

# Lectures on pentagram maps and KdV hierarchies

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*To the memory of Michael Atiyah, a great mathematician and expositor*

**ABSTRACT.** We survey definitions and integrability properties of the pentagram maps on generic plane polygons and their generalizations to higher dimensions. We also describe the corresponding continuous limit of such pentagram maps: in dimension  $d$  it turns out to be the  $(2, d + 1)$ -equation of the KdV hierarchy, generalizing the Boussinesq equation in 2D.

## 1. Introduction

The pentagram map was defined by R. Schwartz in [19] on plane convex polygons considered modulo projective equivalence. Figure 1 explains the definition: for a generic  $n$ -gon  $P \subset \mathbb{RP}^2$  the image under the pentagram map is a new  $n$ -gon  $T(P)$  spanned by the “shortest” diagonals of  $P$ . It turns out that:

- i)* for  $n = 5$  the map  $T$  is the identity (hence the name of a pentagram map): the pentagram map sends a pentagon  $P$  to a projectively equivalent  $T(P)$ , i.e.  $T = id$ ;
- ii)* for  $n = 6$  the map  $T$  is an involution: for hexagons  $T^2 = id$ ;
- iii)* for  $n \geq 7$  the map  $T$  is quasiperiodic: iterations of this map on classes of projectively equivalent polygons manifest quasiperiodic behaviour, which indicates hidden integrability [19, 20].

**Remark 1.1.** The fact that  $T = id$  for pentagons can be seen as follows. Recall that the cross-ratio of 4 points in  $\mathbb{P}^1$  is given by

$$[t_1, t_2, t_3, t_4] = \frac{(t_1 - t_2)(t_3 - t_4)}{(t_1 - t_3)(t_2 - t_4)},$$

where  $t$  is any affine parameter. Such a cross-ratio is an invariant of 4 points in  $\mathbb{P}^1$  under any projective transformation. Similarly, for any 4 lines passing through the same point  $O$  in  $\mathbb{P}^2$  one defines their cross-ratio as the cross-ratio of the 4 points of intersection of those 4 lines with any other line not passing through the point  $O$ . This cross-ratio of 4 lines will not depend on the choice of the intersecting line and is an invariant of the quadruple of lines under any projective transformation of  $\mathbb{P}^2$ .

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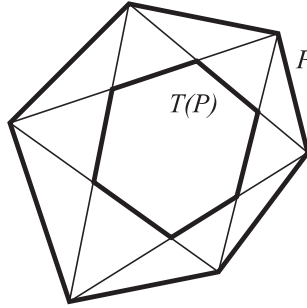


FIGURE 1. The image  $T(P)$  of a pentagon  $P$  under the 2D pentagram map.

Note that the space of projectively equivalent classes of pentagons in  $\mathbb{P}^2$  is two-dimensional. So modulo projective equivalence, any pentagon  $P = (v_1, \dots, v_5)$  is defined by two continuous parameters. These two parameters  $\alpha_i$  can be taken as the cross-ratios of 4 lines that are the two sides and two diagonals passing through  $v_i$ , i.e.,

$$\alpha_i := [(v_i, v_{i+1}), \dots, (v_i, v_{i+4})]$$

for two of the pentagon vertices, for instance, with indices  $i = 1, 2$  (and all indices are understood mod 5). The cross-ratio  $\alpha_i$  of the four lines for each vertex  $v_i$  of any pentagon  $P$  coincides (by definition) with the cross-ratios of 4 points of intersection with the diagonal  $(v_{i-1}, v_{i+1})$ , which in turn coincides with the cross-ratio  $\alpha'_i$  for the new pentagon  $T(P)$ . Thus not only two, but all five coordinate cross-ratios of the image pentagon coincide with cross-ratios of the initial one, which implies that the image pentagon is projectively equivalent to the initial one.

Furthermore, it turns out that for any  $n$  the image polygon  $T(P)$  under the pentagram map is projectively equivalent to the initial polygon  $P$ , provided that  $P$  is Poncelet, i.e., it is inscribed into a conic and circumscribed about a conic [22]. (Note that any pentagon is Poncelet.) Moreover, surprisingly, the converse statement is true: being Poncelet is a necessary and sufficient condition for a polygon  $P$  with an odd number of vertices to be projectively equivalent to  $T(P)$ , see [9]! Other beautiful facts about behaviour of inscribed polygons under iterations of pentagram maps can be found in [23].

**Remark 1.2.** As we mentioned, the pentagram map is quasiperiodic on  $n$ -gons (modulo projective equivalence) for any  $n \geq 7$ . Its integrability was proved in [17] for the pentagram map in 2D on a larger class of the so called twisted polygons, which are piecewise linear curves with a fixed monodromy relating their ends, see Section 4. Closed polygons correspond to the monodromy given by the identity transformation, and in these lecture notes most of the time we confine ourselves to the case of closed polygons. It turned out that there is an invariant Poisson structure for the pentagram map and it has sufficiently many invariant quantities [21, 17]. Moreover, this map turned out to be related to a variety of mathematical domains, including cluster algebras [5, 7], frieze patterns, and

integrable systems of mathematical physics [3, 10]: in particular, its continuous limit in 2D is the classical Boussinesq equation [17]. Integrability of the pentagram map for 2D closed polygons was established in [24, 18], while a more general framework related to surface networks was presented in [4].

There seem to be no natural generalization of the pentagram map to polytopes in higher dimension  $d \geq 3$ . Indeed, the initial polytope should be simple for its diagonal hyperplanes to be well defined as those passing through the neighbouring vertices. In order to iterate the pentagram map the dual polytope has to be simple as well. Thus iterations could be defined only for  $d$ -simplices, which are all projectively equivalent.

Below we describe integrable higher-dimensional generalizations for the pentagram map on space polygons. The main difficulty in higher dimensions is that diagonals of a polygon are generically skew and do not intersect. One can either confine oneself to special polygons (e.g., corrugated ones, [4]) to retain the intersection property or one has too many possible choices for using hyperplanes as diagonals, where it is difficult to find integrable ones, cf. [14, 12]. It turns out that an analogue of the 2D diagonals for a generic polygon in a projective space  $\mathbb{RP}^d$  are “diagonal hyperplanes” passing through  $d$  vertices of a space polygon. However, before we describe those generalizations, we discuss in detail the continuous limit of the pentagram map.

## 2. Continuous limit of pentagram maps

### 2.1. The Boussinesq equation in 2D

The continuous limit of generic  $n$ -gons in  $\mathbb{RP}^2$  is the limit as  $n \rightarrow \infty$ , and it can be viewed as a smooth parameterized curve  $\gamma : \mathbb{R} \rightarrow \mathbb{RP}^2$  (its continuous parameter  $x \in \mathbb{R}$  replaces the vertex index  $i \in \mathbb{Z}$ ). The genericity assumption, requiring every three consecutive points of an  $n$ -gon to be in general position, corresponds to the assumption that  $\gamma$  is a non-degenerate curve in  $\mathbb{RP}^2$ , i.e., the vectors  $\gamma'(x)$  and  $\gamma''(x)$  are linearly independent for all  $x \in \mathbb{R}$ .

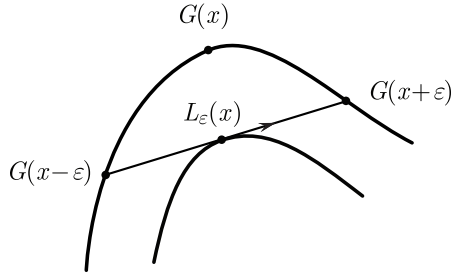


FIGURE 2. Constructing the envelope  $L_\epsilon(x)$  in 2D.

A continuous analogue of the pentagram map is obtained by the following construction. First fix some  $\epsilon > 0$ . Given a smooth non-degenerate curve  $x \mapsto \gamma(x) \in \mathbb{RP}^2$ , we draw the chord  $(\gamma(x - \epsilon), \gamma(x + \epsilon))$  for each point  $x$ . Consider the envelope  $\ell_\epsilon(x)$  of these chords. (Figure 2 shows their lifts from  $\mathbb{RP}^2$  to  $\mathbb{R}^3$ : chords  $(G(x - \epsilon), G(x + \epsilon))$  and their envelope  $L_\epsilon(x)$ .) The nondegeneracy condition on the curve  $\gamma$  guarantees that the envelope  $\ell_\epsilon(x)$  is smooth for sufficiently small  $\epsilon$  and depends smoothly on it. Expand the envelope  $\ell_\epsilon(x)$  in  $\epsilon$ . Since the construction is invariant under  $\epsilon \mapsto -\epsilon$ , the expansion will be only in even powers of  $\epsilon$ , and schematically can be written as follows:

$$\ell_\epsilon(x) = \gamma(x) + \epsilon^2 b_\gamma(x) + \mathcal{O}(\epsilon^4),$$

where  $b_\gamma(x)$  is a certain differential operator on  $\gamma$ . Now consider the evolution of the curve  $\gamma$  in the direction of this envelope  $\ell_\epsilon$ , regarding  $\epsilon^2$  as the time parameter.

**Theorem 2.1** ([17]). *The evolution equation  $\partial_t \gamma(x) = b_\gamma(x)$  is equivalent to the classical Boussinesq equation  $u_{tt} + (u^2)_{xx} + u_{xxxx} = 0$ .*

Before we explain how the curve evolution is related to the Boussinesq equation we describe a higher-dimensional version of this statement.

## 2.2. Evolution of curves in any dimension

Let  $\gamma : \mathbb{R} \rightarrow \mathbb{RP}^d$  be a parametrized nondegenerate curve, i.e., a map satisfying the condition that the vector-derivatives  $\gamma'(x), \gamma''(x), \dots, \gamma^{(d)}(x)$  are linearly independent in  $\mathbb{RP}^d$  for all  $x \in \mathbb{R}$ .

Fix a small  $\epsilon > 0$  and a set of real numbers  $\kappa_1 < \kappa_2 < \dots < \kappa_d$ , such that  $\sum_j \kappa_j = 0$ . Consider the hyperplane  $H_\epsilon(x)$  passing through  $d$  points  $\gamma(x + \kappa_1 \epsilon), \dots, \gamma(x + \kappa_d \epsilon)$  on the curve  $\gamma$ . Let  $\ell_\epsilon(x)$  be the envelope curve for the family of hyperplanes  $H_\epsilon(x)$  sliding along  $\gamma$ , i.e., for a fixed  $\epsilon$  and changing  $x$ . The envelope condition means that  $H_\epsilon(x)$  are the osculating hyperplanes of the curve  $\ell_\epsilon(x)$ , that is the point  $\ell_\epsilon(x)$  belongs to the plane  $H_\epsilon(x)$ , while the vector-derivatives  $\ell'_\epsilon(x), \dots, \ell_\epsilon^{(d-1)}(x)$  span this plane  $H_\epsilon(x)$  for each  $x$ , (see Figure 3 for  $d = 3$  and  $\kappa_1 = -1, \kappa_2 = 0, \kappa_3 = 1$ ). An equivalent definition of the envelope is that the family of hyperplanes  $H_\epsilon(x)$  defines a curve in the dual space  $\mathbb{RP}^{d*}$ . Now consider the family of osculating hyperplanes to this dual curve, which can be regarded as the curve in  $(\mathbb{RP}^{d*})^* = \mathbb{RP}^d$  itself. This is the envelope curve  $\ell_\epsilon(x)$ . Again, the nondegeneracy condition on the initial curve  $\gamma$  guarantees smoothness and smooth dependence on  $\epsilon$  of the envelope  $\ell_\epsilon(x)$ , and we expand it in  $\epsilon$ . One can check that, thanks to the centered condition  $\sum_j \kappa_j = 0$ , there will be no linear in  $\epsilon$  term:

$$\ell_\epsilon(x) = \gamma(x) + \epsilon^2 b_\gamma(x) + \mathcal{O}(\epsilon^3).$$

**Theorem 2.2** ([11, 13]). *For any real constants  $\kappa_1 < \kappa_2 < \dots < \kappa_d$  satisfying  $\sum_j \kappa_j = 0$  the evolution equation  $\partial_t \gamma(x) = b_\gamma(x)$  is equivalent to the  $(2, d + 1)$ -KdV flow of the Adler-Gelfand-Dickey hierarchy.*

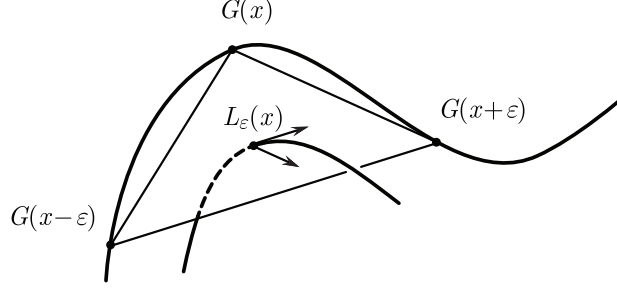


FIGURE 3. The envelope  $L_\epsilon(x)$  in 3D. The point  $L_\epsilon(x)$  and the vectors  $L'_\epsilon(x)$  and  $L''_\epsilon(x)$  belong to the plane  $(G(x), G(x+\epsilon), G(x-\epsilon))$ .

**Definition 2.1.** The  $(m, d+1)$ -KdV flows are defined on linear differential operators

$$\mathcal{L} = \partial^{d+1} + u_{d-1}(x)\partial^{d-1} + u_{d-2}(x)\partial^{d-2} + \dots + u_1(x)\partial + u_0(x)$$

of order  $d+1$  with (periodic) coefficients  $u_j(x)$ , where  $\partial^k$  stands for  $d^k/dx^k$ , see e.g. [1]. For the differential operator  $\mathcal{L}$  one can define its root  $Q := \mathcal{L}^{1/d+1} = \partial + \sum_{j=1}^{\infty} a_j(x)\partial^{-j}$  as a pseudo-differential operator such that  $Q^{d+1} = \mathcal{L}$ . For any positive integer  $m$  the fractional power  $\mathcal{L}^{m/d+1}$  is a pseudo-differential operator of order  $m$ , and one can take its purely differential part  $Q_m := (\mathcal{L}^{m/d+1})_+$ , which is a differential operator of order  $m$ . Then the  $(m, d+1)$ -KdV equation is the evolution equation on (the coefficients of)  $\mathcal{L}$  given by  $\partial_t \mathcal{L} = [Q_m, \mathcal{L}]$ . In particular, for  $m=2$  one has  $Q_2 = \partial^2 + \frac{2}{d+1}u_{d-1}(x)$ , and the corresponding  $(2, d+1)$ -KdV flow is given by

$$\partial_t \mathcal{L} = [\partial^2 + \frac{2}{d+1}u_{d-1}(x), \mathcal{L}].$$

**Remark 2.1.** For  $d=2$  the  $(2,3)$ -KdV equation is the classical Boussinesq equation, found in [17]. Namely, one starts with linear differential operators of the third order:

$$\mathcal{L} = \partial^3 + u_1(x)\partial + u_0(x)$$

with coefficients  $u_0$  and  $u_1$ . Then the evolution of the differential operator

$$\partial_t \mathcal{L} = [\partial^2 + (2/3)u_1(x), \mathcal{L}]$$

stands for the pair of differential equations, the evolution of its coefficients  $du_i/dt = \dots$ ,  $i=0,1$ . After getting rid of  $u_0$  this becomes a single equation involving the second derivative of  $u_1$  in  $t$ . This way the classical Boussinesq equation  $u_{tt} + (u^2)_{xx} + u_{xxx} = 0$  on the function  $u = u_1$  emerges as the  $(2,3)$ -flow in the KdV hierarchy of integrable equations.

**Remark 2.2.** Now we relate the evolution of the curve  $\gamma$  (modulo projective equivalence) in  $\mathbb{RP}^d$  and the evolution of differential operators  $\mathcal{L}$  of order  $d+1$ . For a linear differential operator  $\mathcal{L} = \partial^{d+1} + u_{d-1}(x)\partial^{d-1} + u_{d-2}(x)\partial^{d-2} + \dots + u_1(x)\partial + u_0(x)$  consider the

corresponding differential equation  $\mathcal{L}G = 0$  on the line. It has  $d + 1$  linearly independent solutions  $G_j : \mathbb{R} \rightarrow \mathbb{R}$ . Write the corresponding fundamental system of solutions in the form of a vector function

$$x \mapsto G(x) = (G_1(x), \dots, G_{d+1}(x)) \in \mathbb{R}^{d+1}.$$

Then the corresponding projectivization provides the homogeneous coordinates for the curve

$$x \mapsto \gamma(x) = (G_1(x) : \dots : G_{d+1}(x)) \in \mathbb{RP}^d.$$

The fact that  $G(x)$  has a nonzero Wronskian for all  $x$  implies that the curve  $\gamma$  is nondegenerate. Since the differential operator  $\mathcal{L}$  does not have the subleading term, the Wronskian of  $G$  is constant in  $x$  and can be normalized by  $\det |G(x), G'(x), \dots, G^{(d)}(x)| = 1$  for all  $x \in \mathbb{R}$ . Furthermore, given a parametrized curve  $\gamma$  in  $\mathbb{RP}^d$ , i.e. the homogeneous coordinates of solutions to  $\mathcal{L}G = 0$ , this extra normalization condition allows one to reconstruct the solutions  $G(x)$  themselves. Finally, the  $SL_{d+1}(\mathbb{R})$  ambiguity in the choice of a fundamental system of solutions  $G$  translates into the fact that the curve  $\gamma$  is defined only modulo projective equivalence. Thus coefficients of  $\mathcal{L}$  are coordinates on the space of classes of projectively equivalent curves.

The  $\epsilon$ -expansions above are to be written for the lifts  $G(x)$  and  $L_\epsilon(x)$  in  $\mathbb{R}^{d+1}$  (where there is a linear structure) of the curves  $\gamma(x)$  and  $\ell_\epsilon(x)$  in  $\mathbb{RP}^d$ , see Figures above and details in [11]. One obtains the  $(2, d + 1)$ -KdV equation for a large class of pentagram maps [13, 8], as well as for some maps defined by taking intersections of various planes, rather than the envelopes [14].

**Remark 2.3.** Curiously, the  $(1, d + 1)$ -KdV flow is given by the equation

$$\partial_t \mathcal{L} = [\partial, \mathcal{L}],$$

since  $Q_1 = Q_+ = \partial$ . The commutator of  $\partial$  and  $\mathcal{L}$  in the right-hand side boils down (according to the Leibniz rule) to differentiation in  $x$  of all coefficients of  $\mathcal{L}$ :  $[\partial, \mathcal{L}] = \mathcal{L}'$ , and then the above equation means that  $\partial_t \mathcal{L} = \mathcal{L}'$ , i.e., that the time variable  $t$  of the  $(1, d + 1)$ -KdV flow is equivalent to the space variable  $x$ .

On the other hand, in the above construction of the curve evolution towards its envelope, if we allowed a set of real numbers  $\kappa_1 < \kappa_2 < \dots < \kappa_d$  with  $\sum_j \kappa_j \neq 0$ , then direct computations show that the corresponding expansion of the envelope were

$$\ell_\epsilon(x) = \gamma(x) + \epsilon \left( \sum_j \kappa_j \right) \gamma'(x) + \epsilon^2 C(\kappa) b_\gamma(x) + \mathcal{O}(\epsilon^3).$$

Here the linear in  $\epsilon$  term is present and equal to the derivative  $\gamma'(x)$  (modulo a factor). This  $\gamma'$ -term for the curve evolution is equivalent to the  $\mathcal{L}'$ -term for the evolution of the differential operator. Therefore, the  $(1, d + 1)$ -KdV flow naturally arises in the linear in  $\epsilon$  expansion of the envelope!

As we observed above, once we impose the centered condition  $\sum_j \kappa_j = 0$ , the  $(2, d + 1)$ -KdV equation arises. It is an open question to see if other equations of the  $(m, d)$ -KdV

hierarchy appear in the evolution of curves towards their envelopes in projective spaces. This would give a completely geometric interpretation of the full KdV hierarchy, cf. [1], while the pentagram maps discussed below would give a natural discretization of these ubiquitous equations of mathematical physics.

### 3. Example: an integrable pentagram map in 3D

The continuous case gives a good intuition to define their discrete analogues, pentagram maps in higher dimensions. We start by defining one example of a pentagram map on polygons in  $\mathbb{RP}^3$ , which turns out to be a discrete integrable system.

A closed  $n$ -gon in a projective space  $\mathbb{RP}^3$  is a map  $v : \mathbb{Z} \rightarrow \mathbb{RP}^3$ , such that  $v_{k+n} = v_k$  for each  $k \in \mathbb{Z}$ . We assume that the vertices  $v_k$ ,  $k \in \mathbb{Z}$ , are in general position, i.e., in particular, no four consecutive vertices of an  $n$ -gon belong to one and the same 2-dimensional plane in  $\mathbb{RP}^3$ . Two  $n$ -gons  $v$  and  $\tilde{v}$  are equivalent if there is a transformation  $g \in PSL_4(\mathbb{R})$  such that  $g \circ v = \tilde{v}$ . The following pentagram map  $T$  is generically defined on the space  $\mathcal{P}_n$  of  $n$ -gons considered up to the above equivalence:

**Definition 3.1.** Given an  $n$ -gon  $(v_k)$  in  $\mathbb{RP}^3$ , for each  $k \in \mathbb{Z}$  consider the two-dimensional *short-diagonal plane*  $P_k := (v_{k-2}, v_k, v_{k+2})$  passing through 3 vertices  $v_{k-2}, v_k, v_{k+2}$ . Take the intersection point of the three consecutive planes  $P_{k-1}, P_k, P_{k+1}$  and call it the image of the vertex  $v_k$  under the *space pentagram map*  $T$ :

$$Tv_k := P_{k-1} \cap P_k \cap P_{k+1},$$

see Figure 4. We assume general position, so that every three consecutive planes  $P_k$  for the given  $n$ -gon intersect at a point.

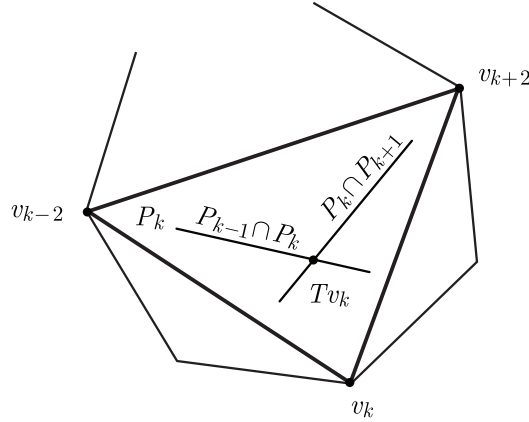


FIGURE 4. The image  $Tv_k$  of the vertex  $v_k$  in  $\mathbb{P}^3$ .

**Theorem 3.1** ([11]). *The space pentagram map on closed  $n$ -gons is a discrete completely integrable system: it admits a Lax representation with a spectral parameter. A Zariski open subset of the complexified space  $\mathcal{P}_n$  is a fibration whose fibres are Zariski open subsets of tori. These tori are invariant with respect to the space pentagram map and have dimensions  $3\lfloor n/2 \rfloor - 6$  for odd  $n$  and  $3(n/2) - 9$  for even  $n$ .*

**Remark 3.1.** Recall that for a smooth dynamical system the *Lax form* (or *Lax representation*) is a differential equation of type  $\partial_t \mathcal{L} = [\mathcal{A}, \mathcal{L}]$  on an operator  $\mathcal{L}$ . (For instance, the KdV hierarchy discussed above can be understood in this form for  $\mathcal{A} = Q_m$ .) Such a form of the evolution equation implies that the eigenvalues of the operator  $\mathcal{L}$  are invariant. In particular, if  $\mathcal{L}$  is a matrix, the evolution of  $\mathcal{L}$  changes it to a similar matrix, thus preserving its eigenvalues. If the matrix  $\mathcal{L}$  depends on a parameter,  $\mathcal{L} = \mathcal{L}(\lambda)$ , then the corresponding eigenvalues as functions of parameter  $\lambda$  do not change and in many cases provide sufficiently many first integrals for complete integrability of such a system.

Similarly, an analogue of the Lax form for differential operators of type  $\partial_x - \mathcal{L}$  is a *zero curvature equation*  $\partial_t \mathcal{L} - \partial_x \mathcal{A} = [\mathcal{A}, \mathcal{L}]$ . This is a compatibility condition which provides the existence of an auxiliary function  $\psi = \psi(t, x)$  satisfying a system of equations

$$\begin{cases} \partial_x \psi = \mathcal{L} \psi \\ \partial_t \psi = \mathcal{A} \psi. \end{cases}$$

In the discrete case, the *Lax form* or *zero-curvature equation* (with a spectral parameter) is an equation of the form

$$\mathcal{L}_{i,t+1}(\lambda) = \mathcal{A}_{i+1,t}(\lambda) \mathcal{L}_{i,t}(\lambda) \mathcal{A}_{i,t}^{-1}(\lambda), \quad (1)$$

which represents a dynamical system. (Note the replacement of a continuous variable  $x$  with a discrete index  $i$ .) This equation may be regarded as a compatibility condition of an over-determined system of equations:

$$\begin{cases} \Psi_{i+1,t} = \mathcal{L}_{i,t} \Psi_{i,t} \\ \Psi_{i,t+1} = \mathcal{A}_{i,t} \Psi_{i,t}, \end{cases}$$

for an auxiliary function  $\Psi_{i,t} = \Psi_{i,t}(\lambda)$ . The algebraic-geometric integrability of the pentagram map is based on such a representation, see details in [11, 13, 8].

**Remark 3.2.** One can see that intersections of diagonal planes for a polygon in  $\mathbb{RP}^d$  as  $n \rightarrow \infty$  tends to the envelope of the limiting curve. One can also try to trace how the Lax representation from [11] tends to the Lax form of the  $(2, d+1)$ -KdV equation.

## 4. More general integrable pentagram maps

When discussing general pentagram maps, one usually deals with a larger space of so-called twisted  $n$ -gons. They are discrete analogues of solution curves  $G$  of linear differential equations  $\mathcal{L}G = 0$  with periodic coefficients. Namely, solutions of periodic equations are not necessarily periodic, but can have a certain monodromy:  $G(x + 2\pi) = \tilde{M} \circ G(x)$

for  $x \in \mathbb{R}$  and  $\widetilde{M} \in SL_{d+1}(\mathbb{R})$ . Respectively, the projectivization curve  $\gamma$  is quasiperiodic: one has  $\gamma(x + 2\pi) = M \circ \gamma(x)$  for  $x \in \mathbb{R}$  and some  $M \in PSL_{d+1}(\mathbb{R})$ .

Discrete analogues of quasiperiodic curves are twisted  $n$ -gons in  $\mathbb{RP}^d$ . Define a *twisted  $n$ -gon* in a projective space  $\mathbb{RP}^d$  with a monodromy  $M \in PSL_{d+1}(\mathbb{R})$  as a doubly-infinite sequence of points  $v_k \in \mathbb{RP}^d$ ,  $k \in \mathbb{Z}$  such that  $v_{k+n} = M \circ v_k$  for each  $k \in \mathbb{Z}$ . We assume that the vertices  $v_k$  are in general position (i.e., no  $d+1$  consecutive vertices lie in the same hyperplane in  $\mathbb{RP}^d$ ), and denote by  $\widetilde{\mathcal{P}}_n$  the space of generic twisted  $n$ -gons considered up to the projective equivalence. General pentagram maps are defined as follows.

**Definition 4.1.** We define two types of diagonal hyperplanes for a given twisted polygon  $(v_k)$  in  $\mathbb{RP}^d$ . The *short-diagonal hyperplane*  $P_k^{sh}$  is defined as the hyperplane passing through  $d$  vertices of the  $n$ -gon by taking every other vertex starting with  $v_k$ :

$$P_k^{sh} := (v_k, v_{k+2}, v_{k+4}, \dots, v_{k+2(d-1)}).$$

Next, given two positive integers  $p$  and  $q$  define the *deep dented diagonal hyperplane*  $P_k^{dd}$  for a fixed  $m = 1, 2, \dots, d-1$  is the hyperplane passing through  $p$  vertices in a row starting with  $v_k$ , then skipping  $q$  vertices, and then again passing through the remaining  $d-p$  vertices in a row, thus ending at the vertex  $v_{k+q+d-1}$ :

$$P_k^{dd} := (v_k, v_{k+1}, \dots, v_{k+p-1}, v_{k+p+q}, v_{k+p+q+1}, \dots, v_{k+q+d-1}).$$

Now the corresponding *short-diagonal* and *deep dented pentagram maps*  $T^{sh}$  and  $T^{dd}$  on twisted polygons  $(v_k)$  in  $\mathbb{RP}^d$  are defined by intersecting  $d$  consecutive diagonal hyperplanes:

$$Tv_k := P_k \cap P_{k+1} \cap \dots \cap P_{k+d-1},$$

where for  $T^{sh}$  and  $T^{dd}$  one uses the definition of the hyperplanes  $P_k^{sh}$  and  $P_k^{dd}$  respectively. These pentagram maps are generically defined on the classes of projective equivalence of twisted polygon:  $T : \widetilde{\mathcal{P}}_n \rightarrow \widetilde{\mathcal{P}}_n$ .

**Example 4.1.** For  $d = 2$  and  $p = q = 1$  both definition coincide with the classical 2D pentagram map in [19]. For  $d = 3$  the map  $T^{sh}$  uses the short diagonal planes  $P_k^{sh} := (v_k, v_{k+2}, v_{k+4})$ , which differ only by an index shift from the example of Section 3. Also for  $d = 3$  and, e.g., for  $p = q = 1$  the deep dented diagonals are the planes  $P_k^{dd} = (v_k, v_{k+2}, v_{k+3})$ .

**Theorem 4.1.** *The short-diagonal  $T^{sh}$  and deep dented  $T^{dd}$  pentagram maps on both twisted and closed  $n$ -gons in any dimension  $d$  and any  $p$  and  $q$  are discrete integrable systems in the sense that they admit Lax representations with a spectral parameter.*

This theorem summarizes results of many works in the area. Namely, integrability for these maps in 2D was proved in [17, 18], while its Lax representation was found in [24]. For short-diagonal pentagram maps their Lax representations with a spectral parameter were found in [11]. They were based on a scale invariance of such maps proved in [11] for 3D and in [15] for higher  $d$ . For the dented pentagram maps their Lax representations

and scale invariance in any dimension were described in [13]. The Lax representation provides first integrals (as the coefficients of the corresponding spectral curve) and allows one to use algebraic-geometric machinery to prove various integrability properties.

There is a broader class of “long-diagonal” maps which unifies the above two cases, and (together with its dual) delivers the most general class of pentagram maps presently known to be integrable.

**Definition 4.2** ([8]). Given a positive integer  $m$  consider the *long-diagonal hyperplane*

$$P_k^{lg} = (v_k, v_{k+m}, \dots, v_{k+(d-1)m})$$

spanned by  $d$  vertices of the  $m$ -arithmetic progression. Now, when defining the corresponding *long-diagonal pentagram map*  $T^{lg}$  we intersect not  $d$  consecutive hyperplanes  $P_k^{lg}$ , but  $d$  hyperplanes with indices that constitute two arithmetic sequences of step  $m$ :

$$T^{lg} v_k := (P_k \cap P_{k+m} \cap \dots \cap P_{k+mp}) \cap (P_{k+q} \cap P_{k+q+m} \cap \dots \cap P_{k+q+m(d-p-2)}) ,$$

assuming that all the indices are different (i.e. for instance,  $q$  is not divisible by  $m$  or  $q > m$  for sufficiently large  $n$ ).

The above definition generalizes the previously known examples [4, 11, 13, 15, 19]. For instance, the short diagonal case corresponds to  $m = 2$  and  $q = 1$ , thus taking separately hyperplanes with even and odd numbers. The deep dented case corresponds to the dual map, where  $m = 1$ , but numbers of the vertices and planes interchanged, see details in [8].

**Theorem 4.2** ([8]). *The long-diagonal pentagram maps  $T^{lg}$  are completely integrable discrete dynamical systems on generic twisted  $n$ -gons in  $\mathbb{RP}^d$ .*

The main tool to prove integrability in this general setting is the refactorization of difference operators introduced in [8]. It also allowed one to describe the corresponding Poisson structures and first integrals in a unified way related to an appropriate Lie-Poisson group of difference operators.

There are many other developments in the world of pentagram maps. To mention just a few:  $Y$ -meshes and integrable networks [6, 4], integrability of corrugated pentagram and leap-frog maps [4], cluster algebra structures [5], pentagrams on Grassmannians [2, 16], etc.

**Remark 4.2.** Here is a word of caution: not all pentagram maps are integrable. Numerical nonintegrability, manifested as a doubly-exponential growth of denominators for rational coordinates of initial polygons was observed, for instance, in 3D for pentagram maps obtained by intersecting every three consecutive planes  $P_k = (v_k, v_{k+2}, v_{k+5})$ , see [12]. Note that this case is one of the simplest not covered by Theorem 4.1: the planes  $P_k = (v_k, v_{k+2}, v_{k+4})$  or  $P_k = (v_{k+1}, v_{k+2}, v_{k+5})$  would correspond, respectively, to the short-diagonal or deep-dented cases and would define integrable maps. To find the exact border between integrable and nonintegrable pentagram maps seems to be a formidable task.

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