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# Notes on the invariants of Legendrian submanifolds and exact Lagrangian cobordisms

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*Dedicated to the memory of Michael Atiyah*

Abstract. These are the notes from the mini-course on Legendrian submanifolds and exact Lagrangian cobordisms given by the author at the Summer Workshop 2019 at Gökova Geometry-Topology Institute. In the mini-course we discuss classical invariants and modern invariants (coming from Chekanov-Eliashberg algebra) of Legendrian submanifolds in both low and high dimensions and the obstructions to the existence of exact Lagrangian cobordisms between Legendrian submanifolds coming from such Legendrian invariants. We do not claim originality of the material in these notes.

## 1. Basic notions

In this section we introduce the basic notions of contact geometry and provide some examples of contact manifolds

A pair $(M, \xi)$, where $M$ is a $(2n+1)$-dimensional manifold and $\xi$ is a smooth maximally non-integrable hyperplane field, is called a *contact manifold.* Maximal non-integrability of $\xi$ means that locally $\xi = \ker \alpha$ for some 1-form $\alpha$ such that

$$(\alpha \wedge (d\alpha)^n)|_x \neq 0,$$

where $x$ is in the domain of definition of $\alpha$. In other words, $\alpha \wedge (d\alpha)^n$ is a volume form on the domain where $\alpha$ is defined. We call a smooth maximally non-integrable hyperplane field $\xi$ a *contact structure* and $\alpha$ is a *contact form* which locally defines $\xi$. The contact condition can also be expressed as $d\alpha|_{\xi_x}$ is a symplectic form for all $x$ in the domain of $\alpha$.

Clearly, there are many contact 1-forms which locally define one contact structure. More precisely, multiplying a local contact 1-form $\alpha$ by a nowhere zero function $f$, we see that $f\alpha$ is again a contact 1-form which defines

$$\xi = \ker \alpha = \ker f\alpha.$$

It follows from the following calculation:

$$(f\alpha \wedge (d(f\alpha))^n)|_x = (f\alpha \wedge (df \wedge \alpha + f d\alpha)^n)|_x = (f^{n+1}\alpha \wedge (d\alpha)^n)|_x \neq 0.$$

From now on we will consider the case when $\xi$ is given as a kernel of a global 1-form $\alpha$.

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Given a contact 1-form $\alpha$, there is a vector field $R_\alpha$ solving the following two equations: $\alpha(R_\alpha) = 1$ and $i_{R_\alpha} d\alpha = 0$. $R_\alpha$ is called the *Reeb vector field corresponding to* $\alpha$. Note that the first equation implies that $R_\alpha$ is transverse to the contact structure. In order to interpret the second equation, consider the flow $\phi_t$ of $R_\alpha$ and observe that from Cartan's formula it follows that

$$\frac{d}{dt}\phi_t^*\alpha = \phi_t^*(\mathcal{L}_{R_\alpha}\alpha) = \phi_t^*(di_{R_\alpha}\alpha + i_{R_\alpha}d\alpha) = 0.$$

Since $\phi_0 = id$, the flow of $R_\alpha$ preserves contact form $\alpha$.

Contact manifolds are considered up to the action of so-called contactomorphisms. We call a diffeomorphism $\varphi : (M_1, \xi_1) \to (M_2, \xi_2)$ satisfying the property that $\varphi_*\xi_1 = \xi_2$ *contactomorphism.*

We discuss a few examples of contact manifolds that will play an important role in this notes:

*Example* 1.1. We start with the most basic example of a contact manifold. Consider $(\mathbb{R}^{2n+1}, \xi_{st}) = (\mathbb{R}^{2n+1}, \ker(\alpha_{st}))$, where $\alpha_{st} = dz - \sum_i y_i dx_i$,

$$\xi_{st} = \mathbb{R} < \partial_{y_i}, \partial_{x_i} + y_i\partial_z >$$

and the Reeb vector filed associated to $\alpha_{st}$ is $\partial_z$. See Figure 1 for the schematic picture of $\xi_{st}$ on $\mathbb{R}^3$.

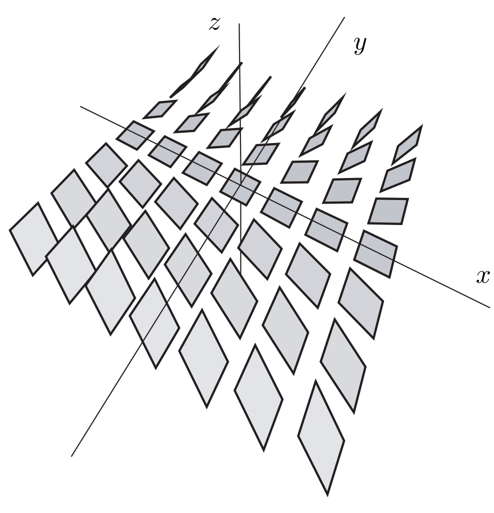



FIGURE 1. The standard contact structure $\xi_{st} = \ker(dz - ydx)$ on $\mathbb{R}^3$.

*Example* 1.2. This example is slightly more general than the previous one. Take $(J^1(M), \xi) = (T^*M \times \mathbb{R}, \ker(dz - pdq))$, where $q$ is a coordinate on $M$, $p$ is a coordinate on the fiber $T^*_q M$ and $z$ is a coordinate on the $\mathbb{R}$-factor. Here the Reeb vector field is $\partial_z$.

*Example* 1.3. This example generalizes the previous two. Consider $(M, \xi) = (P \times \mathbb{R}, \ker \alpha = \ker(dz + \theta))$, where $(P, d\theta)$ is a symplectic manifold and $z$ is a coordinate on the $\mathbb{R}$-factor. Here $R_\alpha = \partial_z$.

*Example* 1.4. Finally, we take $S^{2n+1}$ with $\alpha = i^*(\sum x_i dy_i - \sum y_i dx_i)$, where $i : S^{2n+1} \to \mathbb{R}^{2n+2}$ is the inclusion map. Note that $\ker i^*\alpha$ is a contact structure on $S^{2n+1}$ and that $(S^{2n+1} - \{point\}, \ker i^*\alpha)$ is contactomorphic to the standard Euclidean contact vector space $(\mathbb{R}^{2n+1}, \xi_{st})$.

The next theorem (Darboux's theorem) says that every contact manifold locally looks like a contact Euclidean vector space, and hence contact geometry has no local invariants. More precisely:

**Theorem 1.5** (Darboux)**.** *Let $(M, \xi)$ be a contact $(2n+1)$-dimensional manifold. For all $x \in M$ there are open neighborhoods $x \in U \subset (M, \xi)$, $0 \in V \subset (\mathbb{R}^{2n+1}, \xi_{st})$ and a contactomorphism $\phi : U \to V$ satisftying $\phi(x) = 0$.*

The most interesting submanifolds of a contact manifold are Legendrian submanifolds. A submanifold $\Lambda$ of a $(2n+1)$-dimensional contact manifold $(M, \xi)$ is called *Legendrian* if

- $\Lambda$ is $n$-dimensional and
- $\Lambda$ is everywhere tangent to the contact structure $\xi$, i.e. $T_x\Lambda \subset \xi_x$ for all $x \in \Lambda$.

*Remark* 1.6. Legendrian submanifolds $\Lambda$ of $(M, \xi = \ker \alpha)$ are submanifolds of maximal dimension satisfying the property that $T_x\Lambda \subset \xi_x$ for all $x \in \Lambda$.

*Proof.* Note that given two vector fields $X_1, X_2 \subset \xi$, $[X_1, X_2]$ is a section of $\xi$ if and only if $d\alpha(X_1, X_2) = 0$. This follows from $d\alpha(X_1, X_2) = X_1\alpha(X_2) - X_2\alpha(X_1) - \alpha([X_1, X_2]) = \alpha([X_1, X_2])$. We now observe that maximal dimension of a subspace of $\xi$ satisfying this property is $n$ (Lagrangian subspace of a symplectic vector space $(\xi_x, d\alpha|_{\xi_x})$). □

Legendrian submanifolds of a contact manifold are considered up to Legendrian isotopy. Given two Legendrian submanifolds $\Lambda_0$ and $\Lambda_1$ of a contact manifold $(M, \xi)$, $\Lambda_0$ is said to be Legendrian isotopic to $\Lambda_1$ if there exists a map $f : \Lambda \times [0, 1] \to M$ satisfying the following properties:

(i) $f(\Lambda \times \{0\}) = \Lambda_0$, $f(\Lambda \times \{1\}) = \Lambda_1$, and
(ii) $f|_{\Lambda \times \{t\}}$ is a Legendrian embedding for all $t \in [0, 1]$.

Legendrian submanifolds are often represented by their projections. There are two natural projections:

- the front projection is defined to be a map $\Pi_F : J^1(M) = T^*M \times \mathbb{R} \to M \times \mathbb{R}$ defined by $\Pi_F(q, p, z) = (q, z)$,

- the Lagrangian projection $\Pi_L : (P\times\mathbb{R}, \ker(dz+\theta)) \to P$ defined by $\Pi_F(x,z) = x$, where $x \in P$ and $z \in \mathbb{R}$.

To every Legendrian submanifold $\Lambda \subset P \times \mathbb{R}$ we can associate the set of integral trajectories of the Reeb vector field which start and end on $\Lambda$. Usually such trajectories are called *Reeb chords*, and the set of Reeb chords of $\Lambda$ is denoted by $Q(\Lambda)$. A Legendrian $\Lambda$ is called *chord generic* if the only self-intersections of $\Pi_L(\Lambda)$ are transverse double points. Observe that this condition is open and dense. From now on we will consider only chord generic Legendrian submanifolds. For a closed Legendrian submanifold being chord generic implies that the set of transverse double points of $\Pi_L(\Lambda)$ is finite. Observe that $Q(\Lambda)$ is in one-to-one correspondence with the set of transverse double points of $\Pi_L(\Lambda)$, and hence for a closed chord generic Legendrian submanifold $|Q(\Lambda)| < \infty$.

## 2. Legendrian knots in $\mathbb{R}^3$

In this section we discuss the theory of Legendrian knots in $(\mathbb{R}^3, \ker(dz - ydx))$. Note that in this case $\Pi_F : \mathbb{R}^3 \to \mathbb{R}^2$, $\Pi_F(x,y,z) = (x,z)$ and $\Pi_L : \mathbb{R}^3 \to \mathbb{R}^2$, $\Pi_L(x,y,z) = (x,y)$.

### 2.1. Front projections of Legendrian knots

Let $\Lambda$ be a Legendrian knot in $\mathbb{R}^3$ parametrized by $\phi : S^1 \to \mathbb{R}^3$ with $\phi(t) = (x(t), y(t), z(t))$. The fact that $\Lambda$ is tangent to $\xi_{st}$ can be written as

$$z'(t) = y(t)x'(t). \tag{1}$$

From Equation (1) it follows that $z'(t)$ vanishes whenever $x'(t)$. This in particular implies that $\Pi_F \circ \phi$ is not an immersion, because if $\Pi_F \circ \phi$ is an immersion $x'(t)$ will never vanish and hence $\Pi_F(\Lambda)$ has no vertical tangencies which contradicts the fact that any immersion of $S^1 \to \mathbb{R}^2$ has vertical tangencies. We just showed that $\Pi_F(\Lambda)$ has no vertical tangencies. Now we investigate why $\Pi_F \circ \phi$ can't be an immersion. Equation (1) can be rewritten as

$$y(t) = \frac{z'(t)}{x'(t)}$$

and hence $y$-coordinate of $\phi$ can be recovered from $\Pi_F \circ \phi$ whenever $x'(t) \neq 0$. We then define

$$y(t_0) = \lim_{t\to t_0} \frac{z'(t)}{x'(t)}$$

for the case when $x'(t_0) = 0$. For a generic smooth Legendrian embedding in $\mathbb{R}^3$, $x'(t)$ may vanish only at isolated points, where there is a well-defined tangent line in the front projection. Thus $\Pi_F \circ \phi$ is an immersion except at a finite number of points (generalized cusps), where there is a well-defined tangent line.

Finally, we summarize this discussion by saying that every $xz$-knot diagram

- where the only non-smooth points are generalized cusps,
- without vertical tangencies, and

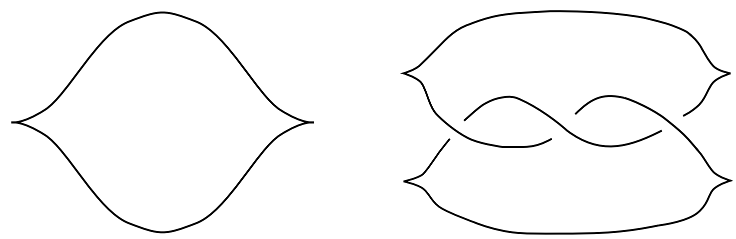


Figure 2. Front projections of unknot (left) and right-handed trefoil (right).

- where at each crossing the slope of the overstrand is more negative than of the understrand

is the front projection of a Legendrian knot. The last condition comes from the fact that in order to have the standard orientation on $\mathbb{R}^3$ the positive $y$-axis should go into the page. See Figure 2 for the examples of front projections of Legendrian knots.

Similar to the Reidemeister moves for knot diagrams of topological knots, there are Reidemeister moves for front projections of Legendrian knots in $\mathbb{R}^3$.

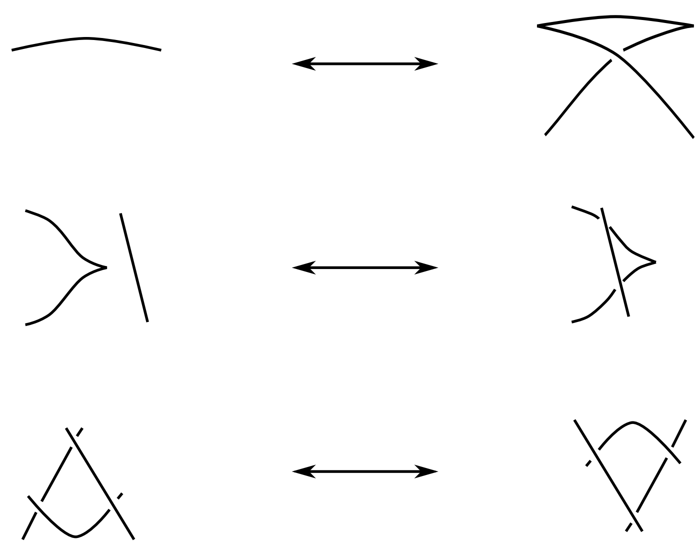


Figure 3. Legendrian Reidemeister moves.

**Theorem 2.1** ([37]). *Two front projections of Legendrian knots in* $(\mathbb{R}^3, \xi_{st})$ *define the Legendrian isotopic Legendrian knots if and only if they are related by regular homotopy* *and a sequence of moves shown in Figure 3 together with the moves obtained by rotating these moves by 180 degrees about each coordinate axis.*

Then we say a few words about approximation of smooth knots by Legendrian knots.

**Theorem 2.2.** *Any smooth knot $K$ in a contact 3-manifold can be $C^0$-approximated by a Legendrian knot.*

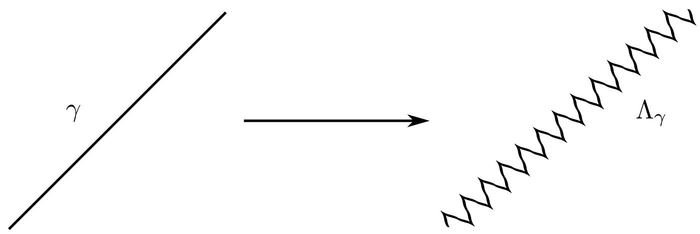



FIGURE 4. $C^0$-approximation by a Legendrian arc.

*Proof.* Using Darboux's theorem it is enough to prove that given an arc $\gamma$ in $(\mathbb{R}^3, \xi_{st})$, there exists a $C^0$-close Legendrian approximation $\Lambda_\gamma$ of $\gamma$ with the endpoints fixed. Take a parametrization $\gamma(t) = (x(t), y(t), z(t))$, $\Pi_F \circ \gamma(t) = (x(t), z(t))$ and approximate $\Pi_F \circ \gamma$ by a zig-zag curve $\Pi_F \circ \Lambda_\gamma(t) = (\tilde{x}(t), \tilde{z}(t))$ with the same endpoints which is $C^0$-close to $\Pi_F \circ \gamma$ and whose derivative is close to $y(t)$. For a schematic description of the approximation see Figure 4.

$\square$

## 2.2. Lagrangian projections of Legendrian knots

We start with a Legendrian knot $\Lambda$ parametrized by $\phi : S^1 \to \mathbb{R}^3$ with $\phi(t) = (x(t), y(t), z(t))$ which leads to the parametrization $\Pi_L \circ \phi(t) = (x(t), y(t))$, $t \in S^1$, of $\Pi_L(\Lambda)$.

Note that there is a way to recover $\Lambda$ from $\Pi_L(\Lambda)$ (up to $z$-translation): let $z(0) = z_0$ for some $z_0$, then

$$z(t) = z_0 + \int_0^t y(s)x'(s)ds. \tag{2}$$

The only freedom we have is a choice of $z_0$.

The Lagrangian projection $\Pi_L(\Lambda)$ is always parameterized by an immersion (unlike $\Pi_F(\Lambda)$). We now investigate the restrictions on it. We take an arbitrary immersion

$\psi : S^1 \to \mathbb{R}^2, t \to (x(t), y(t))$. Note that in order to use Formula (2) to recover $z$ we need to know that $z(0) = z(2\pi)$ which leads to

$$\int_0^{2\pi} y(s)x'(s)ds = 0. \tag{3}$$

Formula (3) guarantees that $\psi$ lifts to an immersion $S^1 \to \mathbb{R}^3$ whose image is tangent to $\xi_{st}$. In order to make sure that $\psi$ lifts to an embedding we need one extra condition: we need to assume that double points in the image of $\psi$ lift to have different $z$-coordinates.

Finally, we summarize by saying that every $xy$-knot diagram satisfying

$$\int_0^{2\pi} y(s)x'(s)ds = 0 \text{ and} \int_{t_0}^{t_1} y(s)x'(s)ds \neq 0 \text{ for all } t_0 \neq t_1 \text{ with } \psi(t_0) = \psi(t_1)$$

lifts to an embedded Legendrian knot.

There is a simple way to get a Lagrangian projection of a Legendrian knot out of its front projection:

**Theorem 2.3** ([25]). *Given a Legendrian knot $\Lambda \subset (\mathbb{R}^3, \xi_{st})$, one gets an immersed curve isotopic to the Lagrangian projection of a Legendrian isotopic knot by applying the local changes described in Figure 5 to $\Pi_F(\Lambda)$.*

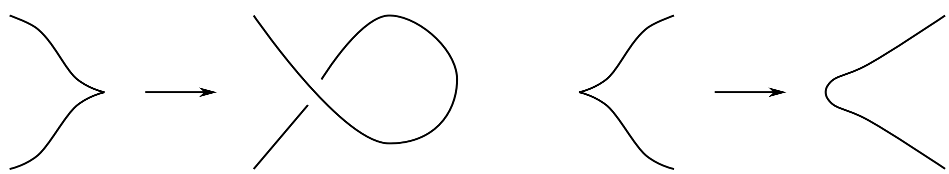


FIGURE 5. The way to get $\Pi_L(\Lambda)$ from $\Pi_F(\Lambda)$.

## 3. Classical invariants

There are three classical invariants of Legendrian submanifolds: topological isotopy type, Thurston-Bennequin number and rotation class.

### 3.1. Thurston-Bennequin number

The Thurston–Bennequin invariant (or number) for Legendrian knots in the standard contact 3-dimensional vector space was originally defined by Bennequin [2] and independently by Thurston. Tabachnikov generalized it to higher dimensions, see [38]. Informally speaking it describes how contact structure twists about Legendrian submanifold.

Let $(M, \xi = \ker\alpha)$ be an oriented contact $(2n+1)$-manifold and $\Lambda \subset (M, \xi)$ be an orientable connected Legendrian submanifold. Here we define the Thurston-Bennequin invariant of $\Lambda$, when $\Lambda$ is homologically trivial in $M$ (whenever $M = \mathbb{R}^{2n+1}$, this poses no additional constraints). We define the *Thurston–Bennequin number of* $\Lambda$ denoted by $tb(\Lambda)$ in the following way. Choose an orientation on $\Lambda$ and push $\Lambda$ slightly off of itself along the Reeb vector field $R_\alpha$ to get another oriented submanifold $\Lambda'$ disjoint from $\Lambda$. Then, we define the Thurston-Bennequin number of $\Lambda$ to be the linking number of $\Lambda$ and $\Lambda'$, i.e.

$$tb(L) := lk(\Lambda, \Lambda'). \tag{4}$$

We now assume that $\Lambda$ is a Legendrian submanifold of $(\mathbb{R}^{2n+1}, \xi_{st})$. Let $c \in Q(\Lambda)$, $c$ has end points $c_+$ and $c_-$ with $z(c_+) > z(c_-)$. Let $V_{c_-} = d\Pi_L(T_{c_-}\Lambda)$ and $V_{c_+} = d\Pi_L(T_{c_+}\Lambda)$. Fix an orientation on $\Lambda$, then $V_{c_-}$ and $V_{c_+}$ are $n$-dimensional transverse subspaces in $\mathbb{R}^{2n}$. Define $sign(c)$ to be 1 if the orientation of $V_{c_-} \oplus V_{c_+}$ agrees with $\mathbb{R}^{2n}$ and $-1$ if the orientation of $V_{c_-} \oplus V_{c_+}$ is opposite to $\mathbb{R}^{2n}$. This leads to the following formula of $tb(\Lambda)$:

$$tb(\Lambda) = \sum_{c \in Q(\Lambda)} sign(c). \tag{5}$$

In order compare Formula (5) with Formula (4) we push $\Lambda$ off of itself using Reeb vector field $\partial_z$ and pick the cycle $C$ as the cone over $\Lambda$ through some point with a large negative $z$-coordinate.

Then we observe that for $\Lambda \subset (\mathbb{R}^{2n+1}, \xi_{st})$, where $n$ is even, $tb(\Lambda)$ is a topological invariant:

**Proposition 3.1** ([21]). *If $\Lambda$ is a Legendrian submanifold of $(\mathbb{R}^{2n+1}, \xi_{st})$, where $n$ is even, then*

$$tb(\Lambda) = (-1)^{\frac{n}{2}+1}\chi(\Lambda). \tag{6}$$

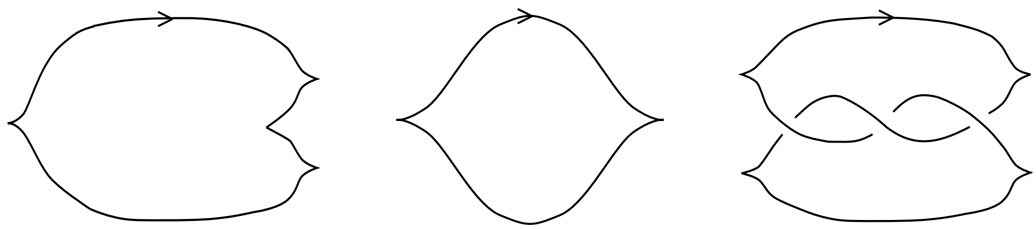
In the case when $\Lambda$ is a Legendrian knot in $(\mathbb{R}^3, \xi_{st})$, we can compute $tb(\Lambda)$ in terms of $\Pi_F(\Lambda)$ and $\Pi_L(\Lambda)$:

$$\begin{aligned} tb(\Lambda) &= \text{writhe}(\Pi_F(\Lambda)) - \frac{1}{2}\#(\text{ of cusps of } \Pi_F(\Lambda)) \qquad (7) \\ tb(\Lambda) &= \text{writhe}(\Pi_L(\Lambda)), \end{aligned}$$

where writhe of the diagram is defined to be the number of positive crossings of the diagram minus the number of negative crossings of the diagram. For some basic calculations, see Figure 6.

### 3.2. Rotation class/number

We start with an arbitrary contact manifold $(M, \xi = \ker \alpha)$ and a Legendrian immersion $f : \Lambda \to (M, \xi)$. Take any complex structure $J$ on $\xi$ which is compatible with its symplectic structure (i.e. $d\alpha|_\xi(Jv, Jw) = d\alpha|_\xi(v, w)$ and $d\alpha|_\xi(v, Jv) > 0$ for all $v, w \in \xi$) and the complexification of $Tf$ which is a fiber bundle isomorphism $Tf_{\mathbb{C}} : T\Lambda \otimes \mathbb{C} \to \xi$. The *rotation class of* $f$ denoted by $r(f)$ is the homotopy class of $(f, Tf_{\mathbb{C}})$ in the space



$tb(\Lambda) = -2$ $\quad$ $tb(\Lambda) = -1$ $\quad$ $tb(\Lambda) = 1$

Figure 6. Thurston-Bennequin numbers of some Legendrian knots.

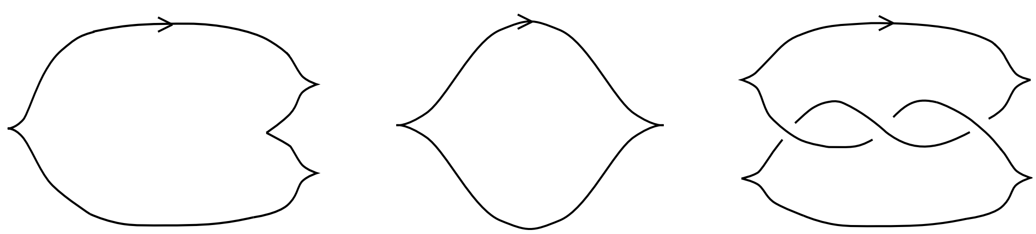
of complex fiberwise isomorphisms $T\Lambda \otimes \mathbb{C} \to \xi$. If $\Lambda$ is a submanifold embedded by inclusion, then it is called the *rotation class of* $\Lambda$ and is denoted by $r(\Lambda)$.

In the case of Legendrian spheres $\Lambda = S^n$ in $(\mathbb{R}^{2n+1}, \xi_{st})$, $r(\Lambda) \in \pi_n(U(n))$. Since $\pi_n(U(n)) \simeq \mathbb{Z}$ when $n$ is odd and $\pi_n(U(n)) \simeq 0$ when $n$ is even, we call $r(\Lambda)$ the *rotation number of* $\Lambda$.

When $\Lambda$ is a Legendrian knot in $(\mathbb{R}^3, \xi_{st})$, there is a way to compute $r(\Lambda)$ from $\Pi_F(\Lambda)$ and $\Pi_L(\Lambda)$:

$$\begin{aligned} r(\Lambda) &= \frac{1}{2}(\#\text{ of downward cusps of } \Pi_F(\Lambda) - \#\text{ of upward cusps of } \Pi_F(\Lambda)), \\ r(\Lambda) &= \text{winding number}(\Pi_L(\Lambda)). \end{aligned} \tag{8}$$

A few computations are done in Figure 7.



$r(\Lambda) = 1$ $\quad$ $r(\Lambda) = 0$ $\quad$ $r(\Lambda) = 0$

Figure 7. Rotation numbers of some Legendrian knots.

### 3.3. Legendrian simple knot types and Chekanov's examples

For some topological knot types rotation and Thurston-Bennequin numbers are enough to classify Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ up to Legendrian isotopy. These knot types are called Legendrian simple. There are a few known examples of Legendrian simple knot types:

(1) Eliashberg and Fraser in [22] proved that unknot is Legendrian simple;
(2) Etnyre and Honda in [26] showed that torus knots and figure-eight knot are Legendrian simple;
(3) a few extra knot types are Legendrian simple, for more details we refer to the works of Etnyre-Honda [27] and of Etnyre-Ng-Vértesi [28].

There exist knot types that are not Legendrian simple. Chekanov proved that the mirror of the $5_2$ knot is not Legendrian simple. More precisely, the following theorem holds:

**Theorem 3.2** ([9]). *There are two Legendrian representatives $\Lambda_1, \Lambda_2$ (whose Lagrangian projections appear in Figure 8) of the mirror of the $5_2$ knot that are not Legendrian isotopic, but have the same Thurston-Bennequin number and rotation number.*

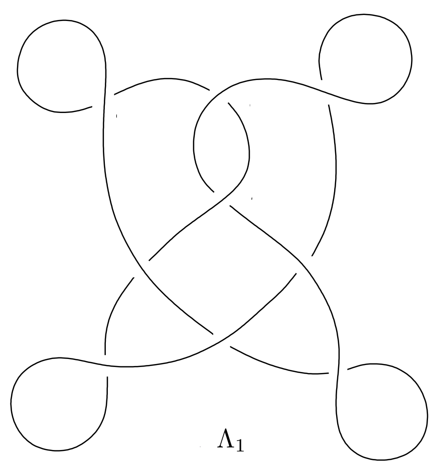



FIGURE 8. Lagrangian projections of Chekanov's knots.

## 3.4. Stabilized and loose Legendrian submanifolds

Stabilization is a procedure that transforms a Legendrian knot in a contact 3-manifold into another smoothly isotopic Legendrian knot with smaller Thurston–Bennequin number. In the case of Legendrians $\Lambda$ in $(\mathbb{R}^3, \xi_{st})$, stabilisation corresponds to adding a zig-zag to $\Pi_F(\Lambda)$. We can extend this definition to an arbitrary contact 3-manifold by using Darboux's theorem. More precisely, there are two versions of stabilization, and for an oriented Legendrian knot $\Lambda$, its positive (negative) stabilization $S_+(\Lambda)$ $(S_-(\Lambda))$ is an operation of adding a zigzag to $\Pi_F(\Lambda)$ the way described in Figure 9.

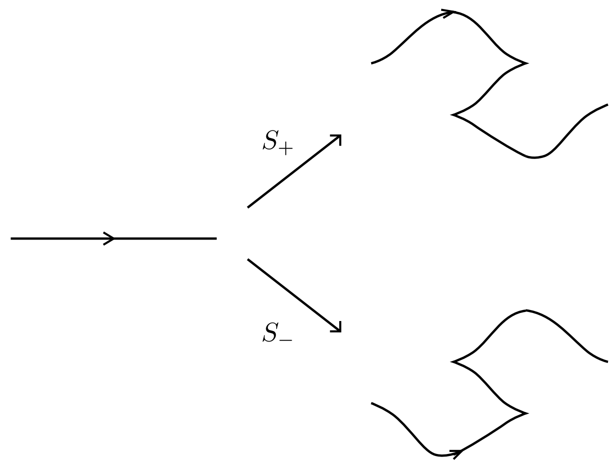



FIGURE 9. Stabilization of a Legendrian knot in the front projection.

From Formulas (7) and (8) it follows that

$$tb(S_\pm(\Lambda)) = tb(\Lambda) - 1,$$
$$r(S_\pm(\Lambda)) = r(\Lambda) \pm 1.$$

Note that stabilization does not depend at what point the stabilization is done. It has been proven for a general contact 3-manifold in [26] and in the case of $(\mathbb{R}^3, \xi_{st})$ in [29].

Fuchs and Tabachnikov proved the following important fact about stabilizations which is based on the observation that topological Reidemeister moves can be performed via Legendrian Reidemeister moves after sufficiently many stabilizations:

**Theorem 3.3** ([29])**.** *Any two topologically isotopic Legendrian knots $\Lambda_1, \Lambda_2 \subset (\mathbb{R}^3, \xi_{st})$ become Legendrian isotopic after each has been stabilized some number of times.*

Loose Legendrian submanifolds have been introduced by Murphy in [32]. Roughly speaking, they can be seen as analogues of stabilized Legendrian knots in high dimensions. They are defined using the following local model:

We say that a Legendrian submanifold $\Lambda \subset (M^{2n+1}, \xi)$ is loose if there exists a neighborhood contactomorphic to a standard loose chart $(B_0, \Lambda_0)$, where

$$(B_0, \Lambda_0) = (U_a \times D_\delta, \lambda \times \gamma_\delta),\ a < \frac{\delta^2}{2}.$$

Here

$$D_\delta = \{(x, y) \in \mathbb{R}^{2n-2}\ :\ |x| \leq \delta, |y| \leq \delta\},$$
$$\gamma_\delta = \{(x, y) \in \mathbb{R}^{2n-2}\ :\ |x| \leq \delta, y = 0\} \subset D_\delta,$$

and $\lambda$ is the Legendrian arc in

$$U_a = \{(x_0, y_0, z) \in \mathbb{R}^3 \ : \ -\frac{a}{2} < z < \frac{a}{2}\},$$

whose front projection is described in Figure 10.

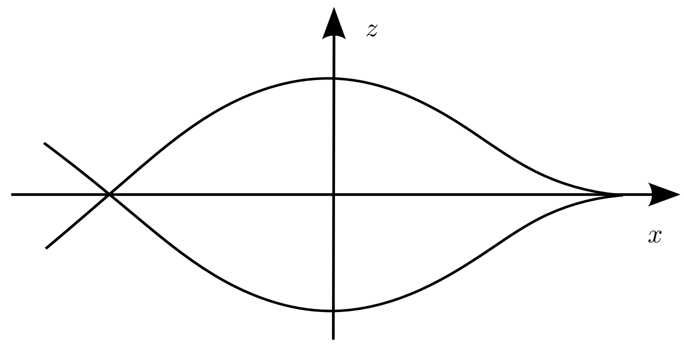



FIGURE 10. Front projection of $\lambda$.

Murphy proved that loose Legendrian submanifolds satisfy h-principle (are flexible). More precisely, Murphy's flexibility result [32] says that that two formally isotopic loose Legendrian submanifolds (i.e. loose Legendrian submanifolds isotopic as smooth submanifolds through an isotopy that can be covered by a homotopy of injective linear maps from the tangent spaces of the submanifold to the tangent spaces of the ambient contact manifold such that the image is everywhere an isotropic subspace) are Legendrian isotopic.

In the next sections we will define the modern invariant called Legendrian contact homology. As it turns out, and as we will see in Example 4.2, the Legendrian contact homology of stabilized Legendrian knots (as well as of loose Legendrian submanifolds) vanish.

## 4. Chekanov-Eliashberg algebra: combinatorial definition

The examples of Chekanov from Theorem 3.2 have been distinguished by the modern invariant which is called Legendrian contact homology. It is a homology of the so-called Chekanov-Eliashberg differential graded algebra (DGA). In this section we provide a combinatorial definition of this algebra given for Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ by Chekanov [9].

Given a Legendrian knot in $\Lambda \subset (\mathbb{R}^3, \xi_{st})$, we know that generically $\Lambda$ has a finite set of Reeb chords $Q(\Lambda)$ which are in one-to-one correspondence with the set of transverse double points of $\Pi_L(\Lambda)$.

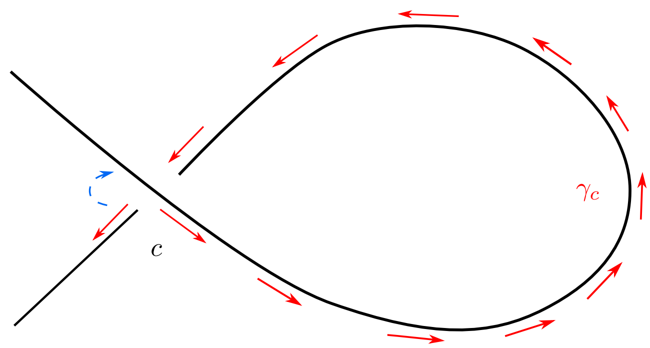



FIGURE 11. The way we define grading on $c \in Q(\Lambda)$, $|c| = 1$.

The Chekanov-Eliashberg algebra of $\Lambda$ (denoted by $\mathcal{A}(\Lambda)$) is the unital non-commutative DGA over a unital commutative ring $R$ freely generated by $\Pi_L(\Lambda)$. Note that $1 \in \mathcal{A}(\Lambda)$ corresponds to the empty word. For simplicity, we will assume that $R = \mathbb{Z}_2$.

There is a $\mathbb{Z}/2r(\Lambda)\mathbb{Z}$-grading on $\mathcal{A}(\Lambda)$. On $\Pi_L(\Lambda)$ we define it the following way. For a transverse double point $c \in \Pi_L(\Lambda)$ there is a corresponding Reeb chord that we still denote by $c \in Q(\Lambda)$ with two endpoints $c_+$ and $c_-$ such that $z(c_+) > z(c_-)$. We take a regular path $\gamma_c : [0, \pi] \to \Pi_L(\Lambda)$ which starts at $c_+$ and ends at $c_-$ (i.e. on $\Pi_L(\Lambda)$ we take a path which starts on the upper strand at $c$ and ends on the lower strand at $c$). Note that there are always two choices of such $\gamma_c$. We define $\Gamma_c : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{RP}^1$ by first taking $\gamma_c'(t)$ for $t \in [0, \pi]$ and then the clockwise rotation from $[\gamma_c'(\pi)]$ to $[\gamma_c'(0)]$ for $t \in [\pi, 2\pi]$. We visualize this procedure in Figure 11. Now we set

$$|c| := \text{degree}(\Gamma_c) \in \mathbb{Z}/2r(\Lambda)\mathbb{Z}. \tag{9}$$

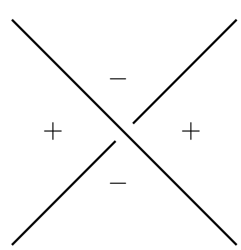



FIGURE 12. Labelling at $c$.

Then we define the differential $\partial_{\mathcal{A}(\Lambda)}$ on $\mathcal{A}(\Lambda)$. We will first define it on $\Pi_L(\Lambda)$ and then will extend it to $\mathcal{A}(\Lambda)$ by linearity and the Leibniz rule:

$$\partial_{\mathcal{A}(\Lambda)}(ab) = \partial_{\mathcal{A}(\Lambda)}(a)b + (-1)^{|a|}a\partial_{\mathcal{A}(\Lambda)}(b).$$

Since in this note we will deal mostly with $\mathbb{Z}_2$ coefficients, the coefficient $(-1)^{|a|}$ does not look necessary. On the other hand, when one considers $\mathcal{A}(\Lambda)$ over a more general ring, the coefficient $(-1)^{|a|}$ plays an essential role.

Let $c$ be a transverse double point of $\Pi_L(\Lambda)$. The neighborhood of $c \in \mathbb{R}^2$ is divided into four quadrants by $\Pi_L(\Lambda)$. We label two of the quadrants by "$-$" and two by "$+$", see Figure 12.

We take transverse double points $c^+, c_1^-, \dots, c_k^- \in \Pi_L(\Lambda)$ and let $\mathbf{c}^- := c_1^- \dots c_k^- \in \mathcal{A}(\Lambda)$. Take a $(k+1)$-sided polygon $P_{k+1}$ with vertices labelled counterclockwise $v_0, v_1, \dots, v_k$. Then we define

$$\mathcal{M}(c^+, \mathbf{c}^-) := \frac{\{u : (P_{k+1}, \partial P_{k+1}) \to (xy\text{-plane}, \Pi_L(\Lambda)) \mid u \text{ is immersion satisfing (i), (ii)}\}}{\text{reparametrizations}},$$

where

(i) $u(v_0) = c^+$ and the image of a small neighborhood of $v_0$ near $c^+$ covers a "$+$" quadrant,

(ii) $u(v_i) = c_i^-$ and the image of a small neighborhood of $v_i$ near $c_i^-$ covers a "$-$" quadrant, $i = 1, \dots, k$.

Finally, define

$$\partial_{\mathcal{A}(\Lambda)}(c^+) = \sum_{\mathbf{c}^-} \#_{\mathbb{Z}_2}\mathcal{M}(c^+, \mathbf{c}^-)\mathbf{c}^-.$$

Here sum is taken over all words $\mathbf{c}^- \in \mathcal{A}(\Lambda)$ and $\#_{\mathbb{Z}_2}\mathcal{M}(c^+, \mathbf{c}^-)$ denotes $\mathbb{Z}_2$-count of elements of $\mathcal{M}(c^+, \mathbf{c}^-)$.

**Theorem 4.1** ([9])**.** $\partial^2_{\mathcal{A}(\Lambda)} = 0$ *and* $\partial_{\mathcal{A}(\Lambda)}$ *decreases the degree by* 1.

The homology of the Chekanov-Eliashberg algebra $H(\mathcal{A}(\Lambda), \partial_{\mathcal{A}(\Lambda)})$ is called Legendrian contact homology of $\Lambda$ and is denoted by $LCH(\Lambda)$.

We now compute Chekanov-Eliashberg algebras for a few Legendrian knots.

*Example* 4.2. Consider Legendrian unknot $\Lambda_1$ whose Lagrangian projection appears in Figure 13.

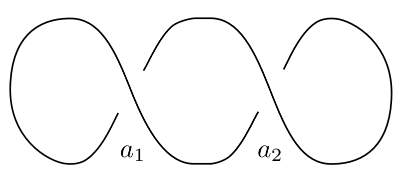



FIGURE 13. Lagrangian projection of $\Lambda_1$.

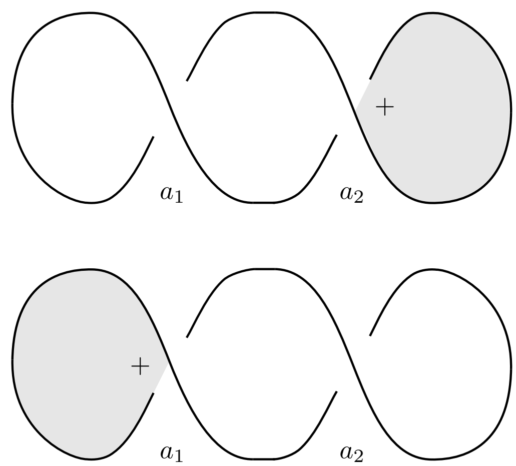



FIGURE 14. Top: 1-gon contributing to $\partial_{\mathcal{A}(\Lambda_1)}(a_2)$, bottom: 1-gon contributing to $\partial_{\mathcal{A}(\Lambda_1)}(a_1)$.

First observe that the set of transverse double points of $\Pi_L(\Lambda_1)$ consists of $a_1, a_2$ and $|a_1| = |a_2| = 1$. Then we see that

$$\partial_{\mathcal{A}(\Lambda_1)}(a_1) = \partial_{\mathcal{A}(\Lambda_1)}(a_2) = 1,$$

where the polygons non-trivially contributing to the differential $\partial_{\mathcal{A}(\Lambda_1)}$ are visualized in Figure 14. Since there are generators on which $\partial_{\mathcal{A}(\Lambda_1)}$ gives 1, $LCH_i(\Lambda_1) = 0$.

*Remark* 4.3. Note that $\Lambda_1$ is stabilized. It turns out that in the Chekanov-Eliashberg algebra of every stabilized Legendrian knot there is a generator on which the differential gives 1, and hence the Legendrian contact homology of it vanishes.

*Example* 4.4. Then take Legendrian knot $\Lambda_2$ with $\Pi_L(\Lambda_2)$ in Figure 15. The set of transverse double points is $\{b_1, b_2, b_3\}$. The computation of $|b_i|$ is left to the reader.

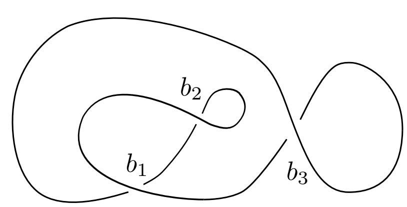



FIGURE 15. Lagrangian projection of $\Lambda_2$.

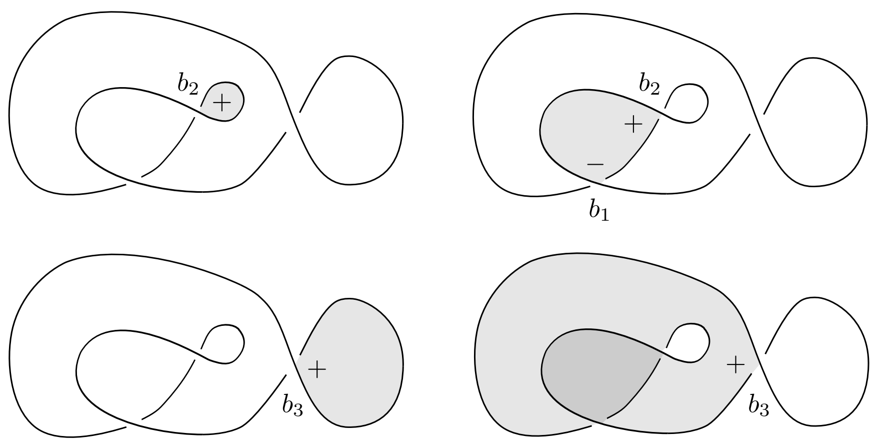



FIGURE 16. Top: the polygons contributing to $\partial_{\mathcal{A}(\Lambda_2)}(b_2)$. Bottom: the polygons contributing to $\partial_{\mathcal{A}(\Lambda_2)}(b_3)$.

The differential $\partial_{\mathcal{A}(\Lambda_2)}$ is given by

$$\begin{aligned}\partial_{\mathcal{A}(\Lambda_2)}(b_1) &= 0,\\ \partial_{\mathcal{A}(\Lambda_2)}(b_2) &= 1 + b_1,\\ \partial_{\mathcal{A}(\Lambda_2)}(b_3) &= 1 + 1 = 0,\end{aligned}$$

where the non-trivial contributions of $\partial_{\mathcal{A}(\Lambda_2)}(b_2)$ and $\partial_{\mathcal{A}(\Lambda_2)}(b_3)$ are described in Figure 16.

*Example* 4.5. Finally, consider right-handed Legendrian trefoil $\Lambda_3$ whose Lagrangian projection appears in Figure 17.

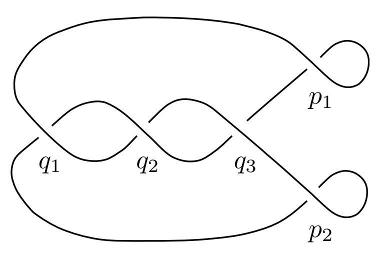



FIGURE 17. Lagrangian projection of $\Lambda_3$.

Note that the set of double points of $\Pi_L(\Lambda_3)$ is $\{p_1, p_2, q_1, q_2, q_3\}$ and $|p_1| = |p_2| = 1$, $|q_1| = |q_2| = |q_3| = 0$. From the fact that $\partial_{\mathcal{A}(\Lambda_3)}$ decreases the grading by 1 (Theorem

4.1) it follows that

$$\partial_{\mathcal{A}(\Lambda_3)}(q_1) = \partial_{\mathcal{A}(\Lambda_3)}(q_2) = \partial_{\mathcal{A}(\Lambda_3)}(q_3) = 0.$$

We compute

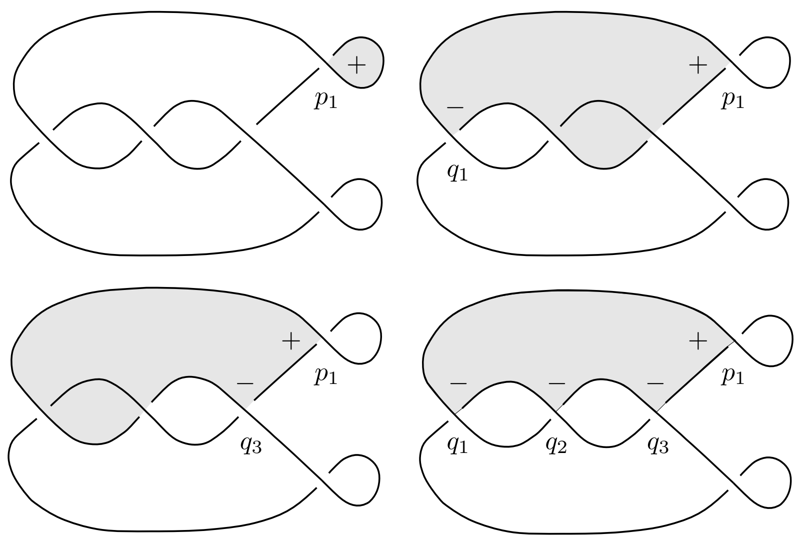



FIGURE 18. Polygons contributing to $\partial_{\mathcal{A}(\Lambda_3)}(p_1)$.

$$\partial_{\mathcal{A}(\Lambda_3)}(p_1) = 1 + q_1 + q_3 + q_1q_2q_3,$$

where the polygons contributing to the differential $\partial_{\mathcal{A}(\Lambda_3)}(p_1)$ appear in Figure 18.

Finally, we compute

$$\partial_{\mathcal{A}(\Lambda_3)}(p_2) = 1 + q_1 + q_3 + q_3q_2q_1,$$

where the polygons contributing to the differential $\partial_{\mathcal{A}(\Lambda_3)}(p_2)$ appear in Figure 19.

## 4.1. Invariance of Chekanov-Eliashberg algebra up to Legendrian isotopy

We first provide a few abstract algebraic definitions. Consider a unital DGA freely generated by $c_1, \dots, c_k$ that we denote by $\mathcal{A}(c_1, \dots, c_k)$. We call a (graded) automorphism of $\mathcal{A}(c_1, \dots, c_k)$ *elementary* if it is of the form

$$\phi(c_i) = \begin{cases} \pm c_j + u, & \text{if } i = j \text{ and } u \in \mathcal{A}(c_1, \dots, c_{j-1}, c_{j+1}, \dots, c_k); \\ c_i, & \text{if } i \neq j \end{cases}$$

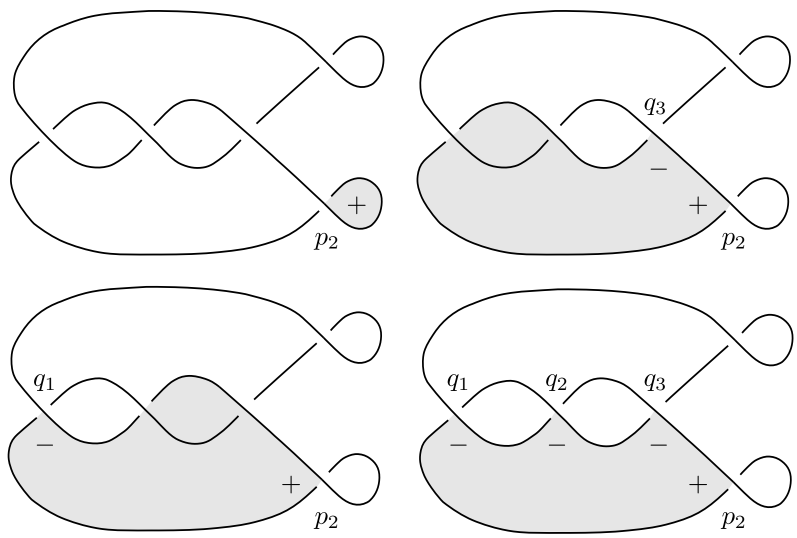



FIGURE 19. Polygons contributing to $\partial_{\mathcal{A}(\Lambda_3)}(p_2)$.

for some $j$, and we call a composition of elementary automorphisms *tame*. Finally, if we compose an identification of the generators of two unital DGAs with a tame isomorphism, the composition is called a *tame isomorphism*.

Then we define the notion of stabilization. A stabilization of a DGA $\mathcal{A}(c_1, \ldots, c_k)$ is the graded algebra $\mathcal{A}(c_1, \ldots, c_k, p, q)$, where $|p| = |q| - 1 = i$ for some $i$, and

$$\partial_{\mathcal{A}(c_1,\ldots,c_k,p,q)}(c_i) = \partial_{\mathcal{A}(c_1,\ldots,c_k)}(c_i) \text{ for all } i,$$
$$\partial_{\mathcal{A}(c_1,\ldots,c_k,p,q)}(p) = 0,$$
$$\partial_{\mathcal{A}(c_1,\ldots,c_k,p,q)}(q) = p.$$

Two unital differential graded algebras are called *stably tame isomorphic* if after stabilizing each of them several times (possibly different number of times) they become tame isomorphic.

**Theorem 4.6** ([9])**.** *If $\Lambda$ is Legendrian isotopic to $\Lambda'$, then $(\mathcal{A}(\Lambda), \partial_{\mathcal{A}(\Lambda)})$ is stable tame isomorphic to $(\mathcal{A}(\Lambda'), \partial_{\mathcal{A}(\Lambda')})$. In addition, $LCH(\Lambda)$ is a Legendrian invariant, i.e. does not change under Legendrian isotopy.*

## 5. Geometric definition of the Chekanov-Eliashberg algebra

Here we sketch another (geometric) definition of the Chekanov-Eliashberg algebra which works in more general settings. Let $\Lambda \subset (P^{2n} \times \mathbb{R}, \alpha := dz + \theta)$ be a closed chord-generic Legendrian submanifold. We define $\mathcal{A}(\Lambda)$ to be a non-commutative unital differential graded algebra over $\mathbb{Z}_2$ (or a general field $\mathbb{F}$) freely generated by elements of $Q(\Lambda)$.

### 5.1. Grading

Assume that $\Lambda$ is connected and that the first Chern class of the tangent bundle of $P$ equipped with any almost complex structure compatible with $\omega$ that we denote by $c_1(P,\omega)$ vanishes. In addition, assume that $H_1(P)$ is free. In case when $H_1(P)$ has torsion, the grading on the Chekanov-Eliashberg algebra can be taken to be a $\mathbb{Z}_2$-grading, see [23].

Now we discuss the definition of a Maslov index. Consider the Grassman manifold of Lagrangian subspaces in the symplectic vector space $(\mathbb{C}^n, \omega_{st})$ and denote it by $L_n$. Observe that $H_1(L_n) = \pi_1(L_n) = \mathbb{Z}$. Now fix a Lagrangian subspace $L \subset \mathbb{C}^n$ and define $\Sigma_k(L) \subset L_n$ to be the subset of Lagrangian spaces that intersects $L$ in a $k$-dimensionsional subspace. Then the Maslov cycle is defined to be $\Sigma := \cup_{i=1}^n \Sigma_i(L)$. Let $\gamma : [0,1] \to L_n$ be a loop through $L$. There is a standard isomorphism $\mu : H_1(L_n) \to \mathbb{Z}$, where $\mu(\gamma)$ is given by the intersection number of $\gamma$ and Maslov cycle $\Sigma$, and $\mu(\gamma)$ is called the Maslov index of $\gamma$.

Then we define Conley-Zehnder index. Let $c \in Q(\Lambda)$ with two endpoints $c_+$ and $c_-$, where $z(c_+) > z(c_-)$. We choose a capping path $\gamma_c : [0,1] \to \Lambda$ that runs from $c_+$ to $c_-$. We fix embedded circles $a_1, \dots, a_k$ whose homology classes generate $H_1(P)$, and fix a symplectic trivialization of $TP$ over each $a_i$. Since $H_1(P)$ is torsion-free, there is a surface $\Sigma_c$ and a uniquely determined collection $n_1, \dots, n_k$ such that $\partial\Sigma_c = \Pi_L(\gamma_c) - \sum_i n_i a_i$. There is a unique trivialization of $TP$ over $\Sigma_c$ that extends the chosen trivializations over $a_i$. With respect to this trivialization of the symplectic bundle $TP$ over $\Pi_L(\gamma_c)$ we can describe $\Gamma(t) := d\Pi_L(T_{\gamma_c(t)}\Lambda) \subset TP$, $0 \le t \le 1$, as a path of Lagrangian subspaces in $\mathbb{C}^n$. Since $c \in Q(\Lambda)$ corresponds to the transverse double point of $\Pi_L(\Lambda)$, $\Gamma(t)$ is not a closed loop. Then we close $\Gamma(t)$. Let $V_i = \Gamma(i)$, $i = 0, 1$. Choose a complex structure $I$ on $\mathbb{C}^n$ such that $I(V_1) = V_0$ and such that it is compatible with the standard symplectic form on $\mathbb{C}^n$. Then we define $\Gamma(V_1, V_0)(t) := e^{tI}V_1$, $t \in [0, \frac{\pi}{2}]$. Take the concatenation $\Gamma * \Gamma(V_1, V_0)$ which is a closed loop and define Conley-Zehnder index of $c$, denoted by $\mu_{CZ}(c)$, to be the Maslov index of this loop, i.e.

$$\mu_{CZ}(c) := \mu(\Gamma * \Gamma(V_1, V_0)).$$

The grading of $c$ is $|c| = \mu_{CZ}(c) - 1$.

### 5.2. Differential

In order to define $\partial_{\mathcal{A}(\Lambda)}$, we will first define it on the elements of $Q(\Lambda)$ and then will extend it by Leibniz rule and linearity. Now we take the symplectization

$$S(P \times \mathbb{R}) := (\mathbb{R} \times P \times \mathbb{R}, d(e^t(\alpha))),$$

where $t$ is a coordinate on the first $\mathbb{R}$ multiple, and an almost complex structure $J$ on $S(P \times \mathbb{R})$ such that

- $J : \xi \to \xi$, where $\xi = \ker \alpha$, and $J$ is compatible with the symplectic form,

- $J$ is invariant under translation in $t$-direction,
- $J(\partial_t) = \partial_z$.

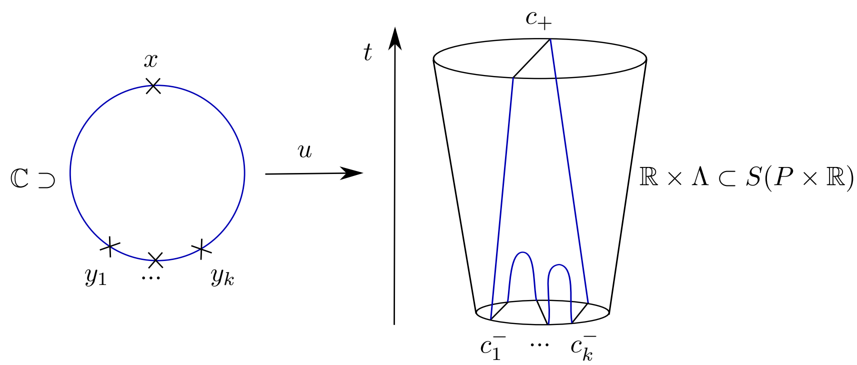



FIGURE 20. An element of $u \in \mathcal{M}_{S(P\times\mathbb{R})}(c_+, c_1^- \dots c_k^-)$.

Let $D^2 \subset (\mathbb{C}, i)$ be the closed unit disc and let $\{x, y_1, \dots, y_k\} \subset \partial D^2$ be the set of points in the cyclic order with respect to the orientation of the boundary. Then we define the moduli space of $J$-holomorphic punctured discs $\mathcal{M}_{S(P\times\mathbb{R})}(c^+, c_1^- \dots c_k^-)$ which consists of

$$u := (a, f) : (D^2 \setminus \{x, y_1, \dots, y_k\}, \partial D^2 \setminus \{x, y_1, \dots, y_k\}) \to (\mathbb{R} \times P \times \mathbb{R}, \mathbb{R} \times \Lambda)$$

such that

- $du + J \circ du \circ i = 0$,
- $\lim\limits_{p\to x} a(p) = \infty$, $\lim\limits_{p\to y_i} a(p) = -\infty$,
- $\lim\limits_{p\to x} f(p) = c_+$, $\lim\limits_{p\to y_i} f(p) = c_i^-$ in the appropriate sense, $i = 1, \dots, k$.

In addition, two solutions are identified if they differ by a biholomorphism of the domain. We refer to $x$ as a positive boundary puncture asymptotic to $c^+$, and to $y_i$ as a negative boundary puncture asymptotic to $c_i^-$, $i = 1, \dots, k$. The dimension of $\mathcal{M}_{S(P\times\mathbb{R})}(c^+, c_1^- \dots c_k^-)$ for a generic $J$ is given by

$$\dim \mathcal{M}_{S(P\times\mathbb{R})}(c^+, c_1^- \dots c_k^-) = |c_+| - |c_1^- \dots c_k^-|.$$

Then define

$$\partial_{\mathcal{A}(\Lambda)}(c_+) = \sum_{\substack{c_1^- \dots c_k^- \\ |c_+| - |c_1^- \dots c_k^-| = 1}} \#_{\mathbb{Z}_2} \frac{\mathcal{M}_{S(P\times\mathbb{R})}(c^+, c_1^- \dots c_k^-)}{\mathbb{R}} c_1^- \dots c_k^-.$$

Here the $\mathbb{R}$-action on $\mathcal{M}_{S(P\times\mathbb{R})}(c^+, c_1^- \dots c_k^-)$ is given by translations of $a$ by a constant. Gromov-Hofer compactness (see [3] and [19]) implies that the above count makes sense.

*Remark* 5.1. For a Legendrian submanifold $\Lambda \subset P \times \mathbb{R}$ the Chekanov-Eliashberg algebra $\mathcal{A}(\Lambda)$ comes in two different flavors:

- It can be described the way we do it in this section. First it has been constructed this way by Ekholm, who following the philosophy of symplectic field theory of Eliashberg-Givental-Hofer [23] constructed relative symplectic field theory [13]. The advantage of such description is that it fits directly into the algebraic framework of symplectic field theory.
- There is another description, where $\mathcal{A}(\Lambda)$ is generated by the transverse double points of $\Pi_L(\Lambda)$, and where the differential $\partial_{\mathcal{A}(\Lambda)}$ counts rigid pseudoholomorphic polygons in $P$ with the boundary on $\Pi_L(\Lambda)$, with one positive boundary-puncture at the double point of $\Pi_L(\Lambda)$ and possibly many negative boundary-punctures that are mapped to the double-points $\Pi_L(\Lambda)$ where the boundary-punctures appear in this cyclic order with respect to the orientation of the boundary. In other words, this description coincides with the combinatorial description we made for Legendrian knots in $(\mathbb{R}^3, \xi_{st})$

For the detailed comparison of these two descriptions we refer to [10].

## 6. Augmentations and representations

### 6.1. Augmentations

Even though Chekanov-Eliashberg algebra is a powerful invariant, it is quite difficult to deal with because it is not of a finite rank even in a fixed degree. The same difficulty persists in homology: the graded pieces of the Legendrian contact homology are quite often infinite dimensional. However, sometimes one can linearize Chekanov-Eliashberg algebra and Legendrian contact homology. The way to do it (due to Chekanov) is to use an augmentation of the Chekanov-Eliashberg differential graded algebra.

A (graded) augmentation of the Chekanov-Eliashberg differential graded algebra is a unital graded algebra map $\varepsilon : \mathcal{A}(\Lambda) \to \mathbb{Z}_2$ such that

- $\varepsilon \circ \partial_{\mathcal{A}(\Lambda)} = 0$ and
- $\varepsilon(c) = 0$ for $|c| \neq 0$.

Observe that existence of an augmentation is a restrictive condition, i.e. there exist Legendrian submanifolds which do not admit augmentations. This in particular follows from the fact that existence of an augmentation of $(\mathcal{A}(\Lambda), \partial_{\mathcal{A}(\Lambda)})$ implies non-vanishing of $\oplus_i LCH_i(\Lambda) \neq 0$. Stabilized and loose Legendrian submanifolds have vanishing Legendrian contact homology, and hence their Chekanov-Eliashberg algebras do not admit augmentations.

If there is an augmentation $\varepsilon : \mathcal{A}(\Lambda) \to \mathbb{Z}_2$ of the Chekanov-Eliashberg DGA , then the linearization is defined the following way. Consider the graded isomorphism $\varphi^\varepsilon : \mathcal{A}(\Lambda) \to \mathcal{A}(\Lambda)$ defined by $\varphi^\varepsilon_{\mathcal{A}(\Lambda)}(c) = c + \varepsilon(c)$ for all $c \in Q(\Lambda)$. This map defines a new differential

$$\partial^\varepsilon_{\mathcal{A}(\Lambda)}(c) := \varphi^\varepsilon \circ \partial_{\mathcal{A}(\Lambda)} \circ (\varphi^\varepsilon)^{-1}(c).$$

Take the following filtration

$$\mathcal{A}(\Lambda) = \mathcal{A}_0(\Lambda) \supset \mathcal{A}_1(\Lambda) \supset \cdots \supset \mathcal{A}_n(\Lambda) \supset \dots,$$

where $\mathcal{A}_n(\Lambda)$ is the subalgebra of $\mathcal{A}(\Lambda)$ generated as a vector space by all words in $Q(\Lambda)$ of length bigger or equal than $n$, and observe that $\partial^{\varepsilon}_{\mathcal{A}(\Lambda)}$ preserves it. Finally, we define linearized Legendrian contact homology $LCH^{\varepsilon}_i(\Lambda) := H_i(\mathcal{A}_1(\Lambda)/\mathcal{A}_2(\Lambda), \partial^{\varepsilon}_{A(\Lambda)})$, where $\partial^{\varepsilon}_{A(\Lambda)} := (\partial^{\varepsilon}_{\mathcal{A}(\Lambda)})_1$.

Equivalently, one defines $LCH^{\varepsilon}_i(\Lambda) := H_i(A(\Lambda), \partial^{\varepsilon}_{A(\Lambda)})$, where $A(\Lambda)$ is a $\mathbb{Z}_2$-vector space generated by $Q(\Lambda)$, and

$$\partial^{\varepsilon}_{A(\Lambda)}(c^+) = \sum_{\substack{c_1^- \dots c_k^- \\ |c_+| - |c_1^- \dots c_k^-| = 1}} \sum_i \#_{\mathbb{Z}_2} \frac{\mathcal{M}_{S(P\times\mathbb{R})}(c^+, c_1^- \dots c_k^-)}{\mathbb{R}} \varepsilon(c_1^-) \dots c_i^- \dots \varepsilon(c_k^-).$$

**Theorem 6.1** ([9]). *The collection*

$$\{LCH^{\varepsilon}_i(\Lambda) \mid \varepsilon \textit{ is an augmentation of } \mathcal{A}(\Lambda)\}$$

*is a Legendrian invariant.*

*Example* 6.2. Consider the $tb = -1$ Legendrian unknot $\Lambda \subset (\mathbb{R}^3, \xi_{st})$, whose Lagrangian projection is in Figure 21.

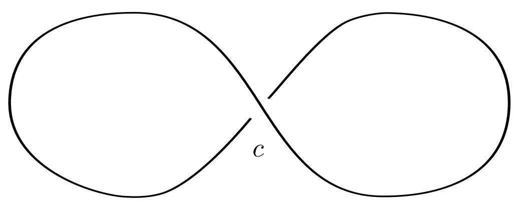



FIGURE 21. Lagrangian projection of $\Lambda$.

$\Pi_L(\Lambda)$ has a unique transverse double point $c$ which corresponds to the unique Reeb chord of $\Lambda$ that we still denote by $c$ with $|c| = 1$. Assume that $\varepsilon : \mathcal{A}(\Lambda) \to \mathbb{Z}_2$ is an augmentation of $\mathcal{A}(\Lambda)$. Since $|c| = 1$, $\varepsilon(c) = 0$. It is not difficult to see that $\varepsilon : \mathcal{A}(\Lambda) \to \mathbb{Z}_2$ defined on $Q(\Lambda)$ by $\varepsilon(c) = 0$ is a graded augmentation of $\mathcal{A}(\Lambda)$.

*Example* 6.3. We discuss the set of augmentations of $(\mathcal{A}(\Lambda_3), \partial_{\mathcal{A}(\Lambda_3)})$ from Example 4.5. Given an augmentation $\varepsilon$ of $(\mathcal{A}(\Lambda_3), \partial_{\mathcal{A}(\Lambda_3)})$, observe that from the fact that $|p_1| = |p_2| = 1$ it follows that $\varepsilon(p_i) = 0$ for $i = 1, 2$. In addition, from the condition $\varepsilon \circ \partial_{\mathcal{A}(\Lambda_3)} = 0$ together with the formula for $\partial_{\mathcal{A}(\Lambda)}(p_1)$ (or equivalently $\partial_{\mathcal{A}(\Lambda)}(p_2)$) we get the following equation

$$1 + \varepsilon(q_1) + \varepsilon(q_3) + \varepsilon(q_1)\varepsilon(q_2)\varepsilon(q_3) = 0,$$

which admits the following solutions:

(i) $\varepsilon(q_1) = 1$, $\varepsilon(q_2) = 1$, $\varepsilon(q_3) = 1$,
(ii) $\varepsilon(q_1) = 0$, $\varepsilon(q_2) = 1$, $\varepsilon(q_3) = 1$,
(iii) $\varepsilon(q_1) = 0$, $\varepsilon(q_2) = 0$, $\varepsilon(q_3) = 1$,
(iv) $\varepsilon(q_1) = 1$, $\varepsilon(q_2) = 1$, $\varepsilon(q_3) = 0$,
(v) $\varepsilon(q_1) = 1$, $\varepsilon(q_2) = 0$, $\varepsilon(q_3) = 0$.

### 6.2. Representations

Note that augmentations to $\mathbb{Z}_2$ (or to an arbitrary field $\mathbb{F}$) can be seen as one-dimensional representations of the Chekanov-Eliashberg DGA. More generally, one may consider high-dimensional representations, i.e. augmentations to $M_k(\mathbb{Z}_2)$ (or to $M_k(\mathbb{F})$). An advantage of considering high-dimensional representations is that this allows us to use the non-commutativity of the Chekanov-Eliashberg algebra.

Similar to the discussion we made about the non-existence of augmentations to a field, there are Legendrian submanifolds which do not admit $k$-dimensional representations for any $k$. In particular, if there exists a $k$-representation of $(\mathcal{A}(\Lambda), \partial_{\mathcal{A}(\Lambda)})$ for some $k$, then the Legendrian contact homology of $\Lambda$ does not vanish. Since stabilized and loose Legendrian submanifolds have vanishing Legendrian contact homology, their Chekanov-Eliashberg algebras do not admit representations. As it turns out, there are examples of Legendrian submanifolds with non-acyclic Chekanov-Eliashberg algebras which do not admit $k$-dimensional representations for any $k$. Sivek in [36] proved that Legendrian representative $\Lambda_{m(10_{132})}$ of $m(10_{132})$ whose front projection appears in Figure 22 is of this type.

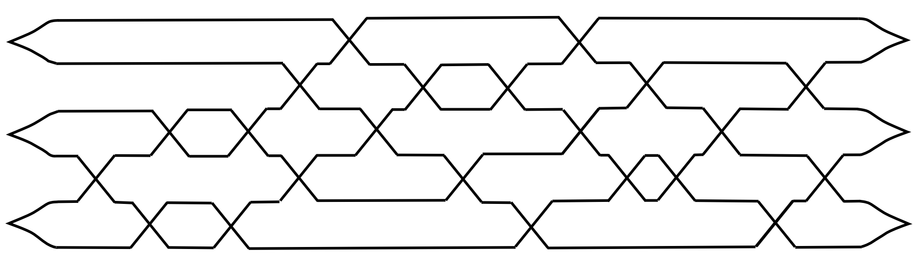


FIGURE 22. Front projection of $\Lambda_{m(10_{132})}$.

It is interesting to study representations of minimal dimension. Sivek in [36] showed that Legendrian knot $\Lambda_{p,-q}$, which is a Legendrian representative of $(p,-q)$-torus knot with $p \geq 3$ odd and $q > p$, see Figure 23, has a property that Chekanov-Eliashberg algebra admits two-dimensional representations over $\mathbb{Z}_2$, but not one-dimensional representations. For more examples we refer to [11]. On the other hand, so far it is not known if there exists a Legendrian whose Chekanov-Eliashberg algebra admits a three-dimensional representation, but does not admit a two-dimensional representation. In some situations,

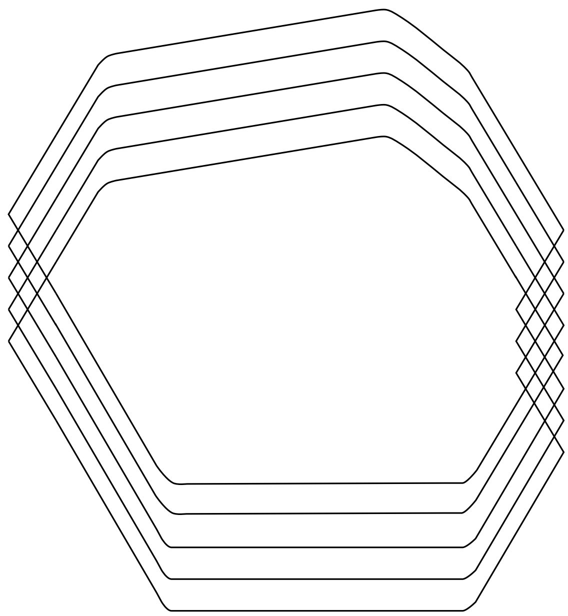


FIGURE 23. Front projection of $\Lambda_{5,-8}$.

the work of Ng and Rutherford [31] allows to say when a representation of dimension $k$ does not exist. More precisely, for Legendrian knots Ng and Rutherford in [31] proved that there is a correspondence between $k$-dimensional representations of the Chekanov-Eliashberg DGA $(\mathcal{A}(\Lambda), \partial_{\mathcal{A}(\Lambda)})$ and 1-dimensional representations (augmentations) of a Legendrian link $tw(\Lambda^k)$, where $tw(\Lambda^k)$ is the $k$-component Legendrian link whose front projection consists of $k$ copies of the front projection of $\Lambda$, pushed off from each other by small perturbations in $\partial_z$ direction, with one segment of the $k$ parallel fronts replaced by a full positive twist.

**Theorem 6.4** ([31]). $(\mathcal{A}(\Lambda), \partial_{\mathcal{A}(\Lambda)})$ *admits a $k$-dimensional representation over $\mathbb{Z}_2$ if and only if $(\mathcal{A}(tw(\Lambda^k)), \partial_{\mathcal{A}(tw(\Lambda^k))})$ admits an augmentation to $\mathbb{Z}_2$.*

Theorem 6.4 in particular says that non-existence of an augmentation of $tw(\Lambda^k)$ implies non-existence of a $k$-dimensional representation of $\Lambda$.

## 7. Exact Lagrangian cobordisms and fillings

We start with the definition of exact Lagrangian cobordism in the symplectization of a contact manifold.

Let $\Lambda_-, \Lambda_+ \subset (P^{2n} \times \mathbb{R}, \alpha := dz + \theta)$ be closed (not necessarily connected) Legendrian submanifolds. We say that that $\Lambda_-$ is *exact Lagrangian cobordant to* $\Lambda_+$ and write $\Lambda_- \prec_L$

$\Lambda_+$ if there is a properly embedded $(n+1)$-dimensional submanifold $L \hookrightarrow S(P\times\mathbb{R})$ satisfying

(1) $L|_{(-\infty,-T_L]\times P\times\mathbb{R}} = (-\infty,-T_L]\times\Lambda_-$, $L|_{[T_L,\infty)\times P\times\mathbb{R}} = [T_L,\infty)\times\Lambda_+$ for some $T_L \gg 0$;

(2) $e^t\alpha|_{TL} = df$ for some $f\in C^\infty(L,\mathbb{R})$, where $f$ is constant on $(-\infty,-T_L]\times\Lambda_-$.

If $L_\Lambda$ is an exact Lagrangian cobordism with empty $-\infty$-end and whose $+\infty$-end equals $\Lambda$, then $L_\Lambda$ is called an *exact Lagrangian filling of* $\Lambda$ and we write $\emptyset \prec_{L_\Lambda} \Lambda$. If $\Lambda_1 \prec_{L_{1,2}} \Lambda_2$ and $\Lambda_2 \prec_{L_{2,3}} \Lambda_3$, then we denote by $L_{1,2}\odot L_{2,3}$ the exact Lagrangian cobordism obtained by gluing the positive end of $L_{1,2}$ to the negative end of $L_{2,3}$ so that $\Lambda_1 \prec_{L_{1,2}\odot L_{2,3}} \Lambda_3$. Sometimes $L_{1,2}\odot L_{2,3}$ is called the concatenation of $L_{1,2}$ and $L_{2,3}$. Note that the condition that $f$ is constant on $(-\infty,-T_L]\times\Lambda_-$ is needed in order to glue (concatenate) exact Lagrangian cobordisms, see the work of Chantraine [6].

*Example* 7.1. For a Legendrian submanifold $\Lambda\subset(P\times\mathbb{R},\ker(dz+\theta))$,

$$\mathbb{R}\times\Lambda\subset(\mathbb{R}\times P\times\mathbb{R}, d(e^t(dz+\theta))) = S(P\times\mathbb{R})$$

is an exact Lagrangian cobordism which is sometimes called the trivial cobordism of $\Lambda$.

*Example* 7.2. There is a generalization of a cobordism from Example 7.1. If $\Lambda_-$ is Legendrian isotopic to $\Lambda_+$, then there is an exact Lagrangian cobordism $L$ diffeomorphic to a cylinder (such a cobordism is usually called concordance) such that $\Lambda_- \prec_L \Lambda_+$. For several constructions of $L$ we refer to [5, 13, 24, 30].

The notion of exact Lagrangian cobordism defines a relation on Legendrian submanifolds. From Example 7.2 it follows that this relation is reflexive and the fact that we can concatenate cobordisms implies that this relation is transitive. Chantraine in [6] proved that even in the case of exact Lagrangian concordances this relation is not symmetric. More precisely, Chantraine proved that there is an exact Lagrangian concordance from $tb=-1$ unknot $\Lambda_0$, see Figure 21, to the Legendrian representative $\Lambda$ of the knot $m(9_{46})$ whose Lagrangian projection appears in Figure 24 and there is no exact Lagrangian cobordism of any topology (in particular, diffeomorphic to a cylinder) from $\Lambda$ to $\Lambda_0$.

As it turns out, Thurston-Bennequin invariant provides a natural obstruction to the existence of exact Lagrangian cobordisms between Legendrian submanifolds of $(\mathbb{R}^{2n+1},\xi_{st})$:

**Theorem 7.3** ([5]). *Given Legendrian submanifolds $\Lambda_-,\Lambda_+\subset\mathbb{R}^{2n+1}$ such that $\Lambda_-\prec_L \Lambda_+$, then*

$$tb(\Lambda_+) - tb(\Lambda_-) = (-1)^{\frac{1}{2}(n^2-3n)}\chi(L,\Lambda_+).$$

In the case whe $n=1$, Theorem 7.3 has been proven by Chantraine in [5]. The proof of Chantraine admits a natural extension to the case when $n\geq 1$.

Define a category of exact Lagrangian cobordisms between Legendrian submanifolds $\mathcal{C}ob$ to be a category whose objects are Legendrian submanifolds $\Lambda\subset P\times\mathbb{R}$ and for a given two Legendrian submanifolds $\Lambda_-,\Lambda_+\subset P\times\mathbb{R}$ the space of morphisms $Hom_{\mathcal{C}ob}(\Lambda_-,\Lambda_+)$ is a space of exact Lagrangian cobordisms $L$ from $\Lambda_-$ to $\Lambda_+$. The composition map in $\mathcal{C}ob$

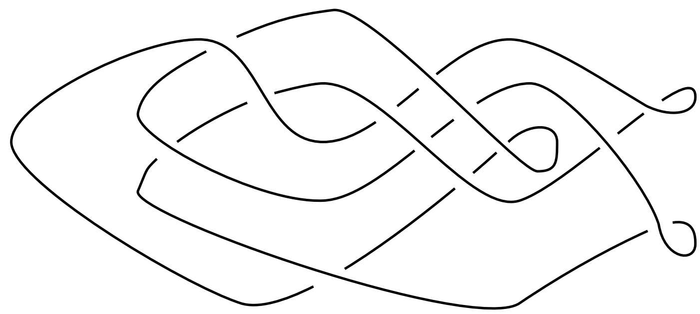


Figure 24. Lagrangian projection of $\Lambda$.

is given by gluing (concatenating) exact Lagrangian cobordisms. Using this terminology, Legendrian contact homology admits a contravariant functor $\Phi$ from the category of exact Lagrangian cobordisms $\mathcal{C}ob$ to the category of unital differential graded algebras $\mathcal{DGA}$.

This functor has been investigated by Ekholm, Honda and Kálmán [20].

**Theorem 7.4** ([20])**.** *Given $\Lambda_- \prec_L \Lambda_+$, there is a unital differential graded algebra map*

$$\Phi_L : (\mathcal{A}(\Lambda_+), \partial_{\mathcal{A}(\Lambda_+)}) \to (\mathcal{A}(\Lambda_-), \partial_{\mathcal{A}(\Lambda_-)})$$

*such that*

- *if $\Lambda_1 \prec_{L_{1,2}} \Lambda_2$ and $\Lambda_2 \prec_{L_{2,3}} \Lambda_3$, then $\Phi_{L_{1,2} \odot L_{2,3}} = \Phi_{L_{1,2}} \circ \Phi_{L_{2,3}}$,*
- *if $\Lambda_- \prec_L \Lambda_+$ and $\Lambda_- \prec_{L'} \Lambda_+$ and $L$ and $L'$ are isotopic through exact Lagrangian cobordisms, then $\Phi_L$ and $\Phi_{L'}$ are chain homotopic,*
- $\Phi_{\mathbb{R}\times\Lambda} = id_{\mathcal{A}(\Lambda)}$.

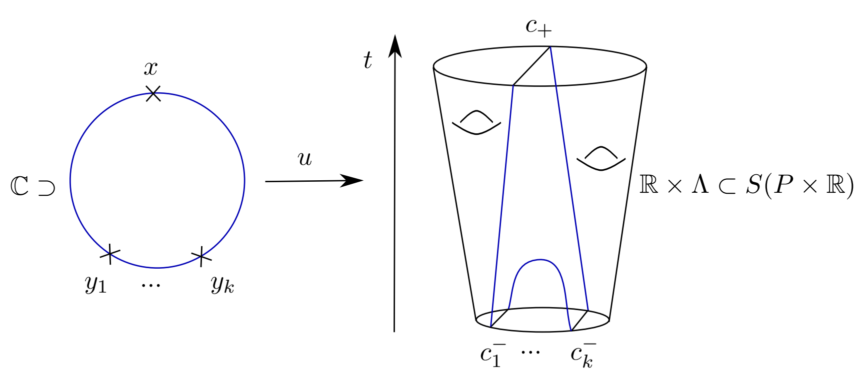



Figure 25. An element $u \in \mathcal{M}^L_{S(P\times\mathbb{R})}(c^+, c_1^- \dots c_k^-)$.

$\Phi_L$ is defined the following way. First, take the moduli space $\mathcal{M}^L_{S(P\times\mathbb{R})}(c^+, c_1^- \dots c_k^-)$ for $c^+ \in Q(\Lambda_+)$ and $c_1^-, \dots, c_k^- \in Q(\Lambda_-)$. Take a closed unit disc $D^2 \subset (\mathbb{C}, i)$ and $\{x, y_1, \dots, y_k\} \subset \partial D^2$ the set of points in this cyclic order with respect to the orientation of the boundary. The moduli space of $J$-holomorphic punctured discs $\mathcal{M}^L_{S(P\times\mathbb{R})}(c^+, c_1^- \dots c_k^-)$ consists of

$$u := (a, f) : (D^2 \setminus \{x, y_1, \dots, y_k\}, \partial D^2 \setminus \{x, y_1, \dots, y_k\}) \to (\mathbb{R} \times P \times \mathbb{R}, L)$$

such that

- $du + J \circ du \circ i = 0$,
- $\lim\limits_{p\to x} a(p) = \infty$, $\lim\limits_{p\to y_i} a(p) = -\infty$,
- $\lim\limits_{p\to x} f(p) = c_+$, $\lim\limits_{p\to y_i} f(p) = c_i^-$ in the appropriate sense, $i = 1, \dots, k$.

In addition, two solutions are identified if they differ by a biholomorphism of the domain. See Figure 25.

$$\Phi_L(c_+) = \sum_{\substack{c_1^- \dots c_k^- \\ \dim \mathcal{M}^L_{S(P\times\mathbb{R})}(c^+, c_1^- \dots c_k^-) = 0}} \#_{\mathbb{Z}_2} \mathcal{M}^L_{S(P\times\mathbb{R})}(c^+, c_1^- \dots c_k^-) c_1^- \dots c_k^-,$$

where

$$\dim \mathcal{M}^L_{S(P\times\mathbb{R})}(c^+, c_1^- \dots c_k^-) = |c^+| - |c_1^- \dots c_k^-|$$

for a generic $J$. Observe that if $\varepsilon_-$ is an augmentation of $(\mathcal{A}(\Lambda_-), \partial_{\mathcal{A}(\Lambda_-)})$ and $\Lambda_- \prec_L \Lambda_+$, then there is a pull-back augmentation of $(\mathcal{A}(\Lambda_+), \partial_{\mathcal{A}(\Lambda_+)})$ induced by $\varepsilon_-$ and $L$, i.e.

$$\varepsilon_+ := \Phi_L^*(\varepsilon_-) := \varepsilon_- \circ \Phi_L.$$

When $L = L_\Lambda$ is an exact Lagrangian filling of a Legendrian $\Lambda$,

$$\Phi_L = \Phi_{L_\Lambda} : (\mathcal{A}(\Lambda_+), \partial_{\mathcal{A}(\Lambda_+)}) \to (\mathbb{Z}_2, 0)$$

defines the augmentation induced by $L_\Lambda$ by count of punctured holomorphic disks in $S(P\times\mathbb{R})$ with the boundary on $L_\Lambda$ and positive ends at one Reeb chord, i.e. for $c \in Q(\Lambda)$

$$\Phi_{L_\Lambda}(c) = \#_{\mathbb{Z}_2} \mathcal{M}^{L_\Lambda}_{S(P\times\mathbb{R})}(c),$$

see Figure 26.

We call a transverse double point $c$ of $\Pi_L(\Lambda)$ of a Legendrian link $\Lambda$ in $\mathbb{R}^3$ contractible if the length of the corresponding Reeb chord that we still denote by $c$ can be made arbitrary small by a Legendrian isotopy of $\Lambda$ that corresponds to a planar isotopy of $\Pi_L(\Lambda)$.

**Theorem 7.5** ([20]). *For Legendrian links $\Lambda -$, $\Lambda_+$ in $\mathbb{R}^3$ there is an exact Lagrangian cobordism from $\Lambda_-$ to $\Lambda_+$ for the following:*

(i) *Legendrian isotopy;*
(ii) *removing a $tb = -1$ unknot component of $\Lambda_+$ that is contractible in the complement of the remainder of $\Lambda_+$;*

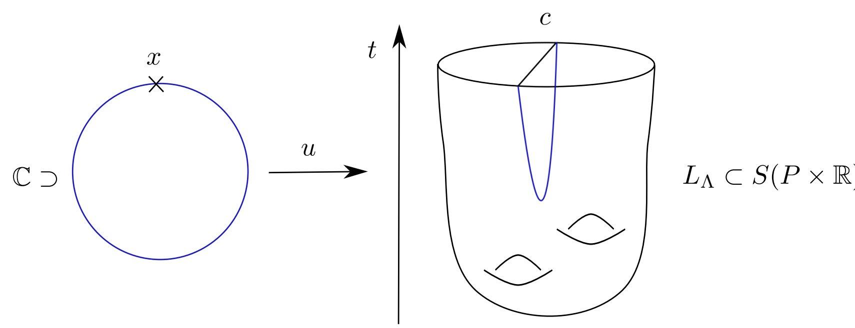



FIGURE 26. An element of $u \in \mathcal{M}^{L_\Lambda}_{S(P\times\mathbb{R})}(c)$.

(iii) *a saddle (or pinch) move in the front projection described in Figure 27.*

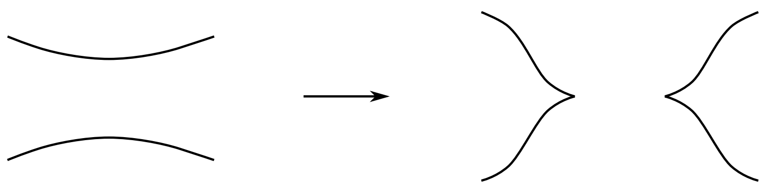


FIGURE 27. Saddle (or pinch) move: left side corresponds to the positive end of the cobordism and right side corresponds to the negative end of the cobordism.

These days it is an open question whether every exact Lagrangian cobordim between Legendrian links in $(\mathbb{R}^3, \xi_{st})$ can be described as a concatenation of the cobordisms from Theorem 7.5.

The sketch of proof of the following isomorphism first appeared in the work of Ekholm [14] (inspired by the ideas of Seidel) and then it has been completely proven with $\mathbb{Z}_2$-coefficients by Dimitroglou Rizell in [10]

**Theorem 7.6** (Seidel, [14, 10])**.** *Let $\Lambda$ be a Legendrian submanifold of $P \times \mathbb{R}$ which admits an exact Lagrangian filling $L_\Lambda$. Then*

$$H_{n-i}(L_\Lambda) \simeq LCH^i_\varepsilon(\Lambda),$$

*where $\varepsilon$ is the augmentation induced by $L_\Lambda$.*

As a consequence of Seidel's isopmorphism, there are the following long exact sequences that can be seen as the natural obstructions to the existence of exact Lagrangian cobordisms:

**Theorem 7.7** ([30]). *Let $\Lambda_-, \Lambda_+ \subset (P\times\mathbb{R}, \ker(dz+\theta))$ be Legendrians such that $\emptyset \prec_{L_{\Lambda_-}} \Lambda_-$ and $\Lambda_- \prec_L \Lambda_+$ . Then there are the following long exact sequences*

$$\cdots \to H_i(\Lambda_-) \to H_i(L) \oplus LCH^{n-i}_{\varepsilon_-}(\Lambda_-) \to LCH^{n-i}_{\varepsilon_+}(\Lambda_+) \to H_{i-1}(\Lambda_-) \to \dots \quad (10)$$

$$\cdots \to LCH^{n-i}_{\varepsilon_-}(\Lambda_-) \to LCH^{n-i}_{\varepsilon_+}(\Lambda_+) \to H_i(L,\Lambda_-) \to LCH^{n-i+1}_{\varepsilon_-}(\Lambda_-) \to \dots \quad (11)$$

*Here $\varepsilon_-$ is the augmentation induced by $L_{\Lambda_-}$ and $\varepsilon_+ = \Phi^*_L(\varepsilon_-)$.*

*Proof.* Long exact sequence (10) is typically called the Mayer-Vietoris long exact sequence and it is obtained from the standard topological Mayer-Vietoris long exact sequence for the spaces $L_{\Lambda_-}, L \subset L_{\Lambda_-} \odot L$ and $L\cap L_{\Lambda_-} \simeq \mathbb{R}\times\Lambda_-$, where $H_i(L_{\Lambda_-}) \simeq LCH^{n-i}_{\varepsilon_-}(\Lambda_-)$ and $H_i(L_{\Lambda_-}\odot L) \simeq LCH^{n-i}_{\varepsilon_+}(\Lambda_+)$ by Seidel's isomorphism from Theorem 7.6. Long exact sequence (11) is typically called the long exact sequence of a pair and it is obtained from the standard topological long exact sequence of the pair $(L_{\Lambda_-}\odot L, L_{\Lambda_-})$, where $H_{i-1}(L_{\Lambda_-}) \simeq LCH^{n-i+1}_{\varepsilon_-}(\Lambda_-)$ and $H_i(L_{\Lambda_-}\odot L) \simeq LCH^{n-i}_{\varepsilon_+}(\Lambda_+)$ by Seidel's isomorphism from Theorem 7.6 and $H_i(L_{\Lambda_-}\odot L, L_{\Lambda_-}) \simeq H_i(L,\Lambda_-)$ by the excision theorem. □

Then together with Chantraine, Dimitroglou Rizell and Ghiggini we developed Floer homology for exact Lagrangian cobordisms in [8] which allows to generalize Theorem 7.7 to the case when $\varepsilon_-$ is not necessarily induced by an embedded exact Lagrangian filling. More precisely, the following has been proven:

**Theorem 7.8** ([8]). *Let $\Lambda_-, \Lambda_+ \subset P\times\mathbb{R}$ be Legendrians such that $\Lambda_- \prec_L \Lambda_+$ and $\mathcal{A}(\Lambda_-)$ admits an augmentation $\varepsilon_-$. Then there are the following long exact sequences*

$$\cdots \to H_i(\Lambda_-) \to H_i(L) \oplus LCH^{n-i}_{\varepsilon_-}(\Lambda_-) \to LCH^{n-i}_{\varepsilon_+}(\Lambda_+) \to H_{i-1}(\Lambda_-) \to \dots \quad (12)$$

$$\cdots \to LCH^{n-i}_{\varepsilon_-}(\Lambda_-) \to LCH^{n-i}_{\varepsilon_+}(\Lambda_+) \to H_i(L,\Lambda_-) \to LCH^{n-i+1}_{\varepsilon_-}(\Lambda_-) \to \dots \quad (13)$$

$$\cdots \to LCH^{i}_{\varepsilon_+}(\Lambda_+) \to LCH_{n-i-1}^{\varepsilon_-}(\Lambda_-) \to H_{n-i-1}(L) \to LCH^{i+1}_{\varepsilon_+}(\Lambda_+) \to \dots . \quad (14)$$

*Here $\varepsilon_+ = \Phi^*_L(\varepsilon_-)$.*

Long exact sequence (14) is usually called the duality long exact sequence. A version of it which corresponds to the case when $\Lambda = \Lambda_- = \Lambda_+$ and $L = \mathbb{R}\times\Lambda$ appeared in the work of Ekholm, Etnyre and Sabloff [15].

Recall that Whitney sphere is a $k$-dimensional Legendrian sphere $W^k \subset (\mathbb{R}^{2k+1}, \xi_{st})$ with the unique Reeb chord, which is the Legendrian lift of the Whitney immersion in $\mathbb{R}^{2n}$, see Figure 28.

Using long exact sequences from Theorem 7.8 and their refinements to coefficients twisted by the fundamental group, in [8] together with Chantraine, Dimitroglou Rizell and Ghiggini we found topological restrictions on exact Lagrangian endocobordisms (cobordisms whose positive and negative ends are the same) of Whitney spheres. We proved the following:

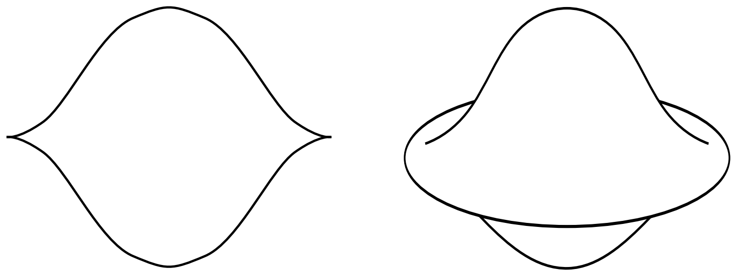


FIGURE 28. Left: $\Pi_F(W^1)$, right: $\Pi_F(W^2)$.

**Theorem 7.9** ([8]). *Any exact Lagrangian cobordism $L$ from the $n$-dimensional Whitney sphere to itself inside $S(\mathbb{R}^{2n+1})$ is an $h$-cobordism. In particular, given that $n \geq 4$, it follows that $L$ is diffeomorphic to a cylinder.*

*Remark* 7.10.
(i) Besides embedded exact Lagrangian cobordisms one can also study immersed exact Lagrangian cobordisms and their functorial properties, see [31].
(ii) One can study some quantitative properties of exact Lagrangian cobordisms such as minimal length, see [35].
(iii) There are obstructions to the existence of exact Lagrangian cobordisms coming from other invariants such as generating family homology [34] and knot Floer homology [1].

## 8. Arnold conjecture for Legendrian submanifolds

Legendrian contact homology has been used by Ekholm, Etnyre and Sullivan [18], and Ekholm, Etnyre and Sabloff [15] to prove a special case of the Arnold Conjecture:

**Theorem 8.1** ([18, 15]). *Given a Legendrian submanifold $\Lambda \subset (P \times \mathbb{R}, \ker(dz + \theta))$ such that $\Pi_L(\Lambda)$ can be displaced by the time-one map of a Hamiltonian vector field $X_H$ on $(P, d\theta)$, the first Chern class of $TP$ vanishes, that the Maslov number of $\Lambda$ equals zero and such that $(\mathcal{A}(\Lambda), \partial_{\mathcal{A}(\Lambda)})$ admits an augmentation. Then*

$$\dim H_i(\Lambda, \mathbb{Z}_2) \leq c_i + c_{n-i}$$

*for all $n$ and $i$ such that $n \geq i \geq 0$. Here $c_i$ is the number of Reeb chords of $\Lambda$ of degree $i$.*

*Proof.* The proof of it follows from the duality long exact sequence (14) if one applies rank inequality to the consecutive three terms of it with $\Lambda_- = \Lambda_+ = \Lambda$ and $L = \mathbb{R} \times \Lambda$, i.e.

$$\begin{aligned}\dim H_{n-i-1}(L, \mathbb{Z}_2) &\leq \dim LCH^{\varepsilon_-}_{n-i-1}(\Lambda_-) + \dim LCH^{i+1}_{\varepsilon_+}(\Lambda_+)\\ &= \dim LCH^{\varepsilon_-}_{n-i-1}(\Lambda_-) + \dim LCH^{\varepsilon_+}_{i+1}(\Lambda_+) \leq c_{n-i-1} + c_{i+1}\end{aligned}$$

for all $i$. $\square$

One can naturally extend Theorem 8.1 to the case when $(\mathcal{A}(\Lambda), \partial_{\mathcal{A}(\Lambda)})$ admits a $k$-representation for some $k$.

**Theorem 8.2** ([11]). *Given a Legendrian submanifold $\Lambda \subset (P \times \mathbb{R}, \ker(dz + \theta))$ satisfying the assumptions of the previous theorem and such that $(\mathcal{A}(\Lambda), \partial_{\mathcal{A}(\Lambda)})$ admits a $k$-representation to $M_k(\mathbb{Z}_2)$. Then the following inequality holds:*

$$\dim H_i(\Lambda, \mathbb{Z}_2) \leq c_i + c_{n-i}$$

*for all $n$ and $i$ such that $n \geq i \geq 0$. Here $c_i$ is the number of Reeb chords of $\Lambda$ of degree $i$.*

If we assume that $\Lambda$ admits an exact Lagrangian filling $L_\Lambda$, and hence Theorems 8.1 and 8.2 hold, one can get a stronger estimate which is sometimes called the stable version of the strong Arnold conjecture.

**Theorem 8.3** ([12]). *Assume that $\Lambda \subset (P \times R, dz+\theta)$ is a chord generic closed Legendrian submanifold admitting an exact Lagrangian filling $L_\Lambda$ which is spin and has vanishing Maslov number. Then*

$$|Q(\Lambda)| \geq \text{stableMorse}(L_\Lambda),$$

*where*

$$\text{stableMorse}(\mathrm{L}_\Lambda) := \min_{k \geq 0, f} \{\#Crit(f) \mid f : L_\Lambda \times \mathbb{R}^k \to \mathbb{R} \quad \textit{Morse, almost quadratic at infinity}\}.$$

The condition that $f$ is almost quadratic at infinity is needed in order to avoid critical points at infinity. The precise meaning of this condition is the following. Given a compact manifold $M$ (possibly with non-empty boundary), we say that a function $f : M \times \mathbb{R}^k \to \mathbb{R}$ is *almost quadratic at infinity* if there exists is a non-degenerate quadratic form $q$ on $\mathbb{R}^k$ together with a Riemannian metric $g$ on $M$ such that

- $\|df - dq\|_{C^0}$ is bounded, where the $C^0$-norm on $T^*(M \times \mathbb{R}^k)$ is induced by the metric $g \oplus g_{Eucl}$ on $M \times \mathbb{R}^k$, where $g_{Eucl}$ is the Euclidean metric on $\mathbb{R}^k$; and
- If $t \in (1/2, 1]$ is a coordinate on a collar neighborhood $(1/2, 1] \times \partial M \times \mathbb{R}^k$ of $\partial M$, then $df(\partial_t) > 0$ on $(1/2, 1] \times \partial M \times \mathbb{R}^k$.

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