exotic *if it cannot be mapped to a product torus by a symplectomorphism/Hamiltonian diffeomorphism.*

Remark 2.22. (Product tori in general manifolds) In conjunction with Darboux's theorem (see Proposition 2.9), this example shows that Lagrangian tori are abundant in every symplectic manifold. Indeed, let $\varphi \colon U \to V \subset M$ be a Darboux chart. Then for small enough $a \in \mathbb{R}^n_{>0}$, the product torus $T(a)$ is contained in U and hence its image $\varphi(T(a)) \subset M$ is Lagrangian. These tori were studied for example by Chekanov–Schlenk [11].

Let $L \subset (M, \omega)$ be a Lagrangian. Then to every disk with boundary on L, one can associate its symplectic area. Due to the Lagrangian condition and the closedness of the symplectic form, this yields a well-defined morphism on the homotopy classes of such disks,

$$\mathrm{Area}_L \colon \pi_2(M, L) \to \mathbb{R}, \quad D \mapsto \int_D \omega. \tag{33}$$

Let us call this the *area class* of L. The area class is a symplectic invariant in the sense that if there is another Lagrangian L' which is the image of L under a symplectomorphism, then

$$\mathrm{Area}_{L'} = \mathrm{Area}_L \circ (\phi|_L)_*, \quad (\phi|_L)_* \colon \pi_2(M, L) \to \pi_2(M, L'). \tag{34}$$

There is another morphism of this type, called the *Maslov class* of L which takes integer values and has the same symplectic invariance property as the area class. We will not go into details about the definition of the Maslov class here, see for example the standard reference [21].

Definition 2.23. *A Lagrangian submanifold $L \subset M$ is called* monotone *if the area and Maslov classes are positively proportional.*

Monotonicity is a symplectic invariant in the sense that it is preserved under symplectomorphisms. This follows immediately from (34) and the analogous property of the Maslov class.

Example 2.24. (Monotonicity of product tori) Let $M = \mathbb{C}^n$ and $L = T(a)$ be a product torus. Then $\pi_2(\mathbb{C}^n, T(a)) \cong \pi_1(T(a)) \cong \mathbb{Z}^n$ with generators given for example by the disks D_i bounding the circles $S^1(a_i)$ in the factors of $\mathbb{C}^n$ equipped with the natural orientation. The Maslov index of these disks is 2, independently of a. For the area class, we compute $\int_{D_i} \omega_0 = a_i$. We conclude that a product torus is monotone if and only if all a_i are equal. Monotone product tori are also sometimes called *Clifford tori*.

2.6. Background: Commuting Hamiltonians

Recall from Proposition 2.1 that an (autonomous) Hamiltonian function F is preserved by the Hamiltonian flow it generates. Infinitesimally, this can be expressed as $dF(X_F) = 0$. Two Hamiltonians are said to commute if one is preserved by the flow of the other.

Definition 2.25. *Two Hamiltonians F, G on a symplectic manifold* (M, ω) (Poisson-) commute *if $dF(X_G) = 0$. The Hamiltonians $F_1, \dots, F_n$* commute *if they commute pairwise.*

First note that this relation is symmetric, meaning that F is preserved under the flow of G if and only if G is preserved under the flow of F. This follows immediately from Hamilton's equation and anti-symmetry of the symplectic form,

$$dF(X_G) = \omega(X_G, X_F) = -dG(X_F). \tag{35}$$

Remark 2.26. (Poisson brackets) Setting $\{F, G\} = \omega(X_F, X_G)$ induces a structure on the space of smooth functions on the symplectic manifold. This structure is called a *Poisson bracket* and it can be given by a set of axioms the study of which is of independent interest. Hence the term *Poisson-commute* for Hamiltonians satisfying $\{F, G\} = 0$.

If two Hamiltonians commute, then their corresponding Hamiltonian flows commute,

$$\phi_t^F \circ \phi_s^G = \phi_s^G \circ \phi_t^F, \quad \text{if } \omega(X_F, X_G) = 0. \tag{36}$$

This follows from the fact that $\omega(X_G, X_F)$ agrees with the *Lie bracket of vector fields* $[X_F, X_G]$ and the latter vanishes if and only if the corresponding flows commute. The converse is not true, but almost. If the flows commute, then $\omega(X_F, X_G)$ is locally constant as a function on the symplectic manifold.

Let us now consider the special case where the two commuting Hamiltonians F, G generate Hamiltonian circle actions. By commutativity of the flows (36), we can define an induced action of the two-torus $T^2 = (\mathbb{R}/\mathbb{Z})^2$,

$$\psi\colon T^2 \times M \to M, \quad (\theta = (\theta_1, \theta_2), p) \mapsto \psi_\theta(p) = (\phi^F_{\theta_1} \circ \phi^G_{\theta_2})(p). \tag{37}$$

This leads us to the following definition.

Definition 2.27. *An action ψ of a smooth torus $T^k = (\mathbb{R}/\mathbb{Z})^k$ on a symplectic manifold (M, ω) is called* Hamiltonian *if there is a commuting set of Hamiltonains $F_1, \ldots, F_k$ each of which generates a Hamiltonian circle action such that $\psi_\theta = \phi^{F_1}_{\theta_1} \circ \ldots \circ \phi^{F_k}_{\theta_k}$ for all $\theta = (\theta_1, \ldots, \theta_k) \in T^k$.*

A map $\mu = (F_1, \ldots, F_k)\colon M \to \mathbb{R}^k$ generating a Hamiltonian T^k-action is called *moment map*. Moment maps have very interesting geometric properties. For example note that the orbits of the torus action are contained in the level sets of $\mu\colon M \to \mathbb{R}^k$ since the Hamiltonians are preserved by the Hamiltonian flows. Hence we often think of moment maps as *singular fibrations* which are compatible with the group action. Note furthermore that the moment map is not unique. For one, we can add constant vectors to it. Furthermore, we can replace for example F_1 by an integral linear combination $F_1 + k_2F_2 + \ldots + k_nF_n$ without changing the torus action. More generally, one can post-compose it with elements in $\mathrm{GL}(k; \mathbb{Z})$, which corresponds to changing the basis on the torus T^k which acts on the space. Therefore, it is useful to formulate the theory of Hamiltonian group actions and moment maps in a more invariant way. However, we will not go into details here, since we only need the following example of torus actions for the definition of the Chekanov torus.

Example 2.28. (Standard torus action on $\mathbb{C}^n$) The main example for us is the standard T^n-action on $\mathbb{C}^n$, which we have already mentioned in §1.3. It is generated by $F_i = \pi|z_i|^2$ and given by rotation in each of the factors of $\mathbb{C}^n$,

$$\psi_\theta(z_1, \ldots, z_n) = (e^{2\pi i\theta_1} z_1, \ldots, e^{2\pi i\theta_n} z_n), \quad \theta \in T^n. \tag{38}$$

The orbits of this action are the product tori from Example 2.20 as well as lower-dimensional tori on which some of the coordinates z_i vanish. This yields another proof of the fact that the product tori are Lagrangian. Indeed, their tangent spaces are spanned by the Hamiltonian vector fields $X_{F_1}, \ldots, X_{F_n}$ and since F_i, F_j commute, we deduce $\omega(X_{F_i}, X_{F_j}) = 0$.

We can obtain other fairly obvious examples by taking products of the two-dimensional Examples 2.4, 2.5 and 2.6. All examples obtained in this way are of a very special type, since the torus that acts has half the dimension of the ambient space. As it turns out, this is the maximal case, meaning that there cannot be more than n independent commuting Hamiltonians on a $2n$-dimensional symplectic manifold. Such maximal Hamiltonian torus actions are called *toric* and they have very special properties, see Section 6.

2.7. Symplectic reduction of Hamiltonian circle actions

In the setting of smooth manifolds, one can divide out a free smooth action of a compact Lie group to obtain a quotient manifold. *Symplectic reduction* is the analogue of this operation for Hamiltonian group actions where the quotient manifold inherits a symplectic structure. Although symplectic reduction can be carried out in a much more general setting, we restrict our attention to Hamiltonian circle actions. Note that *a priori* it is not at all obvious how to get a symplectic structure on the quotient space of group actions. For one, the orbit space M^{2n}/S^1 of a Hamiltonian circle action (M, ω, H) has odd dimension and hence cannot carry a symplectic structure. This problem is resolved by not taking a global quotient, but rather restricting to a level set of the S^1-Hamiltonian H. We denote level sets by

$$Z_c = H^{-1}(c), \quad c \in \mathbb{R}. \tag{39}$$

Theorem 2.29. (Marsden–Weinstein) *Let (M, ω, H) be a Hamiltonian circle action and $c \in \mathbb{R}$ such that S^1 acts freely on the level set $Z_c = H^{-1}(c)$. Then c is a regular value and the quotient $M_c = Z_c/S^1$ carries a natural symplectic form ω_c such that $\pi_c^*\omega_c = \omega|_{Z_c}$. By $\pi_c \colon Z_c \to M_c$, we denote the quotient map.*

First note that the circle action generated by H automatically preserves the level sets Z_c of H. For the theorem to apply, we only need this action to be free. Furthermore, the fact that $c \in \mathbb{R}$ is a regular value follows directly from Hamilton's equation (10). Indeed, if the circle action on Z_c is free, then the Hamiltonian vector field does not vanish on Z_c and so neither does dH. This proves that Z_c is a submanifold. Symplectic reduction is best summarized by a *reduction diagram*,

$$(M_c, \omega_c) \xleftarrow{\pi_c} Z_c = H^{-1}(c) \hookrightarrow (M, \omega). \tag{40}$$

Sketch of proof. By the hypotheses of the theorem, the pair $(Z_c, \omega|_{Z_c})$ is an $(2n-1)$-dimensional manifold equipped with a degenerate closed two-form and a free circle action. Note that the circle action preserves $\omega|_{Z_c}$ since Hamiltonian flows preserve symplectic forms. Since the quotient map is a submersion, this proves that there is a closed two-form ω_c on $M_c = Z_c/S^1$ such that $\pi_c^*\omega_c = \omega|_{Z_c}$. The hard part is to prove that ω_c is non-degenerate and hence symplectic. For this, note that Z_c is a *coisotropic submanifold*, meaning that the symplectic complement $T_pZ_c^\omega$ is contained in T_pZ_c for every $p \in Z_c$. In fact, we will prove that the symplectic complement is spanned by the Hamiltonian vector field X_H, which is tangent to Z_c. This is the *miracle of symplectic reduction*, since it shows that we can get rid of the degeneracy of the form $\omega|_{Z_c}$ by dividing out the group action and hence the form ω_c on the quotient is non-degenerate. To conclude, we prove that

$$TZ_c^\omega = \operatorname{span}\{X_H\}. \tag{41}$$

Since $Z_c = H^{-1}(c)$, the tangent bundle can be written as $TZ_c = \ker dH$. Therefore, $\omega(X_H, Y) = dH(Y) = 0$ for all $Y \in TZ_c$ and thus $X_H \in TZ_c^\omega$. On the other hand TZ_c^ω is one-dimensional and hence there is equality. □

There is a very natural relationship between the symplectic geometry of the reduced space (M_c, ω_c) and the S^1-equivariant symplectic geometry of (M, ω). We crucially exploit this relationship in our discussion of the Chekanov torus in Section 3. For the remainder of this subsection, let (M, ω, H) be a Hamiltonian circle action which admits symplectic reduction at the level $c \in \mathbb{R}$ and take the notation from (40).

Proposition 2.30. *There is a one-to-one correspondence between Lagrangian submanifolds $L_c \subset M_c$ and Lagrangian submanifolds $L \subset H^{-1}(c) \subset M$.*

Proof. Given $L_c \subset M_c$, set $L = \pi_c^{-1}(L_c)$. It follows from $\pi_c^* \omega_c = \omega$ that L is Lagrangian. On the other hand, given $L \subset H^{-1}(c)$, it follows from the Lagrangian condition that L is invariant under the circle action. This is actually a general fact: Lagrangian submanifolds contained in a level set of a Hamiltonian are invariant under the flow of that Hamiltonian. Indeed, due to the Lagrangian condition, it suffices to show that $\omega(Y, X_H) = 0$ for all $Y \in TL$ in order to conclude that $X_H \in TL$. Note however that $\omega(Y, X_H) = dH(Y) = 0$ due to that fact that $TH^{-1}(c) = \ker dH$. We obtain a Lagrangian in the symplectic quotient by setting $L_c = \pi_c(L) = L/S^1$. □

Now let (M, ω, F) be another (not necessarily periodic) Hamiltonian system on the same space as the circle action H. We say that F is *H-equivariant* if the Hamiltonians H, F commute. The commutativity of flows in (36) explains this terminology.

Proposition 2.31. *Every H-equivariant Hamiltonian system (M, ω, F) projects to a unique Hamiltonian system (M_c, ω_c, F_c), meaning that*

$$\pi_c \circ \phi_t^F = \phi_t^{F_c} \circ \pi_c, \quad t \in \mathbb{R}. \tag{42}$$

Conversely, a given Hamiltonian system (M_c, ω_c, F_c) can be lifted to a (non-unique) Hamiltonian system (M, ω, F) for which (42) *holds.*

Proof. For the first statement note that the H-equivariance $\omega(X_H, X_F) = dF(X_H) = 0$ implies that F is constant on the orbits of the circle action H and hence that the restriction $F|_{Z_c}$ descends to a well-defined function F_c in the quotient. Using $\pi_c^* \omega_c = \omega$ and $F_c \circ \pi_c = F|_{Z_c}$, it is straightforward to check that F_c generates the desired Hamiltonian system. For the converse, lift the function F_c given on M_c to the level set Z_c and extend it to M, for example by a cut-off in a tubular neighbourhood. The Hamiltonian system we obtain depends on the choice of this extension and is hence not unique. □

2.8. Example: Complex projective plane

As an example of the utility of symplectic reduction, we describe the complex projective plane $\mathbb{C}P^2$ as a symplectic quotient of $\mathbb{C}^3$. This can be seen as a symplectic version of the Hopf fibration which naturally equips $\mathbb{C}P^2$ with the Fubini–Study form $\omega_{\mathbb{C}P^2}$. The construction can be carried over to $\mathbb{C}P^n$ verbatim. Let $\mathbb{C}^3$ be equipped with the symplectic form ω_0 and let

$$H(z_1, z_2, z_3) = \pi(|z_1|^2 + |z_2|^2 + |z_3|^2). \tag{43}$$

This Hamiltonian defines the diagonal circle action,

$$\phi_t^H(z_1, z_2, z_3) = (e^{2\pi i t} z_1, e^{2\pi i t} z_2, e^{2\pi i t} z_3). \tag{44}$$

The circle action is free on $H^{-1}(c)$ for all $c > 0$ and hence we can perform symplectic reduction, for example for $c = 3$,

$$(\mathbb{C}P^2, \omega_{\mathbb{C}P^2}) \xleftarrow{\pi} S^5(3) = H^{-1}(3) \hookrightarrow (\mathbb{C}^3, \omega_0). \tag{45}$$

Obviously, the choice $c = 3$ is quite arbitrary. Any other convention yields the same symplectic form on $\mathbb{C}P^2$ up to scaling. Let us illustrate the lifting/projection constructions of §2.7 in this example. Proposition 2.30 tells us that there is a one-to-one correspondence between Lagrangian submanifolds of $\mathbb{C}P^2$ and Lagrangian submanifolds of $\mathbb{C}^3$ contained in $S^5(3)$. Take for example the Lagrangian torus $T(1,1,1) = \{|z_i| = 1\} \subset \mathbb{C}^3$. It is contained in $H^{-1}(3)$ and hence automatically invariant under the circle action. Therefore it projects to a Lagrangian torus in $\pi(T(1,1,1)) = T_{\mathrm{Cl}} \subset \mathbb{C}P^2$, known as the *Clifford torus*. The same can be done for all product tori (see Example 5.5),

$$T(a_1, a_2, a_3) \subset \mathbb{C}^3, \quad a_1 + a_2 + a_3 = 3. \tag{46}$$

Let's illustrate the inverse process of lifting Lagrangians from the reduced space by taking real projective space $\mathbb{R}P^2 \subset \mathbb{C}P^2$ obtained as the real locus in homogeneous coordinates. Lifting $\mathbb{R}P^2$ to $S^5 \subset \mathbb{C}^3$ yields an interesting Lagrangian $L = \pi^{-1}(\mathbb{R}P^2)$, which has the structure of a (non-trivial) principal S^1-bundle over $\mathbb{R}P^2$. In order to identify the topological type of L, note that it can be written as

$$L = \{(e^{2\pi i t} x_1, e^{2\pi i t} x_2, e^{2\pi i t} x_3) \subset \mathbb{C}^3 \,|\, e^{2\pi i t} \in S^1,\, x_i \in \mathbb{R},\, x_1^2 + x_2^2 + x_3^2 = 3\}. \tag{47}$$

This shows that L can be identified with $S^2 \times_{\mathbb{Z}_2} S^1$, where $\mathbb{Z}_2$ acts on S^2 and S^1 by the antipodal map. Using the projection $S^2 \times_{\mathbb{Z}_2} S^1 \to \mathbb{R}P^1 \cong S^1$, we see that L is the mapping torus of the antipodal map $x \mapsto -x$ on the two-sphere, meaning that L is a higher dimensional analogue of the Klein bottle.

As an illustration of Proposition 2.31, we construct a Hamiltonian T^2-action on $\mathbb{C}P^2$. In order to do so, let us first take a step back and note that (44) is in some sense induced by the standard torus action on $\mathbb{C}^3$ described in Example 2.28. Indeed, we obtain this circle action by restricting to the diagonal circle $S^1 \hookrightarrow T^3$. As it turns out, this is equivalent to the fact that we can write H in terms of the Hamiltonians F_i generating the torus action,

$$H = F_1 + F_2 + F_3, \quad F_i = \pi |z_i|^2. \tag{48}$$

This implies that H commutes with F_1, F_2, F_3, meaning that we obtain three reduced Hamiltonian systems on $\mathbb{C}P^2$ by Proposition 2.31 which we will denote by $f_1, f_2, f_3 \colon \mathbb{C}P^2 \to \mathbb{R}$. These reduced systems still generate circle actions and they still commute in the quotient. However, note that there is some redundancy. Indeed, we have restricted to the level set $H = 3$ in the reduction and hence we can use (48) to write $f_3 = 3 - f_1 - f_2$, meaning that the flow of f_3 can be written as a composition of the flows of f_1, f_2. Forgetting about f_3, we obtain a Hamiltonian T^2-action on $\mathbb{C}P^2$. This is just the *residual action* of the T^3-action in the sense that there is a natural action of the quotient $T^2 \cong T^3/S^1$ by the diagonal circle on $\mathbb{C}P^2$.

Remark 2.32. (Quotients of Hamiltonian torus actions) Note that the above discussion applies to a much more general setting. Whenever we have a Hamiltonian torus action of T^k and we can perform symplectic reduction with respect to a subtorus $K \subset T^k$, then the quotient carries a residual Hamiltonian action by T^k/K. This follows from the fact that the actions of T^k and K commute since T^k is abelian. One application of this is the *Delzant construction* which proves one direction of Theorem 6.3. Indeed, Delzant noticed that every toric manifold can be constructed as a quotient of some $(\mathbb{C}^N, \omega_0)$ equipped with the standard T^N-action such that the toric structure on the quotient comes as the residual torus action.

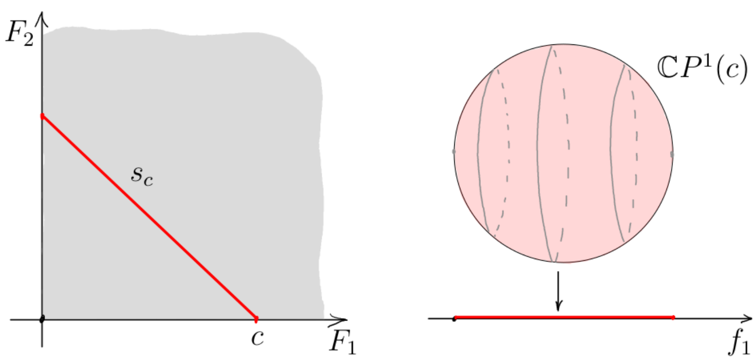

FIGURE 5. The symplectic reduction described in Remark 2.33 at the level $c > 0$.

Remark 2.33. (The case of $\mathbb{C}P^1$) Let us take a closer look at the same construction for $\mathbb{C}P^1$, since we will use an analogous idea for the construction of the Chekanov torus in Section 3. In that case, reduction is performed on one of the sets

$$H = F_1 + F_2 = \pi(|z_1|^2 + |z_2|^2) = c, \quad c > 0, \tag{49}$$

to obtain $\mathbb{C}P^1(c)$. In the image $\mathbb{R}^2_{\geqslant 0}$ of the moment map $\mu = (F_1, F_2)$, these levels project to segments $s_c = \{F_1 + F_2 = c\} \cap \mathbb{R}^2_{\geqslant 0}$ as in Figure 5. As above, the circle actions generated by F_1, F_2 yield induced circle actions f_1, f_2 on $\mathbb{C}P^1$. The image of the map (f_1, f_2) is s_c. However, there is again some redundancy and we may pick f_1 as generator of the induced circle action on $\mathbb{C}P^1(c)$. This shows that the image of the Hamiltonian f_1 is the segment of length c coming from the projection of s_c onto the first coordinate axis. This allows us to apply Theorem 2.17 to conclude that $(\mathbb{C}P^1(c), \omega_c, f_1)$ is equivariantly symplectomorphic to the circle action $(S^2, \alpha\omega_{S^2}, \alpha G)$ from Example 2.5 with normalization $\alpha = \frac{c}{4\pi}$.

3. Constructing the Chekanov torus in $\mathbb{R}^4$

The construction of the Chekanov torus we give here is essentially due to Eliashberg–Polterovich [14]. The picture in terms of reduced spaces can also be found in [10], from which we have taken the depiction in Figure 7. Another elucidating account from a slightly different point of view is given by Auroux, for example in [4].

3.1. A family of symplectic reductions

One of the key elements in this construction of the Chekanov torus is the Hamiltonian

$$F\colon \mathbb{C}^2 \to \mathbb{R}, \quad (z_1, z_2) \mapsto \pi(|z_1|^2 - |z_2|^2). \tag{50}$$

It induces the following Hamiltonian circle action

$$\psi_\theta(z_1, z_2) = (e^{2\pi i\theta} z_1, e^{-2\pi i\theta} z_2), \quad \theta \in S^1 = \mathbb{R}/\mathbb{Z}. \tag{51}$$

All of our constructions will be invariant or equivariant with respect to this circle action. Roughly speaking, we will perform a ψ-equivariant perturbation of the standard toric structure on $\mathbb{C}^2$. For this we first discuss the symplectic quotients obtained at different levels of F.

Let us denote the level sets of F by $Z_c = F^{-1}(c)$ for all $c \in \mathbb{R}$. First of all note that for all $c \neq 0$, this level set is smooth. In fact, it is diffeomorphic to $S^1 \times \mathbb{C}$. The only critical value of F is $0 \in \mathbb{R}$ and its only critical point is $0 \in \mathbb{C}^2$. Thus ψ has $0 \in \mathbb{C}^2$ as its only fixed point and thus we can perform symplectic reduction away from $c = 0$. We first treat the case $c \neq 0$ and then move on to the critical value $c = 0$.

Proposition 3.1. *Let $c \neq 0$. The symplectic quotient $M_c = F^{-1}(c)/S^1 = Z_c/S^1$ by the circle action is symplectomorphic to the standard symplectic plane $(\mathbb{C}, \omega_0)$.*

Before proving this claim, let us first take a step back and interpret the Hamiltonian S^1-action in terms of the toric structure on $\mathbb{C}^2$. Recall from §1.3 and Example 2.28 that the moment map of the standard toric structure on $\mathbb{C}^2$ is

$$\mu\colon \mathbb{C}^2 \to \mathbb{R}^2_{\geqslant 0}, \quad (z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2), \tag{52}$$

and that its action is by rotations in the factors of $\mathbb{C}^2$. Now note that our Hamiltonian F is compatible with this structure in the sense that we can write it as $F = \mu_1 - \mu_2$. In particular, this means that the level set Z_c projects to the half-line $\{\mu_1 - \mu_2 = c\} \subset \mathbb{R}^2_{\geqslant 0}$ in the base of the toric fibration. See Figure 6. The key observation for us is that after we quotient out the circle action generated by F, the quotient M_c still carries a residual circle action (coming from the initial toric T^2-action) which turns it into a toric manifold in its own right. Indeed, note that the pair of Hamiltonians F, μ_1 commutes and hence we may apply Proposition 2.31 to obtain a reduced Hamiltonian system $(M_c, \omega_c, (\mu_1)_c)$. For simplicity, let us call

$$g_c = (\mu_1)_c \colon M_c \to \mathbb{R}. \tag{53}$$

The reduced Hamiltonian g_c generates an effective Hamiltonian circle action on M_c. Recall that we have classified all effective Hamiltonian circle actions on symplectic surfaces in Theorem 2.17 in terms of the image of the Hamiltonian function. Here, the image of g_c is a half-line. Indeed, it is the projection of the half-line $\{\mu_1 - \mu_2 = c\} \subset \mathbb{R}^2_{\geqslant 0}$ to the first component. It follows that there is an equivariant symplectomorphism to the harmonic oscillator,

$$(M_c, \omega_c, g_c) \cong (\mathbb{C}, \omega_0, \pi|z|^2), \quad c \neq 0. \tag{54}$$

Note the similarity with Remark 2.33, where we have looked at the reduced spaces of the Hamiltonian $H = \pi(F_1 + F_2)$. In that case, the sets $H^{-1}(c) \cap \mathbb{R}^2_{\geqslant 0}$ are given by closed line segments, allowing us to equivariantly identify the quotients with the standard Hamiltonian circle action on the sphere via Theorem 2.17. Both these cases are instances of a more general principle which we briefly describe in Remark 6.8.

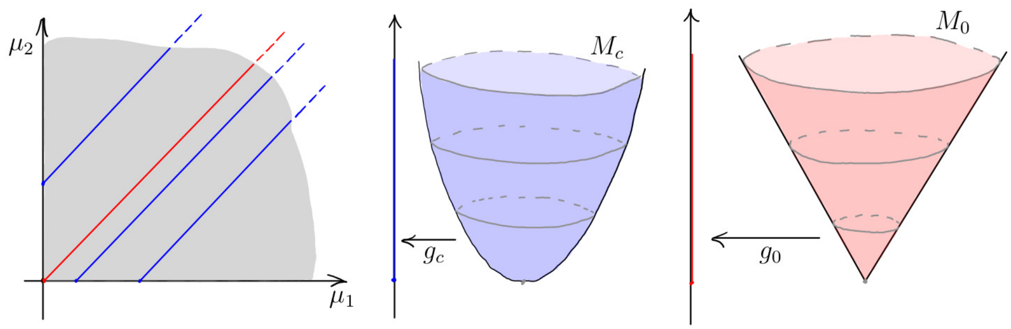

FIGURE 6. The family $\pi_c \colon Z_c \to M_c$ with the residual Hamiltonian circle actions on the quotients M_c.

Let us now turn to the critical value $c = 0$. The only critical point $0 \in \mathbb{C}^2$ is the only fixed point of the circle action ψ restricted to Z_0. This means that we may perform symplectic reduction on its complement, $Z_0 \setminus \{0\}$. Note that there is again a residual circle action on the quotient $(Z_0 \setminus \{0\})/S^1$ and hence we can use this additional structure to determine the quotient as above. This time, the image of the induced Hamiltonian is the open half-line and hence the quotient is symplectomorphic to $(\mathbb{C}^\times, \omega_0)$. Obviously, we can add a point to this space and extend the smooth and symplectic structures. Therefore we set $(M_0, \omega_0) = (\mathbb{C}, \omega_0)$. Let us emphasize that, as opposed to $M_{c\neq 0}$, this space is not defined by symplectic reduction, but it coincides with the symplectic quotient of $Z_0 \setminus \{0\}$ outside of $0 \in \mathbb{C}$. We will see that this distinction is crucial for the construction of the

Chekanov torus. Let us briefly recapitulate. We have now constructed a one-parameter family of quotients

$$(\mathbb{C},\omega_0) \cong (M_c,\omega_c) \overset{\pi_c}{\leftarrow} Z_c = F^{-1}(c) \hookrightarrow (\mathbb{C}^2,\omega_0). \tag{55}$$

with the caveat that for $c = 0$ this is not an honest symplectic reduction. This is summarized by Figure 6. However, we obtain a map $\pi_0\colon Z_0 \to \mathbb{C}$ which restricts to a principal S^1-bundle coming from symplectic reduction over $\mathbb{C}^\times$ but which has a degenerate fibre $\{0\}$ over the origin $0 \in \mathbb{C}$.

Remark 3.2. For now, we will also be a little bit sloppy and identify $(M_c,\omega_c) = (\mathbb{C},\omega_0)$, although this identification is not unique. In fact, for every smooth curve $\gamma\colon [0,+\infty) \to M_c$ with $\gamma(0)$ equal to the fixed point of the circle action g_c and transverse to its orbits, we obtain a symplectomorphism mapping the image of γ to the ray $\mathbb{R}_{\geqslant 0}$. See also the proof of Proposition 2.14.

3.2. The Chekanov torus

We can now turn to the definition of the Chekanov torus. The basic idea is to lift circles from the quotient spaces M_c to tori in $\mathbb{C}^2$ via (55). Since circles in the plane are Lagrangian, such tori are automatically Lagrangian by Proposition 2.30. Let $S^1_{\mathrm{Ch}}(a) \subset \mathbb{C}$ be a simple closed curve bounding area $a > 0$ in the quotient space $(M_0,\omega_0) = (\mathbb{C},\omega_0)$ such that $S^1_{\mathrm{Ch}}(a)$ *does not enclose the origin*, see also Figure 7. From now on, we call such a curve *Chekanov circle*. Then we can define the following one-parameter family of tori.

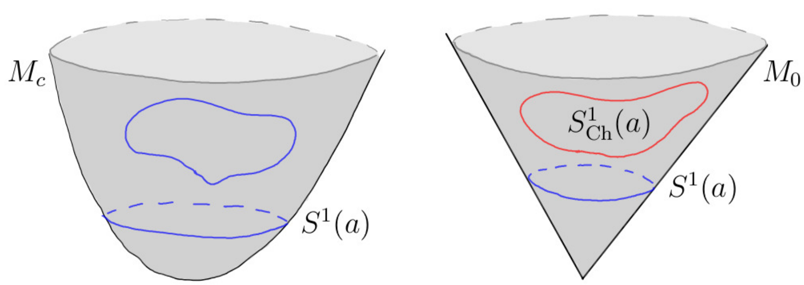

FIGURE 7. On the right: A standard circle $S^1(a)$ and a Chekanov circle $S^1_{\mathrm{Ch}}(a)$. On the left, both circles are Hamiltonian isotopic and so are the lifted tori. Such a Hamiltonian isotopy does not lift to the fibre over the origin in M_0.

Definition 3.3. *The* Chekanov torus of area parameter $a > 0$ *is defined as the lift*

$$T_{\mathrm{Ch}}(a) = \pi_0^{-1}(S^1_{\mathrm{Ch}}(a)) \subset \mathbb{C}^2. \tag{56}$$

This definition does not depend on the choice of the circle $S^1_{\mathrm{Ch}}(a)$ up to applying a Hamiltonian isotopy. Indeed, all circles enclosing area $a > 0$ in $\mathbb{C}^\times$ which do not enclose the origin are Hamiltonian isotopic by a Hamiltonian isotopy inside $\mathbb{C}^\times$, and such a Hamiltonian isotopy can be lifted by Proposition 2.31. Note that $S^1_{\mathrm{Ch}}(a)$ is Hamiltonian isotopic to the standard circle $S^1(a)$ (which does enclose the origin) through a Hamiltonian isotopy supported in $\mathbb{C}$ but *it is not* through a Hamiltonian isotopy supported in $\mathbb{C}^\times$. Now recall that the quotient map $\pi_0 \colon Z_0 \to \mathbb{C}$ is singular at the origin and hence a Hamiltonian isotopy whose support contains $0 \in \mathbb{C}$ cannot be lifted. In fact, a smooth circle that passes through the origin lifts to a *pinched Lagrangian torus* in $\mathbb{C}^2$. See also Figure 8. We will see now that the standard circles $S^1(a)$ in the reduced spaces lift to product tori in $\mathbb{C}^2$. This motivates Definition 3.3 of the Chekanov torus, since it means that there are two kinds of circles in the reduced space $M_0 = \mathbb{C}$, those of standard type $S^1(a)$ (enclosing the origin) and those of Chekanov type $S^1_{\mathrm{Ch}}(a)$ (not enclosing the origin). Both types come in a one-parameter family indexed by the area they contain. Note that in all other reduced spaces with $c \neq 0$, this is not so, since the reduction $\pi_c \colon Z_c \to \mathbb{C}$ is defined everywhere.

Proposition 3.4. *Let $\pi_c \colon Z_c \to M_c$ be the symplectic reduction induced by $F = \pi(|z_1|^2 - |z_2|^2) = c$ on $\mathbb{C}^2$ for $c \neq 0$. Then the lifts of standard circles $S^1(a) = \{\pi|z|^2 = a\} \subset M_c = \mathbb{C}$ are product tori,*

$$\pi_c^{-1}(S^1(a)) = \begin{cases} T(a, a-c) & \text{if } c < 0, \\ T(a+c, a) & \text{if } c > 0. \end{cases} \tag{57}$$

Furthermore, the lift of any simple closed curve is Hamiltonian isotopic to a product torus.

Proof. Since $S^1(a)$ is the orbit of the residual Hamiltonian circle action (53), it lifts to an orbit of the T^2-action on $\mathbb{C}^2$, and this orbit is a product torus. The identity (57) follows from the identification of the toric moment map image of g_c with $\{\mu_1 - \mu_2 = c\} \cap \mathbb{R}^2_{\geqslant 0}$, see the discussion surrounding (54) and Figure 6. Let us now prove that lifts of simple closed curves from the spaces $M_{c\neq 0}$ yield product tori up to applying a Hamiltonian isotopy. This follows from the fact that every such curve in $(\mathbb{C}, \omega_0)$ is Hamiltonian isotopic to some $S^1(a)$ (see the discussion in §1.1) together with the fact that Hamiltonian isotopies can be lifted through symplectic reductions by Proposition 2.31. □

In particular, we have proven that we cannot come up with exotic tori by lifting circles from the reduced spaces M_c for $c \neq 0$. For $c = 0$, however, the situation is different, as we have seen in Definition 3.3. Similarly as above, we have

$$\pi_0^{-1}(S^1(a)) = T(a, a), \quad a > 0. \tag{58}$$

Remark 3.5. (Equivariant equivalence) Since the two types of circles $S^1(a), S^1_{\mathrm{Ch}}(a) \subset \mathbb{C}^\times$ are not Hamiltonian isotopic, we have shown that there is no ψ-equivariant Hamiltonian

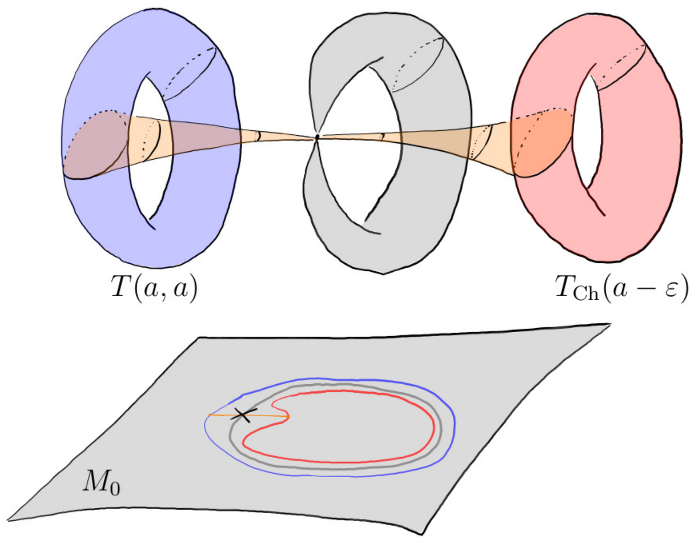

Figure 8. The outer circle lifts to a product torus. The inner circle lifts to a Chekanov torus. There is an intermediate circle lifting to a pinched torus, since the circle fibres of $\pi_0\colon Z_0 \to M_0$ degenerate to a point fibre over the point marked by a cross. The lift of the transverse segment is the union of two Lagrangian disks meeting at the singular point of the pinched torus.

isotopy mapping $T_{\mathrm{Ch}}(a)$ to the product torus $T(a,a)$. In Section 5, we will see that there is no general such Hamiltonian isotopy, i.e. that the Chekanov torus is exotic.

Remark 3.6. (Monotonicity of the Chekanov torus) Proposition 3.4 allows us to compute the area and Maslov class of the Chekanov torus $T_{\mathrm{Ch}}(a)$. The area and Maslov classes depend continuously on the Lagrangian we choose (in the C^1-topology, see for example Remark 4.1). We can define the sequence of Lagrangian tori $L_n = \pi_{1/n}^{-1}(S^1_{\mathrm{Ch}}(a))$ which converges to the Chekanov torus $T_{\mathrm{Ch}}(a)$. However by the second statement of Proposition 3.4, we obtain $L_n \cong T(a+\frac{1}{n}, a)$ and hence the Chekanov torus has the same area and Maslov class as the product torus $T(a,a)$. Therefore, it is monotone. In particular, we cannot use these invariants to prove that the Chekanov torus is exotic.

3.3. An exotic Lagrangian torus fibration

Let us look once more at the quotient map $\pi_0\colon Z_0 \to \mathbb{C}$ coming from the reduction of the circle action generated by $F = \pi(|z_1|^2 - |z_2|^2)$. The standard Lagrangian torus fibration by product tori descends to the fibration of $\mathbb{C}^\times$ by standard circles as on the left hand side of Figure 9. Conversely, the lift of the fibration by circles lifts to a Lagrangian torus fibration of, at the very least, the set $Z_0 \setminus \{0\}$. Let us now try to obtain a fibration of $Z_0 \setminus \{0\}$ by Chekanov tori instead of product tori. The previous paragraph §3.2 suggests how to do this: We should lift a fibration of $\mathbb{C}^\times$ by circles which do not enclose the origin, as these lift to Chekanov tori. A moment's reflection shows that this is impossible. The best we can do is shown on the right hand side of Figure 9, which shows a circle fibration (with one singular point where the circles degenerate to a point) of the plane from which we removed a set which is diffeomorphic to a half-line with endpoint in the origin. Note that we are only interested in Hamiltonian isotopy classes and hence it does not matter what this set looks like precisely, nor what the circle fibration on its complement looks like as long as it has only one singular fibre. Thus from now on, we assume that the set we remove is $\mathbb{R}_{\geqslant 0}$ and we denote the slit plane by $\mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\geqslant 0}$. Furthermore, we take a circle fibration which is standard outside of an ε-neighbourhood of the slit $\mathbb{R}_{\geqslant 0}$ as in Figure 9. We give a concrete such fibration in Lemma 3.8, but for now we consider it given. For simplicity, we encode this circle fibration as the level sets of a function $h \in C^\infty(\mathbb{C}^-)$. Note that we may even assume, by Remark 2.15, that h generates a Hamiltonian circle action; this then determines h up to an additive constant. Let us set

$$h\colon \mathbb{C}^- \to \mathbb{R}, \quad h(z) = \text{area contained by the circle fibre through } z. \tag{59}$$

We obtain the Chekanov tori as lifts,

$$T_{\mathrm{Ch}}(a) = \pi_0^{-1}(h^{-1}(a)), \quad a > 0. \tag{60}$$

Thus we obtain a fibration of $\pi_0^{-1}(\mathbb{C}^-)$ by Chekanov tori. Note that the lift $\pi_0^{-1}(\mathbb{R}_{\geqslant 0})$ is given (in a suitable identification of $(Z_0 \setminus \{0\})/S^1$ with $\mathbb{C}^\times$) by the Lagrangian plane

$$P = \{(z, \overline{z}) \in \mathbb{C}^2 \mid z \in \mathbb{C}\}. \tag{61}$$

We extend the Lagrangian torus fibration of $Z_0 \setminus P$ to a Lagrangian torus fibration of $\mathbb{C}^2 \setminus P$.

Theorem 3.7. *There is a smooth map $p\colon \mathbb{C}^2 \setminus P \to \mathbb{R} \times \mathbb{R}_{\geqslant 0}$ such that the fibres over $\mathbb{R} \times \mathbb{R}_{>0}$ are Lagrangian tori and*

(1) *the fibres over the ray $\{0\} \times \mathbb{R}_{>0}$ are Chekanov tori,*

$$p^{-1}(0, a) = T_{\mathrm{Ch}}(0, a), \quad a > 0, \tag{62}$$

(2) *all other fibres are Hamiltonian isotopic to product tori,*

$$p^{-1}(c, a) \cong \begin{cases} T(a, a - c) & \text{if } c < 0, \\ T(a + c, a) & \text{if } c > 0. \end{cases} \tag{63}$$

In order to extend the Lagrangian torus fibration of $Z_0 \setminus P$, we need to work across different reduced spaces, since $\mathbb{C}^2$ is decomposed into slices $\mathbb{C}^2 = \cup_{c \in \mathbb{R}} Z_c$ and each of the slices reduces to one of the M_c. Recall that up until now, we were sloppy about how we identify $M_c = \mathbb{C}$ but now we need to make this precise. For this, we use an idea going back to Eliashberg–Polterovich [14] and featuring prominently in Auroux [4], namely we consider the following holomorphic function

$$f\colon \mathbb{C}^2 \to \mathbb{C}, \quad (z_1, z_2) \mapsto z_1 z_2. \tag{64}$$

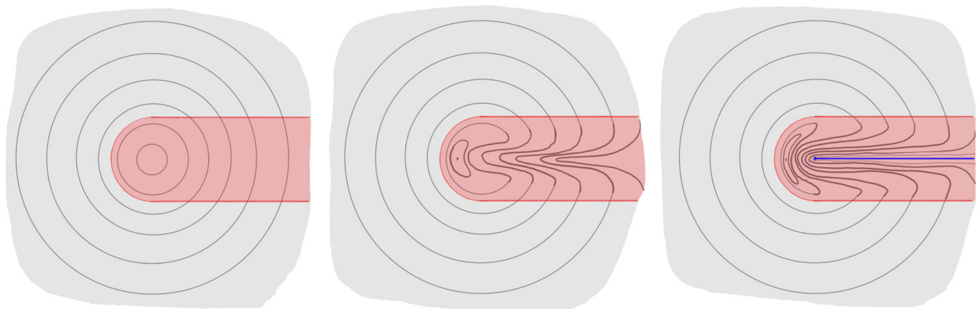

FIGURE 9. A deformation of the standard circle action on $\mathbb{C}$ (on the left) to one on the slit plane $\mathbb{C}^- = \mathbb{C} \setminus \mathbb{R}_{\geqslant 0}$ (on the right). The latter has only Chekanov circles as orbits, as none of them encloses the origin. Note that the orbits are standard outside an ε-neighbourhood of the slit.

Note that this function is invariant under the circle action ψ generated by the Hamiltonian F. Therefore f factors through the reduced spaces. Recall that the quotients carry a residual circle action induced by rotation in the first factor, $\theta.(z_1, z_2) = (e^{2\pi i\theta} z_1, z_2)$. Since f is equivariant with respect to this action and the standard action on $\mathbb{C}$, we deduce that the identification $M_c = \mathbb{C}$ induced by f is equivariant. Following [17, Section 7], we note that this induces a map

$$\pi \colon \mathbb{C}^2 \to \mathbb{C} \times \mathbb{R}, \quad (z_1, z_2) \mapsto (z_1 z_2, F(z_1, z_2) = \pi(|z_1|^2 - |z_2|^2)), \tag{65}$$

which is ψ-invariant and such that $\pi|_{Z_c} = \pi_c$. In other words, π is a global quotient map for the Hamiltonian S^1-action which realizes symplectic reduction on the slices Z_c.

The main idea in order to extend the fibration on $Z_0 \setminus P$ is to define a one-parameter family $\{h^t\}_{t \in [0,1]}$ of Hamiltonian circle actions on $\mathbb{C}$ (or $\mathbb{C}^-$, respectively) which interpolate between the standard circle action and the one defined by (59) and take the $\mathbb{R}$-component in the image of (65) as a parameter to lift this family of systems. More precisely, let us construct this one-parameter family.

Lemma 3.8. *There is a family of functions $h^s \in C^\infty(\mathbb{C})$ for $s \in [0,1)$ and $h^1 \in C^\infty(\mathbb{C}^-)$ which generate Hamiltonian S^1-actions on the respective spaces with the following properties.*

1. *For $s = 0$, we have $h^0(z) = \pi|z|^2$;*
2. *The family depends smoothly on s, meaning $s \mapsto h^s$ is smooth for all $s \in [0,1)$ and $s \mapsto h^s|_{\mathbb{C}^-}$ is smooth for all $s \in [0,1]$;*
3. *For all $\varepsilon > 0$, we can choose h^s such that their orbits are circles centered in $0 \in \mathbb{C}$ outside of a ε-neighbourhood of $\mathbb{R}_{\geqslant 0} \subset \mathbb{C}$.*

See Figure 9 for an illustration of what the orbits of this family of functions looks like. Actually, one way of finding such a family of functions is starting with the corresponding families of orbits as in the figure and defining h^s as in (59).

Proof of Lemma 3.8. Pick a smooth family of $\{\tilde{h}^s\}_{s\in[0,1]}$ with $\tilde{h}^s \in C^\infty(\mathbb{C})$ for $s \in [0,1)$ and $\tilde{h}^1 \in C^\infty(\mathbb{C}^-)$ satisfying properties (1)–(3), but which do not necessarily have periodic flows. This can be explicitly done as follows. For $\varepsilon > 0$, let U_ε be the ε-neighbourhood of the slit $\mathbb{R}_{\geqslant 0} \subset \mathbb{C}$, i.e. all points at Euclidean distance less than ε from $\mathbb{R}_{\geqslant 0}$. Let $\chi_\varepsilon \colon \mathbb{C} \to [0,1]$ be a smooth characteristic function of the slit $\mathbb{R}_{\geqslant 0}$ with

$$\operatorname{supp}(\chi_\varepsilon) \subset U_\epsilon, \quad \chi_\varepsilon|_{\mathbb{R}_{\geqslant 0}} = 1, \quad \chi_\varepsilon|_{\mathbb{C}^-} < 1. \tag{66}$$

Define the family of functions

$$\tilde{h}^s(z) = \pi\,|z|^2 + \frac{s\chi_\varepsilon(z)}{1 - s\chi_\varepsilon(z)}. \tag{67}$$

Note that $\tilde{h}^1$ is not defined on the slit (as expected) and that for a suitable choice of χ_ε, any $\tilde{h}^s$ has a unique critical point in $(-\varepsilon, 0]$. This implies that all level sets are diffeomorphic to circles. This family satisfies properties (1)–(3) but it does not define circle actions in general. However, Remark 2.15 allows us to conclude by setting the value of h^s equal to the symplectic area contained by the level sets of $\tilde{h}^s$. □

Proof of Theorem 3.7. Let $\varepsilon > 0$ be a small parameter and let $b \colon \mathbb{R} \to [0,1]$ be a bump function with support in $(-\varepsilon, \varepsilon)$ and with $b(0) = 1$ as its unique maximum. Let $\{h^s\}_{s\in[0,1]}$ be the family of functions from Lemma 3.8 with ε the parameter used for (3) and set

$$\tilde{H} \colon \mathbb{C} \times \mathbb{R} \setminus (\mathbb{R}_{\geqslant 0} \times \{0\}) \to \mathbb{R}, \quad (z, r) \mapsto h^{b(r)}(z). \tag{68}$$

Let us define $H = \tilde{H} \circ \pi$ as the lift of $\tilde{H}$ under the map (65). We claim that $p = (F, H)$ is the desired Lagrangian torus fibration. First note that H is not defined on the preimage of the set $\mathbb{R}_{\geqslant 0} \times \{0\}$ and this preimage is precisely the Lagrangian plane P. Now let us look at a level set $p^{-1}(c, a) = \{F = c, H = a\}$. Such a level set is given as the lift of a circle $\tilde{H}^{-1}(a) \cap \{r = c\}$ under π. Recall that $\pi|_{Z_c} \colon Z_c \to \{r = c\}$ is the symplectic reduction at level $F = c$. Thus we can use Proposition 2.30 to conclude that the fibre $p^{-1}(c, a)$ is a Lagrangian torus. For $c = 0$, it is a Chekanov torus. This follows from the definition of Chekanov tori and our definition of $\tilde{H}$. Indeed, all circles $\tilde{H}^{-1}(a) \cap \{r = 0\}$ do not enclose the origin, since $\tilde{H}(z, 0) = h^{b(0)}(z) = h^1(z)$. The Hamiltonian isotopy (63) of all other fibres with product tori follows form Proposition 3.4. □

Remark 3.9. (Is it a moment map?) Although we suspect this to be true, the map $p = (F, H)$ we have constructed in §3.3 is a priori *not* a moment map of a Hamiltonian T^2-action on $\mathbb{C}^2 \setminus P$. The Hamiltonians commute by construction, $dF(X_H) = 0$, and F generates a circle action, but it is non-trivial to show that the second Hamiltonian H generates a circle action. Although the corresponding flows in the quotients M_c are circle actions, this does not imply that the lifted flow is periodic. Indeed, recall from the proof of Proposition 2.31 that lifted flows depend on the extension of the lifted Hamiltonian.

3.4. The Chekanov torus and almost toric fibrations

This section is somewhat speculative. It merely serves to pique the interest of the reader who is familiar with almost toric geometry. Its content does not appear anywhere else in the present manuscript (except for the discussion in 6.3) and hence it may be skipped by the reader who is unfamiliar with or uninterested in almost toric fibrations. In §3.3, we have passed from the standard toric structure

$$\mu\colon \mathbb{C}^2 \to \mathbb{R}^2_{\geqslant 0}, \quad (z_1, z_2) \mapsto (\pi|z_1|^2, \pi|z_2|^2), \tag{69}$$

defining a Lagrangian torus fibration by product tori to an *exotic* fibration, i.e. a fibration having Chekanov tori as some of its fibres,

$$p\colon \mathbb{C}^2 \setminus P \to \mathbb{R} \times \mathbb{R}_{\geqslant 0}, \quad (z_1, z_2) \mapsto (F(z_1, z_2) = \pi(|z_1|^2 - |z_2|^2), H(z_1, z_2)). \tag{70}$$

We briefly explain the relationship of this operation with certain operations from *almost toric geometry*. Almost toric geometry is a generalization (in dimension four) of toric geometry, where one allows for more complicated singularities in the fibration, namely so-called *focus-focus singularities*. We do not give details on almost toric geometry here, nor do we actually *prove* some kind of equivalence of the construction from §3.3 and almost toric operations, since this would be outside the scope of the present exposition.

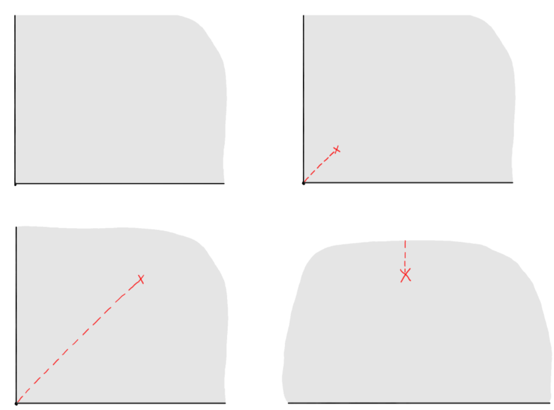

FIGURE 10. An illustration of the three operations in almost toric geometry for $\mathbb{C}^2$. The first diagram is the toric base $\mathbb{R}^2_{\geqslant 0}$ of $\mathbb{C}^2$. The second is obtained from the first by a *nodal trade*. The third is obtained from the second by a *nodal slide*. The fourth is obtained from the third by a *mutation*.

The almost toric geometry package includes three basic operations: *nodal trades*, *nodal slides* and *mutations*. Here we use the terminology of [15]. These operations *do not change the symplectic geometry of the manifold under consideration*, they merely give different Lagrangian fibrations of a fixed manifold. The nodal trade operation [15, Section 8.2] is the most important one. In terms of singularities, it exchanges an *elliptic-elliptic* singularity with a *focus-focus* singularity. The former is of toric type and simply corresponds to a vertex in a toric moment map image, and the latter is the only non-toric singularity of the almost-toric world. We illustrate the operations by the example of $\mathbb{C}^2$, see Figure 10. Passing from the first picture to the second in Figure 10 combinatorially encodes the nodal trade. The base point marked by a cross in Figure 10 is called a *node* and its corresponding fibre is a pinched torus. The main conceptual difference with the toric case is that the base of regular fibres of the fibration is not simply-connected anymore. Indeed, this is crucial since the torus fibration around a node has non-trivial *monodromy*. This monodromy is combinatorially encoded by the dashed line, we again refer to Sections 6, 7 and 8 in [15] for details. One can slide the node along the direction of the dashed line, this is called a nodal slide [15, Section 8.3]. Passing from the second to the third picture in Figure 10 represents a nodal slide. The mutation operation [15, Section 8.4] is represented by passing from the third to the fourth picture, and geometrically it corresponds to choosing a different fundamental domain of the covering associated to the non simply-connected base.

Let us now describe an (equivalent?) procedure using the approach from §3.3. For this, we consider a parametric version $\{p_t\}_{t\in[0,1]}$ interpolating between the standard fibration (69) and the exotic one (70). Define

$$p_t = (F, H_t), \quad H_t = \tilde{H}_t \circ \pi, \quad \tilde{H}_t(z,r) = h^{tb(r)}(z), \tag{71}$$

where we use the same construction and conventions as in the proof of Theorem 3.7, see also the discussion around (68). Note that $p_1 = p$. Since the bump function b takes values in $[0,1]$ and the functions h^s from Lemma 3.8 are defined on $\mathbb{C}$ for all $s \in [0,1)$, we deduce that p_t is defined on all of $\mathbb{C}^2$ for each $t \in [0,1)$, in contrast to $p = p_1$. Another difference with the case $s = 1$ is the presence of a singular fibre in the interior of the base $\mathbb{R} \times \mathbb{R}_{\geqslant 0}$. Indeed, note that $\tilde{H}_t|_{\{r=0\}} = h^t$ and h^t has a level set passing through the origin $(0,0) \in \mathbb{C} \times \mathbb{R}$. Recall that the fibration (65) is singular at $(0,0)$ and thus the corresponding fibre is a pinched torus as shown in Figure 8. Thus the basis of this fibration is as depicted in Figure 11 for different values of $s \in [0,1]$. The cross here depicts a pinched torus fibre. This fibre gets pushed off to infinity in $t = 1$, at the cost of removing a Lagrangian plane from the total space, see also Figure 9. This suggests that the family p_t encodes a nodal trade when we pass from p_0 to p_t for small $t > 0$ and a nodal slide with increasing t.

Remark 3.10. (Nodal trade at infinity) We have described a process to pass from a Lagrangian torus fibration on $\mathbb{C}^2$ to one on $\mathbb{C}^2 \setminus P$. This suggests that there is an inverse procedure which should consist of adding a Lagrangian plane to the latter space and extending the Lagrangian torus fibration over it. This should maybe be viewed as a

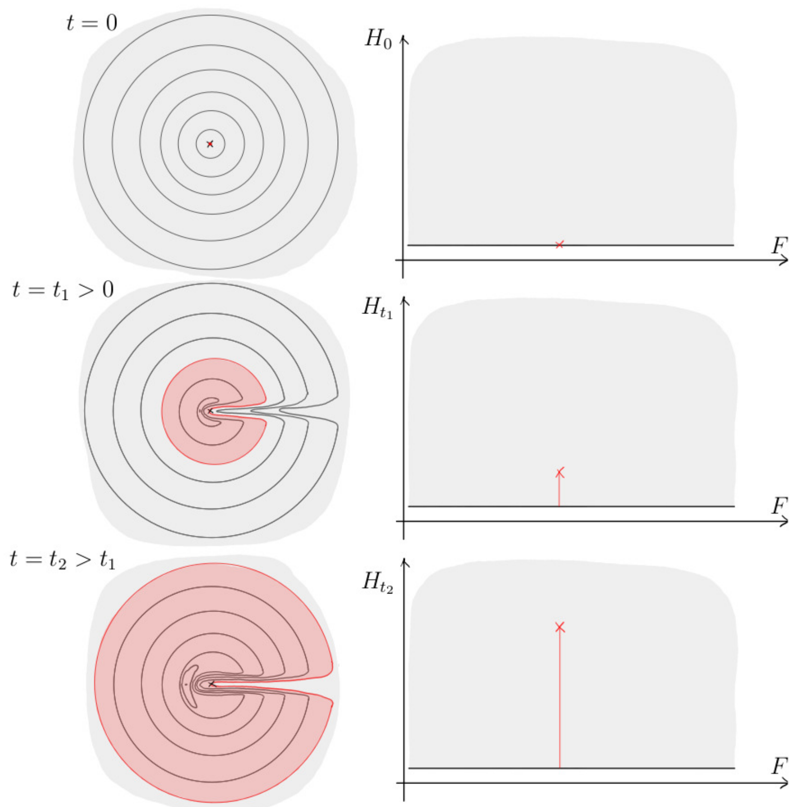

FIGURE 11. The image of the map $p_t = (F, H_t)$ for different values of the parameter t on the right. On the left, the fibrations induced on the reduced space M_0. The level set passing through the origin lifts to a pinched torus, as in Figure 8. The image of the pinched torus is marked by a cross on the right. The fibres over the vertical segment under the cross are Chekanov tori.

nodal trade at infinity, since it is inverse to pushing off a node to infinity. This should correspond to the *integrable Lagrangian surgery* appearing in [17, Section 7.4].

4. Versal deformations of Lagrangians

In this section, we describe small deformations of a given Lagrangian submanifold using Weinstein's theorem 4.2. Up to (locally supported) Hamiltonian isotopies, the deformation space of a given Lagrangian L is parametrized by a small neighbourhood in $H^1(L;\mathbb{R})$. This yields a method of strengthening symplectic invariants of Lagrangian manifolds, since one can evaluate a given symplectic invariant on the deformations of L instead of only on L. These deformations are known as *versal deformations* and the idea of using them to strengthen given invariants goes back to Chekanov [8], see also Chekanov–Schlenk [10], [11].

4.1. Deformations of smooth submanifolds

We start by recalling smooth deformations. Let $N \subset M$ be a smooth submanifold. We would like to describe the ways in which N can be smoothly deformed and for this we need the normal bundle. Recall that the *normal bundle* $\nu N \to N$ of N is defined by the short exact sequence of vector bundles over N

$$0 \to TN \to TM|_N \to \nu N \to 0. \tag{72}$$

In other words, the normal bundle is the collection of directions which are transversal to $TN \subset TM|_N$. Despite its somewhat misleading name, we do not need to pick a Riemannian metric (or something along those lines) to define the normal bundle. The *tubular neighbourhood theorem* states that a neighbourhood of N in M looks like a neighbourhood of the zero section in νN. Let $U \subset \nu N$ and $V \subset M$ be such neighbourhoods with a diffeomorphism $\varphi\colon U \to V$. We obtain a bijection

$$\Gamma(\nu N) \supset \mathcal{U} \to \mathcal{V} \subset \{\text{submanifolds of } M \text{ in } V\}, \quad s \mapsto \varphi(\Gamma_s). \tag{73}$$

By $\Gamma(\nu N)$, we denote the sections of the normal bundle, by $\Gamma_s \subset \nu N$ the graph of a section s and by $\mathcal{U}$ the set of sections which lie in U. The set $\mathcal{V}$ is defined as the submanifolds inside V *which are graphs* in the corresponding tubular neighbourhood. The map (73) is thus tautologically a bijection.

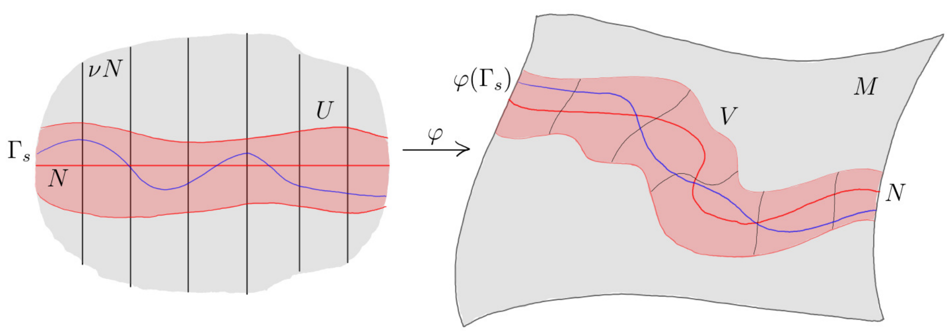

FIGURE 12. Deformations of submanifolds correspond to sections of the normal bundle via the tubular neighbourhood theorem.

Remark 4.1. (C^1-topology) A section of a vector bundle is small in the C^1-topology if its values *and* its derivatives are small. Although one needs to choose a C^1-norm in the process, the corresponding topology is well-defined. Via the tubular neighbourhood theorem and the bijection (73), this induces a topology on the set of submanifolds which are smoothly isotopic to $N \subset M$. We call this the C^1-topology, as well. By definition, one can choose deformations C^1-small enough so that they correspond to a graph in a tubular neighbourhood.

4.2. Cotangent bundles and Weinstein's theorem

Now let $L \subset (M, \omega)$ be a Lagrangian submanifold of a symplectic manifold. Then there is a natural identification (i.e. a vector bundle isomorphism) of the normal bundle νL with the cotangent bundle T^*L. Indeed, the symplectic form yields a surjective homomorphism

$$TM|_L \to T^*L, \quad X \mapsto \iota(X)\omega|_{TL}. \tag{74}$$

By the definition of Lagrangian submanifolds, the kernel of this homomorphism is given by $TL \subset TM|_L$ and hence a comparison with (72) proves that $\nu L \cong T^*L$ by the map

$$\nu L = TM|_L/TL \to T^*L, \quad [X] \mapsto \iota(X)\omega|_{TL}. \tag{75}$$

This is a well-defined explicit identification. The tubular neighbourhood theorem applied to Lagrangian submanifolds thus states that a neighbourhood of L looks like a neighbourhood of the zero-section $0_L \subset T^*L$. *Weinstein's (Lagrangian neighbourhood) theorem* upgrades the diffeomorphism in this claim to a symplectomorphism.

Theorem 4.2. (Weinstein's theorem) *Let $L \subset (M, \omega)$ be a Lagrangian submanifold. Then there is a neighbourhood $U \subset T^*L$ of the zero–section, a neighbourhood $V \subset M$ of L and a symplectomorphism*

$$\varphi\colon (T^*L, \omega_{\mathrm{can}}) \supset U \to V \subset (M, \omega), \quad \varphi^*\omega = \omega_{\mathrm{can}} \tag{76}$$

mapping the zero–section to L.

Note that we tolerate a small abuse of notation here, since L refers to an abstract manifold on one hand and to a Lagrangian submanifold of M on the other. We call such a symplectomorphism $\varphi\colon U \to V$ a *Weinstein chart*. The symplectic form ω_{can} is the canonical symplectic form on cotangent bundles. It is an exact form, $\omega_{\mathrm{can}} = -d\lambda$, where λ is defined as $\sum p_i dq_i$ in a bundle chart of the cotangent bundle coming from a chart of the underlying manifold. See any textbook on symplectic geometry for more details on λ and ω_{can} and for a proof of Weinstein's theorem, for example [21, §3].

4.3. Versal deformations

We have seen that *smooth* deformations of a Lagrangian submanifold are sections of the cotangent bundle, i.e. one-forms. We will see that *Lagrangian* deformations of a Lagrangian submanifold correspond to closed one-forms. We first restrict our attention to the case of the cotangent bundle T^*L of a compact manifold L. This will serve as model, via Weinstein's theorem, for a general Lagrangian. Recall the following classical fact about graphs of one-forms.

Proposition 4.3. *Let T^*L be the cotangent bundle of a compact smooth manifold L equipped with its canonical symplectic form $\omega_{\mathrm{can}} = -d\lambda$. Then we have*

(1) *The graph $\Gamma_\alpha \subset T^*L$ of a one-form $\alpha \in \Omega^1(L)$ is Lagrangian if and only if the one-form is closed;*
(2) *Two graphs $\Gamma_\alpha, \Gamma_\beta \subset T^*L$ are Hamiltonian isotopic if and only if the forms are cohomologous, i.e. there is a smooth function $f \in C^\infty(L)$ for which $\alpha - \beta = df$.*

It follows that the set of C^1-small Lagrangian perturbations of the zero-section (viewed a Lagrangian in T^*L) corresponds to a neighbourhood of the one-form $0 \in \Omega^1(L)$ in the space of closed one-forms. Furthermore, we obtain a bijection between such perturbed Lagrangians up to Hamiltonian isotopy and small elements in the first cohomology $H^1(L;\mathbb{R})$. This readily follows from the definition of de Rham-cohomology as the space of closed forms quotiented by the space of exact forms.

Now let $L \subset M$ be a compact Lagrangian in a symplectic manifold (M, ω) and take a Weinstein chart, $\varphi\colon U \to V$ with conventions as in (76). By Proposition 4.3 this yields a homeomorphism (in the respective C^1-topologies),

$$\Omega^1_{\mathrm{cl}}(L) \supset \widehat{\mathcal{U}} \to \widehat{\mathcal{V}} \subset \{\text{Lagrangians in } M\}, \quad \alpha \mapsto \varphi(\Gamma_\alpha). \tag{77}$$

By $\widehat{\mathcal{U}}$ we have denoted the set of closed sections contained in U and by $\widehat{\mathcal{V}}$ the set of Lagrangians in V which are graphs in the Weinstein chart. Furthermore, we may quotient out by exact one-forms on the left-hand side and by Hamiltonian isotopies supported in V on the right-hand side to obtain an induced homeomorphism. Note that the submanifolds in the image of (77) are all Lagrangian isotopic to L within the Weinstein neighbourhood V.

Proposition 4.4. *Let $L \subset M$ be a compact Lagrangian submanifold. Every Weinstein chart $\varphi\colon U \to V$ yields a C^1-homeomorphism*

$$w_L\colon H^1(L;\mathbb{R}) \supset \mathcal{U} \to \mathcal{V} \subset \{\text{Lagrangians in } M\}/\!\sim, \quad [\alpha] \mapsto [\varphi(\Gamma_\alpha)]. \tag{78}$$

The equivalence relation on Lagrangian submanifolds is given by Hamiltonian isotopies which are supported in V. Furthermore, this homeomorphism is independent of the choice of Weinstein chart.

The first claim of the proposition follows from the above discussion. We postpone the proof of the independence of the Weinstein chart to Remark 4.9, where we construct an inverse to w_L that does not depend on the choice of the Weinstein chart.

Definition 4.5. *Let $L \subset (M, \omega)$ be a compact Lagrangian. Then the associated map w_L from* (78) *is called the* versal deformation *of L.*

Example 4.6. Let $L \subset (M, \omega)$ be a Lagrangian sphere. Since $H^1(L;\mathbb{R}) = 0$, the versal deformation is trivial. Indeed, any Lagrangian isotopy of a sphere is automatically Hamiltonian.

Example 4.7. Let (S^2, ω_{S^2}) be the two-sphere and as Lagrangian submanifold we choose the equator $L = G^{-1}(0)$, which we have viewed as orbit of the Hamiltonian system described in Example 2.5. We identify $(T^*S^1, -d\lambda_{\text{can}})$ with the cylinder $(\mathbb{R} \times S^1, dx \wedge dt)$ and define a Weinstein chart omitting the North and South Poles $N, S \in S^2$,

$$\varphi\colon (-2\pi, 2\pi) \times S^1 \to S^2 \setminus \{N, S\}, \quad (x, t) \mapsto \phi_t^G(\gamma(x)). \tag{79}$$

We have chosen a smooth curve $\gamma\colon [-2\pi, 2\pi] \to S^2$ such that $\gamma(-2\pi) = S$, and $\gamma(2\pi) = N$, and $G(\gamma(x)) = x$. It follows that φ is a symplectomorphism as in the proof of Proposition 2.14. Since L is one-dimensional, every one-form is closed and the homomorphism

$$\Omega^1(S^1) \to \mathbb{R}, \quad \alpha \mapsto \int_{[S^1]} \alpha \tag{80}$$

descends to an isomorphism $H^1(L; \mathbb{R}) \cong \mathbb{R}$. The generator $[S^1] \in H_1(S^1; \mathbb{Z})$ here carries the orientation coming from the circle action generated by G. The integral (80) corresponds to the area contained between the zero-section and the graph of α. By applying the Weinstein chart (which preserves area), we obtain that the versal deformation of L by a small deformation parameter $a \in H^1(L; \mathbb{R})$ can be represented by any Lagrangian circle $L(a)$ in S^2 such that the cylinder bounded by L and $L(a)$ contains symplectic area a. For convenience, we can choose constant one-forms. Under our Weinstein chart, these correspond to orbits of the Hamiltonian circle action G, meaning that the map

$$H^1(L; \mathbb{R}) \cong \mathbb{R} \supset (-2\pi, 2\pi) \to \{\text{Lagrangians in } S^2\}, \quad a \mapsto G^{-1}(a) \tag{81}$$

induces a versal deformation. We will see that it is not a coincidence that one can choose nearby orbits of a Hamiltonian circle action as Lagrangian neighbours of the orbit $G^{-1}(0)$. See for example §4.4.

We would usually like to think of versal deformations as explicit families of Lagrangian submanifolds,

$$H^1(L; \mathbb{R}) \supset \mathcal{U} \to \{\text{Lagrangians in } M\}, \quad a \mapsto L(a). \tag{82}$$

However, this way of thinking is a little dangerous, since by far not all families of Lagrangian submanifolds are versal deformations, i.e. come from Weinstein neighbourhoods. For one, the family needs to be continuous and satisfy $L(0) = L$, where L is the original Lagrangian we want to deform. But there are more subtle conditions, which we discuss now. For clarity, we illustrate these by Example 4.7.

Our Lagrangian in Example 4.7 was the equatorial circle $L \subset S^2$. We realized its versal deformation by the set of circles of constant height $G^{-1}(a)$. This choice is obviously not unique, since there are many Lagrangians which represent the same Hamiltonian isotopy class. However not every one-parameter family of Lagrangians yields a versal deformation. For one, a family $a \mapsto L(a)$ has to be surjective onto Hamiltonian isotopy classes. In Example 4.7, one can easily imagine a one-parameter family of circles for which the areas of the two disks they bound are constant. All such circles are Hamiltonian isotopic, meaning that curve $a \mapsto [L(a)]$ is constant on the level of Hamiltonian isotopy classes. This is clearly not a versal deformation. But it gets more subtle, since versal deformations

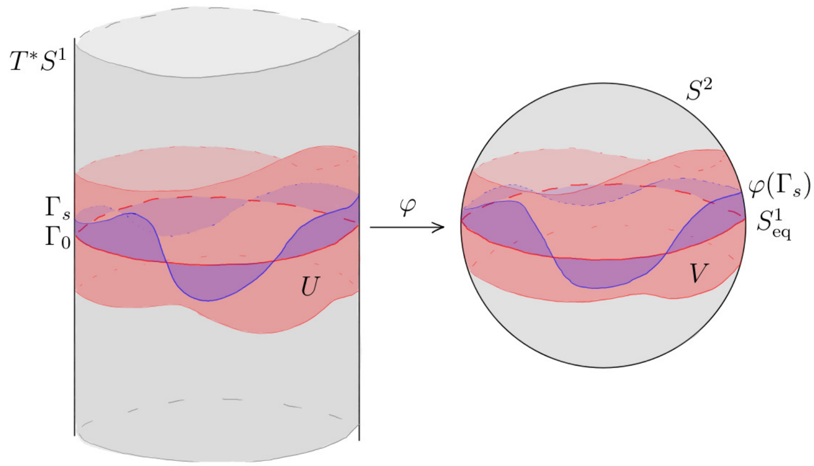

FIGURE 13. A Weinstein chart $\varphi\colon U \to V$ of the equator $S^1_{\text{eq}} \subset S^2$. The area bounded by the zero-section $\Gamma_0 \subset T^*S^1$ and the graph $\Gamma_s \subset T^*S^1$ is equal to the area bounded by the equator $S^1_{\text{eq}} \subset S^2$ and $\varphi(\Gamma_s) \subset S^2$.

encode certain quantitative properties of the symplectic form. Note, again in Example 4.7, that the deformation parameter encodes the symplectic area contained between $G^{-1}(a)$ and its deformation. For example the family of circles of constant height given by

$$H^1(S^1;\mathbb{R}) \supset (-2\pi, 2\pi) \to \{\text{Lagrangians in } S^2\}, \quad a \mapsto G^{-1}(a^3) \tag{83}$$

exhausts all Hamiltonian classes of neighbourhing Lagrangians, but it is not a versal deformation of L. Indeed the symplectic area between L and $G^{-1}(a^3)$ is a^3 instead of a. We have the following general fact about versal deformations.

Proposition 4.8. *Let $L \subset (M, \omega)$ be a compact Lagrangian submanifold and let $a \mapsto L(a)$ be a versal deformation of L, meaning that every Lagrangian $L(a)$ for $a \in \mathcal{U} \subset H^1(L;\mathbb{R})$ is of the form $L(a) = \varphi(\Gamma_\alpha)$ for a Weinstein chart φ and a closed one-form α with $[\alpha] = a$. Then the Lagrangians $L(a)$ have the following property*

$$\int_{Z_\xi(L(a))} \omega = \langle a, \xi \rangle, \qquad \textit{for all } \xi \in H_1(L). \tag{84}$$

Before discussing the proof, we need to explain the notation in (84), see also Figure 14. Let $a \in \mathcal{U}$ and $\xi \in H_1(L)$, then by $Z_\xi(L(a))$ we denote a cylinder with one boundary component on L and one boundary component on $L(a)$ such that the boundary component on L is in the homology class of $\xi \in H_1(L)$. Note that (84) implicitly states that the value of the integral on the left-hand side is independent of all the choices we made to get a cylinder $Z_\xi(L(a))$. This will become apparent in the proof.

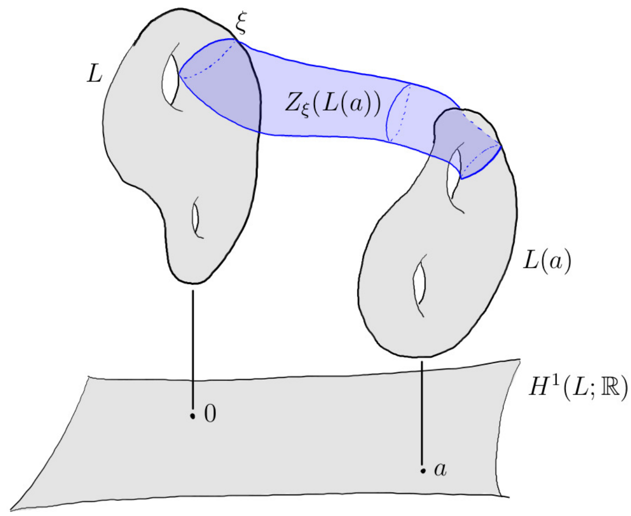

Figure 14. An illustration of Proposition 4.8. If $L(a)$ is a versal deformation of L for $a \in H^1(L;\mathbb{R})$, then the area of a cylinder over a cycle representing $\xi \in H_1(L)$ is equal to $\langle a, \xi \rangle$.

Proof of Proposition 4.8. The proof is a computation. The first crucial ingredient is the fact that we work in a Weinstein chart $\varphi\colon U \to V$ and hence the symplectic form ω is exact when restricted to V. Indeed, by (76), we have $\varphi^*\omega|_V = -d\lambda$, where λ is the Liouville one-form. For simplicity, we will make the computation in the cotangent bundle $U \subset T^*L$. The second crucial ingredient is the following property of the Liouville one-form

$$\alpha^*\lambda = \alpha, \quad \text{for all } \alpha \in \Omega^1(L). \tag{85}$$

This is the property which bridges the gap between purely cohomological data on the one hand and quantitative data about the symplectic form ω on the other. Now let $L(a) = \varphi(\Gamma_\alpha)$ for $a = [\alpha]$. Let $\xi \in H_1(L)$ and let $c\colon S^1 \to L \subset T^*L$ be a curve representing ξ. Then

$$\int_{Z_\xi(L(a))} \omega = \int_{\varphi^{-1}(Z_\xi(L(a)))} \varphi^*\omega = \int_{\alpha(c)} \lambda - \int_c \lambda = \int_c \alpha^*\lambda = \int_c \alpha = \langle a, \xi \rangle. \tag{86}$$

For the second identity, we have used Stokes and the fact that $L(a) = \varphi(\Gamma_\alpha)$. For the third identity, we have used that λ vanishes on the zero-section. This follows from the expression in coordinates $\lambda = \sum_i p_i dq_i$ or also from property (85). $\square$

Remark 4.9. The identity (84) can be used to construct an inverse to the map w_L from (78). Indeed, to a Lagrangian L' which is Lagrangian isotopic to L inside a tubular neighbourhood V of L, one can associate the corresponding cohomology class $a_{L'} \in H^1(L;\mathbb{R})$ by measuring the symplectic areas of cylinders between L and L' as in (84),

$$\mathcal{V} \to \mathcal{U} \subset H^1(L;\mathbb{R}), \quad [L'] \mapsto a_{L'}, \quad \int_{Z_\xi(L')} \omega = \langle a_{L'}, \xi \rangle, \quad \forall \xi \in H_1(L). \tag{87}$$

This map is well-defined and it depends only on the Hamiltonian isotopy class of L'. The computation in the proof of Proposition 4.8 shows that it is an inverse to (78). Since this map does not depend on a Weinstein chart, this proves that (78) is independent of the choice of Weinstein chart.

4.4. Versal deformations of product tori

Note the similarity between the area condition (84) for versal deformations and the one in (21) for Hamiltonian circle actions. Indeed, an effective Hamiltonian circle action (Σ, ω, H) yields an associated versal deformation of a circle $H^{-1}(c)$ by setting

$$(-\varepsilon, \varepsilon) \to \{\text{Lagrangians in } \Sigma\}, \quad a \mapsto H^{-1}(c + a), \tag{88}$$

where $\varepsilon > 0$ is chosen small enough so that there are no critical values in $(c - \varepsilon, c + \varepsilon)$. Indeed, we can use exactly the same argument as in Example 4.7. In fact, the same argument can be used for all *toric* Hamiltonian torus actions. Since we have not yet introduced toric structures, we restrict our attention to the case of product tori in $\mathbb{C}^n$, see also Example 2.20. The proof we give can be carried over *verbatim* to get a versal deformation of general toric fibres[1]. Indeed, we will make use of the commuting Hamiltonians

$$F_i(z) = \pi |z_i|^2, \quad \omega(X_{F_i}, X_{F_j}) = 0, \quad i, j \in \{1, \ldots, n\}. \tag{89}$$

These fit together to a moment map $\mu = (F_1, \ldots, F_n) \colon \mathbb{C}^n \to \mathbb{R}^n_{\geqslant 0}$ the fibres of which are precisely the product tori,

$$\mu^{-1}(x) = T(x), \quad (x_1, \ldots, x_n) \in \mathbb{R}^n_{>0}. \tag{90}$$

See also Example 2.28. In the general toric case, there is a similar moment map and the same arguments apply to yield a versal deformation of its Lagrangian fibres. Note that $H^1(T(x), \mathbb{R}) \cong \mathbb{R}^n$ and there is a canonical identification $H^1(T(x), \mathbb{R}) = \mathbb{R}^n$ by choosing a dual basis to the circle orbits of the F_i. The versal deformation is simply given by varying the base point of $\mu \colon \mathbb{C}^n \to \mathbb{R}^n_{\geqslant 0}$.

Proposition 4.10. *Let $c \in \mathbb{R}^n_{>0}$, then we obtain a versal deformation of the product torus $T(c) \subset \mathbb{C}^n$ by setting*

$$H^1(T(c), \mathbb{R}) \supset (\mathbb{R}^n_{>0} - c) \to \{\textit{Lagrangians in } \mathbb{C}^n\}, \quad a \mapsto T(c + a). \tag{91}$$

By $\mathbb{R}^n_{>0} - c$ we denote the positive orthant shifted by $-c \in \mathbb{R}^n$.

---

[1] Note that for a product torus $T(c) \subset \mathbb{C}^n$, everything splits into factors and so one could just use the product of versal deformations of the factors $S^1(c_i) \subset \mathbb{C}$. In the general toric setting this is no longer true and we choose an approach, which may seem too complicated for $\mathbb{C}^n$, but which generalizes to the toric setting.

Proof. We denote the abstract n-torus by $T^n = (\mathbb{R}/\mathbb{Z})^n$. Its cotangent bundle $(T^*T^n, \omega_{\text{can}} = d\lambda)$ can be naturally identified with

$$\mathbb{R}^n \times T^n = \{(x,\theta) = (x_1, \dots, x_n, \theta_1, \dots, \theta_n) \,|\, x_i \in \mathbb{R},\, \theta_i \in S^1\}, \quad \omega = \sum_{i=1}^n dx_i \wedge d\theta_i. \tag{92}$$

This is the n-fold product of Example 2.6 and hence there is a natural T^n-action on the space (92) given by rotation in the T^n-coordinate. It is generated by the commuting set of Hamiltonians $G_i(x,\theta) = x_i$. Define

$$\varphi\colon (\mathbb{R}^n_{>0} - c) \times T^n \to \mathbb{C}^n, \quad (x,\theta) \mapsto (\phi^{F_1}_{\theta_1} \circ \dots \circ \phi^{F_n}_{\theta_n})(f(c+x)), \tag{93}$$

where $f\colon \mathbb{R}^n_{>0} \to \mathbb{R}^n_{>0} \subset \mathbb{C}^n$ is a diffeomorphism. We claim that, for a suitable choice of f, this map is a T^n-equivariant symplectomorphism onto its image intertwining the action by rotation on the cotangent bundle with the standard action on $\mathbb{C}^n$. This claim finishes the proof, since it shows that φ is a Weinstein chart for $T(c)$ such that the graphs of constant one-forms, $\{a\} \times T^n$ are mapped to $T(c+a)$. The T^n-equivariance of (93) is obvious by definition and does not depend on the choice of f. We now need to choose an f for which φ is symplectic. Note the similarity of φ with the symplectomorphism h constructed in the proof of Proposition 2.14. The role of the curve γ as *transverse coordinate* to the orbits of the Hamiltonian system is played here by f. The crucial property we need in order for φ to be symplectic is the analogue of (22),

$$F_i(f(x)) = x_i, \quad i \in \{1, \dots, n\}. \tag{94}$$

Equivalently, f is a section if we view the moment map $\mu\colon \mathbb{C}^n \to \mathbb{R}^n_{\geqslant 0}$ as a (singular) fibration. From (94) we deduce that the suitable definition for f is $f_i(x) = \frac{\sqrt{x_i}}{\pi}$. In order for φ to be symplectic, it suffices to check that

$$\omega_0(\varphi_*\partial_{x_i}, \varphi_*\partial_{x_j}) = 0, \quad \omega_0(\varphi_*\partial_{\theta_i}, \varphi_*\partial_{\theta_j}) = 0, \quad \omega_0(\varphi_*\partial_{x_i}, \varphi_*\partial_{\theta_j}) = \delta_{ij}. \tag{95}$$

Note that $\varphi_*\partial_{x_i}$ is contained in the Lagrangian subset $\mathbb{R}^n \subset \mathbb{C}^n$, which proves the first set of equations. For the second set of equations, note that $\varphi_*\partial_{\theta_i} = X_{F_i}$ and use the commutativity of the Hamiltonians F_i, F_j. Apply Hamilton's equation to to the third set of equations to find

$$\omega_0(\varphi_*\partial_{x_i}, X_{F_j}) = dF_j(\varphi_*\partial_{x_i}) = \delta_{ij}. \tag{96}$$

In the last equality we have used the fact that the Hamiltonian flows $\phi^{F_k}_t$ preserve the functions F_j along with (94). $\square$

5. Displacement energy and exoticity

We use a symplectic invariant called *displacement energy* to prove that the Chekanov torus is exotic. The displacement energy $e(M,A) \in \mathbb{R} \cup \{\infty\}$ is a numerical invariant attached to any compact subset $A \subset M$. It essentially measures how much *Hofer energy* is needed for the set A to be displaced from itself by a Hamiltonian isotopy. The displacement energy by itself is not sufficient to distinguish the Chekanov torus T^2_{Ch} from the product torus $T(1,1)$. Indeed, we will see that

$$e(\mathbb{C}^2, T_{\mathrm{Ch}}) = 1 = e(\mathbb{C}^2, T(1,1)). \tag{97}$$

Chekanov's crucial insight was that the displacement energy invariant may be strengthened by evaluating it on the versal deformation of the Lagrangians in question. This indeed yields a means of distinguishing the Lagrangian tori in question.

5.1. The group of Hamiltonian diffeomorphisms

Let (M,ω) be a symplectic manifold. Recall from §2.1 that to any smooth function H on M, we can associate a Hamiltonian vector field X_H via Hamilton's equation $dH = -\iota(X_H)\omega$. By integrating X_H, we obtain a Hamiltonian flow ϕ^H_t. For the previous sections, it was sufficient to think about *autonomous* Hamiltonians flows, i.e. those flows which are generated by a vector field which is *constant in time*. However, in order to define the full group of Hamiltonian diffeomorphisms, we need to allow for time-dependent Hamiltonian vector fields.

Definition 5.1. *A smooth family of diffeomorphisms $t \mapsto \phi_t$ with $\phi_0 = \mathrm{id}$ is called* Hamiltonian flow *if there is a smooth family of Hamiltonian functions $t \mapsto H_t$ such that the vector field tangent to ϕ_t at time t is the Hamiltonian vector field associated to H_t via Hamilton's equation* (10)*.*

Furthermore, we define the *group of Hamiltonian diffeomorphisms* (or simply *Hamiltonian group*) $\mathrm{Ham}(M,\omega)$ as the set of diffeomorphisms of M that can be realized as the time-one map of a Hamiltonian flow. One can verify that this is indeed a group. It is useful to think of this group as an infinite-dimensional Lie group with Lie algebra given by the Hamiltonian vector fields. Hamiltonian vector fields are in bijection with smooth functions up to adding a constant to the function, therefore we may think of the Lie algebra of Hamiltonian diffeomorphisms as $C^\infty(M)/\mathbb{R}$. A big difference between autonomous and time-dependent Hamiltonian flows is that the latter do not preserve the Hamiltonian function generating them. However, time-dependent Hamiltonian flows still preserve the symplectic form, meaning that we obtain an inclusion

$$\mathrm{Ham}(M,\omega) \subseteq \mathrm{Symp}(M,\omega). \tag{98}$$

In fact, the Hamiltonian group is even contained in the identity component of the symplectic group, $\mathrm{Symp}_0(M,\omega)$. In case the first homology $H^1(M;\mathbb{R})$ vanishes, this inclusion is an equality. Indeed, a vector field X generates a symplectic isotopy if and only if $\mathfrak{L}_X\omega = d\iota(X)\omega = 0$. In case the first cohomology vanishes, the closed form $\iota(X)\omega$ is automatically exact meaning that there is a function H with $dH = -\iota(X)\omega$. On more general manifolds, the difference between elements in $\mathrm{Symp}_0(M,\omega)$ and Hamiltonian diffeomorphisms is measured by the *symplectic flux*. We refer to [23, Chapter 14] for more details.

Remark 5.2. (Compact support) Note that we have not assumed that the Hamiltonian functions (and hence the Hamiltonian flows) have compact support, as is the case for example in [23]. For example, the harmonic oscillator $F(z) = \pi|z|^2$ does not have compact support. This means that we may run into trouble sometimes, since some Hamiltonian flows may not be defined for all times if we do not assume compact support.

5.2. Displacement energy

We will only give the key definitions and examples here. For a detailed introduction to the Hamiltonian group and the geometry induced on it by the Hofer norm, we again refer to Polterovich [23]. Let $H = \{H_t\}_{t\in[0,1]}$ be a time-dependent Hamiltonian function on a symplectic manifold (M,ω). For the following discussion we either need to assume that M is compact or that H_t is compactly supported. The *Hofer norm* of H_t is defined as

$$\|H\| = \int_0^1 \left(\max_{x\in M} H_t(x) - \min_{x\in M} H_t(x) \right) dt. \tag{99}$$

Definition 5.3. *Let $A \subset M$ be a compact subset. Its* displacement energy *is defined as*

$$e(M,A) = \inf \left\{ \|H\| \;\middle|\; \phi_1^H(A) \cap A = \varnothing \right\}. \tag{100}$$

If the set in (100) *is empty, we set $e(M,A) = \infty$ by convention. In that case A is called* non-displaceable.

Recall that by ϕ_t^H we denote the Hamiltonian flow at time t generated by H. The displacement energy is a symplectic invariant in the sense that

$$e(M,\phi(A)) = e(M,A), \quad \text{for all } \phi \in \mathrm{Symp}(M,\omega). \tag{101}$$

Although the displacement energy is defined for general subsets, we will be exclusively interested in the displacement energy of Lagrangians. We thus obtain an invariant to distinguish Lagrangian submanifolds. For all of our applications, it is enough to know the displacement energy of product tori in $\mathbb{C}^n$, see also Example 2.20.

Example 5.4. (Circles in the plane) The circle $S^1(x) \subset \mathbb{C}$ containing area $x > 0$ has displacement energy

$$e(\mathbb{C}, S^1(x)) = x. \tag{102}$$

The upper bound $e(\mathbb{C}, S^1(x)) \leqslant x$ can be proven by explicitly writing down a Hamiltonian which displaces $S^1(x)$ using energy $x + \varepsilon$ for all $\varepsilon > 0$. The lower bound can be obtained either by the calculus of variations of the action functional or by J-holomorphic curve techniques, see Theorem 5.6. This illustrates that, despite its simplicity, this result is non-trivial.

Example 5.5. (Product tori) By taking products of the previous example we obtain product tori $T(x_1,\dots,x_n) \subset \mathbb{C}^n$. Note that a Hamiltonian which displaces a factor $S^1(x_i) \subset \mathbb{C}$ of the product torus displaces the product torus when completed with the identity in the remaining components and when suitably cut off. Therefore, we obtain $\min\{x_1, \ldots, x_n\}$ as an upper bound. As it turns out, this bound is sharp,

$$e(\mathbb{C}^n, T(x_1, \ldots, x_n)) = \min\{x_1, \ldots, x_n\}. \tag{103}$$

Again, the lower bound is non-trivial and follows for example from Theorem 5.6.

A classical result by Chekanov [9], see also Oh [22], gives a lower bound on the displacement energy in terms of the minimal area of non-constant J-holomorphic curves. Let $\mathcal{J}(M, \omega)$ be the space of ω-tame almost complex structures and define

$$\sigma(M, L, J) = \inf \left\{ \int_D u^*\omega \,\middle|\, u\colon (D, \partial D) \to (M, L) \ J\text{-holomorphic, non-constant} \right\}. \tag{104}$$

By tameness and Gromov compactness, this value is attained and thus strictly positive. Now set

$$\sigma(M, L) = \sup_{J \in \mathcal{J}(M,\omega)} \sigma(M, L, J). \tag{105}$$

Theorem 5.6. (Chekanov)

$$\sigma(M, L) \leqslant e(M, L). \tag{106}$$

In practice, we can thus *choose* a suitable tame almost complex structure J and compute $\sigma(M, L, J)$ to obtain a lower bound. For $S^1(a) \subset \mathbb{C}$, see Example 5.4, we can pick the standard complex structure on $\mathbb{C}$ and see that the area-minimal non-constant disk has area a. A similar argument works in Example 5.5.

5.3. Displacement energy and symplectic reduction

As it turns out, symplectic reduction is a very useful tool for computing displacement energy. This is due to the fact that if one can displace a certain set A in a symplectic quotient (M_c, ω_c) of (M, ω) by a Hamiltonian isotopy, then one can displace its lifted set $\pi_c^{-1}(A)$ by the lifted Hamiltonian isotopy.

Proposition 5.7. *Let (M, ω, H) be a Hamiltonian circle action which admits symplectic reduction at the level $c \in \mathbb{R}$. Then, in the notation of* (40)*, we have*

$$e(M, \pi_c^{-1}(A)) \leqslant e(M_c, A), \quad A \subset M_c. \tag{107}$$

Sketch of proof. If the set A is non-displaceable, there is nothing to prove. Assume it is displaceable. Let $G_c = \{G_{c,t}\}_{t\in[0,1]}$ be a (time-dependent) Hamiltonian with Hofer norm $g > 0$ which displaces the set A. Then any lifted Hamiltonian $G = \{G_t\}_{t\in[0,1]}$ as in Proposition 2.31 displaces the set $\pi^{-1}(A)$. Furthermore, we can choose a suitable cut-off to extend G_t to all of M so that it also has Hofer norm g. $\square$

Remark 5.8. (General symplectic reduction) Although in §2.7 we have discussed symplectic reduction only for Hamiltonian circle actions, an analogous statement to Proposition 5.7 holds for symplectic reduction by any group.

The relationship between symplectic reduction and displacement energy was for example used in [1] and [5]. Note also that the method of probes by McDuff [20] follows from Proposition 5.7. As in Remark 2.33 and §3.1, performing symplectic reduction on a segment yields a two-dimensional reduced space (in the case of probes it is a disk) in which invariant Lagrangians project to a curve. In a second step, displacements of a curve can be lifted to displacements of the original Lagrangian.

5.4. Displacement energy germs

We now combine versal deformations and displacement energy. Let $L \subset (M,\omega)$ be a compact Lagrangian and $w_L \colon \mathcal{U} \to \mathcal{V}$ a versal deformation of L as in Proposition 4.4. Recall that $\mathcal{V}$ is a set of Lagrangians up to locally supported Hamiltonian isotopies. Since the displacement energy is invariant under Hamiltonian isotopies by (101), we can consider the displacement energy of the image of versal deformations,

$$\mathcal{E}_L \colon H^1(L;\mathbb{R}) \supset \mathcal{U} \to \mathbb{R} \cup \{\infty\}, \quad a \mapsto e(M, w_L(a)). \tag{108}$$

Definition 5.9. *The function $\mathcal{E}_L$ is called* displacement energy germ *of the compact Lagrangian $L \subset (M,\omega)$.*

As the terminology suggests, we should think of $\mathcal{E}_L$ as a germ instead of a function (although we will be sloppy here concerning this point). Indeed, the domain of the function $\mathcal{E}_L$ we have defined depends on the choice of the Weinstein neighbourhood used to define the versal deformation. Hence the domain of definition may shrink when considering the intersection for two different choices. The germ of the function however is well-defined, by Proposition 4.4. In view of the symplectic invariance of displacement energy, we obtain the following invariance of $\mathcal{E}_L$ on the intersection of domains of definition,

$$\mathcal{E}_{\phi(L)} = \mathcal{E}_L \circ (\phi_L)^*, \quad \text{where } (\phi_L)_* \colon H_1(L;\mathbb{Z}) \to H_1(\phi(L),\mathbb{Z}). \tag{109}$$

From now on, denote by $L = T(1,1) \subset \mathbb{C}^2$ the Clifford torus and by $L' = T_{\mathrm{Ch}}(1) \subset \mathbb{C}^2$ the Chekanov torus. The arguments for general area parameter $a > 0$ are strictly analogous. We will show that

$$\mathcal{E}_L(a,b) = \min\{1+a, 1+b\}, \quad \mathcal{E}_{L'}(s,t) = 1+t \tag{110}$$

on some neighbourhoods of the origin. Here we have chosen suitable linear identifications $H^1(L;\mathbb{R}) \cong \mathbb{R}^2 \cong H^1(L';\mathbb{R})$. By the invariance (109), this shows that the Chekanov torus is exotic.

Theorem 5.10. (Chekanov [8], Eliashberg–Polterovich [14]) *The Chekanov torus $L' = T_{\mathrm{Ch}}(1) \subset \mathbb{C}^2$ is exotic, i.e. there is no symplectomorphism of $\mathbb{C}^2$ mapping L' to a product torus.*

For the product torus L, recall that we have already determined an explicit versal deformation of product tori in Proposition 4.10,

$$(-1,\infty) \times (-1,\infty) \to \{\text{Lagrangians in } \mathbb{C}^2\}, \quad (a,b) \mapsto T(1+a, 1+b) \tag{111}$$

We have identified $H^1(L;\mathbb{R}) \cong \mathbb{R}^2$ by the identification dual to $H_1(L;\mathbb{Z}) = \mathbb{Z}\langle[\gamma_1],[\gamma_2]\rangle$, where γ_1 is the orbit of the Hamiltonian circle action F_1 and γ_2 the orbit of F_2. Combining this versal deformation with Example 5.5, we obtain the first equation from (110).

5.5. The Chekanov torus

Recall from Remark 3.6 that it is enough to distinguish L and L' in order to prove Theorem 5.10. Indeed, if L' is Hamiltonian isotopic to a product torus, then this product torus must be monotone of the same proportionality constant between area and Maslov class. Recall from the same remark that the area and Maslov classes are insufficient to distinguish L and L'. Before turning to versal deformations, let us first show that displacement energy by itself is also insufficient. Indeed, recall from Example 5.5 that $e(\mathbb{C}^2, L) = 1$.

Proposition 5.11. *The displacement energy of the Chekanov torus is* $e(\mathbb{C}^2, L') = 1$.

Proof. Let us first prove the upper bound $e(\mathbb{C}^2, L') \leqslant 1$. Recall that, by Definition 3.3, the Chekanov torus is a lift of a circle $S^1_{\mathrm{Ch}}(1)$ from the reduced space $\pi_0\colon Z_0 \setminus \{0\} \to \mathbb{C}^\times$. By Proposition 5.7 we can estimate,

$$e(\mathbb{C}^2, L') \leqslant e(\mathbb{C}, S^1_{\mathrm{Ch}}(1)) \leqslant 1. \tag{112}$$

For the lower bound, we note that displacement energy is upper semi-continuous in the C^1-topology on the space of Lagrangian manifolds. This follows from the fact that if a Hamiltonian diffeomorphism displaces a given compact Lagrangian L, then it also displaces a small Weinstein neighbourhood of L and in particular all the Lagrangian neighbours contained in that neighbourhood. Now take the sequence L_n (with $a = 1$) as in Remark 3.6 which converges to L' and satisfies $L_n \cong T(1+\frac{1}{n}, 1)$. This implies $e(L_n) = 1$ by Example 5.5, which allows us to conclude by upper semi-continuity. $\square$

The construction of the Chekanov torus as lift $\pi_0^{-1}(S^1_{\mathrm{Ch}}(1)) \subset \mathbb{C}^2$, see also Definition 3.3, suggests to use the following family of Lagrangians to find a versal deformation,

$$(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon) \to \{\text{Lagrangians in } \mathbb{C}^2\}, \quad (s,t) \mapsto L'(s,t) = \pi_s^{-1}(S^1_{\mathrm{Ch}}(1+t)). \tag{113}$$

This is indeed a versal deformation as can be shown by choosing an equivariant Weinstein chart with respect to the action $e^{2\pi i\theta}(z_1,z_2) = (e^{2\pi i\theta}z_1, e^{-2\pi i\theta}z_2)$ on $\mathbb{C}^2$. A deformation of the form $L'(0,t)$ is a Chekanov torus and, as in Proposition 5.11, its displacement energy is equal to $1+t$. By Proposition 3.4, we see that any $L'(s,t)$ for $s \neq 0$ is Hamiltonian isotopic to a product torus,

$$L'(s,t) \cong \begin{cases} T(1+t, 1+t-s) & \text{if } s<0, \\ T(1+t+s, 1+t) & \text{if } s>0. \end{cases} \tag{114}$$

Note that in both cases of (114), the displacement energy is equal to $1+t$ and hence the second identity from (110) follows. This finishes the proof of the exoticity of L'.

6. Outlook : Toric manifolds

6.1. Overview

Let us first pick up the discussion of Hamiltonian torus actions from §2.6. Recall that a Hamiltonian T^k-action is a smooth group action $\theta \mapsto \psi_\theta$ of the k-torus on a symplectic manifold (M, ω) which can be written as a composition of Hamiltonian flows,

$$\psi_\theta = \phi_{\theta_1}^{F_1} \circ \ldots \circ \phi_{\theta_k}^{F_k}, \quad \theta = (\theta_1, \ldots, \theta_k) \in T^k = (\mathbb{R}/\mathbb{Z})^k \tag{115}$$

for a set of k commuting Hamiltonians F_i, each of which generates a circle action. The map $\mu = (F_1, \ldots, F_k)\colon M \to \mathbb{R}^k$ is called a *moment map* of the action ψ. Note that the moment map determines the torus action, but the converse is not true. Indeed, one can add constants to the F_i and apply base changes in $\mathrm{GL}(k; \mathbb{Z})$ to μ without changing the action. We call elements in the group $\mathrm{Aff}_k(\mathbb{Z}) = \mathrm{GL}(k; \mathbb{Z}) \ltimes \mathbb{R}^k$ *affine integral transformations*. Atiyah [2] and Guillemin–Sternberg [19] proved that the moment map of a Hamiltonian torus action has the following remarkable property.

Theorem 6.1. (Atiyah/Guillemin–Sternberg) *The image $\mu(M) \subset \mathbb{R}^k$ of the moment map μ of a Hamiltonian torus action on a compact symplectic manifold (M, ω) is a rational convex polytope given by the convex hull of the images of fixed points.*

The image $\Delta = \mu(M)$ is called the *moment polytope* of (M, ω, μ). The proof of the theorem uses Morse-theoretic arguments akin to the discussion from §2.3. In fact, one of the crucial ingredients is the fact that Hamiltonians generating circle actions are Morse-Bott functions all of whose critical submanifolds have even Morse index. Now one can ask the following classification question:

Can we reconstruct (M, ω, μ) from the moment polytope $\Delta \subset \mathbb{R}^n$?

In general, the answer is *no*. Note however that the question is reasonable, since we have proved in Theorem 2.17 that the answer is *yes* in the (very) special case of circle actions on symplectic surfaces. As it turns out, the appropriate subset of Hamiltonian torus actions to look at is the *toric* ones.

Definition 6.2. *A Hamiltonian torus action ψ on a symplectic manifold (M, ω) is called* toric *if it is effective and the torus has half the dimension of M.*

We obtain examples of toric symplectic manifolds by taking products of symplectic surfaces carrying a Hamiltonian circle action, such as the plane $\mathbb{C}$, the sphere S^2 and the cylinder T^*S^1, see Examples 2.4, 2.5 and 2.6. The respective moment polytopes are products of the images of the Hamiltonians generating the circle actions. From now on, we assume that M has dimension $2n$, meaning that the acting torus has dimension n. Delzant [12] proved that *compact* toric manifolds are classified by their moment polytopes.

Theorem 6.3. (Delzant) *There is a bijective correspondence between* Delzant polytopes *up to affine integral transformations and compact toric symplectic manifolds (M, ω, μ) up to equivariant symplectomorphisms.*

By *Delzant polytopes* we mean all convex polytopes that can be realized as moment polytopes of toric systems. They have the following nice characterization. A convex polytope $\Delta \subset \mathbb{R}^n$ is called *Delzant* if all edges have rational slope and at every vertex of Δ the set of *primitive vectors* v_i defined by the outgoing edges at that vertex form a basis of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. The *primitive vector* associated to a directed rational line is the first vector lying in the lattice $\mathbb{Z}^n$. In coordinates, this just means that all coordinates are integers and have no common divisors. In particular, all vertices of a Delzant polytope $\Delta \subset \mathbb{R}^n$ have exactly n outgoing edges. Note that n lattice vectors $v_1, \dots, v_n \in \mathbb{Z}^n$ are primitive and form a basis of $\mathbb{Z}^n$ if and only if $\det(v_1, \dots, v_n) = \pm 1$. Let us make some examples.

Example 6.4. (The Delzant condition) The first two vertices in Figure 15 are Delzant, the third one is not.

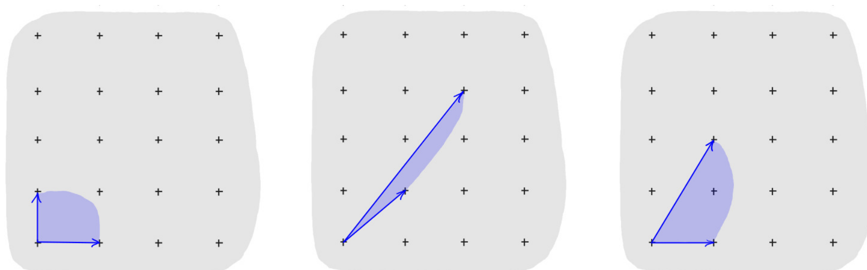

FIGURE 15. The first two vertices are Delzant, the last one is not.

Remark 6.5. (Delzant construction) Note that Theorem 6.3 states in particular that for a given Delzant polytope $\Delta \subset \mathbb{R}^n$, one can always find a toric manifold (M, ω, μ) having Δ as moment polytope. In fact, one can give an explicit construction of the corresponding toric manifold as a symplectic quotient of some $\mathbb{C}^N$. The toric structure on the quotient M is induced by the standard toric structure (see Example 2.28) on $\mathbb{C}^N$ as a residual action. In fact, we have encountered an example of the Delzant construction in §2.8, where we have represented $\mathbb{C}P^2$ as symplectic quotient of $\mathbb{C}^3$ by the diagonal circle action. The pair (f_1, f_2) is the induced toric moment map and its image is the following Delzant polytope

$$\Delta_{\mathbb{C}P^2} = \{(f_1, f_2) \in \mathbb{R}^2 \mid f_1, f_2 \geqslant 0, f_1 + f_2 \leqslant 3\}. \tag{116}$$

Remark 6.6. (Integrable systems) From a dynamical point of view, toric manifolds are a special case of *integrable systems*. An integrable Hamiltonian system on a $2n$-dimensional symplectic manifold (M, ω) is a set of commuting Hamiltonians $F_1, \dots, F_n$ such that the map $(F_1, \dots, F_n)\colon M \to \mathbb{R}^n$ is proper and regular (meaning that $dF_1, \dots, dF_n$ are linearly independent) on an open and dense subset of M. Note that here we do not suppose that the F_i generate circle actions. Still, the Arnold–Liouville theorem states that locally, after a change of variables, one can get a Hamiltonian torus action that has the same torus-fibres as the original Hamiltonian system. Such coordinates are called *action-angle coordinates*, the action coordinates being the values of the components of the local moment map and the angle coordinates the corresponding coordinates on the torus fibre. Toric symplectic manifolds can thus be seen as integrable systems for which the action-angle coordinates extend globally. This is not the case for all integrable systems. For example for the spherical pendulum, one needs at least two *action-angle charts*. See the classical reference [13] for more details.

It is very useful to view the moment map $\mu\colon M \to \Delta$ of a toric manifold as a singular fibration. Indeed, recall that for general Hamiltonian torus actions, orbits of the action are included in a fibre of the moment map (since commuting Hamiltonians are preserved under their flows). In the toric case, *the orbits are precisely equal to the fibres of the moment map*. This follows purely from dimensional reasons. Furthermore, the orbits are always *isotropic submanifolds* of (M, ω), meaning that the symplectic form vanishes on their tangent space. In particular, the orbits of maximal dimension are Lagrangian. One can also show that the dimension of the orbit $\mu^{-1}(x) \subset M$ is equal to the dimension of the maximal facet containing $x \in \Delta$. See Figure 16 for a sketch of the fibration structure. This leads us to the following definition.

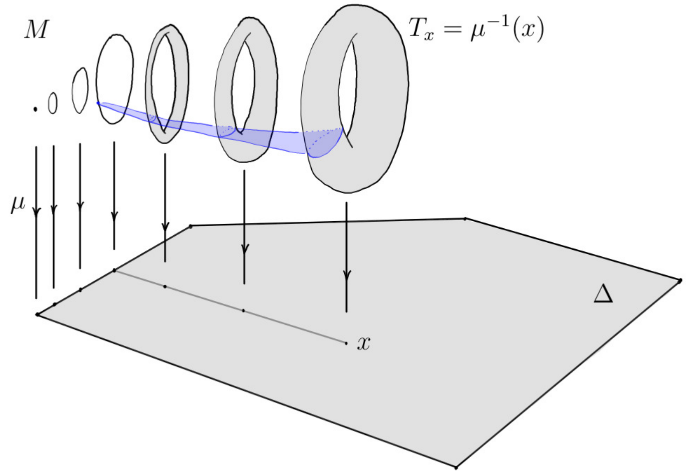

FIGURE 16. The singular fibration structure of the moment map $\mu\colon M \to \Delta$ of a toric manifold. The fibres over a k dimensional face of Δ are k-dimensional isotropic tori.

Definition 6.7. *Let (M, ω, μ) be a toric manifold and $x \in \operatorname{int}(\Delta)$ a point in the interior of the moment polytope. Then the (Lagrangian) fibre $T_x = \mu^{-1}(x) \subset M$ is called* toric fibre.

This is a generalization of product tori. Indeed, a product torus $T(x) \subset \mathbb{C}^n$ is the toric fibre $\mu^{-1}(x)$ of the standard toric structure from Example 2.28. In light of this, we call a Lagrangian torus in a toric manifold *exotic* if it is not symplectomorphic/Hamiltonian isotopic to a toric fibre.

6.2. Chekanov tori in toric manifolds

The construction of the Chekanov torus we discussed in Section 3 can be easily adapted to toric manifolds. In favourable situations, we can distinguish it from toric fibres by the methods of Section 5 and thus we get an exotic torus in toric manifolds. This was first done by Chekanov–Schlenk [10] for $S^2 \times S^2$. We refer to [5] for details of the content of this subsection, where we carry out these ideas in the general toric context.

Recall that the construction in Section 3 came from a one-parameter family of symplectic reductions with two-dimensional reduced spaces. To write down the Chekanov torus, it was sufficient to look at one particular (singular) reduced space, namely M_0. The family $\pi_c \colon Z_c \to M_c$ was needed to determine the versal deformation of the Chekanov torus and to distinguish it from the product torus. Here, it will be similar, except that we will need to consider an $(n-1)$-parametric family of two-dimensional reduced spaces.

First we pick a vertex $w \in \Delta$ at which we carry out the construction. Let $v_1, \dots, v_n \in \mathbb{Z}^n$ be the primitive vectors associated to the edges emanating from w. The family of two-dimensional reduced spaces will come from a family of lines with directional vector $v_1 + \ldots + v_n$. Note that the situation in Figure 6 is a special case of this. Without loss of generality, assume

$$w = 0 \in \mathbb{R}^n, \quad v_1 = e_1, \dots, v_n = e_n, \tag{117}$$

for the standard basis vectors $e_i \in \mathbb{Z}^n$. This is a normal form at the vertex v, which can be achieved by applying an element of the group of affine integral transformations $\operatorname{Aff}_n(\mathbb{Z}) = \mathrm{GL}(n;\mathbb{Z}) \ltimes \mathbb{R}^n$. Now let $U \subset \mathbb{R}^{n-1}$ be a neighbourhood of $0 \in \mathbb{R}^{n-1}$. The family of reductions will be indexed by elements $c \in U$ and we will view $U \subset \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$. Define the family of line segments and the corresponding pre-images by

$$l_c = (c + \mathbb{R}(e_1 + \ldots + e_n)) \cap \Delta, \quad Z_c = \mu^{-1}(l_c). \tag{118}$$

We want to perform symplectic reduction on the $\{Z_c\}_{c \in U}$ with respect to the following T^{n-1}-action,

$$T^{n-1} = \{(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_{n-1}}, e^{2\pi i(-\theta_1 - \ldots - \theta_{n-1})}) \,|\, \theta_1, \dots, \theta_{n-1} \in S^1\} \subset T^n. \tag{119}$$

This torus action is not chosen at random. In fact, it is the *orthogonal complement* of the directional vector $e_1 + \ldots + e_n$ of the l_c. This is a special case of the following remark.

Remark 6.8. (Toric reduction) Note that reduction with respect to the full T^n-action on a toric manifold is quite uninteresting. Indeed, the reduced space will be a point. However, we may pick a subtorus $K \subset T^n$ and try to reduce with respect to the action of K.

In that case, the quotient carries an induced action of T^n/K and one can show that this induced action is also Hamiltonian. There is a nice way to interpret this construction on the Delzant polytope of the toric manifold at hand. Indeed, picking a subtorus $K \subset T^n$ is equivalent to picking a rational affine vector subspace V_K in the image space $\mathbb{R}^n$ of the moment map $\mu\colon M \to \mathbb{R}^n$. In fact, under some natural identifications, the subspace V_K is the orthogonal complement of the Lie algebra $\mathrm{Lie}(K) \subset \mathrm{Lie}(T^n) = \mathbb{R}^n$. In the case that the symplectic reduction on the intersection $V_K \cap \Delta$ by K is regular, the quotient is automatically toric by the induced action of T^n/K, and its moment polytope is given by the intersection $V_K \cap \Delta$. Furthermore, one can determine from the pair (Δ, V_K) whether symplectic reduction is admissible. This means that (as is common in the toric world) everything is determined by the geometry of the moment polytope. See the forthcoming [6] for more details. Note that we have already encountered several examples of toric reduction, namely in §2.8 and in §3.1. In fact, the Delzant construction is a special case of toric reduction, see Remark 6.5.

Note that the action of T^{n-1} is automatically free on $l_c \cap \mathrm{int}\,\Delta$ and hence we only need to worry about the endpoints of the segments l_c. In fact, let us restrict our attention to the half-open segments l'_c obtained by keeping the boundary point of l_c closest to the vertex $v = 0 \in \mathbb{R}^n$ and removing the other one. We remove this boundary point, since what happens away from the vertex v depends heavily on the geometry of Δ, see also Example 6.15. Close to the vertex v, the situation is however quite simple. Whenever l'_c intersects the boundary $\partial\Delta$ in a codimension one facet, the action is free and when it does not, then the action is not free. In the latter case we obtain a singular reduced space as in the discussion in §3.1. We denote the quotient maps by $\pi_c\colon Z'_c = \mu^{-1}(l'_c) \to M_c$. Since the l_c are half open segments, the reduced spaces are symplectomorphic to symplectic disks (some of them having a singularity at the origin) equipped with the standard symplectic form and a natural circle action. This follows from the same arguments as in the case of $\mathbb{C}^2$ using Theorem 2.17. As for $\mathbb{C}^2$, the toric fibres project to the orbits of the residual circle action on the disks. Again, as for $\mathbb{C}^2$, Chekanov tori can be defined by making use of the points where symplectic reduction is not defined, by taking circles $S^1_{\mathrm{Ch}}(a)$ which are not Hamiltonian isotopic to the standard circles.

Definition 6.9. *The* Chekanov torus *of area parameter $a > 0$ in (M, ω) with respect to the vertex v is defined as the lift $\pi_0^{-1}(S^1_{\mathrm{Ch}}(a))$.*

By our choice of directional vector $e_1 + \ldots + e_n$, we obtain the following.

Proposition 6.10. *For an open and dense subset $U_{\mathrm{reg}} \subset U$, the reduction $\pi_c\colon Z'_c = \mu^{-1}(l'_c) \to M_c$ with $c \in U_{\mathrm{reg}}$ is regular and the reduced space is symplectomorphic to a disk. In particular, the circles $S^1(a)$ and $S^1_{\mathrm{Ch}}(a)$ are Hamiltonian isotopic in M_c for $c \in U_{\mathrm{reg}}$ and so are the corresponding lifted tori.*

As in the case of $\mathbb{C}^2$, this will turn out to be crucial when trying to show that the Chekanov torus is exotic, since it proves that most neighbouring tori in the versal deformation are Hamiltonian isotopic to toric fibres and hence we can compute their displacement energy. In fact, we make the following assumption about the displacement energy of toric fibres of a toric manifold $\mu\colon M \to \Delta$.

Assumption 6.11. *Assume the displacement energy of a toric fibre $T_x = \mu^{-1}(x) \subset M$ is given by the affine distance of x to $\partial\Delta$,*

$$e(M, T_x) = d_{\text{aff}}(x, \partial\Delta) \tag{120}$$

for x in an open and dense subset of Δ.

See [5, Section 3] for the definition of affine distance and a discussion of why this is a reasonable assumption. For example, one can prove that $e(M, T_x) \geqslant d_{\text{aff}}(x, \partial\Delta)$ for all toric fibres $T_x \subset M$ and without any additional assumptions on the toric symplectic manifold M. For our purposes, it is enough to have (120) on an open and dense subset. For example for $\mathbb{C}P^2$ and $S^2 \times S^2$, the level sets of the function $x \mapsto e(M, T_x)$ are given in Figure 17. Note that the level sets are just scalings of the boundary of the corresponding moment polytopes. Let us now restrict our attention to the case where M is *monotone*. We will not give the general definition of a monotone symplectic manifold here, since in the toric setting monotonicity can be easily read off from the moment polytope. A toric symplectic manifold M is *monotone* if there is a *central point* $x(\Delta) \in \Delta$, meaning a point which is equidistant (in terms of affine distance) to all facets of Δ. The toric fibre $T_{x(\Delta)}$ over the central point is a monotone Lagrangian torus. In both cases in Figure 17, the central point is the point where the level set degenerates to a point.

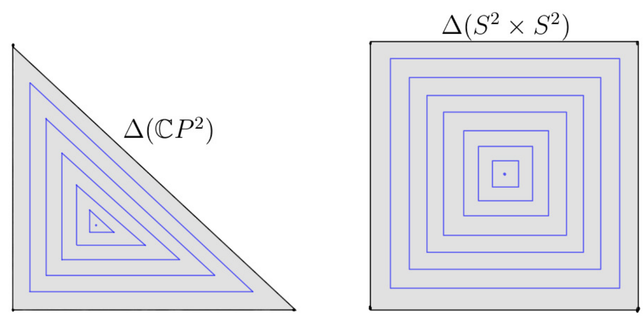

Figure 17. The level sets of the displacement energy function $x \mapsto e(M, T_x)$ of toric fibres for $M = \mathbb{C}P^2$ on the left and for $M = S^2 \times S^2$ on the right. The central point $x(\Delta)$ is at the central dot and the corresponding fibre is non-displaceable.

Let us now compute the displacement energy germ of the central fibre $T_{x(\Delta)}$. A versal deformation of $T_{x(\Delta)}$ is given by $b \mapsto T_{x(\Delta)+b}$ for small enough b. We have proved this in the special case of product tori in $\mathbb{C}^n$ in Proposition 4.10, and exactly the same argument goes through in the case of toric fibres. In conjunction with Assumption 6.11, this yields

$$\mathcal{E}_{T_{x(\Delta)}}(b) = d_{\mathrm{aff}}(x(\Delta) + b, \partial\Delta), \tag{121}$$

on an open dense subset in a neighbourhood of the origin $0 \in H^1(T^n;\mathbb{R})$. The point here is that the level sets of $\mathcal{E}_{T_{x(\Delta)}}$ are given by scalings of Δ. From the symplectic invariance of $\mathcal{E}_{T_{x(\Delta)}}$ we deduce that the $\mathrm{GL}(n;\mathbb{Z})$-equivalence class of Δ is a symplectic invariant of $T_{x(\Delta)}$.

Remark 6.12. (The central fibre) In particular, the displacement energy germ of the central fibre $T_{x(\Delta)}$ in a monotone toric manifold *determines* its ambient space up to the scaling of the symplectic form. Indeed, we have seen that $\mathcal{E}_{T_{x(\Delta)}}$ determines Δ up to scaling and that Δ determines (M,ω) by Delzant's theorem 6.3.

Let $T^v_{\mathrm{Ch}} \subset M$ be the *monotone* Chekanov torus with respect to a given vertex $v \in \Delta$. This means that we lift the circle $S^1_{\mathrm{Ch}}(a)$ in the reduced space M_0, where the area parameter $a > 0$ is chosen so that $\pi_0(T_{x(\Delta)}) = S^1(a)$, i.e. so that T^v_{Ch} is in the same area class as the central fibre $T_{x(\Delta)}$. We get a versal deformation of T^v_{Ch} by using the family of reduced spaces $\{M_c\}_{c\in U}$ as constructed above. Indeed, the assignment

$$(b_1,\dots,b_n) \mapsto \pi^{-1}_{(b_1,\dots,b_{n-1})}(S^1_{\mathrm{Ch}}(a+b_n)), \quad b \in U \times (-\varepsilon,\varepsilon) \tag{122}$$

is a versal deformation of T^v_{Ch} for small enough $(-\varepsilon,\varepsilon)$. The tori in (122) are Hamiltonian isotopic to toric fibres for b in the open and dense subset $U_{\mathrm{reg}} \times (-\varepsilon,\varepsilon)$ coming from Proposition 6.10. Using this, we can explicitly determine $\mathcal{E}_{T^v_{\mathrm{Ch}}}$ on an open and dense subset. Its level sets are given by scaling of a polytope

$$\Delta^v_{\mathrm{Ch}} = \Psi(\Delta). \tag{123}$$

The map Ψ is a piecewise linear homeomorphism of $\mathbb{R}^n$. In the normal form (117), it is given by

$$\Psi\colon \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_n \\ \vdots \\ x_{n-1} - x_n \\ \min\{x_1,\dots,x_n\} \end{pmatrix}. \tag{124}$$

We conclude that if Δ and Δ^v_{Ch} are not $\mathrm{GL}(n;\mathbb{Z})$-equivalent, then the Chekanov torus T^v_{Ch} is exotic.

Theorem 6.13. ([5]) *Let (M,ω) be a monotone toric symplectic manifold satisfying Assumption 6.11. Then M contains an exotic Chekanov torus.*

Remark 6.14. (Choice of the vertex) There are some vertices $v \in \Delta$ of Delzant polytopes for which the polytope Δ^v_{Ch} is $\mathrm{GL}(n;\mathbb{Z})$-equivalent to Δ, see Example 6.15. However, for a given Delzant polytope Δ, there is always a vertex v for which T^v_{Ch} is exotic. Indeed, let v be adjacent to the largest facet (in terms of integral affine $(n-1)$-volume) of Δ. Since Ψ merges all facets adjacent to v into one facet, the volume of the largest facet strictly increases when we pass from Δ to Δ^v_{Ch}.

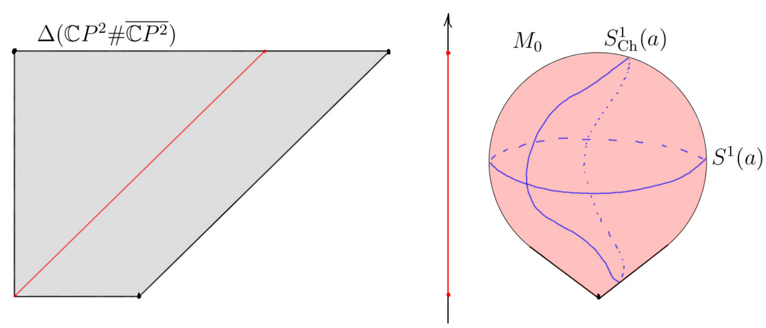

FIGURE 18. The polytope of the monotone blow-up of $\mathbb{C}P^2$ is shown on the left. Performing symplectic reduction on the indicated segment, we obtain the space sketched on the right. The Chekanov circle $S^1_{\mathrm{Ch}}(a)$ is Hamiltonian isotopic to the standard circle $S^1(a)$.

Example 6.15. (A non-exotic Chekanov torus) Let us illustrate that the Chekanov torus T^v_{Ch} is not exotic for a certain choice of v and Δ. Let $\Delta(\mathbb{C}P^2\#\overline{\mathbb{C}P^2})$ be the polytope given in Figure 18 and $v=0$ the vertex at the origin. This is the Delzant polytope of the monotone blow-up of $\mathbb{C}P^2$. Then the reduced space M_0 is smooth at the endpoint opposite to 0. This means that we can isotope Chekanov circles $S^1_{\mathrm{Ch}}(a)$ to standard circles $S^1(a)$ and hence the lifted Chekanov torus is Hamiltonian isotopic to a toric fibre.

6.3. Discussion and an open question

Let us look at some examples of the polytopes Δ^v_{Ch} appearing as level sets of displacement energy germs. We again take the examples $\mathbb{C}P^2$ and $S^2\times S^2$ and apply the map Ψ to the Delzant polytopes to obtain the corresponding Chekanov polytopes, see Figure 19. The reader who is familiar with almost toric geometry recognizes these as the almost-toric base polytopes of $\mathbb{C}P^2$ and $S^2\times S^2$ obtained from their Delzant polytopes after one mutation, see [25], [26] or [15, Section 8.4]. The idea to use mutations to find exotic Lagrangian tori first appeared as a conjecture in [16] and was then proved by Vianna. As was noticed by Chekanov–Schlenk, one can also use the approach of displacement energy germs to distinguish Vianna's tori and, as in the classical toric case, the level sets of the displacement energy germs coincide with the almost toric base polytopes. Hence distinguishing them boils down to distinguishing polytopes up to the action of $\mathrm{Aff}_2(\mathbb{Z})$.

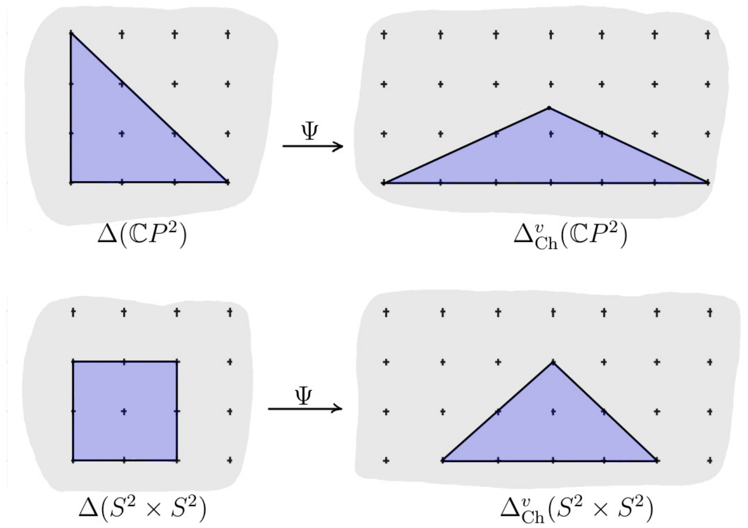

FIGURE 19. The Delzant polytope $\Delta(M)$ on the left and the associated Chekanov polytope $\Delta^v_{\mathrm{Ch}}(M)$ on the right for $M = \mathbb{C}P^2$ and for $M = S^2 \times S^2$.

Note that our approach from §6.2 starts at the other end: We know how to construct an exotic Lagrangian torus T^v_{Ch} in a toric manifold and find, via displacement energy germs, an associated polytope $\Delta^v_{\mathrm{Ch}} = \Psi(\Delta)$. This begs the following question about toric symplectic manifolds, see also Remark 6.12.

Question 6.16. Is there some kind of *almost toric fibration* on M having (a suitably scaled version of) the Chekanov polytope Δ^v_{Ch} as basis?

See Figure 20 for a depiction of the Delzant polytopes and the Chekanov polytopes Δ^v_{Ch} associated to $\mathbb{C}P^3$ and $S^2 \times S^2 \times S^2$. These are arguably the simplest 6-dimensional toric manifolds one should think about. The corresponding Chekanov polytopes are very far from Delzant, as the smoothness condition is not verified by any of their vertices and the Chekanov polytope for $S^2 \times S^2 \times S^2$ even has a vertex at which 6 edges meet. Note that we do not even know what *almost toric fibration* means in dimensions $\geqslant 6$, since almost toric fibrations are defined by the kinds of singularities we admit in the fibration and adding the so-called *focus-focus singularities* to the list of admissible ones seems to be a particularity of dimension four. This leads to the question: *What kind of singularities should we admit in higher dimensional almost toric fibrations?* Furthermore, one may wonder whether there is a meaningful notion of mutation which would allow us to produce more than one exotic fibration (and thus more than one exotic torus). Note that our approach *by* *hand* in Section 3 suggests where to start to construct this kind of mutation. Namely one can look at a family of symplectic reductions exhausting all of M and perturb the residual Hamiltonian circle action. In fact, this is what we have done locally when we have constructed the versal deformation of the Chekanov torus in §6.2. The almost toric fibration should appear as an extension of this local construction to all of M. Although the higher dimensional analogue of the construction by hand seems tricky, we will try to carry this out in future work.

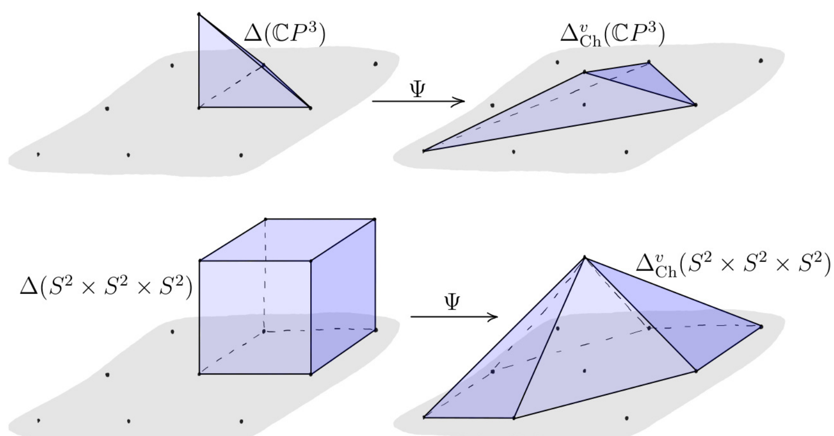

FIGURE 20. The Delzant polytope $\Delta(M)$ on the left and the associated Chekanov polytope $\Delta^v_{\mathrm{Ch}}(M)$ on the right for $M = \mathbb{C}P^3$ and for $M = S^2 \times S^2 \times S^2$.

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References

[1] M. Abreu and L. Macarini, *Remarks on Lagrangian intersections in toric manifolds.* Trans. Amer. Math. Soc. **365(7)** (2013), 3851–3875.

[2] M. F. Atiyah, *Convexity and commuting Hamiltonians.* Bull. London Math. Soc. **14(1)** (1982), 1–15.

[3] M. Audin, *Torus actions on symplectic manifolds.* Revised edition, Birkhäuser Verlag, (2004).

[4] D. Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor.* J. Gökova Geom. Topol. GGT **1** (2007), 51–91.

[5] J. Brendel, *Real lagrangian tori and versal deformations,* preprint.
[6] J. Brendel, *On the symplectic classification of toric fibres,* in preparation.
[7] A. Cannas da Silva, *Symplectic toric manifolds,* unpublished book.
[8] Y. Chekanov, *Lagrangian tori in a symplectic vector space and global symplectomorphisms.* Math. Z. **223(4)** (1996), 547–559.
[9] Y. Chekanov, *Lagrangian intersections, symplectic energy, and areas of holomorphic curves.* Duke Math. J. **95(1)** (1998), 213–226.
[10] Y. Chekanov and F. Schlenk, *Notes on monotone Lagrangian twist tori.* Electron. Res. Announc. Math. Sci. **17** (2010), 104–121.
[11] Y. Chekanov and F. Schlenk, *Lagrangian product tori in symplectic manifolds.* Comment. Math. Helv. **91(3)** (2016), 445–475.
[12] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment.* Bull. Soc. Math. France **116(3)** (1988), 315–339.
[13] J. J. Duistermaat, *On global action-angle coordinates.* Comm. Pure Appl. Math. **33(6)** (1980), 687–706.
[14] Y. Eliashberg and L. Polterovich, *The problem of Lagrangian knots in four-manifolds.* In *Geometric topology (Athens, GA, 1993).* Volume 2, Amer. Math. Soc. (1997), 313–327.
[15] J. D. Evans, *Lectures on lagrangian torus fibrations,* unpublished book.
[16] S. Galkin and U. Alexandr, *Laurent phenomenon for Landau–Ginzburg potential,* preprint.
[17] Y. Groman and U. Varolgunes, *Locality of relative symplectic cohomology for complete embeddings,* preprint.
[18] V. Guillemin, *Moment maps and combinatorial invariants of Hamiltonian T^n-spaces.* Birkhäuser Boston, Inc., (1994).
[19] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping.* Invent. Math. **67(3)** (1982), 491–513.
[20] D. McDuff, *Displacing Lagrangian toric fibers via probes.* In *Low-dimensional and symplectic topology.* Volume 82, Amer. Math. Soc., (2011).
[21] D. McDuff and D. Salamon, *Introduction to symplectic topology.* Third edition, Oxford University Press, (2017).
[22] Y.-G. Oh, *Gromov-Floer theory and disjunction energy of compact Lagrangian embeddings.* Math. Res. Lett. **4(6)** (1997) 895–905.
[23] L. Polterovich, *The geometry of the group of symplectic diffeomorphisms.* Birkhäuser Verlag, (2001).
[24] M. Symington, *Four dimensions from two in symplectic topology.* In *Topology and geometry of manifolds (Athens, GA, 2001).* Volume 71, Amer. Math. Soc. (2003), 153–208.
[25] R. V. Vianna. *Infinitely many exotic monotone Lagrangian tori in* $\mathbb{CP}^2$. J. Topol. **9(2)** (2016) 535–551.
[26] R. Vianna. *Infinitely many monotone Lagrangian tori in del Pezzo surfaces.* Selecta Math. (N.S.) **23(3)** (2017) 1955–1996.

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