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Braid group actions on matrix factorizations

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Abstract. Let X be a smooth scheme with an action of a reductive algebraic group G over an algebraically closed field k of characteristic zero. We construct an action of the extended affine Braid group on the G-equivariant absolute derived category of matrix factorizations on the Grothendieck variety times T^*X with potential given by the Grothendieck-Springer resolution times the moment map composed with the natural pairing.

1. Introduction

The present paper is a follow up of the paper [AK]. Recall the setting in that paper.

1.1. Coherent Hecke category for the derived loop group and affine Hecke category.

Let X be a regular Noetherian scheme with an action of a reductive algebraic group G. Fix a Borel subgroup $B \subset G$. In the introduction to [AK] we explained that it is natural to consider the Hecke category for the pair of the derived group schemes $(L_{\mathrm{top}}(G), L_{\mathrm{top}}(B))$ of derived loops with values in G (resp., in B).

The category $\mathrm{QCHecke}(L_{\mathrm{top}}(G), L_{\mathrm{top}}(B))$ is a monoidal triangulated category via the convolution functor. It is expected to act naturally on the category of $L_{\mathrm{top}}(B)$-equivariant coherent sheaves on the derived scheme of derived loops with values in the scheme X. Formal definitions of the categories above would require the heavy machinery of Derived Algebraic Geometry and the authors prefer to avoid this. Thus we use the picture as a source of inspiration and act by analogy with the case of B-equivariant coherent sheaves on X. Our actual construction follows a different direction.

In [BN] Ben-Zvi and Nadler make precise sense of $\mathrm{QCHecke}(L_{\mathrm{top}}(G), L_{\mathrm{top}}(B))$ in classical algebro-geometric terms. Namely they identify the category with the affine Hecke category of Bezrukavnikov and prove that

$$\mathrm{QCHecke}(L_{\mathrm{top}}(G), L_{\mathrm{top}}(B)) \xrightarrow{\sim} D^b \operatorname{Coh}^G(St).$$

Here St denotes (the group version of) the Steinberg variety for the group G.

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1.2. Equivariant matrix factorizations.

In [AK] we gave a precise definition of the module category discussed above and understood informally as $\mathrm{Coh}^{L_{\mathrm{top}}(B)}(L_{\mathrm{top}}(X))$. Namely, we considered the moment map

$$h : T^*X \times Lie(B) \to \mathbb{A}^1$$

and the corresponding equivariant absolute derived category of matrix factorizations $D^{\mathrm{abs}}\,\mathrm{Coh}^B(T^*X \times Lie(B), h)$.

In our paper, we construct a monoidal action of the category $D^b\,\mathrm{Coh}^G(St)$ on this category. For technical reasons it is easier to work with geometric objects (coherent sheaves etc) equivariant with respect to the reductive group G.

We replace the category $D^{\mathrm{abs}}\,\mathrm{Coh}^B(T^*X \times \mathrm{Lie}(B), h)$ by the equivalent category of G-equivariant matrix factorizations on

$$T^*X \times \frac{\mathrm{Lie}(B) \times G}{B}.$$

Notice that $\tilde{\mathfrak{g}} := \frac{Lie(B)\times G}{B}$ is one of the constructions of the Grothendieck variety for G. The potential for the category of matrix factorizations is given by the composition

$$T^*X \times \tilde{\mathfrak{g}} \to \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{A}^1.$$

Here the first map is a product of the canonical projection with the moment map for the G-action. The second map is the canonical bilinear pairing.

The main result of the paper is the following statement.

Theorem 1.1. *There exists a natural monoidal action of the affine Hecke category $D^b\,\mathrm{Coh}^G(St)$ on the derived category of equivariant matrix factorizations $D^{\mathrm{abs}}\,\mathrm{Coh}^G(T^*X\times \tilde{\mathfrak{g}}, h)$.*

Another action of the affine braid group on the (non-derived) category of matrix factorizations has been constructed independently by Oblomkov and Rozansky in [OR] and [OR2]. Their affine braid group action is given by explicit matrices and does not use the action of Bezrukavnikov and Riche. It is not known wether our construction is equivalent to theirs.

1.3. The structure of the paper.

Section 2 recalls the construction by Bezrukavnikov and Riche [BR] of a categorical action of the affine braid group on the derived category of the dg-category of $G \times \mathbb{G}_m$-equivariant coherent sheaves on $\tilde{\mathfrak{g}}$, $\mathcal{D}^b\,\mathrm{Coh}^{G\times\mathbb{G}_m}(\tilde{\mathfrak{g}})$. The action is constructed by defining a monoidal action of another category on $\mathcal{D}^b\,\mathrm{Coh}^{G\times\mathbb{G}_m}(\tilde{\mathfrak{g}})$ and then identifying elements in this category whose convolution satisfy affine braid group relations.

The basic definitions of matrix factorizations and their absolute derived category is recalled in section 3 and in section 4 we state the main theorem. Section 5 introduces the functors of derived push-forward, pull-back and tensor product for equivariant matrix factorizations. We prove some relations between these functors which are needed in the subsequent sections.

In section 6 we define a convolution product. Let X and Y be smooth G-schemes and V a vector space. Assume that we have equivariant morphisms $\mu : X \to V^*$ and $\nu : Y \to V$. This defines two functions

$$w : Y \times X \to k, \qquad (y, x) \mapsto \mu(x)(\nu(y)), \tag{1}$$

$$h : Y \times Y \times V^* \to k, \qquad (y_1, y_2, v) \mapsto v(\nu(y_1) - \nu(y_2)). \tag{2}$$

Using a convolution, which is a modified analogy of the convolution product used in [BR], we define a monoidal structure on the category $D^{\text{abs}}(\text{QCoh}^G(Y \times Y \times V^*), h)$. We also define a monoidal action of this category on the category $D^{\text{abs}}(\text{QCoh}^G(Y \times X), w)$. Finally, we prove that imposing some support condition on $D^{\text{abs}}(\text{QCoh}^G(Y \times Y \times V^*), h)$ this induces a monoidal action on $D^{\text{abs}}(\text{Coh}^G(Y \times X), w)$.

In section 7 we finish the proof of 1.1 by constructing a monoidal functor from a subcategory of the monoidal category used by Bezrukavnikov and Riche to $D^{\text{abs}}(\text{QCoh}^G(Y \times Y \times V^*), h)$. The subcategory contains the generators for the B_{af}-action and the image of these under the constructed functor lies in the subcategory of $D^{\text{abs}}(\text{QCoh}^G(Y \times Y \times V^*), h)$ preserving $D^{\text{abs}}(\text{Coh}^G(Y \times X), w)$. Thus, convolution with the images of the generators produces a B_{af}-action on $D^{\text{abs}}(\text{Coh}^G(Y \times X), w)$.

2. The braid group action of Bezrukavnikov and Riche

In this section we recall the construction of an action of the (extended) affine braid group by Bezrukavnikov and Riche in [BR] and [Ric]. Let G be a reductive algebraic group over an algebraically closed field k. Their proof works both in characteristic zero and when the characteristic is bigger than the Coxeter number. Fix a maximal torus T and Borel subgroup B containing it. Recall that the extended affine braid group has the following presentation.

Theorem 2.1. [Ric, Thm 1.1.3] *The extended affine braid group, B_{af}, admits a presentation with generators $\{T_\alpha \mid \alpha \in \Pi\} \cup \{\theta_x \mid x \in \mathbb{X}\}$ and relations:*

(1) $T_\alpha T_\beta T_\alpha \cdots = T_\beta T_\alpha T_\beta \cdots$ with $m(\alpha, \beta)$ factors on each side.
(2) $\theta_x \theta_y = \theta_{x+y}$.
(3) $T_\alpha \theta_x = \theta_x T_\alpha$ if $\langle x, \alpha \rangle = 0$, i.e. $s_\alpha(x) = x$.
(4) $\theta_x = T_\alpha \theta_{x-\alpha} T_\alpha$ if $\langle x, \alpha \rangle = 1$, i.e. $s_\alpha(x) = x - \alpha$.

We sketch their construction. Let X and Y be G-varieties. Denote the projections $X \times Y \to X$ and $X \times Y \to Y$ by p_X and p_Y respectively. These projections are not assumed to be proper, so push-forward might not take coherent sheaves to coherent sheaves. This problem is fixed by introducing the following full subcategory

$$D^b_{\text{prop}}(\text{Coh}(X \times Y)) \subset D^b(\text{Coh}(X \times Y))$$

in the following way. An object in $D^b(\mathrm{Coh}(X \times Y))$ belongs to $D^b_{\mathrm{prop}}(\mathrm{Coh}(X \times Y))$ if its cohomology sheaves are topologically supported on a closed subscheme $Z \subset X \times Y$ such that the restrictions of p_X and p_Y to Z are proper. They define a convolution product

$$* : D^b_{\mathrm{prop}}(\mathrm{Coh}^G(Y \times Z)) \times D^b_{\mathrm{prop}}(\mathrm{Coh}^G(X \times Y)) \to D^b_{\mathrm{prop}}(\mathrm{Coh}^G(X \times Z)), \quad (3)$$

$$\mathcal{F} * \mathcal{G} := Rp_{X,Z*}(p^*_{X,Y}\mathcal{F} \otimes^L_{X \times Y \times Z} p^*_{Y,Z}\mathcal{G}), \quad (4)$$

where $p_{X,Y}, p_{Y,Z}$ and $p_{X,Z}$ are the projections from $X \times Y \times Z$ to the listed factors. Any $\mathcal{F} \in D^b_{\mathrm{prop}}(\mathrm{Coh}^G(X \times Y))$ defines a functor

$$F^{\mathcal{F}}_{X \to Y} : D^b(\mathrm{Coh}^G(X)) \to D^b(\mathrm{Coh}^G(Y)), \quad (5)$$

$$\mathcal{M} \mapsto Rp_{Y*}(\mathcal{F} \otimes^L_{X \times Y} p^*_X \mathcal{M}). \quad (6)$$

Lemma 2.2. [Ric, Lemma 1.2.1] *Let* $\mathcal{F} \in D^b_{prop}(\mathrm{Coh}^G(X \times Y))$ *and* $\mathcal{G} \in D^b_{prop}(\mathrm{Coh}^G(Y \times Z))$*. Then*

$$F^{\mathcal{G}}_{Y \to Z} \circ F^{\mathcal{F}}_{X \to Y} \simeq F^{\mathcal{G} * \mathcal{F}}_{X \to Z}.$$

The categorical action of B_{af} to be constructed will be a weak action. Recall that a weak action of a group A on a category $\mathcal{C}$ is a group morphism from A to the group of isomorphism classes of auto-equivalences of the category $\mathcal{C}$. Note that no compatibility conditions are imposed on the morphisms. In particular, to construct an action of B_{af} on $D^b(\mathrm{Coh}^G(X))$ it suffices to find objects in $(D^b_{\mathrm{prop}}(\mathrm{Coh}^G(X \times X)), *)$, whose convolution with each other satisfy affine braid group relations.

In this construction the variety is going to be the Grothendieck variety $\tilde{\mathfrak{g}}$. Recall that $\tilde{\mathfrak{g}}$ is smooth and that $\nu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is proper. It follows that the base change morphisms $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ are proper, so we have a full monoidal subcategory

$$D^b_{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}})) \subset D^b_{\mathrm{prop}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}})),$$

whose objects are supported on $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ (i.e. their restriction to $\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \backslash \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ is 0). It is a monoidal category with $*$. We call this monoidal category the affine Hecke category

$$\mathrm{Hecke}_{\mathrm{af}}(G, B) := (D^b_{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}})), *).$$

Consider the composition

$$\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \hookrightarrow \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \twoheadrightarrow \mathcal{B} \times \mathcal{B}.$$

For $w \in W$ we denote by Z_w the closure of the inverse image of the orbit of the point $(B/B, w^{-1}B/B)$ for the diagonal action of G. For $x \in \mathbb{X}$ we have the canonical line bundle $\mathcal{O}_{\mathcal{B}}(x)$. We define $\mathcal{O}_{\Delta(\tilde{\mathfrak{g}})}(x)$ to be the pull-back of $\mathcal{O}_{\mathcal{B}}(x)$ along the projection of the diagonal $\Delta(\tilde{\mathfrak{g}}) \to \mathcal{B}$. We can now state the main result of Bezrukavnikov and Riche.

Theorem 2.3. [BR, Theorem 1.3.2] *There is a categorical* B_{af}*-action on* $D^b(\mathrm{Coh}^G(\tilde{\mathfrak{g}}))$ *in which* T_{α_i} *acts by convolution with* $\mathcal{O}_{Z_{s_{\alpha_i}}}$ *and* θ_x *acts by convolution with* $\mathcal{O}_{\Delta(\tilde{\mathfrak{g}})}(x)$*.*

Moreover, they prove that for a reduced expression $w = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_n}}$

$$\mathcal{O}_{Z_{s_{\alpha_{i_1}}}} * \cdots * \mathcal{O}_{Z_{s_{\alpha_{i_n}}}} \simeq \mathcal{O}_{Z_w}.$$

In [BR, Section 4 and 5] Bezrukavnikov and Riche also lift this construction to DG-categories. Recall the following definition from [MR3]:

Definition 2.4. Let $\mathcal{A}$ be a G-equivariant sheaf of DG-algebras on X.

(1) A G-equivariant quasi-coherent dg-sheaf $\mathcal{F}$ on $(X, \mathcal{A})$ is a G-equivariant $\mathcal{A}$-dg-module such that $\mathcal{F}^i$ is a G-equivariant quasi-coherent $\mathcal{O}_X$-module for all $i \in \mathbb{Z}$. The category of G-equivariant quasi-coherent dg-modules over $\mathcal{A}$ is denoted by $\mathcal{C}\,\mathrm{QCoh}^G(\mathcal{A})$ and its derived category by $\mathcal{D}\,\mathrm{QCoh}^G(\mathcal{A})$ (The definition of the equivariant derived category is analogous to the non-equivariant derived category as defined in [BL]).
(2) A coherent dg-module $\mathcal{M}$ over $\mathcal{A}$ is a quasi-coherent dg-sheaf whose cohomology sheaf $\mathcal{H}(\mathcal{M})$ is coherent over $\mathcal{H}(\mathcal{A})$. The full subcategory of $\mathcal{D}\,\mathrm{QCoh}^G(\mathcal{A})$ whose objects are coherent is denoted by $\mathcal{D}\,\mathrm{Coh}^G(\mathcal{A})$.
(3) The full subcategory of $\mathcal{D}\,\mathrm{QCoh}^G(\mathcal{A})$ consisting of objects whose cohomology is bounded and coherent as a $\mathcal{O}_X$-module is denoted by $\mathcal{D}^{bc}\,\mathrm{QCoh}^G(\mathcal{A})$.

Remark 2.5. Consider $\mathcal{O}_X$ as a dg-algebra with $\mathcal{O}_X$ in degree zero and 0 elsewhere. Then

$$\mathcal{D}\,\mathrm{QCoh}^G(\mathcal{O}_X) \simeq D^b(\mathrm{QCoh}^G(X)), \qquad \mathcal{D}\,\mathrm{Coh}^G(\mathcal{O}_X) \simeq D^b(\mathrm{Coh}^G(X)).$$

To be able to define the derived functors we will assume the following property:

$$\text{For any } \mathcal{F} \in \mathrm{Coh}^G(X), \text{ there exists } \mathcal{P} \in \mathrm{Coh}^G(X) \tag{7}$$

$$\text{which is flat over } \mathcal{O}_X \text{ and a surjection } \mathcal{P} \twoheadrightarrow \mathcal{F} \text{ in } \mathrm{Coh}^G(X). \tag{8}$$

This is satisfied e.g. when X admits an ample family of line bundles in the sense of [VV, Definition 1.5.3] or when X is normal and quasi-projective (see [CG, Proposition 5.1.26]).

Definition 2.6. Let $\mathcal{A}$ be an equivariant sheaf of dg-algebras on X. If $\mathcal{A}$ is quasi-coherent, non-positively graded and graded-commutative then the pair $(X, \mathcal{A})$ is called a dg-scheme.

Consider two $G \times \mathbb{G}_m$-equivariant dg-algebras $X = (X_0, \mathcal{A}_X)$ and $Y = (Y_0, \mathcal{A}_Y)$ and an equivariant DG-morphism of sheaves of DG-algebras $f : X \to Y$. Then one can define the derived fiber product $X \times^R_Y X$ and Bezrukavnikov and Riche considers the DG-categories

$$\mathcal{K}_{X,Y} := \mathcal{D}\,\mathrm{QCoh}^{G \times \mathbb{G}_m}(X \times^R_Y X).$$

and its full subcategory $\mathcal{K}^{\mathrm{Coh}}_{X,Y}$, whose objects are complexes with only finitely many non-zero cohomology sheaves, each of which is a coherent sheaf on $X \times_Y X$.

Proposition 2.7. [BR, Prop. 4.2.1] *Assume that X and Y are ordinary schemes and that f is proper. Then $\mathcal{K}^{\mathrm{Coh}}_{X,Y}$ is a monoidal category with the restricted convolution product and an action of $\mathcal{K}^{\mathrm{Coh}}_{X,Y}$ on $\mathcal{D}\,\mathrm{QCoh}^{G \times \mathbb{G}_m}(X)$ preserving the full subcategory $\mathcal{D}^b\,\mathrm{Coh}^{G \times \mathbb{G}_m}(X)$.*

In this setting there is a "direct image under closed embedding" functor

$$\mathcal{K}^{\mathrm{Coh}}_{X,Y} \to \mathcal{D}^b \operatorname{Coh}_{X\times_Y X}(X \times X)$$

This functor is monoidal as remarked in [MR3, Section 4.1]. The generators of the braid group action from theorem 2.3 are all schemes on $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ so they can naturally be considered as objects in $\mathcal{K}^{\mathrm{Coh}}_{\tilde{\mathfrak{g}},\mathfrak{g}}$. Hence,

Theorem 2.8. [BR] *There is a (weak) categorical action of B_{af} on $\mathcal{K}^{\mathrm{Coh}}_{\tilde{\mathfrak{g}},\mathfrak{g}}$.*

In the case where Y is a vector space V we can write $\mathcal{O}_{X\times^R_V X}$ explicitly in terms of the Koszul resolution. Consider the function $\rho : Y \times Y \to V$ given by $(y_1, y_2) \mapsto \nu(y_2) - \nu(y_1)$. It induces a map $\rho^\sharp : V^* \to \mathcal{O}_{Y\times Y}$. Since $Y \times_V Y = \rho^{-1}(0)$ we have a resolution

$$\cdots \longrightarrow \mathcal{O}_{Y\times Y} \otimes \wedge^2 V^* \longrightarrow \mathcal{O}_{Y\times Y} \otimes V^* \xrightarrow{\theta} \mathcal{O}_{Y\times Y} \to \mathcal{O}_{Y\times_V Y} \to 0,$$

where $\theta(f \otimes s) = f\rho^\sharp(s)$ and the differential is extended by Leibniz rule. Hence,

$$\mathcal{D}(\operatorname{Coh}^G(Y \times^R_V Y)) = \mathcal{D}(\operatorname{Coh}^G(Y \times_V Y, \mathcal{O}_{Y\times Y} \otimes \Lambda(V^*)))$$

In the special case when $X = \tilde{\mathfrak{g}}$, $V = \mathfrak{g}$ and f is the Grothendieck-Springer resolution the derived fiber product equals the non-derived fiber product (see [Bez, Section 1.2.2]). This is not the case when $X = \tilde{\mathcal{N}}$ and $V = \mathfrak{g}$. The convolution is defined as follows. Consider the derived projections $\bar{p}_{ij} : X \times^R_V X \times^R_V X \to X \times^R_V X$. The convolution product is defined as

$$\mathcal{M} * \mathcal{N} := R\bar{p}_{13*}(L\bar{p}^*_{12} \otimes_{Y\times^R_V Y \times^R_V Y} L\bar{p}^*_{23})$$

This translates into the following picture

$$\begin{tikzcd}
 & \mathcal{O}_{Y\times Y\times Y} \otimes \Lambda(V^*) \otimes \Lambda(V^*) & \\
\mathcal{O}_{Y\times Y\times Y} \otimes \Lambda(V^*) \arrow[ur, "q_{12}"] & \mathcal{O}_{Y\times Y\times Y} \otimes \Lambda(V^*) \arrow[u, "q_{13}"] & \mathcal{O}_{Y\times Y\times Y} \otimes \Lambda(V^*) \arrow[ul, "q_{23}"'] \\
\mathcal{O}_{Y\times Y} \otimes \Lambda(V^*) \arrow[u, "p^*_{12}"] & \mathcal{O}_{Y\times Y} \otimes \Lambda(V^*) \arrow[u, "p^*_{13}"] & \mathcal{O}_{Y\times Y} \otimes \Lambda(V^*) \arrow[u, "p^*_{23}"]
\end{tikzcd}$$

Explicitly, the maps are given by $q_{12}(f\otimes v) = f\otimes v\otimes 1, q_{23}(f\otimes v) = f\otimes 1\otimes v, q_{13}(f\otimes v) = f \otimes v \otimes 1 + f \otimes 1 \otimes v$. The map p^*_{ij} is the pull-back corresponding to the projection to the (i, j)'th factor of Y and identity on V^*. The formula can then be written as

$$\mathcal{M} * \mathcal{N} = Rp_{13*}Rq_{13*}(q^*_{12}p^*_{12}\mathcal{M} \otimes^L_{\mathcal{O}_{Y\times Y\times Y}\otimes\Lambda(V^*)\otimes\Lambda(V^*)} q^*_{23}p^*_{23}\mathcal{N})$$

3. Matrix factorizations

Let X be a separated, Noetherian G-scheme with enough G-equivariant vector bundles-bundles, i.e. every coherent sheaf on X is the quotient sheaf of a G-equivariant locally free sheaf of finite rank. Let $w : X \to k$ be a regular G-invariant function.

Definition 3.1.

(1) A matrix factorization with potential w is a quadruple $(\mathcal{M}^{-1}, \mathcal{M}^0, d_{-1}, d_0)$, where $\mathcal{M}^{-1}$ and $\mathcal{M}^0$ are coherent sheaves on X with morphisms $d_{-1} : \mathcal{M}^{-1} \to \mathcal{M}^0$ and $d_0 : \mathcal{M}^0 \to \mathcal{M}^{-1}$ satisfying that $d_0 \circ d_{-1}$ and $d_{-1} \circ d_0$ are multiplication by $w \in \mathcal{O}(X)$.

(2) A matrix factorization is G-equivariant if the sheaves $\mathcal{M}^{-1}$ and $\mathcal{M}^0$ are G-equivariant and the morphisms d_{-1}, and d_0 are G-equivariant.

The category of matrix factorizations on X with potential w has a DG-category structure with Hom complex

$$\mathrm{Hom}^n(\mathcal{M}, \mathcal{N}) := \mathrm{Hom}_{\mathrm{Coh}^G(X)}(\mathcal{M}^{-1}, \mathcal{N}^{-1+n \bmod 2}) \oplus \mathrm{Hom}_{\mathrm{Coh}^G(X)}(\mathcal{M}^0, \mathcal{N}^{n \bmod 2}),$$

with differential

$$d(f)_i(m) := d_N(f_i(m)) - (-1)^i f_i(d_M(m)).$$

We denote the homotopy category by $H^0(\mathrm{Coh}^G(X), w)$. The corresponding category where we only require the terms in the matrix factorization to lie in $\mathrm{QCoh}^G(X)$ is denoted by $H^0(\mathrm{QCoh}^G(X), w)$. The DG-categories allow shifts and cones so the homotopy categories are triangulated. Let

$$\mathcal{M}^\bullet = \cdots \to \mathcal{M}_i \xrightarrow{g_i} \mathcal{M}_{i+1} \to \ldots$$

be a complex ($g_i \circ g_{i-1} = 0$) of matrix factorizations with potential w. The total object of $\mathcal{M}^\bullet$ is the matrix factorization

$$\mathrm{Tot}(\mathcal{M}^\bullet) = \Big(\bigoplus_{2i} \mathcal{M}_{2i}^{-1} \oplus \bigoplus_{2i+1} \mathcal{M}_{2i+1}^0, \bigoplus_{2i} \mathcal{M}_{2i}^0 \oplus \bigoplus_{2i+1} \mathcal{M}_{2i+1}^{-1}, d_{-1}, d_0 \Big) \tag{9}$$

$$d = \sum_n d_{\mathcal{M}_n} + (-1)^{\text{vertical degree}} g_n \tag{10}$$

There is a notion of absolutely derived category due to Positselski

Definition 3.2.

(1) Objects in $H^0(\mathrm{Coh}^G(X), w)$ are called equivariant absolutely acyclic if they belong to the minimal thick subcategory of $H^0(\mathrm{Coh}^G(X), w)$ containing all total complexes of short exact sequences of matrix factorizations with potential w.

(2) The equivariant absolute derived category $D^{\mathrm{abs}}(\mathrm{Coh}^G(X), w)$ is the quotient category of $H^0(\mathrm{Coh}^G(X), w)$ by the thick subcategory of equivariant absolutely acyclic matrix factorizations.

(3) Let $\mathcal{E}$ be an exact subcategory in $\mathrm{QCoh}^G(X)$. Denote the full subcategory of $(\mathrm{QCoh}^G(X), w)$ whose objects have terms in $\mathcal{E}$ by $(\mathrm{QCoh}^G(X)_{\mathcal{E}}, w)$. The relative absolute derived category $D^{\mathrm{abs}}(\mathrm{QCoh}^G(X)_{\mathcal{E}}, w)$ is the quotient of $H^0(\mathrm{QCoh}^G(X)_{\mathcal{E}}, w)$ by the minimal thick subcategory containing the total objects of all short exact sequences whose terms are in $\mathcal{E}$. The category $D^{\mathrm{abs}}(\mathrm{Coh}^G(X)_{\mathcal{E}}, w)$ is defined similarly.

Lemma 3.3. [EP, Rem. 1.3] *Absolute acyclicity is a local notion, i.e. to check that $\mathcal{M}$ is acyclic it is enough to show that $\mathcal{M}$ restricted to each U_α is acyclic, where $\{U_\alpha\}$ is a finite affine open cover of X.*

Proposition 3.4. *Let G be a reductive algebraic group acting on a smooth scheme X. Then the functor $D^{\mathrm{abs}}(\mathrm{Coh}^G(X), w) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w)$ induced by the inclusion $(\mathrm{Coh}^G(X), w) \hookrightarrow (\mathrm{QCoh}^G(X), w)$ is fully faithful.*

Proof. Without equivariance this is [EP, Prop. 1.5(c)]. The proof extends to the equivariant setting. □

4. Braid group action on matrix factorizations coming from Hamiltonian actions

Let G be a reductive algebraic group over an algebraically closed field of characteristic zero and X a smooth G-scheme. Then we have a Hamiltonian action of G on T^*X with moment map $\mu : T^*X \to \mathfrak{g}^*$ which is G-equivariant with the G-action on $\mathfrak{g}^*$ being the coadjoint action. Combining this with the Grothendieck-Springer resolution we get a G-invariant function

$$w : \tilde{\mathfrak{g}} \times T^*X \xrightarrow{\nu\times\mu} \mathfrak{g} \times \mathfrak{g}^* \xrightarrow{\langle\,,\,\rangle} \mathbb{C}.$$

This defines a potential for equivariant matrix factorizations on $\tilde{\mathfrak{g}} \times T^*X$. The theorem we are going to prove is the following.

Theorem 4.1 (Main theorem)**.** *There is a B_{af}-action on $D^{\mathrm{abs}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times T^*X), w)$.*

The idea of the proof is to construct a monoidal category of matrix factorizations and a monoidal action of this category on $D^{\mathrm{abs}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times T^*X), w)$. Then we construct a monoidal functor from a subcategory of the category from Bezrukavnikov and Riche's theorem 2.8, containing the generators of the braid group action, to this monoidal category.

Under the isomorphism $\frac{G\times\mathfrak{b}}{B} \xrightarrow{\sim} \tilde{\mathfrak{g}}$ given by $(g, x) \mapsto (\mathrm{Ad}_g\, x, gB/B)$ the Grothendieck-Springer resolution becomes the map $(g, x) \mapsto \mathrm{Ad}_g\, x$. Thus, we have an equivalence

$$D^{\mathrm{abs}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times T^*X), w) \simeq D^{\mathrm{abs}}(\mathrm{Coh}^{G\times B}(G \times \mathfrak{b} \times T^*X), w'),$$

where w' is the morphism $(g, x, z) \mapsto \langle \mathrm{Ad}_g\, x, \mu(z)\rangle$. The G-action on $\frac{G\times\mathfrak{b}}{B}$ is multiplication on the G-factor. We also have the equivalence

$$D^{\mathrm{abs}}(\mathrm{Coh}^{G\times B}(G \times \mathfrak{b} \times T^*X), w') \simeq D^{\mathrm{abs}}\Big(\mathrm{Coh}^B\Big(\frac{G \times \mathfrak{b} \times T^*X}{G}\Big), \bar{w'}\Big)$$

Under the isomorphism $\frac{G\times\mathfrak{b}\times T^*X}{G} \simeq \mathfrak{b} \times T^*X$ the potential $\bar{w'}$ becomes the map $\mathfrak{b} \times T^*X \to \mathbb{C}$ with $(b, x) \mapsto \langle b, \mu(x)\rangle$. Thus, the theorem gives a B_{af}-action on the equivalent categories approximating $\mathrm{Coh}^{L_{\mathrm{top}}(B)}(L_{\mathrm{top}}(X))$.

$$D^{\mathrm{abs}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times T^*X), w) \simeq D^{\mathrm{abs}}(\mathrm{Coh}^B(\mathfrak{b} \times T^*X), \bar{w'})$$

5. Functors on matrix factorizations

The usual functors on quasi-coherent sheaves we need can be extended to matrix factorizations. Let $\mathcal{M}, \mathcal{N} \in (\mathrm{QCoh}^G(X), w)$ and let $f : X \to Y$ be a G-equivariant morphism.

$$f^* : (\mathrm{QCoh}^G(Y), w \circ f) \to (\mathrm{QCoh}^G(X, w)), \qquad \mathcal{M} \mapsto (f^*\mathcal{M}^{-1}, f^*\mathcal{M}^0, f^*d_{-1}, f^*d_0), \tag{11}$$

$$f_* : (\mathrm{QCoh}^G(X), w \circ f) \to (\mathrm{QCoh}^G(Y), w), \qquad \mathcal{M} \mapsto (f_*\mathcal{M}^{-1}, f_*\mathcal{M}^0, f_*d_{-1}, f_*d_0), \tag{12}$$

$$\mathcal{M} \otimes_X \mathcal{N} := (\mathcal{M}^{-1} \otimes_X \mathcal{N}^0 \oplus \mathcal{M}^0 \otimes_X \mathcal{N}^{-1}, \mathcal{M}^{-1} \otimes_X \mathcal{N}^{-1} \oplus \mathcal{M}^0 \otimes_X \mathcal{N}^0, d_{-1}, d_0) \tag{13}$$

$$d_i(m \otimes n) := d_i^{\mathcal{M}}(m) \otimes b + (-1)^i a \otimes d_i^{\mathcal{N}}(b). \tag{14}$$

The full subcategory of flat (resp. locally free) matrix factorizations is denoted by $(\mathrm{QCoh}^G(X)_{\mathrm{fl}}, w)$ (resp. $(\mathrm{QCoh}^G(X)_{\mathrm{lf}}, w)$) and the full subcategory of $(\mathrm{QCoh}^G(X), w)$ whose terms are injective (resp. finite injective dimension) by $(\mathrm{QCoh}^G(X)_{\mathrm{inj}}, w)$ (resp. $(\mathrm{QCoh}^G(X)_{\mathrm{fid}}, w)$).

Proposition 5.1. *Let G be a reductive algebraic group acting on a smooth scheme X.*

(1) The natural functor

$$H^0(\mathrm{QCoh}^G(X)_{\mathrm{inj}}, h) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), h)$$

is an equivalence of triangulated categories.

(2) The natural functor

$$D^{\mathrm{abs}}(\mathrm{QCoh}^G(X)_{\mathrm{lf}}, w) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w)$$

is an equivalence of triangulated categories.

Proof. (1) By [EP, Lemma 1.7(a)] there is an equivalence of triangulated categories induced by the inclusion

$$H^0(\mathrm{QCoh}(X)_{\mathrm{inj}}, w) \simeq D^{\mathrm{abs}}(\mathrm{QCoh}(X)_{\mathrm{fid}}, w).$$

The proof extends to the equivariant setting. For smooth schemes all equivariant quasi-coherent sheaves have finite equivariant injective dimension so the result follows.

(2) [EP, Cor. 2.4(b)+rem.] states that when X is a regular separated Noetherian scheme of finite Krull dimension then the natural functor $D^{\mathrm{abs}}(\mathrm{QCoh}(X)_{\mathrm{lf}}, w) \to D^{\mathrm{abs}}(\mathrm{QCoh}(X), w)$ is an equivalence of triangulated categories. The proof extends to the equivariant setting when X has finite G-equivariant locally free dimension. This is satisfied when X is smooth. □

By the proposition the above functors decent to the absolute derived categories

Corollary 5.2. Assume that X and Y are smooth and let $f : X \to Y$ be a G-equivariant morphism.

(1) There is a tensor product
$\otimes^L_X : D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w_1) \times D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w_2) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w_1 + w_2)$

(2) There is a pull-back
$$Lf^* : D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y), w \circ f).$$

(3) There is a push-forward
$$Rf_* : D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y), w \circ f) \to D^{\mathrm{abs}}(\mathrm{QCoh}^G(X), w)$$

From now on we will always assume that our schemes are smooth.

Proposition 5.3. *Let $f : X \to Y$ be an equivariant morphism of smooth G-schemes. Then Rf_* is right adjoint to Lf^*.*

Proof. In the non-equivariant setting this follows from [EP] Prop. 1.9 and Cor. 2.3(b)+(f). The proof also works in the equivariant case. □

Lemma 5.4. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of smooth schemes. Then*
$$L(g \circ f)^* \simeq Lf^* \circ Lg^* \qquad \text{and} \qquad R(g \circ f)_* \simeq Rg_* \circ Rf_*.$$

Proof. The first part follows from the fact that pull-back takes flat to flat. The second part follows from the first by adjunction. □

Lemma 5.5. *Let $f : X \to Y$ be an equivariant morphism of smooth schemes. There is an isomorphism of functors*
$$Lg^*(-) \otimes^L_X Lg^*(-) \simeq Lg^*(- \otimes^L_X -).$$

Proof. The pull-back takes flat modules to flat modules so $Lg^*(-)\otimes^L_X Lg^*(-) \simeq L(g^*(-)\otimes_X g^*(-))$ and $Lg^*(- \otimes^L_X -) \simeq L(g^*(- \otimes_Y -))$. Hence, it is enough to show that $g^*(-) \otimes_X g^*(-) \simeq g^*(- \otimes_Y -)$

$$\begin{aligned} g^*(\mathcal{F}_1, \mathcal{F}_2) \otimes_X g^*(\mathcal{G}_1, \mathcal{G}_2) & \qquad (15)\\ = (g^*\mathcal{F}_1 \otimes_X g^*\mathcal{G}_2 \oplus g^*\mathcal{F}_2 \otimes_X g^*\mathcal{G}_1, g^*\mathcal{F}_1 \otimes_X g^*\mathcal{G}_1 \oplus g^*\mathcal{F}_2 \otimes_X g^*\mathcal{G}_2) & \qquad (16)\\ \simeq g^*((\mathcal{F}_1, \mathcal{F}_2) \otimes_Y (\mathcal{G}_1, \mathcal{G}_2)). & \qquad (17) \end{aligned}$$

Clearly, the differentials also match. □

Proposition 5.6 (Projection formula)**.** *Let $g : X \to Y$ be an equivariant morphism of smooth G-schemes. Then there are isomorphisms of functors*
$$Rg_*(Lg^*(-) \otimes^L_X -) \simeq - \otimes^L_Y Rg_*(-), \qquad Rg_*(- \otimes^L_X Lg^*(-)) \simeq Rg_*(-) \otimes^L_Y -.$$

Proof. The proof is essentially the same as for the usual derived category of quasi-coherent sheaves (see for example [Stacks, Lemma 20.8.2 or 21.37.1]). We only prove the first formula since the other is similar. Using the adjunction morphism $Lg^*Rg_* \to \mathrm{Id}$ one gets a morphism
$$- \otimes^L_Y Rg_*(-) \to Rg_*(Lg^*(-) \otimes^L_X -).$$

If $\mathcal{F}$ is flat and $\mathcal{I}$ is injective then $\mathcal{F} \otimes_X \mathcal{I}$ is injective by [EP, Lemma 2.5] so restricting to $D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{fl}}, w_Y) \times D^{\mathrm{abs}}(\mathrm{QCoh}(X)_{\mathrm{inj}}, w_X)$ we have $Rg_*(Lg^*(-) \otimes^L_X -) = g_*(g^*(-) \otimes_X -)$ and $- \otimes^L_Y Rg_*(-) = - \otimes_Y g_*(-)$. By [EP, Cor. 2.3(h)] the inclusion of $D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{lf}}, w_Y)$ into $D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{fl}}, w_Y)$ is an equivalence of categories, so we may assume that $\mathcal{F} \in D^{\mathrm{abs}}(\mathrm{QCoh}(Y)_{\mathrm{lf}}, w_Y)$. By Lemma 3.3 proving that we have an isomorphism can be done locally so we may assume that $\mathcal{F} = (\mathcal{O}_Y^{\otimes n}, \mathcal{O}_Y^{\otimes m})$. The result is now a straight forward calculation. $\square$

Proposition 5.7. *Consider a Cartesian square of equivariant morphisms*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X \arrow[r, "g"'] & W \end{tikzcd}$$

Assume that either f or g is flat. Then there is a natural isomorphism between the composition of derived functors

$$Lg^* \circ Rf_* \simeq Rf'_* \circ Lh^*.$$

Proof. The standard proof of tor-independent base change for quasi-coherent sheaves (see for example [Stacks, Lemma 35.17.3]) is mostly abstract using that push-forward is right adjoint to pull-back and the projection formula. Both are also true for matrix factorizations and the standard proof also work in our setting with minimal changes. $\square$

6. Convolution

In this section we define a monoidal action on the category from section 4. As in [BR] the categorical action of the affine braid group will come from convolution with certain elements.

6.1. Definition

Let X and Y be smooth G-schemes and V a G-vector space. Let V^* denote the dual vector space with the following G-action

$$g \cdot f(x) = f(g^{-1}x) \qquad \forall g \in G, f \in \mathrm{Hom}(V, k), x \in V.$$

Assume that we have equivariant morphisms $\mu : X \to V^*$ and $\nu : Y \to V$. These determine a G-invariant section $w \in \mathcal{O}(Y \times X)$ by

$$w : Y \times X \to k, \qquad (y, x) \mapsto \mu(x)(\nu(y)).$$

Remark 6.1. In the case we are interested in X is a cotangent bundle, $Y = \tilde{\mathfrak{g}}$, $V = \mathfrak{g}$, μ is the moment map and ν is the Grothendieck-Springer resolution.

We would like construct an action on $D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times X), w)$ similar to the one for coherent sheaves in [BR]. However, if we use the exact same formula then the potentials will not match and we will land in the wrong category. To correct this, we introduce an additional factor of V^*

$$\begin{tikzcd} & Y \times Y \times X \arrow[dl, "p"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\ Y \times Y \times V^* & Y \times X & Y \times X \end{tikzcd}$$

where $p := \mathrm{Id} \times \mathrm{Id} \times \mu$. Define the potential on $Y \times Y \times V^*$ to be the section given by

$$h : Y \times Y \times V^* \to k, \qquad (y_1, y_2, g) \mapsto g(\nu(y_1) - \nu(y_2)).$$

Notice that $w \circ p_{23} + h \circ p = w \circ p_{13}$ so we can define the action $*$ to be the composition.

$$\begin{tikzcd} D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times V^*), h) \times D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times X), w) \arrow[d, "Lp^* \times p_{23}^*"] \\ D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times X), h \circ p) \times D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times X), w \circ p_{23}) \arrow[d, "\otimes^L_{Y \times Y \times X}"] \\ D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times X), h \circ p + w \circ p_{23} = w \circ p_{13}) \arrow[d, "Rp_{13*}"] \\ D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times X), w) \end{tikzcd}$$

Now we need to define a monoidal structure on $D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times V^*), h)$. Consider the projection maps

$$\begin{tikzcd} & Y \times Y \times Y \times V^* \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"] \arrow[dr, "p_{23}"] & \\ Y \times Y \times V^* & Y \times Y \times V^* & Y \times Y \times V^* \end{tikzcd}$$

Notice that $h \circ p_{12} + h \circ p_{23} = h \circ p_{13}$ so we define the convolution product $*$ as the composition.

$$\begin{array}{c}
D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times V^*), h) \times D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times V^*), h) \\
\Big\downarrow{\scriptstyle p_{12}^* \times p_{23}^*} \\
D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times Y \times V^*), h \circ p_{12}) \times D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times Y \times V^*), h \circ p_{23}) \\
\Big\downarrow{\scriptstyle \otimes^L_{Y \times Y \times Y \times V^*}} \\
D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times Y \times V^*), h \circ p_{12} + h \circ p_{23} = h \circ p_{13}) \\
\Big\downarrow{\scriptstyle Rp_{13*}} \\
D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times V^*), h).
\end{array}$$

The proof that the convolution product is associative is standard.

Proposition 6.2. *Let* $\mathcal{M}_1, \mathcal{M}_2 \in D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times Y \times V^*), h)$ *and* $\mathcal{N} \in D^{\mathrm{abs}}(\mathrm{QCoh}^G(Y \times X), w)$*. Then* $\mathcal{M}_1 * (\mathcal{M}_2 * \mathcal{N}) \simeq (\mathcal{M}_1 * \mathcal{M}_2) * \mathcal{N}$*.*

Proof. Consider the following cartesian diagram of projections

$$\begin{tikzcd}
Y \times Y \times Y \times X \arrow[r, "p_{234}"] \arrow[d, "p_{124}"'] & Y \times Y \times X \arrow[d, "p_{13}"] \\
Y \times Y \times X \arrow[r, "p_{23}"'] & Y \times X
\end{tikzcd}$$

Using the flat base change and the projection formula we get

$$\begin{aligned}
&\mathcal{M}_1 * (\mathcal{M}_2 * \mathcal{N}) \\
\simeq &Rp_{13*}Rp_{124*}(p_{124}^* Lp^* \mathcal{M}_1 \otimes^L_{Y \times Y \times Y \times X} p_{234}^* Lp^* \mathcal{M}_2 \otimes^L_{Y \times Y \times Y \times X} p_{234}^* p_{23}^* \mathcal{N}).
\end{aligned} \tag{18}$$

Since $p_{13} \circ p_{124} = p_{13} \circ p_{134}$ and $p_{23} \circ p_{234} = p_{23} \circ p_{134}$ we get

$$\begin{aligned}
&\mathcal{M}_1 * (\mathcal{M}_2 * \mathcal{N}) \\
\simeq &Rp_{13*}(Rp_{134*}(p_{124}^* Lp^* \mathcal{M}_1 \otimes^L_{Y \times Y \times Y \times X} p_{234}^* Lp^* \mathcal{M}_2) \otimes^L_{Y \times Y \times X} p_{23}^* \mathcal{N}).
\end{aligned} \tag{19}$$

Set $p_4 := \mathrm{Id} \times \mathrm{Id} \times \mathrm{Id} \times \mu$. Notice that $p \circ p_{124} = \pi_{12} \circ p_4$ and $p \circ p_{234} = \pi_{23} \circ p_4$.

$$\begin{aligned}
&\mathcal{M}_1 * (\mathcal{M}_2 * \mathcal{N}) \\
\simeq &Rp_{13*}(Rp_{134*} Lp_4^* (\pi_{12}^* \mathcal{M}_1 \otimes^L_{Y \times Y \times Y \times V^*} \pi_{23}^* \mathcal{M}_2) \otimes^L_{Y \times Y \times X} p_{23}^* \mathcal{N})
\end{aligned} \tag{20}$$

Consider the cartesian diagram

$$\begin{tikzcd} Y \times Y \times Y \times X \arrow[r, "p_4"] \arrow[d, "p_{134}"'] & Y \times Y \times Y \times V^* \arrow[d, "\pi_{13}"] \\ Y \times Y \times X \arrow[r, "p"'] & Y \times Y \times V^* \end{tikzcd}$$

Using flat base change we get the result. □

6.2. Restriction to an action on the coherent category

The category normally referred to as derived equivariant matrix factorizations is the category $D^{\mathrm{abs}}(\mathrm{Coh}^G(Y \times X), w)$ so we would like our action to restrict to this category. The derived pull-back and tensor products both restricts to the coherent category. However, this is not the case for the push-forward along a non-proper map. Bezrukavnikov and Riche solved the corresponding problem for coherent sheaves by introducing a support condition. A similar notion exists in our setting.

Definition 6.3.

(i) The category-theoretical support of an equivariant coherent sheaf $\mathcal{M}$ on X is the minimal closed subset $T \subset X$ such that $\mathcal{M}|_{X\setminus T}$ is absolutely acyclic in $D^{\mathrm{abs}}(\mathrm{Coh}^G(X), w)$.

(ii) For T a closed subset of a scheme X we denote by $D^{\mathrm{abs}}_T(\mathrm{Coh}^G(X), w)$ the quotient category of the homotopy category of coherent matrix factorizations category-theoretically supported inside T by the thick subcategory of matrix factorizations which are absolutely acyclic in $D^{\mathrm{abs}}(\mathrm{Coh}^G(X), w)$.

The category $D^{\mathrm{abs}}_T(\mathrm{Coh}^G(X), w)$ is a full subcategory in $D^{\mathrm{abs}}(\mathrm{Coh}^G(X), w)$. In the non-equivariant setting this is [EP, Prop. 1.10(d)] and the proof extends to the equivariant case. Our functors restrict to the support category.

Lemma 6.4. *Let $\phi : X \to Y$ be a G-equivariant morphism of Noetherian separated G-schemes with enough G-equivariant vector bundles and T a G-invariant closed subset in X.*

(1) If $\phi|_T : T \to Y$ is proper of finite type and S is a closed subset in $\phi(T)$ then $R\phi_*$ restricts to

$$R\phi_* : D^{\mathrm{abs}}_T(\mathrm{Coh}^G(X), w \circ \phi) \to D^{\mathrm{abs}}_S(\mathrm{Coh}^G(Y), w).$$

(2) Let T_1, T_2 be closed subsets of X. Then the tensor product restricts to a functor

$$\otimes^L_X : D^{\mathrm{abs}}_{T_1}(\mathrm{Coh}^G(X), h_1) \times D^{\mathrm{abs}}_{T_2}(\mathrm{Coh}^G(X), h_2) \to D^{\mathrm{abs}}_{T_1 \cap T_2}(\mathrm{Coh}^G(X), h_1 + h_2).$$

(3) Let S be a closed subset of Y. Then the pull-back restricts to a functor

$$L\phi^* : D^{\mathrm{abs}}_S(\mathrm{Coh}^G(Y), h) \to D^{\mathrm{abs}}_{X\setminus \phi^{-1}(Y\setminus S)}(\mathrm{Coh}^G(X), h \circ \phi).$$

In particular, the convolution action restricts to

$$* : D^{\mathrm{abs}}_{Y\times_V Y\times V^*}(\mathrm{Coh}^G(Y\times Y\times V^*),h)\times D^{\mathrm{abs}}(\mathrm{Coh}^G(Y\times X),w)\to D^{\mathrm{abs}}(\mathrm{Coh}^G(Y\times X),w)$$

and the category $D^{\mathrm{abs}}_{Y\times_V Y\times V^*}(\mathrm{Coh}^G(Y\times Y\times V^*),h)$ is monoidal.

7. Koszul duality

The main theorem 4.1 would follow from [BR] if we can construct a monoidal functor

$$D(\mathrm{Coh}^G(Y\times^R_V Y))\to D^{\mathrm{abs}}_{Y\times_V Y\times V^*}(\mathrm{Coh}^G(Y\times Y\times V^*),h),$$

This turns out to be a bit hard so we will only construct the functor on a full subcategory containing the generators of the braid group action. This functor is called Koszul duality.

7.1. Definition of a Koszul duality functor

The first step is to define a functor

$$\mathrm{Coh}^G(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*))\to C\,\mathrm{Coh}^G(\mathcal{O}_{Y\times Y}\otimes\mathrm{Sym}(V),h),$$

where $C\,\mathrm{Coh}^G(\mathcal{O}_{Y\times Y}\otimes\mathrm{Sym}(V),h)$ is the category of differential graded modules, where the differentials square to h instead of zero. One way to construct such a functor is to tensor with a $\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)\otimes\mathrm{Sym}(V)$-bimodule with differential d satisfying $d^2=h$.

Definition 7.1. Pick a basis $(t_1,\dots,t_n)$ for V and a dual basis $(\xi_1,\dots,\xi_n)$ for V^*. We define a grading with $\mathcal{O}_{Y\times Y}$ in degree 0, ξ_i in degree -1 and t_i in degree 2. Consider the complex K with terms

$$K^m := \bigoplus_{m=2i-j}\mathcal{O}_{Y\times Y}\otimes\Lambda^j(\xi_1,\dots,\xi_n)\otimes\mathrm{Sym}^i(t_1,\dots,t_n)$$

and differential

$$d(f\otimes x\otimes y) := f\otimes d_\Lambda(x)\otimes y+\sum_{k=1}^n f\otimes\xi_i x\otimes t_i y,$$

where d_Λ is the usual differential on Λ. We call K the Koszul complex.

Lemma 7.2. *The Koszul complex is in* $C\operatorname{Coh}^G(\mathcal{O}_{Y\times Y}\otimes \operatorname{Sym}(V),h)$.

Proof. We check that $d^2=h$.

$$d^2(f\otimes x\otimes y) = f\otimes d_\Lambda^2 x\otimes y + \sum_{k=1}^n f\otimes d_\Lambda(\xi_k x)\otimes t_k y + \sum_{k=1}^n f\otimes \xi_k(d_\Lambda x)\otimes t_k y \tag{21}$$

$$+\sum_{k,\ell=1}^n f\otimes \xi_k\xi_\ell x\otimes t_k t_\ell y \tag{22}$$

$$=\sum_{k=1}^n f\otimes (d_\Lambda(\xi_k x)+\xi_k d_\Lambda x)\otimes t_k y \tag{23}$$

$$=\sum_{k=1}^n f\otimes \rho^\sharp(\xi_k)x\otimes t_k y. \tag{24}$$

By definition $h^\sharp = (\rho^\sharp\otimes \mathrm{Id})\circ\langle\ ,\ \rangle^\sharp = \sum_{k=1}^n \rho^\sharp(\xi_k)\otimes t_k$ so $d^2=h$. □

Using the lemma we can define the functor

$$\kappa : \operatorname{Coh}^G(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*))\to C\operatorname{Coh}^G(\mathcal{O}_{Y\times Y}\otimes\operatorname{Sym}(V),h), \tag{25}$$

$$\mathcal{M}\mapsto \mathcal{M}\otimes_{\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)} K\simeq \mathcal{M}\otimes\operatorname{Sym}(V). \tag{26}$$

To make it $\mathbb{Z}/2$-graded we take the direct sum of the all the odd terms and all the even terms. Notice that

$$(\mathcal{M}\otimes_{\Lambda(V^*)} K)^m = \bigoplus_{m=i+2s-r} \mathcal{M}^i\otimes_{\Lambda(V^*)}\Lambda^r(V^*)\otimes\operatorname{Sym}^s(V)$$

So $(\mathcal{M}\otimes_{\Lambda(V^*)} K)^{\mathrm{odd}}\simeq \mathcal{M}^{\mathrm{odd}}\otimes\operatorname{Sym}(V)$ and likewise for the even part.

Lemma 7.3. κ *descends to the homotopy categories.*

Proof. Let $f:\mathcal{M}\to\mathcal{N}$ be a homotopy equivalence in $\operatorname{Coh}^G(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*))$. A short calculation shows that the following diagram is commutative

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{odd}}\otimes\operatorname{Sym}(V) \arrow[r, "\prod f\otimes\mathrm{Id}"] \arrow[d, shift right, "d_{\kappa(\mathcal{M})}"'] & \mathcal{N}^{\mathrm{odd}}\otimes\operatorname{Sym}(V) \arrow[d, shift right, "d_{\kappa(\mathcal{N})}"'] \\
\mathcal{M}^{\mathrm{even}}\otimes\operatorname{Sym}(V) \arrow[r, "\prod f\otimes\mathrm{Id}"] \arrow[u, shift right, "d_{\kappa(\mathcal{M})}"'] & \mathcal{N}^{\mathrm{even}}\otimes\operatorname{Sym}(V) \arrow[u, shift right, "d_{\kappa(\mathcal{N})}"']
\end{tikzcd}$$

Hence, $\prod f\otimes\mathrm{Id}:\kappa(\mathcal{M})\to\kappa(\mathcal{N})$ is a homotopy equivalence in $H^0(C\operatorname{Coh}^G(\mathcal{O}_{Y\times Y}\otimes \operatorname{Sym}(V),h))$. □

We want a functor into the absolute derived category of coherent sheaves so we cannot take infinite direct sums of coherent modules. To avoid this we restrict our functor to the following category.

Definition 7.4. Let $\mathcal{A}$ be a dg-scheme. The category $\mathrm{Perf}^G(\mathcal{A})$ is the full subcategory of the dg-category of G-equivariant $\mathcal{A}$-dg-modules whose objects are finite complexes of locally free modules of finite rank.

By the lemma we get a functor

$$\kappa : \mathrm{H}^0(\mathrm{Perf}^G(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*))) \to D^{\mathrm{abs}}(\mathrm{Coh}^G(Y\times Y\times V^*), h) \tag{27}$$

$$\mathcal{M} \mapsto (\mathcal{M}^{\mathrm{odd}}\otimes\mathrm{Sym}(V), \mathcal{M}^{\mathrm{even}}\otimes\mathrm{Sym}(V), d, d), \tag{28}$$

$$d(m\otimes s) := d_{\mathcal{M}} m\otimes s + \sum_{k=1}^{n}\xi_k m\otimes t_k s. \tag{29}$$

We want the functor to descend to the derived category $D_{\mathrm{Perf}}(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)-\mathrm{mod}^G)$. By lemma 3.3 checking that something is acyclic can be done locally so we may assume that Y is affine.

Lemma 7.5. *For Y affine there is an equivalence of triangulated categories*

$$H^0(\mathrm{Perf}^G(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*))) \simeq D_{\mathrm{Perf}}\,\mathrm{Coh}^G(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)).$$

Proof. When Y is affine the categories reduce to categories of modules over a DG-algebra. Set $A := \mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)$. The result would follow from showing that objects in $\mathrm{Perf}^G(A)$ are projective in $A-\mathrm{mod}^G$. Equivalently, $\mathrm{Ext}^1_{A-\mathrm{mod}^G}(P,M)=0$ for all $P\in\mathrm{Perf}^G(A)$ and $M\in A-\mathrm{mod}^G$. Recall that $\mathrm{Ext}^1_{A-\mathrm{mod}^G}(P,M) = (\mathrm{Ext}^1_{A-\mathrm{mod}}(P,M))^G$ so the result now follows from the fact that perfect complexes are projective in $A-\mathrm{mod}$. □

Thus, we have constructed a functor

$$\kappa : D_{\mathrm{Perf}}\,\mathrm{Coh}^G(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)) \to D^{\mathrm{abs}}(\mathrm{Coh}^G(Y\times Y\times V^*), h).$$

However, the full subcategory $D_{\mathrm{Perf}}\,\mathrm{Coh}^G(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*))$ is not preserved by convolution since the derived push-forward along a non-proper maps does not send Perf to Perf. To fix this, we restrict to the full subcategory whose cohomology over $Y\times Y$ is set-theoretically supported on $Y\times_V Y$, so that the final projection is proper on the support.

$$\kappa : D_{\mathrm{Perf},Y\times_V Y}\,\mathrm{Coh}^G(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*)) \to D^{\mathrm{abs}}(\mathrm{Coh}^G(Y\times Y\times V^*), h).$$

7.2. Compatibility with convolution

To prove that κ commutes with convolution we need some preparatory lemmas.

Lemma 7.6. *Let $\mathcal{M},\mathcal{N}\in D_{\mathrm{Perf}}\,\mathrm{Coh}^G(\mathcal{O}_{Y\times Y}\otimes\Lambda(V^*))$. Then $\kappa(\mathcal{M})\boxtimes\kappa(\mathcal{N})\simeq\kappa(\mathcal{M}\boxtimes\mathcal{N})$.*

Proof. The functor $\boxtimes$ is clearly exact and takes $\mathrm{Perf}\times\mathrm{Perf}$ to Perf. First we check that the matrix factorizations agree on terms

$$\kappa(\mathcal{M})\boxtimes\kappa(\mathcal{N}) \tag{30}$$

$$= \begin{pmatrix} \mathcal{M}^{\mathrm{odd}}\otimes\mathrm{Sym}(V)\boxtimes\mathcal{N}^{\mathrm{even}}\otimes\mathrm{Sym}(V)\oplus\mathcal{M}^{\mathrm{even}}\otimes\mathrm{Sym}(V)\boxtimes\mathcal{N}^{\mathrm{odd}}\otimes\mathrm{Sym}(V) \\ \uparrow\ \ \downarrow \\ \mathcal{M}^{\mathrm{odd}}\otimes\mathrm{Sym}(V)\boxtimes\mathcal{N}^{\mathrm{odd}}\otimes\mathrm{Sym}(V)\oplus\mathcal{M}^{\mathrm{even}}\otimes\mathrm{Sym}(V)\boxtimes\mathcal{N}^{\mathrm{even}}\otimes\mathrm{Sym}(V) \end{pmatrix} \tag{31}$$

$$= \begin{pmatrix} (\mathcal{M} \boxtimes \mathcal{N})^{\mathrm{odd}} \otimes \mathrm{Sym}(V) \otimes \mathrm{Sym}(V) \\ \uparrow \ \ \downarrow \\ (\mathcal{M} \boxtimes \mathcal{N})^{\mathrm{even}} \otimes \mathrm{Sym}(V) \otimes \mathrm{Sym}(V) \end{pmatrix} \tag{32}$$

$$= \kappa(\mathcal{M} \boxtimes \mathcal{N}). \tag{33}$$

Now we check the differentials

$$d_{\kappa(\mathcal{M})\boxtimes\kappa(\mathcal{N})}(a \otimes r \boxtimes b \otimes s) = d(a \otimes r) \boxtimes b \otimes s + (-1)^{|a|} a \otimes r \boxtimes d(b \otimes s) \tag{34}$$

$$= (d_{\mathcal{M}} a \otimes r + \sum_k \xi_k a \otimes t_k r) \boxtimes b \otimes s + (-1)^{|a|} a \otimes r \boxtimes (d_{\mathcal{N}} b \otimes s + \sum_k \xi_k b \otimes t_k s) \tag{35}$$

$$d_{\kappa(\mathcal{M}\boxtimes\mathcal{N})}(a \boxtimes b \otimes r \otimes s) \tag{36}$$

$$= d(a \boxtimes b) \otimes r \otimes s + \sum_k (\xi_k, 0) \cdot (a \boxtimes b) \otimes (t_k, 0) \cdot (r \otimes s) \tag{37}$$

$$+ \sum_k (0, \xi_k) \cdot (a \boxtimes b) \otimes (0, t_k) \cdot (r \otimes s) \tag{38}$$

$$= d_{\mathcal{M}} a \boxtimes b \otimes r \otimes s + (-1)^{|a|} a \boxtimes d_{\mathcal{N}} b \otimes r \otimes s \tag{39}$$

$$+ \sum_k \xi_k a \boxtimes b \otimes t_k r \otimes s + \sum_k (-1)^{|a|} a \boxtimes \xi_k b \otimes r \otimes t_k s \qquad \square$$

Lemma 7.7. *Let $\theta : Z_1 \to Z_2$ be a morphism of schemes.*

(1) If Z_3 is a closed subscheme of Z_1 and θ restricted to Z_3 is proper then the following diagram is commutative

$$\begin{tikzcd} D_{\mathrm{Perf}} \mathrm{Coh}^G(\mathcal{O}_{Z_2} \otimes \Lambda(V)) \arrow[d, "\kappa_1"] & D_{\mathrm{Perf},Z_3} \mathrm{Coh}^G(\mathcal{O}_{Z_1} \otimes \Lambda(V)) \arrow[l, "(\theta\times\mathrm{Id})^\sharp_*"'] \arrow[d, "\kappa_2"] \\ D^{\mathrm{abs}}(\mathrm{QCoh}^G(Z_2 \times V), h_2) & D^{\mathrm{abs}}(\mathrm{QCoh}^G(Z_1 \times V), h_1) \arrow[l, "R(\theta\times\mathrm{Id})_*"'] \end{tikzcd}$$

(2) If θ is flat then the following diagram is commutative

$$\begin{tikzcd} D_{\mathrm{Perf}} \mathrm{Coh}^G(\mathcal{O}_{Z_2} \otimes \Lambda(V)) \arrow[r, "(\theta\times\mathrm{Id})^{\sharp *}"] \arrow[d, "\kappa_1"] & D_{\mathrm{Perf}} \mathrm{Coh}^G(\mathcal{O}_{Z_1} \otimes \Lambda(V)) \arrow[d, "\kappa_2"] \\ D^{\mathrm{abs}}(\mathrm{QCoh}^G(Z_2 \times V), h_2) \arrow[r, "(\theta\times\mathrm{Id})^*"] & D^{\mathrm{abs}}(\mathrm{QCoh}^G(Z_1 \times V), h_1) \end{tikzcd}$$

Proof. 1) Since θ is proper on the support the functor $(\theta \times \mathrm{Id})^\sharp_*$ sends Perf to Perf so the composition is well-defined. Proving that $\kappa_1(\theta \times \mathrm{Id})^\sharp_*(\mathcal{M}) \simeq (\theta \times \mathrm{Id})_* \kappa_2(\mathcal{M})$ can be done locally so we may assume that all schemes are affine so push-forward is exact.

$$\kappa_2(\theta\times\mathrm{Id})^\sharp_*(\mathcal{M}) = \begin{pmatrix} ((\theta\times\mathrm{Id})^\sharp_*\mathcal{M})^{\mathrm{odd}}\otimes\mathrm{Sym}(V^*) \\ \uparrow\quad\downarrow \\ ((\theta\times\mathrm{Id})^\sharp_*\mathcal{M})^{\mathrm{even}}\otimes\mathrm{Sym}(V^*)\end{pmatrix} \tag{40}$$

$$\simeq(\theta\times\mathrm{Id})_*\kappa_1(\mathcal{M}). \tag{41}$$

The functor $(\theta\times\mathrm{Id})^\sharp_*$ does not change the action of $\Lambda(V)$ so $d_{\kappa_2(\theta\times\mathrm{Id})^\sharp_*(\mathcal{M})} = d_{(\theta\times\mathrm{Id})_*\kappa_1(\mathcal{M})}$.

2) Pull-back preserve Perf and $(\theta\times\mathrm{Id})^*$ is exact so we have.

$$\kappa_1(\theta\times\mathrm{Id})^{\sharp *}(\mathcal{M}) \simeq \begin{pmatrix} \mathcal{M}^{\mathrm{odd}}\otimes_{Z_2}\mathcal{O}_{Z_1}\otimes\mathrm{Sym}(V^*) \\ {\scriptstyle d}\uparrow\quad\downarrow{\scriptstyle d} \\ \mathcal{M}^{\mathrm{even}}\otimes_{Z_2}\mathcal{O}_{Z_1}\otimes\mathrm{Sym}(V^*)\end{pmatrix} \tag{42}$$

$$\simeq(\theta\times\mathrm{Id})^*\kappa_2(\mathcal{M}). \tag{43}$$

We easily check that the differentials agree. $\square$

Lemma 7.8. *Let $f: V\hookrightarrow V\oplus W$ be an inclusion of vector spaces. Then the following diagram is commutative*

$$\begin{tikzcd} D_{\mathrm{Perf}}\,\mathrm{Coh}^G(\mathcal{O}_Z\otimes\Lambda(V)\otimes\Lambda(W)) \arrow[r, "(\mathrm{Id}\times f)^\sharp_*"] \arrow[d, "\kappa_1"] & D_{\mathrm{Perf}}\,\mathrm{Coh}^G(\mathcal{O}_Z\otimes\Lambda(V)) \arrow[d, "\kappa_2"] \\ D^{\mathrm{abs}}(\mathrm{QCoh}^G(Z\times V\times W), h_1) \arrow[r, "(\mathrm{Id}\times f)^*"] & D^{\mathrm{abs}}(\mathrm{QCoh}^G(Z\times V), h_2) \end{tikzcd}$$

Proof. Since f is injective and $\Lambda(W)$ is a finite complex with a finite dimensional vector space in each degree the functor $(\mathrm{Id}\times f)^\sharp_*$ preserves Perf. On the level of components we have

$$L(\mathrm{Id}\times f)^*\kappa_1(\mathcal{M}) = \begin{pmatrix} \mathcal{M}^{\mathrm{odd}}\otimes\mathrm{Sym}(V^*)\otimes\mathrm{Sym}(W^*)\otimes^L_{\mathrm{Sym}(W^*)\otimes\mathrm{Sym}(V^*)}\mathrm{Sym}(V^*) \\ \uparrow\quad\downarrow \\ \mathcal{M}^{\mathrm{even}}\otimes\mathrm{Sym}(V^*)\otimes\mathrm{Sym}(W^*)\otimes^L_{\mathrm{Sym}(W^*)\otimes\mathrm{Sym}(V^*)}\mathrm{Sym}(V^*)\end{pmatrix} \tag{44}$$

$$=\kappa_2(\mathrm{Id}\times f)^\sharp_*(\mathcal{M}). \tag{45}$$

Let $(\xi^V_k, t^V_k)^n_{k=1}$ be a pair of a basis for V and its dual basis and $(\xi^W_l, t^W_l)^m_{l=1}$ a pair of a basis for W and its dual basis. Then we have

$$d_{L(\mathrm{Id}\times f)^*\kappa_1(\mathcal{M})}(m\otimes 1\otimes 1\otimes s) \tag{46}$$

$$= d_{\mathcal{M}}m\otimes 1\otimes 1\otimes s+\sum_{k=1}^n \xi^V_k m\otimes 1\otimes 1\otimes t^V_k s+\sum_{l=1}^m \xi^W_l m\otimes 1\otimes 1\otimes f^\sharp(t^W_l)s \tag{47}$$

$$= d_{\mathcal{M}}m\otimes 1\otimes 1\otimes s+\sum_{k=1}^n \xi^V_k m\otimes 1\otimes 1\otimes t^V_k s \tag{48}$$

$$\simeq d_{\mathcal{M}} m \otimes s + \sum_{k=1}^{n} \xi_k^V m \otimes t_k^V s \tag{49}$$
$$= d_{\kappa_2(\mathrm{Id}\times f)_*^\sharp(\mathcal{M})}(m \otimes s). \tag{50}$$

This finishes the proof. ☐

Proposition 7.9. *The functor κ is monoidal.*

Proof. Recall the definition of the convolution product in the end of section 2.

$$\mathcal{M} * \mathcal{N} = Rp_{13*}Rq_{13*}(q_{12}^* p_{12}^* \mathcal{M} \otimes^L_{\mathcal{O}_{Y\times Y\times Y}\otimes\Lambda(V^*)\otimes\Lambda(V^*)} q_{23}^* p_{23}^* \mathcal{N}) \tag{51}$$
$$\simeq Rp_{13*}Rq_{13*}((\Lambda(V^*) \otimes p_{12}^*\mathcal{M}) \otimes^L_{\mathcal{O}_{Y\times Y\times Y}\otimes\Lambda(V^*)\otimes\Lambda(V^*)} (p_{23}^*\mathcal{N} \otimes \Lambda(V^*))) \tag{52}$$
$$\simeq Rp_{13*}Rq_{13*}(p_{12}^*\mathcal{M} \otimes^L_{\mathcal{O}_{Y\times Y\times Y}} p_{23}^*\mathcal{N}) \tag{53}$$
$$\simeq Rp_{13*}Rq_{13*}L\Delta_Y^*(p_{12}^*\mathcal{M} \boxtimes p_{23}^*\mathcal{N}), \tag{54}$$

where Δ_Y is the diagonal embedding.

The corresponding diagram on the Koszul dual side is the following

$$\begin{tikzcd}
& Y \times Y \times Y \times V^* \times V^* & \\
Y \times Y \times Y \times V^* \arrow[ur, hook, "\psi_{12}"] \arrow[d, "\pi_{12}"] & Y \times Y \times Y \times V^* \arrow[u, hook, "\psi_{13}"] \arrow[d, "\pi_{13}"] & Y \times Y \times Y \times V^* \arrow[ul, hook', "\psi_{23}"'] \arrow[d, "\pi_{23}"] \\
Y \times Y \times V^* & Y \times Y \times V^* & Y \times Y \times V^*
\end{tikzcd}$$

The morphisms in the Koszul dual picture are

$$\psi_{12}(y_1, y_2, y_3, v) = (y_1, y_2, y_3, v, 0), \qquad \psi_{23}(y_1, y_2, y_3, v) = (y_1, y_2, y_3, 0, v) \tag{55}$$
$$\psi_{13}(y_1, y_2, y_3, v) = (y_1, y_2, y_3, v, v). \tag{56}$$

The π_{ij} is projection to the (i, j)'th factor times identity. Using the above lemmas we get

$$\kappa(\mathcal{M} * \mathcal{N}) \simeq R\pi_{13*}L(\Delta_Y \circ \psi_{13})^*(\pi_{12}^*\kappa(\mathcal{M}) \boxtimes \pi_{23}^*\kappa(\mathcal{N})).$$

Notice that $\Delta_Y \circ \psi_{13}$ is the diagonal embedding Δ so

$$\kappa(\mathcal{M} * \mathcal{N}) \simeq R\pi_{13*}L\Delta^*(\pi_{12}^*\kappa(\mathcal{M}) \boxtimes \pi_{23}^*\kappa(\mathcal{N})) = \kappa(\mathcal{M}) * \kappa(\mathcal{N}). \tag{57}$$

This finishes the proof. ☐

Proposition 7.10. *The functor κ takes the unit to the unit.*

Proof. The unit in $\mathcal{K}_{Y,V}$ is the structure sheaf of the diagonal $\mathcal{O}_{\Delta Y}$ sitting in degree 0 and $\kappa(\mathcal{O}_{\Delta Y}) = (\mathcal{O}_{\Delta Y} \otimes \mathrm{Sym}(V), 0, 0, 0)$. This is a push-forward along the inclusion

$i : Y \times_Y Y \times V^* \hookrightarrow Y \times Y \times V^*$. Let $\mathcal{M} \in D^{\mathrm{abs}}(\mathrm{Coh}^G(Y \times Y \times V^*), h)$. We need to check that $\kappa(\mathcal{O}_{\Delta Y}) * \mathcal{M} \simeq \mathcal{M} \simeq \mathcal{M} * \kappa(\mathcal{O}_{\Delta Y})$. Consider the following commutative diagram

$$\begin{tikzcd}
Y \times Y \times V^* \arrow[r, dash, "\sim"] \arrow[d, "\mathrm{Id}"] & Y \times_Y Y \times Y \times V^* \arrow[r, "p_1"] \arrow[d, "i_4"] & Y \times_Y Y \times V^* \simeq Y \times V^* \arrow[d, "i"] \\
Y \times Y \times V^* & Y \times Y \times Y \times V^* \arrow[l, "p_{23}", "p_{13}"'] \arrow[r, "p_{12}"'] & Y \times Y \times V^*
\end{tikzcd}$$

Using flat base change and the projection formula we get

$$\begin{aligned}
\kappa(\mathcal{O}_{\Delta Y}) * \mathcal{M} &= Rp_{13*}(p_{12}^* Ri_* \kappa(\mathcal{O}_{\Delta Y}) \otimes^L_{Y \times Y \times Y \times V^*} p_{23}^* \mathcal{M}) && (58)\\
&\simeq Rp_{13*}(Ri_{4*} p_1^* \kappa(\mathcal{O}_{\Delta Y}) \otimes^L_{Y \times Y \times Y \times V^*} p_{23}^* \mathcal{M}) && (59)\\
&\simeq Rp_{13*} Ri_{4*}(p_1^* \kappa(\mathcal{O}_{\Delta Y}) \otimes^L_{Y \times_Y Y \times Y \times V^*} Li_4^* p_{23}^* \mathcal{M}) && (60)\\
&\simeq \mathrm{Id}_*(p_1^* \kappa(\mathcal{O}_Y) \otimes^L_{Y \times Y \times V*} \mathrm{Id}^* \mathcal{M}) && (61)\\
&\simeq \mathcal{M}. && (62)
\end{aligned}$$

Similarly, we get $\mathcal{M} * \kappa(\mathcal{O}_{\Delta Y}) \simeq \mathcal{M}$. □

We can now finish the proof of the main theorem.

Proof of main theorem 4.1. From theorem 2.8 we have explicit generators of a braid group action in $\mathcal{K}^{\mathrm{Coh}}_{\tilde{\mathfrak{g}},\mathfrak{g}}$. These are sheaves on closed subschemes sitting in one degree so they lie in the full subcategory $D_{\mathrm{Perf}}\,\mathrm{Coh}^G(\mathcal{O}_{\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}} \otimes \Lambda(\mathfrak{g}^*))$. Since all of them are supported on $\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ the images under the monoidal functor κ land in the full subcategory $D^{\mathrm{abs}}_{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}} \times \mathfrak{g}^*}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}} \times \mathfrak{g}^*), h)$. Thus, they act on $D^{\mathrm{abs}}(\mathrm{Coh}^G(\tilde{\mathfrak{g}} \times T^*X), w)$ and generate the desired geometric braid group action. □

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