Proceedings of 25–26th Gökova
Geometry-Topology Conference
pp. 67 – 89

Wall-crossing for toric mutations

David Nadler

Abstract. We explain how to deduce the wall-crossing formula for toric mutations from the perverse schober of the corresponding local Landau-Ginzburg model. Along the way, we develop a general framework to extract a wall-crossing formula from a perverse schober on the projective line with a single critical point.

1. Introduction

Our aim in this note is to deduce the wall-crossing formula for toric mutations (see Auroux [3], Seidel [9, Sect. 11], for the four-dimensional case, and Pascaleff-Tonkonog [8] for a uniform derivation) from the perverse schober of the corresponding local Landau-Ginzburg model (in the microlocal sheaf realization calculated in [6], or more concretely in the six-dimensional case in [5]). Along the way, we develop a general framework (see Theorem 1.2) to extract a wall-crossing formula from a perverse schober $\mathcal{P}$ on the projective line $\mathbb{CP}^1$ with a single critical point $0 \in \mathbb{CP}^1$. The wall-crossing formula applies to the moduli of objects of the inertia category of the clean objects (see Definition 3.1) in the generic fiber of $\mathcal{P}$.

1.1. Wall-crossing formula

The wall-crossing formula for toric mutations (see Pascaleff-Tonkonog [8, Equation (1.5)]) is the birational map

$$\begin{aligned} y_i &\mapsto y_i, \quad i = 1, \dots, n-1 \\ y_n &\mapsto y_n^{-1}(1 + y_1 + \cdots + y_{n-1}) \end{aligned} \tag{1}$$

associated to the mutation $L \rightsquigarrow \mu_D(L)$ of an n-dimensional Lagrangian torus $L \simeq (S^1)^n$ around a singular thimble $D = \mathrm{Cone}(T^{van})$ attached to a vanishing $(n-1)$-dimensional torus $T^{van} \simeq (S^1)^{n-1} \subset L$.

The variables y_i, $i = 1, \dots, n$, are simultaneously coordinates on the moduli $\mathbb{T}^\vee \simeq (\Gamma_m)^n$ of rank one local systems on the Lagrangian torus $L \simeq (S^1)^n$ as well as its mutation $\mu_D(L) \simeq (S^1)^n$ under a natural identification. In particular, the variables y_i, $i = 1, \dots, n-1$, are simultaneously coordinates on the moduli $\mathbb{T}_0^\vee \simeq (\Gamma_m)^{n-1}$ of rank one local systems on the vanishing torus $T^{van} \simeq (S^1)^{n-1}$ as well as its mutation $\mu_D(T^{van}) \simeq (S^1)^{n-1}$.

Dedicated with admiration to Sir Michael Atiyah.

When $n = 2$, the vanishing torus T^{van} is a circle, the thimble $D = \mathrm{Cone}(T^{van})$ is a smooth disk, and the geometry is that of a traditional "mutation configuration" [8, Def. 4.10]. In general, the geometry is given by the following local model (adapted from [8, Sect. 5.4] to suit our further considerations).

1.1.1. Local geometry

Let z be a coordinate on $\mathbb{C}$, $z_1, \ldots, z_n$ coordinates on $\mathbb{C}^n$, and fix the function $W : \mathbb{C}^n \to \mathbb{C}$, $W = z_1 \cdots z_n$.

Fix once and for all $\epsilon > 0$, and introduce the symplectic manifold $M = \{W \neq \epsilon\} \subset \mathbb{C}^n$.

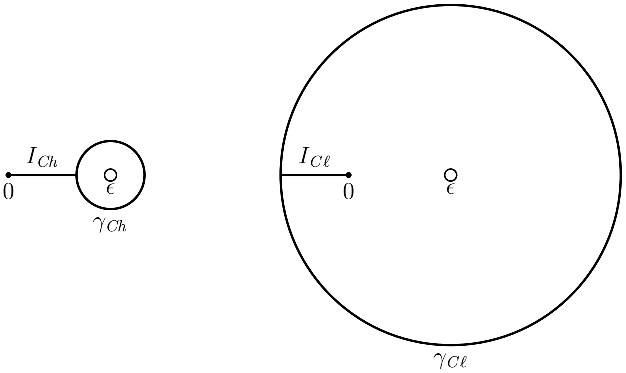

FIGURE 1. Chekanov graph $\Gamma_{\mathcal{C}h}$ and Clifford graph $\Gamma_{C\ell}$.

For a guide to the following constructions, see Figure 1. For any $\rho > 0$, $\rho \neq \epsilon$, consider the circle and corresponding Lagrangian torus

$$\gamma_\rho = \{|z - \epsilon| = \rho\} \simeq S^1 \subset \mathbb{C} \qquad T_\rho = \{W \in \gamma_\rho, |z_1| = \cdots = |z_n|\} \subset M \tag{2}$$

In particular, for fixed $\rho_{\mathcal{C}h} \in (0, \epsilon)$ and $\rho_{C\ell} \in (\epsilon, \infty)$, we have the Chekanov and Clifford circles and corresponding n-dimensional Chekanov and Clifford tori

$$\gamma_{\mathcal{C}h} := \gamma_{\rho_{\mathcal{C}h}} \qquad T_{\mathcal{C}h} := T_{\rho_{\mathcal{C}h}} \qquad \gamma_{C\ell} := \gamma_{\rho_{C\ell}} \qquad T_{C\ell} := T_{\rho_{C\ell}} \tag{3}$$

Consider as well the closed intervals

$$I_{\mathcal{C}h} = [0, \epsilon - \rho_{\mathcal{C}h}] \qquad I_{C\ell} = [\epsilon - \rho_{C\ell}, 0] \tag{4}$$

and the corresponding singular thimbles

$$D_{\mathcal{C}h} = \{W \in I_{\mathcal{C}h}, |z_1| = \cdots = |z_n|\} \simeq \mathrm{Cone}(T^{van}_{\mathcal{C}h}) \tag{5}$$

$$D_{C\ell} = \{W \in I_{C\ell}, |z_1| = \cdots = |z_n|\} \simeq \mathrm{Cone}(T^{van}_{C\ell}) \tag{6}$$

with respective boundaries the vanishing tori

$$\partial D_{\mathcal{C}h} = T^{van}_{\mathcal{C}h} = \{W = \epsilon - \rho_{\mathcal{C}h}, |z_1| = \cdots = |z_n|\} \tag{7}$$

$$\partial D_{C\ell} = T^{van}_{C\ell} = \{W = \epsilon - \rho_{C\ell}, |z_1| = \cdots = |z_n|\} \tag{8}$$

It will also be useful to introduce the Chekanov and Clifford graphs

$$\Gamma_{\mathcal{C}h} = \gamma_{\mathcal{C}h} \cup I_{\mathcal{C}h} \qquad \Gamma_{C\ell} = \gamma \cup I_{C\ell} \tag{9}$$

which are skeleta of $\mathbb{C} \setminus \{\epsilon\} \simeq \mathbb{C}^{\times}$, and the corresponding Chekanov and Clifford skeleta

$$L_{\mathcal{C}h} = \{W \in \Gamma_{\mathcal{C}h}, |z_1| = \cdots = |z_n|\} \qquad L_{C\ell} = \{W \in \Gamma_{C\ell}, |z_1| = \cdots = |z_n|\} \tag{10}$$

of the symplectic manifold $M = \{W \neq \epsilon\}$ itself.

One says the tori $T_{\mathcal{C}h}, T_{C\ell}$ mutate into each other $T_{C\ell} = \mu_{D_{\mathcal{C}h}}(T_{\mathcal{C}h})$, $T_{\mathcal{C}h} = \mu_{D_{C\ell}}(T_{C\ell})$ around the respective thimbles $D_{\mathcal{C}h}, D_{C\ell}$ as the radius ρ passes through ϵ and the circle γ_ρ passes through the critical value $0 \in \mathbb{C}$.

Remark 1.1. We could also work directly with the graph $\Gamma_{crit} = \gamma_\epsilon$ given by the circle of critical radius ϵ, and the corresponding critical skeleton

$$L_{crit} = \{W \in \Gamma_{crit}, |z_1| = \cdots = |z_n|\} \tag{11}$$

It is the union of the two thimbles

$$D^{\pm}_{crit} = \{W \in I^{\pm}_{crit}, |z_1| = \cdots = |z_n|\} \simeq \text{Cone}(T^{van}_{crit}) \tag{12}$$

over the two semi-circles $I^{\pm}_{crit} = \{z \in \Gamma_{crit} | \pm \text{Im}(z) \geq 0\}$, with each thimble the cone over the the same vanishing torus

$$\partial D^{\pm}_{crit} = T^{van}_{crit} = \{W = 2\epsilon, |z_1| = \cdots = |z_n|\} \tag{13}$$

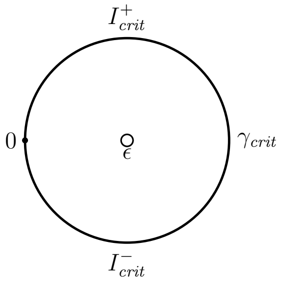

Figure 2. Critical graph Γ_{crit}.

1.2. Landau-Ginzburg model

As proposed by Kapranov-Schechtman [4], one expects the branes in any Landau-Ginzburg A-model to be captured by a perverse schober with singularities at the critical values of the superpotential. In the case of the Landau-Ginzburg A-model $\mathbb{C}^n$ with superpotential $W = z_1 \cdots z_n$, and with branes taken to be microlocal sheaves along natural skeleta, the predicted perverse schober was constructed and calculated in [6].

Following [4], a perverse schober on a disk D with a single critical point at $0 \in D$, in its minimal one-cut realization, is a spherical functor

$$\mathcal{P}_{D,0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi) \tag{14}$$

from a stable "vanishing" dg category $\mathcal{D}_\Phi$ to a stable "nearby" dg category $\mathcal{D}_\Psi$. We recall in Sect. 2 below more details and variants. (Throughout the paper, for concreteness, we will understand stable dg category to mean small pre-trianguated k-linear dg category for a field k, and functor to mean exact k-linear functor. All of the theory discussed also works with small stable ∞-categories, and also stable presentable dg or ∞-categories where more adjunctions might be available.)

For the Landau-Ginzburg A-model of $\mathbb{C}^n$ with superpotential $W = z_1 \cdots z_n$, consider the thimble and and cylinder respectively

$$L = \{W \in \mathbb{R}_{\geq 0}, |z_1| = \cdots = |z_n|\} \subset \mathbb{C}^n \tag{15}$$

$$L^\circ = \{W \in \mathbb{R}_{> 0}, |z_1| = \cdots = |z_n|\} \subset \mathbb{C}^n \tag{16}$$

A main technical result of [6] is that we obtain a perverse schober

$$\mathcal{P}_A = (j^* : \mu Sh_L(\mathbb{C}^n) \longrightarrow \mu Sh_{L^\circ}(\mathbb{C}^n)) \tag{17}$$

by taking dg categories of microlocal sheaves along the thimble and cylinder, with their natural restriction under the open inclusion $j : L^\circ \hookrightarrow L$. As discussed in [6], the arguments work just as well for wrapped microlocal sheaves [7] also giving a perverse schober

$$\mathcal{P}_A^w = (j^* : \mu Sh_L^w(\mathbb{C}^n) \longrightarrow \mu Sh_{L^\circ}^w(\mathbb{C}^n)) \tag{18}$$

Next, let us describe the mirror perverse schobers. Introduce the torus $\mathbb{T}^\vee = (\Gamma_m)^n$, with coordinates $x_1, \ldots, x_n$, and its quotient $\mathbb{T}_0^\vee = \mathbb{T}^\vee/\Gamma_m \simeq (\Gamma_m)^{n-1}$ by the diagonal, with coordinates $y_1 = x_1/x_n, \ldots, y_{n-1} = x_1/x_n$. Consider the inclusion of the pair of pants

$$i : P_{n-2} = \{1 + y_1 + \cdots + y_{n-1} = 0\} \hookrightarrow \mathbb{T}_0^\vee \tag{19}$$

Then it is elementary to check we obtain perverse schobers

$$\mathcal{P}_B = (i_* : \mathrm{Perf}_{prop}(P_{n-2}) \longrightarrow \mathrm{Perf}_{prop}(\mathbb{T}_0^\vee)) \tag{20}$$

$$\mathcal{P}_B^w = (i_* : \mathrm{Coh}(P_{n-2}) \longrightarrow \mathrm{Coh}(\mathbb{T}_0^\vee)) \tag{21}$$

by taking dg categories of perfect complexes with proper support, or alternatively coherent complexes, and the usual pushforward.

A main result of [6] is the following pair of mirror equivalences.

Theorem 1.1 ([6])**.** *There are equivalences of perverse schobers $\mathcal{P}_A \overset{\sim}{\to} \mathcal{P}_B$, $\mathcal{P}_A^w \overset{\sim}{\to} \mathcal{P}_B^w$ i.e. commutative diagrams with vertical equivalences*

$$\begin{tikzcd}
\mu Sh_L(\mathbb{C}^n) \arrow[r, "j^*"] \arrow[d, "\sim"] & \mu Sh_{L^\circ}(\mathbb{C}^n) \arrow[d, "\sim"] & \mu Sh_L^w(\mathbb{C}^n) \arrow[r, "j^*"] \arrow[d, "\sim"] & \mu Sh_{L^\circ}^w(\mathbb{C}^n) \arrow[d, "\sim"] \\
\mathrm{Perf}_{prop}(P_{n-2}) \arrow[r, "i_*"] & \mathrm{Perf}_{prop}(\mathbb{T}_0^\vee) & \mathrm{Coh}(P_{n-2}) \arrow[r, "i_*"] & \mathrm{Coh}(\mathbb{T}_0^\vee)
\end{tikzcd} \tag{22}$$

Remark 1.2. To pin down the equivalences of the theorem, consider the vanishing torus

$$T^{van} = \{W = 1, |z_1| = \cdots = |z_n|\} \subset \mathbb{C}^n \tag{23}$$

and note the natural diffeomorphism $L^\circ \simeq T_{van} \times \mathbb{R}_{>0}$. Note we can canonically identify $\mathbb{T}_0^\vee$ with the moduli of rank one local systems on T^{van}, and thus obtain canonical equivalences $\mu Sh_{L^\circ}(\mathbb{C}^n) \simeq \mathrm{Perf}_{prop}(\mathbb{T}_0^\vee)$, $\mu Sh_{L^\circ}^w(\mathbb{C}^n) \simeq \mathrm{Coh}(\mathbb{T}_0^\vee)$. This fixes the right vertical arrows in (22); it fixes the left vertical arrows, since the functors j^*, i_* are conservative.

1.3. Main result

We state here a general wall-crossing formula (Theorem 1.2 directly below) associated to any perverse schober $\mathcal{P}_{S^2,0}$ on the sphere $S^2 \simeq \mathbb{CP}^1$ with a single critical point $0 \in S^2$. We will regard such a perverse schober as a pair

$$\mathcal{P}_{S^2,0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi, \tau : \mathrm{id} \overset{\sim}{\to} T_{\Psi,r}) \tag{24}$$

of a spherical functor S and a trivialization τ of the monodromy functor $T_{\Psi,r} : \mathcal{D}_\Psi \overset{\sim}{\to} \mathcal{D}_\Psi$ of the nearby category (see Definition 2.1 below). Thus we have a perverse schober $(S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi)$ on a disk around $0 \in S^2$ together with the data τ needed to extend it over $\infty \in S^2$.

Now we will collect the ingredients needed to state Theorem 1.2. We refer the reader to Remark 1.4 and Example 1.1 directly after the theorem for the geometry motivating the various constructions.

Introduce the full subcategory $\mathcal{D}_\spadesuit \subset \mathcal{D}_\Psi$ of *clean objects* of the nearby category: we say an object $Y \in \mathcal{D}_\Psi$ is clean if we have $S^r(Y) \simeq 0$ (equivalently, $S^\ell(Y) \simeq 0$) for the right adjoint S^r (or left adjoint S^ℓ) of S.

Remark 1.3. The terminology "clean" is motivated by the following. If instead of a spherical functor $S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi$, suppose we work with a spherical pair, in the sense of a semi-orthogonal decomposition

$$\begin{tikzcd}
\mathcal{D}_\Phi^- \arrow[r, "I_!"] & \mathcal{D} \arrow[l, bend right, "I^!"'] \arrow[l, bend left, "I^*"] \arrow[r, "J^*"] & \mathcal{D}_\Psi^+ \arrow[l, bend right, "J_*"'] \arrow[l, bend left, "J_!"]
\end{tikzcd} \tag{25}$$

Then an object $Y \in \mathcal{D}_\Psi^+$ is clean if and only if the canonical map $J_! Y \to J_* Y$ is an isomorphism.

Let $\mathcal{LD}_\spadesuit$ denote the *inertia dg category* of pairs (Y, m) consisting of $Y \in \mathcal{D}_\spadesuit$, and an automorphism $m : Y \xrightarrow{\sim} Y$. For any automorphism $\mathfrak{m}$ of the identity functor of $\mathcal{D}_\spadesuit$, we define the translated inverse functor by the formula

$$K_{\mathfrak{m}} : \mathcal{LD}_\spadesuit \xrightarrow{\sim} \mathcal{LD}_\spadesuit \qquad K_{\mathfrak{m}}(Y, m) = (Y, m^{-1} \circ \mathfrak{m}) \tag{26}$$

Finally, consider the restriction

$$\mathcal{P}_{\mathrm{Cyl},0} = \mathcal{P}_{S^2,0}|_{\mathrm{Cyl}} \tag{27}$$

of the perverse schober $\mathcal{P}_{S^2,0}$ to the cylinder $\mathrm{Cyl} = \mathbb{C} \setminus \{\epsilon\} \simeq \mathbb{CP}^1 \setminus \{\epsilon, \infty\}$, and denote by $\Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0})$ its global sections (see Definition 2.2 below).

Here is an abstract statement of our main result.

Theorem 1.2. *There are natural fully faithful Chekanov and Clifford embeddings fitting into a commutative diagram*

$$\begin{tikzcd}
\mathcal{LD}_\spadesuit \arrow[dr, hook, "F_{Ch}"] \arrow[dd, "K_{\mathfrak{m}}"', "\sim"] & \\
& \Gamma(Cyl, \mathcal{P}_{Cyl,0}) \\
\mathcal{LD}_\spadesuit \arrow[ur, hook, "F_{C\ell}"] &
\end{tikzcd} \tag{28}$$

where $\mathfrak{m}$ *is the automorphism of the identity*

$$\mathfrak{m} : \mathrm{id} \xrightarrow{\mathfrak{p}_\Psi} T_{\Psi,r} \xrightarrow{\tau^{-1}} \mathrm{id} \tag{29}$$

given by the composition of the canonical map $\mathfrak{p}_\Psi$ *associated to the spherical functor* S*, and the inverse of the trivialization* τ *of the nearby monodromy functor.*

Remark 1.4. To give geometric context to the theorem, let us return to the two circles γ_{Ch}, $\gamma_{C\ell}$ drawn in Figure 1. Fix the base point in each circle given by the intersection with the ray (ϵ, ∞), and equip each circle with the counterclockwise orientation of the plane.

We can view $\mathcal{LD}_\Psi$ as local systems of objects of $\mathcal{D}_\Psi$ around either based, oriented circle γ_{Ch}, $\gamma_{C\ell}$. These two realizations are the Chekanov and Clifford embeddings of the theorem. They are related in two distinct ways as follows.

First, let us compactify the plane to a sphere S^2 by adding a point ∞. Then keeping the base point fixed, we can move $\gamma_{C\ell}$ across ∞ to a circle around say 2ϵ with *clockwise* orientation. Thus our two realizations of $\mathcal{LD}_\Psi$ are naturally identified by $(Y, m) \mapsto (Y, m^{-1})$.

Second, we can move $\gamma_{C\ell}$ directly to $\gamma_{\mathcal{C}h}$ passing through the critical circle of Figure 2. If we restrict to local systems of clean objects, we obtain a wall-crossing map $(Y, m) \mapsto (Y, m \circ \mathfrak{m})$ relating our two realizations of $\mathcal{L}\mathcal{D}_{\spadesuit}$.

Thus altogether we can view $K_{\mathfrak{m}}$ as the wall-crossing map $(Y, m) \mapsto (Y, m \circ \mathfrak{m})$ under the identification $(Y, m) \mapsto (Y, m^{-1})$.

Corollary 1.3 (Wall-crossing formula)**.** *Under the composition $F_{C\ell}^{-1} \circ F_{\mathcal{C}h}$, objects of the inertia category $\mathcal{L}\mathcal{D}_{\spadesuit}$ undergo the transformation*

$$(Y, m) \longmapsto (Y, m^{-1} \circ \mathfrak{m}) \tag{30}$$

Example 1.1. Recall the perverse schober

$$\mathcal{P}_B = (i_* : \mathrm{Perf}_{prop}(P_{n-2}) \longrightarrow \mathrm{Perf}_{prop}(\mathbb{T}_0^\vee)) \tag{31}$$

on the disk D with a single critical point $0 \in D$.

The monodromy $T_{\Psi,r}$ on the nearby category $\mathcal{D}_\Psi = \mathrm{Perf}_{prop}(\mathbb{T}_0^\vee)$ is given by tensoring with $\mathcal{O}(1)$. The canonical map $\mathfrak{p}_\Psi : \mathrm{id} \to T_{\Psi,r}$ is given by a section $\sigma = x_1 + \cdots + x_n : \mathcal{O} \to \mathcal{O}(1)$ vanishing on $P_{n-2} \subset \mathbb{T}_0^\vee$ written here in homogenous coordinates $x_1, \ldots, x_n$.

The clean objects $\mathcal{D}_{\spadesuit} = \mathrm{Perf}_{prop}(\mathbb{T}_0^\vee)_{\spadesuit} \subset \mathrm{Perf}_{prop}(\mathbb{T}_0^\vee) = \mathcal{D}_\Psi$ are those supported away from P_{n-2}. Thus their inertia category has a simple description

$$\mathcal{L}\,\mathrm{Perf}_{prop}(\mathbb{T}_0^\vee)_{\spadesuit} \simeq \mathrm{Perf}_{prop}((\mathbb{T}_0^\vee \setminus P_{n-2}) \times \Gamma_m) \tag{32}$$

To smoothly extend $\mathcal{P}_B$ to a perverse schober $\tilde{\mathcal{P}}_B$ on the sphere S^2, we must choose a trivialization $\tau : \mathcal{O} \xrightarrow{\sim} \mathcal{O}(1)$. There are n natural choices given by the homogenous coordinates $x_1, \ldots, x_n$. Let us choose say $\tau = x_n$, and write $y_1 = x_1/x_n, \ldots, y_{n-1} = x_{n-1}/x_n$. Let us also set y_n to be a coordinate on the Γ_m factor of the product $(\mathbb{T}_0^\vee \setminus P_{n-2}) \times \Gamma_m$.

Then the wall-crossing formula (30) yields the birational transformation

$$\begin{aligned} y_i &\mapsto y_i, \quad i = 1, \ldots, n-1 \\ y_n &\mapsto y_n^{-1}(1 + y_1 + \cdots + y_{n-1}) \end{aligned} \tag{33}$$

appearing in (1). To deduce this from (30) given the above inputs, note that $m \mapsto m^{-1}$ is simply $y_n \mapsto y_n^{-1}$, and $\mathfrak{m} = \tau^{-1} \circ \sigma : \mathcal{O} \to \mathcal{O}(1) \to \mathcal{O}$ is given by $\mathfrak{m} = (x_1 + \cdots + x_n)/x_n = y_1 + \cdots + y_{n-1} + 1$.

Remark 1.5. The same formula applies as well if we start with the perverse schober

$$\mathcal{P}_B^w = (i_* : \mathrm{Coh}(P_{n-2}) \longrightarrow \mathrm{Coh}(\mathbb{T}_0^\vee)) \tag{34}$$

The only change is the inertia category of clean objects now admits the description

$$\mathcal{L}\,\mathrm{Coh}(\mathbb{T}_0^\vee)_{\spadesuit} \simeq \mathrm{Coh}_{relprop}((\mathbb{T}_0^\vee \setminus P_{n-2}) \times \Gamma_m) \tag{35}$$

as coherent complexes on $(\mathbb{T}_0^\vee \setminus P_{n-2}) \times \Gamma_m$ whose support is proper over $\mathbb{T}_0^\vee \setminus P_{n-2}$.

1.3.1. Back to geometry

By the mirror equivalence of Theorem 1.1, the conclusion of Example 1.1 applies equally well to the perverse schober

$$\mathcal{P}_A = (j^* : \mu Sh_L(\mathbb{C}^n) \longrightarrow \mu Sh_{L^\circ}(\mathbb{C}^n)) \tag{36}$$

We explain here the direct geometric interpretation of the wall-crossing formula (33). We will resume with the constructions and notation introduced in Section 1.1.1 above.

Inside of $\mathrm{Cyl} = \mathbb{C} \setminus \{\epsilon\} \simeq \mathbb{C}^\times$, recall the Chekanov and Clifford graphs

$$\Gamma_{\mathcal{C}h} = \gamma_{\mathcal{C}h} \cup I_{\mathcal{C}h} \qquad\qquad \Gamma_{C\ell} = \gamma \cup I_{C\ell} \tag{37}$$

and inside of the symplectic manifold $M = \{W \neq \epsilon\} \simeq (\mathbb{C}^\times)^n$, the corresponding Chekanov and Clifford skeleta

$$L_{\mathcal{C}h} = \{W \in \Gamma_{\mathcal{C}h}, |z_1| = \cdots = |z_n|\} \qquad\qquad L_{C\ell} = \{W \in \Gamma_{C\ell}, |z_1| = \cdots = |z_n|\} \tag{38}$$

From Theorem 1.2, one can show there are natural equivalences for microlocal sheaves

$$\mu Sh_{L_{\mathcal{C}h}}(M) \simeq \Gamma(\mathrm{Cyl}, \mathcal{P}_A) \simeq \mu Sh_{L_{C\ell}}(M) \tag{39}$$

For the first, we can work with $\Gamma(\mathrm{Cyl}, \mathcal{P}_A)$ in the form given in Definition 2.2. The equivalence is then an easy consequence from Theorem 1.2 and the familiar fact that microlocal sheaves near a trivalent vertex of a planar graph are exact triangles. For the second, we can work with $\Gamma(\mathrm{Cyl}, \mathcal{P}_A)$ in the form given in Definition 2.3 coupled with the identification of Lemma 2.1, then argue similarly.

Via the closed embeddings $T_{\mathcal{C}h} \subset L_{\mathcal{C}h}$, $T_{C\ell} \subset L_{C\ell}$, we have fully faithful functors

$$\mathcal{L}oc(T_{\mathcal{C}h}) \simeq \mu Sh_{T_{\mathcal{C}h}}(M) \hookrightarrow \mu Sh_{L_{\mathcal{C}h}}(M) \tag{40}$$

$$\mathcal{L}oc(T_{C\ell}) \simeq \mu Sh_{T_{C\ell}}(M) \hookrightarrow \mu Sh_{L_{C\ell}}(M) \tag{41}$$

where we write $\mathcal{L}oc$ for the dg category of local systems.

Since $\gamma_{\mathcal{C}h}$ does not wind around $0 \in \mathbb{C}$, we have a canonical splitting $T_{\mathcal{C}h}^{van} \times \gamma_{\mathcal{C}h} \xrightarrow{\sim} T_{\mathcal{C}h}$. The choice of a coordinate, say z_n, from among $z_1, \ldots, z_n$ provides a parallel splitting

$$T_{C\ell}^{van} \times \gamma_{C\ell} \xrightarrow{\sim} T_{C\ell} \qquad\qquad ((z_1, \ldots, z_n), e^{i\theta}) \longmapsto (z_1, \ldots, e^{i\theta} z_k, \ldots, z_n) \tag{42}$$

The splittings in turn provide identifications for inertia stacks

$$\mathcal{L}\mathcal{L}oc(T_{\mathcal{C}h}^{van}) \simeq \mathcal{L}oc(T_{\mathcal{C}h}^{van} \times S^1) \simeq \mathcal{L}oc(T_{\mathcal{C}h}) \tag{43}$$

$$\mathcal{L}\mathcal{L}oc(T_{C\ell}^{van}) \simeq \mathcal{L}oc(T_{C\ell}^{van} \times S^1) \simeq \mathcal{L}oc(T_{C\ell}) \tag{44}$$

More fundamentally, the choice of the coordinate z_n provides an extension

$$\overline{W} : \mathbb{C}^{n-1} \times \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad\qquad \overline{W} = [W, z] \tag{45}$$

where we equip $\mathbb{C}^{n-1}$ with coordinates $z_1, \ldots, z_{n-1}$, and $\mathbb{CP}^1$ with coordinates $[z_n, z]$. This gives a smooth extension of the perverse schober $\mathcal{P}_A$ to the sphere $S^2 \simeq \mathbb{CP}^1$.

The circles $\gamma_{\mathcal{C}h}, \gamma_{C\ell} \subset \mathbb{C}$, with their counterclockwise orientations, are naturally isotopic when regarded within $\mathbb{CP}^1 \setminus \{0\} \simeq \mathbb{C}$, but with opposite orientations. We also obtain an identification $T^{van}_{\mathcal{C}h} \simeq T^{van}_{C\ell}$ by parallel transporting each within $\mathbb{C}^{n-1} \times \mathbb{CP}^1$ above the respective intervals $[\epsilon - \rho_{\mathcal{C}h}, \infty]$, $[-\infty, \epsilon - \rho_{C\ell}]$ to the fiber at ∞.

Thus altogether, we have a canonical identification $T_{\mathcal{C}h} \simeq T_{C\ell}$ compatible with the above splittings, but notably with the inverse map on the factor S^1. Now with the above identifications in hand, we have the following geometric interpretation of Theorem 1.2.

Theorem 1.4. *For the perverse schober $\mathcal{P}_A$, the Chekanov and Clifford functors of Theorem 1.2 factor into the compositions*

$$F_{\mathcal{C}h} : \mathcal{L}\mathcal{D}_{\spadesuit} \xhookrightarrow{\qquad} \mathcal{L}\mathcal{L}oc(T^{van}_{\mathcal{C}h}) \simeq \mathcal{L}oc(T_{\mathcal{C}h}) \xhookrightarrow{\qquad} \mu Sh_{L_{\mathcal{C}h}}(M) \simeq \Gamma(Cyl, \mathcal{P}_A|_{Cyl}) \tag{46}$$

$$F_{C\ell} : \mathcal{L}\mathcal{D}_{\spadesuit} \xhookrightarrow{\qquad} \mathcal{L}\mathcal{L}oc(T^{van}_{C\ell}) \simeq \mathcal{L}oc(T_{C\ell}) \xhookrightarrow{\qquad} \mu Sh_{L_{C\ell}}(M) \simeq \Gamma(Cyl, \mathcal{P}_A|_{Cyl}) \tag{47}$$

Combining the above with Theorem 1.1 and Example 1.1, we obtain the following.

Corollary 1.5. *The wall-crossing formula of Corollary 1.3 is the birational map on moduli of objects for the partially defined functor $\mathcal{L}oc(T_{\mathcal{C}h}) \to \mathcal{L}oc(T_{C\ell})$ given by comparing clean local systems on the Chekanov and Clifford tori as objects in $\Gamma(Cyl, \mathcal{P}_A)$ with coordinates related by the given extension of $\mathcal{P}_A$ to the sphere $S^2 \simeq \mathbb{CP}^1$.*

2. Perverse schobers

We spell out constructions with perverse schobers on some simple but useful complex curves. In particular, we give concrete calculations for the global sections of such perverse schobers on cylinders under the name of Definitions 2.2 and 2.3. We prove the equivalence of these two definitions in Lemma 2.1. In an eventual uniform development of the subject, these formulas can be taken as explicit models of the (co)limits in an invariant notion of global sections.

2.1. On a disk

Following Kapranov-Schechtman [4], a perverse schober on the disk $D = \{|z| < \epsilon\} \subset \mathbb{CP}^1$ with a single critical point at $0 \in D$, in its minimal one-cut realization, is a *spherical functor*

$$\mathcal{P}_{D,0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi) \tag{48}$$

between stable dg categories. By definition, a functor S is spherical if it admits both a left adjoint S^ℓ and right adjoint S^r, so that we have an adjoint triple (S^ℓ, S, S^r) with units and counits of adjunctions denoted by

$$u_r : \mathrm{id}_\Phi \longrightarrow S^r S \qquad c_r : SS^r \longrightarrow \mathrm{id}_\Psi \tag{49}$$

$$u_\ell : \mathrm{id}_\Psi \longrightarrow SS^\ell \qquad c_\ell : S^\ell S \longrightarrow \mathrm{id}_\Phi \tag{50}$$

Moreover, if we form the natural triangles

$$T_{\Phi,r} := \mathrm{Cone}(u_r)[-1] \xrightarrow{\mathfrak{q}_\Phi} \mathrm{id}_\Phi \xrightarrow{u_r} S^r S \tag{51}$$

$$SS^r \xrightarrow{c_r} \mathrm{id}_\Psi \xrightarrow{\mathfrak{p}_\Psi} \mathrm{Cone}(u_r) =: T_{\Psi,r} \tag{52}$$

$$T_{\Psi,\ell} := \mathrm{Cone}(u_\ell)[-1] \xrightarrow{\mathfrak{q}_\Psi} \mathrm{id}_\Psi \xrightarrow{u_\ell} SS^\ell \tag{53}$$

$$S^\ell S \xrightarrow{c_\ell} \mathrm{id}_\Phi \xrightarrow{\mathfrak{p}_\Phi} \mathrm{Cone}(u_\ell) =: T_{\Phi,\ell} \tag{54}$$

then the following properties are required to hold (and in fact, by a theorem of Anno-Logvinenko [1], any two imply all four):

(SF1) $T_{\Psi,r}$ is an equivalence.

(SF2) The natural composition

$$S^r \longrightarrow S^r SS^\ell \longrightarrow T_{\Phi,r}S^\ell[1] \tag{55}$$

is an equivalence.

(SF3) $T_{\Phi,r}$ is an equivalence.

(SF4) The natural composition

$$S^\ell T_{\Psi,r}[-1] \longrightarrow S^\ell SS^r \longrightarrow S^r \tag{56}$$

is an equivalence.

When the above properties hold, $T_{\Phi,\ell}$, $T_{\Psi,\ell}$ are respective inverses of $T_{\Phi,r}$, $T_{\Psi,r}$, and we refer to them as *monodromy functors*.

Example 2.1 (Smooth hypersurfaces)**.** Let X be a smooth variety. Let $\mathcal{L}_X \to X$ be a line bundle and $\sigma : X \to \mathcal{L}_X$ a section transverse to the zero section. Let $Y = \{\sigma = 0\}$ be the resulting smooth hypersurface and $i : Y \to X$ its inclusion.

Let $\mathrm{Coh}(Y)$, $\mathrm{Coh}(X)$ denote the respective dg categories of coherent sheaves. We will regard the line bundle $\mathcal{L}_X$ as an object of $\mathrm{Coh}(X)$, and its restriction $\mathcal{L}_Y = i^*\mathcal{L}_X$ as an object of $\mathrm{Coh}(Y)$. We will regard the section σ as a morphism $\sigma : \mathcal{O}_X \to \mathcal{L}_X$, which by duality gives a morphism $\sigma^\vee : \mathcal{L}_X^\vee \to \mathcal{O}_X$.

We have the adjoint triple $(i^*, i_*, i^!)$ which satisfies functorial identities

$$i^*(-) \simeq \mathcal{O}_Y \otimes_{\mathcal{O}_X} (-) \qquad i^!(-) \simeq \mathcal{L}_Y[-1] \otimes_{\mathcal{O}_X} (-) \tag{57}$$

Set $\mathcal{D}_\Phi = \mathrm{Coh}(Y)$, $\mathcal{D}_\Psi = \mathrm{Coh}(X)$ and $S = i_*$. Then the monodromy functors satisfy

$$T_{\Psi,r}(-) \simeq \mathcal{L}_X \otimes_{\mathcal{O}_X} (-) \qquad T_{\Phi,r}(-) \simeq \mathcal{L}_Y[-2] \otimes_{\mathcal{O}_Y} (-) \tag{58}$$

and hence both are equivalences. Thus (SF1) and (SF3) hold, and so $S = i_*$ is a spherical functor. We will denote this perverse schober by

$$\mathcal{P}_{Y\subset X} = (i_* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)) \tag{59}$$

2.2. On a sphere

Suppose given a perverse schober on the disk $D = \{|z| < \epsilon\} \subset \mathbb{CP}^1$ with a single critical point $0 \in D$ in its realization as a spherical functor

$$\mathcal{P}_{D,0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi) \tag{60}$$

Recall we then have inverse monodromy functors $T_{\Psi,r}$, $T_{\Psi,\ell}$ on the nearby category $\mathcal{D}_\Psi$.

Let us formulate what it means to extend $\mathcal{P}_{D,0}$ smoothly to the sphere $S^2 \simeq \mathbb{CP}^1$.

Definition 2.1. (1) A *framing* for the perverse schober $\mathcal{P}_{D,0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi)$ is a trivialization of the monodromy of the nearby category

$$\tau : \mathrm{id} \xrightarrow{\sim} T_{\Psi,r} \tag{61}$$

(2) A perverse schober on the sphere $S^2 \simeq \mathbb{CP}^1$ with a single critical point $0 \in S^2$ is a pair

$$\mathcal{P}_{S^2,0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi, \tau : \mathrm{id} \overset{\sim}{\to} T_{\Psi,r}) \tag{62}$$

consisting of a spherical functor S and a framing τ.

Example 2.2 (Smooth hypersurfaces)**.** Recall from Example 2.1 the perverse schober

$$\mathcal{P}_{Y \subset X} = (i_* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)) \tag{63}$$

Recall the monodromy of the nearby category is given by

$$T_{\Psi,r}(-) \simeq \mathcal{L}_X \otimes_{\mathcal{O}_X} (-) \tag{64}$$

Thus framings are equivalent to isomorphisms

$$\tau : \mathcal{O}_X \xrightarrow{\sim} \mathcal{L}_X \tag{65}$$

or in other words, non-vanishing sections of $\mathcal{L}_X$.

2.3. On a cylinder

Suppose given perverse schober on the disk $D = \{|z| < \epsilon\} \subset \mathbb{CP}^1$ with a single critical point $0 \in D$ in its realization as a spherical functor

$$\mathcal{P}_{D,0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi) \tag{66}$$

Let us formulate what it means to extend $\mathcal{P}_{D,0}$ smoothly to the cylinder $\mathrm{Cyl} = \mathbb{C} \backslash \{\epsilon\} \simeq \mathbb{CP}^1 \setminus \{\epsilon, \infty\}$.

Definition 2.2. (1) A perverse schober on the cylinder $\mathrm{Cyl} = \mathbb{C} \setminus \{\epsilon\} \simeq \mathbb{CP}^1 \setminus \{\epsilon, \infty\}$ with a single critical point $0 \in \mathrm{Cyl}$ is a pair

$$\mathcal{P}_{\mathrm{Cyl},0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi, M : \mathcal{D}_\Psi \overset{\sim}{\to} \mathcal{D}_\Psi) \tag{67}$$

consisting of a spherical functor S and an additional monodromy functor M.

(2) The *global sections* $\Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0})$ is the dg category of quintuples (X, Y_1, Y_2, Δ, m) consisting of $X \in \mathcal{D}_\Phi$, $Y_1, Y_2 \in \mathcal{D}_\Psi$, an exact triangle

$$\Delta = (Y_1[-1] \xrightarrow{i} S(X) \xrightarrow{p} Y_2 \xrightarrow{\partial} Y_1) \tag{68}$$

and an isomorphism

$$m : Y_1 \xrightarrow{\sim} M(Y_2) \tag{69}$$

Morphisms are maps of diagrams: a morphism $(X, Y_1, Y_2, \Delta, m) \to (X', Y_1', Y_2', \Delta', m')$ consists of maps $f : X \to X'$, $g_1 : Y_1 \to Y_1'$, $g_2 : Y_2 \to Y_2'$, with a lift to a map of triangles

$$\begin{tikzcd}
\Delta = \arrow[d, "g"'] & (Y_1[-1] \arrow[r, "i"] \arrow[d, "{g_1[-1]}"] & S(X) \arrow[r, "p"] \arrow[d, "S(f)"] & Y_2 \arrow[r, "\partial"] \arrow[d, "g_2"] & Y_1) \arrow[d, "g_1"] \\
\Delta' = & (Y_1'[-1] \arrow[r, "i"] & S(X') \arrow[r, "p"] & Y_2' \arrow[r, "\partial"] & Y_1')
\end{tikzcd} \tag{70}$$

and a commutativity isomorphism $M(g_2) \circ m \simeq m' \circ g_1$.

Remark 2.1. It is always possible to extend a perverse schober $\mathcal{P}_{D,0}$ on the disk D with a single critical point $0 \in D$ to one on the cylinder Cyl by taking M to be the identity functor.

Remark 2.2. It is always possible to restrict a perverse schober $\mathcal{P}_{S^2,0}$ on the sphere S^2 with a single critical point $0 \in S^2$ to one on the cylinder Cyl by forgetting the framing τ and taking M to be the identity functor.

Remark 2.3. The notion of global sections $\Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0})$ given in Definition 2.2(2) follows from considering sections supported over the specific Lagrangian skeleton $\Gamma_{\mathcal{C}h} = \gamma_{\mathcal{C}h} \cup I_{\mathcal{C}h} \subset \mathrm{Cyl}$. We will see immediately below an equivalent notion of global sections that follows from considering sections supported over the alternative Lagrangian skeleton $\Gamma_{C\ell} = \gamma_{C\ell} \cup I_{C\ell} \subset \mathrm{Cyl}$.

2.3.1. Mutated global sections

The notion of global sections $\Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0})$ given above in Definition 2.2(2) is one of many equivalent possibilities. We introduce here a key alternative notion and show it is equivalent to the original.

Definition 2.3. The *mutated global sections* $\Gamma^\sharp(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0})$ is the dg category of quintuples $(X^\sharp, Y_1^\sharp, Y_2^\sharp, \Delta^\sharp, m^\sharp)$ consisting of $X^\sharp \in \mathcal{D}_\Phi$, $Y_1^\sharp, Y_2^\sharp \in \mathcal{D}_\Psi$, an exact triangle

$$\Delta^\sharp = (T_{\Psi,r}(Y_1^\sharp)[-1] \xrightarrow{i} S(X^\sharp) \xrightarrow{p} Y_2^\sharp \xrightarrow{\partial} T_{\Psi,r}(Y_1^\sharp)) \tag{71}$$

and an isomorphism

$$m^\sharp : M(Y_1^\sharp) \xrightarrow{\sim} Y_2^\sharp \tag{72}$$

Morphisms are maps of diagrams as in Definition 2.2(2).

Now we will define a mutation equivalence

$$\mathfrak{F}^\sharp : \Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0}) \longrightarrow \Gamma^\sharp(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0}) \qquad \mathfrak{F}^\sharp(X, Y_1, Y_2, \Delta, m) = (X^\sharp, Y_1^\sharp, Y_2^\sharp, \Delta^\sharp, m^\sharp)$$

First, set $Y_1^\sharp := Y_2$, $Y_2^\sharp := Y_1$, and

$$m^\sharp : M(Y_1^\sharp) = M(Y_2) \xrightarrow{m^{-1}} Y_1 = Y_2^\sharp \tag{73}$$

By adjunction, the map $p : S(X) \to Y_2$ provides a map $\tilde{p} : X \to S^r(Y_2)$, and we set $X^\sharp := \mathrm{Cone}(\tilde{p})$. To construct the triangle $\Delta^\sharp$, note that adjunction further provides a commutative diagram

$$\begin{tikzcd} S(X) \arrow[r, "S(\tilde{p})"] \arrow[dr, "p"'] & S(S^r(Y_2)) \arrow[d, "c_r"] \\ & Y_2 \end{tikzcd} \tag{74}$$

Recall as well the natural triangle

$$SS^r \xrightarrow{c_r} \mathrm{id} \xrightarrow{\mathfrak{p}_\Psi} T_{\Psi,r} \longrightarrow SS^r[1] \tag{75}$$

Taking the cone of each map of (74), we obtain another triangle

$$\mathrm{Cone}(S(\tilde{p})) \longrightarrow Y_1 \xrightarrow{\tilde{\mathfrak{p}}_\Psi} T_{\Psi,r}(Y_2) \longrightarrow \mathrm{Cone}(S(\tilde{p}))[1] \tag{76}$$

Now we take $\Delta^\sharp$ to be the rotated triangle

$$\Delta^\sharp = (T_{\Psi,r}(Y_2)[-1] \longrightarrow \mathrm{Cone}(S(\tilde{p})) \longrightarrow Y_1 \xrightarrow{\tilde{\mathfrak{p}}_\Psi} T_{\Psi,r}(Y_2)) \tag{77}$$

using the canonical identification $\mathrm{Cone}(S(\tilde{p})) \simeq S(\mathrm{Cone}(\tilde{p}))$.

Remark 2.4. Let us highlight a key property of the map $\tilde{\mathfrak{p}}_\Psi : Y_1 \to T_{\Psi,r}(Y_2)$ appearing in the triangle $\Delta^\sharp$. By construction, it arises in a commutative diagram

$$\begin{tikzcd} Y_2 \arrow[r, "\partial"] \arrow[dr, "\mathfrak{p}_\Psi"'] & Y_1 \arrow[d, "\tilde{\mathfrak{p}}_\Psi"] \\ & T_{\Psi,r}(Y_2) \end{tikzcd} \tag{78}$$

where ∂ is given in the initial triangle (71), and $\mathfrak{p}_\Psi$ is the canonical map. In particular, if we have $Y_1 = Y_2$, and the map $\partial : Y_2 \to Y_1$ is the identity, then $\tilde{\mathfrak{p}}_\Psi = \mathfrak{p}_\Psi$.

Lemma 2.1. *The mutation functor*

$$\mathfrak{F} : \Gamma(Cyl, \mathcal{P}_{Cyl}) \longrightarrow \Gamma^\sharp(Cyl, \mathcal{P}_{Cyl}) \qquad \mathfrak{F}(X, Y_1, Y_2, \Delta, m) = (X^\sharp, Y_1^\sharp, Y_2^\sharp, \Delta^\sharp, m^\sharp)$$

is an equivalence.

Proof. We can define an inverse to $\mathfrak{F}$ beginning with the evident assignments $Y_1 := Y_2^\sharp$, $Y_1 := Y_2^\sharp$,

$$m : Y_1 = Y_2^\sharp \xrightarrow{m^{-1}} M(Y_1^\sharp) = Y_2 \tag{79}$$

For X and the triangle Δ, we proceed as follows. By adjunction, the map $i : T_{\Psi,r}(Y_1^\sharp)[-1] \to S(X^\sharp)$ provides a map $\tilde{\imath} : S^\ell(T_{\Psi,r}(Y_1^\sharp))[-1] \to X^\sharp$, and we take $X := \mathrm{Cone}(\tilde{\imath})[-1]$.

To construct the triangle Δ, note that adjunction further provides a commutative diagram

$$\begin{tikzcd}
SS^\ell(T_{\Psi,r}(Y_1^\sharp))[-1] \arrow[r, "S(\tilde{\imath})"] & S(X^\sharp) \\
T_{\Psi,r}(Y_1^\sharp)[-1] \arrow[u, "u_\ell"] \arrow[ur, "i"'] &
\end{tikzcd} \tag{80}$$

Recall (SF4) confirms the natural composition is an equivalence.

$$S^\ell T_{\Psi,r}[-1] \longrightarrow S^\ell S S^r \longrightarrow S^r \tag{81}$$

and thus we can write (80) in the form

$$\begin{tikzcd}
SS^r(Y_1^\sharp) \arrow[r, "S(\tilde{\imath})"] & S(X^\sharp) \\
T_{\Psi,r}(Y_1^\sharp)[-1] \arrow[u] \arrow[ur, "i"'] &
\end{tikzcd} \tag{82}$$

Taking the cone of each map of (82), we obtain another triangle

$$Y_1^\sharp \longrightarrow Y_2^\sharp \longrightarrow \mathrm{Cone}(S(\tilde{\imath})) \longrightarrow Y_1^\sharp[-1] \tag{83}$$

We take Δ to be the rotated shifted triangle

$$\Delta = (Y_2^\sharp[-1] \longrightarrow \mathrm{Cone}(S(\tilde{\imath}))[-1] \longrightarrow Y_1^\sharp \longrightarrow Y_2^\sharp) \tag{84}$$

using the canonical identification $\mathrm{Cone}(S(\tilde{\imath})) \simeq S(\mathrm{Cone}(\tilde{\imath}))$.

We leave it to the reader to check the constructed functor is inverse to $\mathfrak{F}$. ☐

3. Abstract wall-crossing

In this section, we will explain how a perverse schober on the sphere $S^2 \simeq \mathbb{CP}^1$ with a single critical point provides a wall-crossing formula.

3.1. Chekanov functor

Suppose given a perverse schober

$$\mathcal{P}_{\mathrm{Cyl},0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi, M : \mathcal{D}_\Psi \xrightarrow{\sim} \mathcal{D}_\Psi) \tag{85}$$

on the cylinder $\mathrm{Cyl} = \mathbb{C} \setminus \{\epsilon\} \simeq \mathbb{CP}^1 \setminus \{\epsilon, \infty\}$ with a single critical point $0 \in \mathrm{Cyl}$. Thus by definition, S is a spherical functor, and M is an invertible functor.

Following Definition 2.2(2), the global sections $\Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0})$ is the dg category of quintuples (X, Y_1, Y_2, Δ, m) consisting of $X \in \mathcal{D}_\Phi$, $Y_1, Y_2 \in \mathcal{D}_\Psi$, an exact triangle

$$\Delta = (Y_1[-1] \xrightarrow{i} S(X) \xrightarrow{p} Y_2 \xrightarrow{\partial} Y_1) \tag{86}$$

and an isomorphism $m : Y_1 \xrightarrow{\sim} M(Y_2)$

Introduce the smooth perverse schober

$$\mathcal{P}_{sm} = (0 : \{0\} \to \mathcal{D}_\Psi, M : \mathcal{D}_\Psi \xrightarrow{\sim} \mathcal{D}_\Psi) \tag{87}$$

on the cylinder with no critical points.

Following Definition 2.2(2), the global sections $\Gamma(\mathrm{Cyl}, \mathcal{P}_{sm})$ is the dg category of quintuples $(0, Y_1, Y_2, \Delta, m)$ consisting of $Y_1, Y_2 \in \mathcal{D}_\Psi$, an exact triangle

$$\Delta = (Y_1[-1] \xrightarrow{i} 0 \xrightarrow{p} Y_2 \xrightarrow{\partial} Y_1) \tag{88}$$

and an isomorphism $m : Y_1 \xrightarrow{\sim} M(Y_2)$.

Thus $\Gamma(\mathrm{Cyl}, \mathcal{P}_{sm}) \subset \Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl}})$ is the full subcategory of quintuples $(0, Y_1, Y_2, \Delta, m)$ with vanishing first entry.

Remark 3.1. There is an evident commutative diagram

$$\begin{tikzcd}
\{0\} \arrow[r, "0"] \arrow[d, "0"'] & \mathcal{D}_\Psi \arrow[d, "\mathrm{id}"] \\
\mathcal{D}_\Phi \arrow[r, "S"] & \mathcal{D}_\Psi
\end{tikzcd} \tag{89}$$

compatible with $M : \mathcal{D}_\Psi \xrightarrow{\sim} \mathcal{D}_\Psi$ that induces the inclusion $\Gamma(\mathrm{Cyl}, \mathcal{P}_0) \subset \Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl}})$. In general, passing to the adjoints to the horizontal maps in (89) does not lead to a commutative diagram. Thus it is not clear whether to consider (89) as a map of perverse schobers $\mathcal{P}_{sm} \to \mathcal{P}_{\mathrm{Cyl}}$. But we will only consider (89) where in the top right we take the full subcategory $\mathcal{D}_\spadesuit \subset \mathcal{D}_\Psi$ of clean objects (see Definition 3.1) for which both adjoints to S vanish, and so indeed fit into a trivially commutative diagram.

Let $\mathcal{L}^M\mathcal{D}_\Psi$ denote the dg path category with objects pairs (Y, m) consisting of $Y \in \mathcal{D}_\Psi$, and a monodromy isomorphism $m : Y \xrightarrow{\sim} M(Y)$; morphisms $(Y, m) \to (Y', m')$ are maps $g : Y \to Y'$ with an isomorphism $M(g) \circ m \simeq m' \circ g$.

Remark 3.2. When $M = \mathrm{id}$, note that $\mathcal{L}^M\mathcal{D}_\Psi$ is the usual inertia dg category $\mathcal{L}\mathcal{D}_\Psi$ of pairs (Y, m) consisting of $Y \in \mathcal{D}_\Psi$, and a monodromy isomorphism $m : Y \xrightarrow{\sim} Y$.

Introduce the *Chekanov functor*

$$F_{\mathcal{C}h} : \mathcal{L}^M\mathcal{D}_\Psi \longrightarrow \Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl}}) \qquad F_{\mathcal{C}h}(Y, m) = (0, Y, Y, \Delta_0, m) \tag{90}$$

where Δ_0 is the split triangle

$$\Delta_0 = (Y[-1] \longrightarrow 0 \longrightarrow Y \xrightarrow{\mathrm{id}} Y) \tag{91}$$

Lemma 3.1. *$F_{\mathcal{C}h}$ is fully faithful with essential image $\Gamma(\mathrm{Cyl}, \mathcal{P}_{sm}) \subset \Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0})$.*

Proof. We may use the isomorphism $\partial : Y_2 \overset{\sim}{\to} Y_1$ in the triangle (96) to identify Y_1, Y_2. Thus any quintuple $(0, Y_1, Y_2, \Delta, m)$ with vanishing first entry is in the essential image of $F_{\mathcal{C}h}$.

Recall morphisms $(0, Y, Y, \Delta, m) \to (0, Y', Y', \Delta', m')$ are maps $g_1 : Y \to Y'$, $g_2 : Y \to Y'$, with a lift to a map of triangles

$$\begin{tikzcd}
\Delta \arrow[d, "g"'] & = (Y[-1] \arrow[r] \arrow[d, "{g_1[-1]}"] & 0 \arrow[r] \arrow[d] & Y \arrow[r, "\mathrm{id}"] \arrow[d, "g_2"] & Y) \arrow[d, "g_1"] \\
\Delta' & = (Y'[-1] \arrow[r] & 0 \arrow[r, "p"] & Y' \arrow[r, "\mathrm{id}"] & Y')
\end{tikzcd} \tag{92}$$

and a commutativity isomorphism $M(g_2) \circ m \simeq m' \circ g_1$. The lift to triangles provides an isomorphism $g_1 \simeq g_2$, and so morphisms are simply maps $g : Y \to Y'$ such that $M(g) \circ m \simeq m' \circ g$ as in the dg path category. □

3.2. Clifford functor

We repeat here the constructions of the preceding section in a parallel mutated form.

Consider again the given perverse schober

$$\mathcal{P}_{\mathrm{Cyl},0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi, M : \mathcal{D}_\Psi \overset{\sim}{\to} \mathcal{D}_\Psi) \tag{93}$$

on the cylinder $\mathrm{Cyl} = \mathbb{C} \setminus \{\epsilon\} \simeq \mathbb{CP}^1 \setminus \{\epsilon, \infty\}$ with a single critical point $0 \in \mathrm{Cyl}$.

Following Definition 2.3, the mutated global sections $\Gamma^\sharp(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0})$ is the dg category of quintuples $(X^\sharp, Y_1^\sharp, Y_2^\sharp, \Delta^\sharp, m^\sharp)$ consisting of $X^\sharp \in \mathcal{D}_\Phi$, $Y_1^\sharp, Y_2^\sharp \in \mathcal{D}_\Psi$, an exact triangle

$$\Delta_0^\sharp = (T_{\Psi,r}(Y_1^\sharp)[-1] \xrightarrow{i} S(X^\sharp) \xrightarrow{p} Y_2^\sharp \xrightarrow{\partial} T_{\Psi,r}(Y_1^\sharp)) \tag{94}$$

and an isomorphism $m^\sharp : M(Y_1^\sharp) \overset{\sim}{\to} Y_2^\sharp$

Consider again the smooth perverse schober

$$\mathcal{P}_{sm} = (0 : \{0\} \to \mathcal{D}_\Psi, M : \mathcal{D}_\Psi \overset{\sim}{\to} \mathcal{D}_\Psi) \tag{95}$$

on the cylinder with no critical points.

Following Definition 2.3, the mutated global sections $\Gamma^\sharp(\mathrm{Cyl}, \mathcal{P}_{sm})$ is the dg category of quintuples $(0, Y_1^\sharp, Y_2^\sharp, \Delta_0^\sharp, m^\sharp)$ consisting of $Y_1^\sharp, Y_2^\sharp \in \mathcal{D}_\Psi$, an exact triangle

$$\Delta_0^\sharp = (T_{\Psi,r}(Y_1^\sharp)[-1] \xrightarrow{i} 0 \xrightarrow{p} Y_2^\sharp \xrightarrow{\partial} T_{\Psi,r}(Y_1^\sharp)) \tag{96}$$

and an isomorphism $m^\sharp : M(Y_1^\sharp) \xrightarrow{\sim} Y_2^\sharp$.

Thus $\Gamma^\sharp(\mathrm{Cyl}, \mathcal{P}_{sm}) \subset \Gamma^\sharp(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl}})$ is the full subcategory of quintuples $(0, Y_1^\sharp, Y_2^\sharp, \Delta^\sharp, m^\sharp)$ with vanishing first entry.

Remark 3.3. The inclusion $\Gamma^\sharp(\mathrm{Cyl}, \mathcal{P}_{sm}) \subset \Gamma^\sharp(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0})$ is induced by the diagram (89).

Let $\mathcal{L}_{M\circ T_{\Psi,r}}\mathcal{D}_\Psi$ denote the dg path category of pairs $(Y^\sharp, m^\sharp)$ consisting of $Y^\sharp \in \mathcal{D}_\Psi$, and a monodromy isomorphism $m^\sharp : M(T_{\Psi,r}(Y^\sharp)) \xrightarrow{\sim} Y^\sharp$; morphisms $(Y^\sharp, m^\sharp) \to ((Y^\sharp)', (m^\sharp)')$ are maps $g : Y \to (Y^\sharp)'$ with an isomorphism $g \circ m^\sharp \simeq (m^\sharp)' \circ M(T_{\Psi,r}(g))$.

Remark 3.4. When $M \circ T_{\Psi,r} = \mathrm{id}$, note that $\mathcal{L}_{M\circ T_{\Psi,r}}\mathcal{D}_\Psi$ is the usual inertia dg category $\mathcal{L}\mathcal{D}_\Psi$.

Introduce the *Clifford functor*

$$F_{C\ell} : \mathcal{L}_{M\circ T_{\Psi,r}}\mathcal{D}_\Psi \longrightarrow \Gamma^\sharp(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0}) \qquad F_{C\ell}(Y^\sharp, m^\sharp) = (0, T_{\Psi,r}^{-1}(Y^\sharp), Y^\sharp, \Delta_0^\sharp, m^\sharp) \tag{97}$$

where $\Delta_0^\sharp$ is the split triangle

$$\Delta_0^\sharp = (Y^\sharp[-1] \longrightarrow 0 \longrightarrow Y^\sharp \xrightarrow{\mathrm{id}} Y^\sharp) \tag{98}$$

Lemma 3.2. *$F_{C\ell}$ is fully faithful with essential image $\Gamma^\sharp(\mathit{Cyl}, \mathcal{P}_{C\ell}) \subset \Gamma^\sharp(\mathit{Cyl}, \mathcal{P}_{\mathit{Cyl}})$.*

Proof. We may use the isomorphism $\partial : Y_2^\sharp \xrightarrow{\sim} T_{\Psi,r}(Y_1^\sharp)$ in triangle (96) to identify $T_{\Psi,r}(Y_1^\sharp), Y_2^\sharp$. Thus any quintuple $(0, Y_1^\sharp, Y_2^\sharp, \Delta^\sharp, m^\sharp)$ with vanishing first entry is in the essential image of $F_{C\ell}$.

The rest of the proof is the same as that of Lemma 3.1. □

3.3. Framing identifications

Now suppose given a perverse schober

$$\mathcal{P}_{S^2,0} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi, \tau : \mathrm{id} \xrightarrow{\sim} T_{\Psi,r}) \tag{99}$$

on the sphere $S^2 \simeq \mathbb{CP}^1$ with a single critical point at $0 \in S^2$.

Let us restrict to a perverse schober

$$\mathcal{P}_{\mathrm{Cyl},0} = \mathcal{P}_{S^2,0}|_{\mathrm{Cyl}} = (S : \mathcal{D}_\Phi \to \mathcal{D}_\Psi, \mathrm{id} : \mathcal{D}_\Psi \xrightarrow{\sim} \mathcal{D}_\Psi) \tag{100}$$

on the cylinder $\mathrm{Cyl} = \mathbb{C} \setminus \{\epsilon\} \simeq \mathbb{CP}^1 \setminus \{\epsilon, \infty\}$ by forgetting the framing τ and taking the additional monodromy functor M to be the identity. Consider the smooth perverse schober

$$\mathcal{P}_{sm} = (0 : \{0\} \to \mathcal{D}_\Psi, \mathrm{id} : \mathcal{D}_\Psi \xrightarrow{\sim} \mathcal{D}_\Psi) \tag{101}$$

again taking the additional monodromy functor M to be the identity.

Observe that the framing $\tau : \mathrm{id} \xrightarrow{\sim} T_{\Psi,r}$ provides a canonical equivalence $\tau : \mathcal{L}\mathcal{D}_\Psi \xrightarrow{\sim} \mathcal{L}_{T_{\Psi,r}}\mathcal{D}_\Psi$. Thus the Chekanov and Clifford functors take the respective forms

$$F_{\mathcal{C}h} : \mathcal{L}\mathcal{D}_\Psi \longrightarrow \Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl}}) \qquad F_{\mathcal{C}h}(Y, m) = (0, Y, Y, \Delta_0, m) \tag{102}$$

$$\Delta_0 = (Y[-1] \longrightarrow 0 \longrightarrow Y \xrightarrow{\mathrm{id}} Y) \tag{103}$$

$$F_{C\ell} : \mathcal{L}\mathcal{D}_\Psi \longrightarrow \Gamma^\sharp(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl}}) \qquad F_{C\ell}(Y^\sharp, m^\sharp) = (0, Y^\sharp, Y^\sharp, \Delta_0^\sharp, m^\sharp) \tag{104}$$

$$\Delta_0^\sharp = (Y^\sharp[-1] \longrightarrow 0 \longrightarrow Y^\sharp \xrightarrow{\mathrm{id}} Y^\sharp) \tag{105}$$

Remark 3.5. Recall the mutation equivalence $\mathfrak{F}^\sharp : \Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0}) \xrightarrow{\sim} \Gamma^\sharp(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0})$. We caution the reader that the following diagram is *not* commutative

$$\begin{tikzcd}
& \Gamma(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0}) \arrow[dd, "\mathfrak{F}^\sharp"] \\
\mathcal{L}\mathcal{D}_\Psi \arrow[ur, "F_{\mathcal{C}h}"] \arrow[dr, "F_{C\ell}"] & \\
& \Gamma^\sharp(\mathrm{Cyl}, \mathcal{P}_{\mathrm{Cyl},0})
\end{tikzcd} \tag{106}$$

We will introduce a correct intertwining relation in the next section.

3.4. Clean objects

Let us continue with the setup of the preceding section.

Definition 3.1. We say an object $Y \in \mathcal{D}_\Psi$ is *clean* if $S^r(Y) \simeq 0$.

We denote by $\mathcal{D}_\spadesuit \subset \mathcal{D}_\Psi$ the full subcategory of clean objects.

Lemma 3.3. *An object $Y \in \mathcal{D}_\Psi$ is clean if and only if any of the following hold:*

1. $S^\ell(Y) \simeq 0$.
2. $\mathfrak{p}_\Psi : \mathrm{id}_\Psi \to T_{\Psi,r}$ *is an isomorphism.*
3. $\mathfrak{q}_\Psi : T_{\Psi,\ell} \to \mathrm{id}_\Psi$ *is an isomorphism.*

Proof. For the equivalence with property (1), recall (SF2): the natural composition

$$S^r \longrightarrow S^r S S^\ell \longrightarrow T_{\Phi,r} S^\ell[1]$$

is an equivalence and (SF3): $T_{\Phi,r}$ is an equivalence.

The equivalences with properties (2) and (3) are immediate from the triangles defining the canonical maps. □

Remark 3.6. Note that the monodromy $T_{\Psi,r}$, and hence also its inverse $T_{\Psi,\ell}$, preserves clean objects by property (1) of the lemma and (SF4).

Remark 3.7. It is important to distinguish between a framing $\tau : \mathrm{id} \xrightarrow{\sim} T_{\Psi,r}$ and the canonical map $\mathfrak{p}_\Psi : \mathrm{id}_\Psi \to T_{\Psi,r}$. The framing τ is an additional structure not intrinsic to the perverse schober, and by definition, it is required to be an isomorphism. The canonical map $\mathfrak{p}_\Psi$ is intrinsic to the perverse schober, but not necessarily an isomorphism when evaluated on some objects.

Example 3.1 (Smooth hypersurfaces)**.** Recall from Example 2.1 the perverse schober

$$\mathcal{P}_{Y\subset X} = (i_* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)) \tag{107}$$

An object $\mathcal{F} \in \mathrm{Coh}(X)$ is clean if and only if $i^*\mathcal{F} \simeq 0$. Equivalently, an object $\mathcal{F} \in \mathrm{Coh}(X)$ is clean if and only if either (hence both) of the maps

$$\mathcal{F} \xrightarrow{\sigma} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}_X \qquad \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}_X^\vee \xrightarrow{\sigma^\vee} \mathrm{id} \tag{108}$$

is an isomorphism. From any of the above descriptions, we see that an object $\mathcal{F} \in \mathrm{Coh}(X)$ is clean if and only if it is supported away from $Y = \{\sigma = 0\}$.

Now introduce the perverse schober

$$\mathcal{P}_\spadesuit = (0 : \{0\} \to \mathcal{D}_\spadesuit, \mathrm{id} : \mathcal{D}_\spadesuit \xrightarrow{\sim} \mathcal{D}_\spadesuit) \tag{109}$$

on the cylinder with no critical points by taking the full dg subcategory $\mathcal{D}_\spadesuit \subset \mathcal{D}_\Psi$ of clean objects and the additional monodromy functor M to be the identity.

Given any automorphism $\mathfrak{m}$ of the identity functor of $\mathcal{D}_\spadesuit$, we have a corresponding translated inverse functor on the dg inertia category

$$K_\mathfrak{m} : \mathcal{L}\mathcal{D}_\spadesuit \xrightarrow{\sim} \mathcal{L}\mathcal{D}_\spadesuit \qquad K_\mathfrak{m}(Y, m) = (Y, m^{-1} \circ \mathfrak{m}) \tag{110}$$

Note we have a canonical identification $m^{-1} \circ \mathfrak{m} \simeq \mathfrak{m} \circ m^{-1}$ by functoriality.

Now we have the intertwining:

Proposition 3.4. *The restrictions of the Chekanov and Clifford functors to clean objects fits into a canonically commutative diagram*

$$\begin{tikzcd}
\mathcal{L}\mathcal{D}_\spadesuit \arrow[r, "F_{Ch}"] \arrow[d, "K_\mathfrak{m}"'] & \Gamma(Cyl, \mathcal{P}_{Cyl}) \arrow[d, "\mathfrak{F}^\sharp"] \\
\mathcal{L}\mathcal{D}_\spadesuit \arrow[r, "F_{C\ell}"] & \Gamma^\sharp(Cyl, \mathcal{P}_{Cyl})
\end{tikzcd} \tag{111}$$

where the automorphism $\mathfrak{m}$ *of the identity functor of* $\mathcal{D}_\spadesuit$ *is the composition*

$$\mathfrak{m} : \mathrm{id} \xrightarrow{\mathfrak{p}_\Psi} T_{\Psi,r} \xrightarrow{\tau^{-1}} \mathrm{id} \tag{112}$$

of the canonical morphism $\mathfrak{p}_\Psi$ *(which is invertible on clean objects) and the inverse of the framing* τ*.*

Proof. For $(Y, m) \in \mathcal{LD}_{\spadesuit}$, we can apply the definitions to find:

$$\mathfrak{F}^\sharp(F_{\mathcal{C}h}(Y, m)) = \mathfrak{F}^\sharp(0, Y, Y, \Delta, m) = (0, Y, Y, \Delta^\sharp, m^{-1}) \tag{113}$$

where $\Delta^\sharp$ is the triangle

$$\Delta^\sharp = (Y[-1] \longrightarrow 0 \longrightarrow Y \xrightarrow{\mathfrak{m}=\tau^{-1}\circ\mathfrak{p}_\Psi} Y) \tag{114}$$

as highlighted in Remark 2.4.

On the other hand, we can also apply the definitions to find:

$$F_{C\ell}(K_{\mathfrak{m}}(Y, m)) = \mathcal{F}_{C\ell}(Y, m^{-1} \circ \mathfrak{m}) = (0, Y, Y, \Delta_0^\sharp, m^{-1} \circ \mathfrak{m}) \tag{115}$$

where $\Delta_0^\sharp$ is the triangle

$$\Delta_0^\sharp = (Y[-1] \longrightarrow 0 \longrightarrow Y \xrightarrow{\text{id}} Y) \tag{116}$$

Thus we seek an isomorphism

$$(0, Y, Y, \Delta^\sharp, m^{-1}) \xrightarrow{\sim} (0, Y, Y, \Delta_0^\sharp, m^{-1} \circ \mathfrak{m}) \tag{117}$$

or in other words, a map of triangles

$$\begin{tikzcd}
\Delta^\sharp \arrow[d, "g"'] & = (Y[-1] \arrow[r] \arrow[d, "{g_1[-1]}"] & 0 \arrow[r] \arrow[d, "0"] & Y \arrow[r, "\mathfrak{m}=\tau^{-1}\circ\mathfrak{p}_\Psi"] \arrow[d, "g_2"] & Y) \arrow[d, "g_1"] \\
\Delta_0^\sharp & = (Y[-1] \arrow[r] & 0 \arrow[r] & Y \arrow[r, "\text{id}"] & Y)
\end{tikzcd} \tag{118}$$

with a commutativity isomorphism $g_2 \circ m^{-1} \simeq m^{-1} \circ \mathfrak{m} \circ g_1$. We take $g_1 = \text{id}$, $g_2 = \mathfrak{m}$, and the canonical isomorphism $g_2 \circ m^{-1} = \mathfrak{m} \circ m^{-1} \simeq m^{-1} \circ \mathfrak{m} \simeq m^{-1} \circ \mathfrak{m} \circ g_1$. ☐

Corollary 3.5. $K_{\mathfrak{m}} \simeq F_{C\ell}^{-1} \circ \mathfrak{F}^\sharp \circ F_{\mathcal{C}h}$.

Example 3.2 (Smooth hypersurfaces)**.** Recall from Example 2.1 the perverse schober

$$\mathcal{P}_{Y \subset X} = (i_* : \text{Coh}(Y) \to \text{Coh}(X)) \tag{119}$$

Recall framings are equivalent to isomorphisms

$$\tau : \mathcal{O}_X \xrightarrow{\sim} \mathcal{L}_X \tag{120}$$

or in other words, non-vanishing sections of $\mathcal{L}_X$. Recall an object $\mathcal{F} \in \text{Coh}(X)$ is clean if and only if $\mathcal{F}$ is supported away from $Y \subset X$. For a clean object $\mathcal{F} \in \text{Coh}(X)$, we have the central automorphism

$$\mathfrak{m} : \mathcal{F} \xrightarrow{\sigma} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}_X \xrightarrow{\tau^{-1}} \mathcal{F} \tag{121}$$

We may view $\mathfrak{m} = \sigma/\tau$ as a function vanishing on $Y = \{\sigma = 0\}$.

4. Application to toric mutation

Now let us return to the local model of toric mutations introduced in Section 1.1.1 and further discussed in 1.3.1. We will adopt the constructions and notation established therein.

We will derived the wall-crossing formula (1) from the formalism developed in Section 2, specifically Corollary 3.5.

As recalled in Section 1.2, there is a mirror equivalence of perverse schobers $\mathcal{P}_A \simeq \mathcal{P}_B$, i.e. a canonically commutative diagram with vertical equivalences

$$\begin{tikzcd}
\mu Sh_L(\mathbb{C}^n) \arrow[r, "j^*"] \arrow[d, "\sim"] & \mu Sh_{L^\circ}(\mathbb{C}^n) \arrow[d, "\sim"] \\
\mathrm{Perf}_{prop}(P_{n-2}) \arrow[r, "i_*"] & \mathrm{Perf}_{prop}(\mathbb{T}_0^\vee)
\end{tikzcd} \tag{122}$$

First, let us extend $\mathcal{P}_A$ to a perverse schober on the cylinder by taking the additional monodromy functor to be the identity. Let us write $\mathcal{P}_{A,sm}$ for the perverse schober on the cylinder with the same generic structure but trivial vanishing category.

Then it is an easy exercise, whose proof we sketch, to deduce the following from Theorem 1.1.

Proposition 4.1. *For the Chekanov and Clifford skeleta $L_{\mathcal{C}h}, L_{C\ell} \subset M$, there are natural equivalences*

$$\mu Sh_{L_{\mathcal{C}h}}(M) \simeq \Gamma(Cyl, \mathcal{P}_A) \qquad \mu Sh_{L_{C\ell}}(M) \simeq \Gamma^\sharp(Cyl, \mathcal{P}_A) \tag{123}$$

Moreover, for the Chekanov and Clifford tori $T_{\mathcal{C}h} \subset L_{\mathcal{C}h}, T_{C\ell} \subset L_{C\ell}$, the above equivalences restrict to full subcategories to give

$$\mathcal{L}oc(T_{\mathcal{C}h}) \simeq \mu Sh_{T_{\mathcal{C}h}}(M) \simeq \Gamma(Cyl, \mathcal{P}_{A,sm}) \qquad \mathcal{L}oc(T_{C\ell}) \simeq \mu Sh_{T_{C\ell}}(M) \simeq \Gamma^\sharp(Cyl, \mathcal{P}_{A,sm})$$

Proof. Let us focus on the first equivalence of (123); the second is similar.

Observe that in a small neighborhood $N \subset \mathbb{C}$ around the circle $\gamma_{\mathcal{C}h}$, we have an isomorphism of pairs $(N, \Gamma_{\mathcal{C}h} \cap N) \simeq (B^*S^1, \Lambda \cap B^*S^1)$, where $B^*S^1 \subset T^*S^1$ is a small neighborhood of the zero-section $S^1 \subset T^*S^1$, and $\Lambda \subset T^*S^1$ is the union of the zero-section $S^1 \subset T^*S^1$ and a single conormal ray $T_p^+S^1 \subset T^*S^1$ based at a point $p \in S^1$. It is a standard exercise to see

$$\mu Sh_{\Lambda \cap B^*S^1}(B^*S^1) \simeq \mathrm{Perf}_{prop}(\mathbb{A}^1) \tag{124}$$

Typically one views objects of $\mathrm{Perf}_{prop}(\mathbb{A}^1)$ as objects $Y \in \mathrm{Perf}(pt)$ together with an endomorphism. Equivalently, we can view objects as quadruples (Y_1, Y_2, ∂, m) consisting of objects $Y_1, Y_2 \in \mathrm{Perf}(pt)$, a map $\partial : Y_2 \to Y_1$, and an isomorphism $m : Y_1 \xrightarrow{\sim} Y_2$. Namely, one uses m to identify Y_2 and Y_1 so that ∂ becomes an endomorphism. Note we can normalize the equivalence (124) so that restriction to the ray $T_p^+S^1$ corresponds to forming the shifted cone $\mathrm{Cone}(\partial)[-1]$.

Next, observe that we also have an isomorphism of pairs $(W^{-1}(N), L_{\mathcal{C}h} \cap W^{-1}(N)) \simeq (B^*S^1, \Lambda \cap B^*S^1) \times (T^*T^{van}_{\mathcal{C}h}, T^{van}_{\mathcal{C}h})$, and hence a natural equivalence

$$\mu Sh_{L_{\mathcal{C}h} \cap W^{-1}(N)}(W^{-1}(N)) \simeq \mathrm{Perf}_{prop}(\mathbb{A}^1) \otimes \mathcal{L}oc(T^{van}_{\mathcal{C}h}) \tag{125}$$

Thus to see the first equivalence of (123), one can apply Theorem 1.1 to see that $\mu Sh_{L_{\mathcal{C}h}}(M)$ classifies quadruples (X, Y_1, Y_2, Δ, m) as in the definition of $\Gamma(\mathrm{Cyl}, \mathcal{P}_A)$. Moreover, the asserted restricted equivalence is evident

$$\mathcal{L}oc(T_{\mathcal{C}h}) \simeq \mu Sh_{T_{\mathcal{C}h} \cap W^{-1}(N)}(W^{-1}(N)) \simeq \mathrm{Perf}_{prop}(\Gamma_m) \otimes \mathcal{L}oc(T^{van}_{\mathcal{C}h}) \simeq \mathcal{L}oc(T_{\mathcal{C}h}) \tag{126}$$

□

Remark 4.1. Following Remark 1.1, we could also consider $\mu Sh_{L_{crit}}(M)$ for the critical skeleton $L_{crit} \subset M$. It is again straightforward to construct a natural equivalence $\mu Sh_{L_{crit}}(M) \simeq \Gamma(\mathrm{Cyl}, \mathcal{P}_A)$ as a consequence of Theorem 1.1. Here it is less evident how to speak about the full subcategories corresponding to $\mu Sh_{T_{\mathcal{C}h}}(M), \mu Sh_{T_{C\ell}}(M)$.

Now let us consider framings for $\mathcal{P}_A$.

On the one hand, the circle $\gamma_{\mathcal{C}h}$ does not wind around $0 \in \mathbb{C}$, and so we have a canonical splitting $T^{van}_{\mathcal{C}h} \times \gamma_{\mathcal{C}h} \xrightarrow{\sim} T_{\mathcal{C}h}$, and hence an identification for the inertia stack

$$\mathcal{L}\mathcal{L}oc(T^{van}_{\mathcal{C}h}) \simeq \mathcal{L}oc(T^{van}_{\mathcal{C}h} \times S^1) \simeq \mathcal{L}oc(T_{\mathcal{C}h}) \tag{127}$$

On the other hand, the circle $\gamma_{C\ell}$ winds once around $0 \in \mathbb{C}$, and so a splitting $T^{van}_{C\ell} \times \gamma_{C\ell} \xrightarrow{\sim} T_{C\ell}$ provides a framing τ. The choice of a coordinate, say z_n, from among $z_1, \ldots, z_n$ gives such a splitting

$$\begin{tikzcd} T^{van}_{C\ell} \times \gamma_{C\ell} \arrow[r, "\sim"] & T_{C\ell} & ((z_1, \ldots, z_n), e^{i\theta}) \arrow[r, mapsto] & (z_1, \ldots, z_{n-1}, e^{i\theta} z_n) \end{tikzcd} \tag{128}$$

and thus a grading τ. The splitting in turn provides an identification for the inertia stack

$$\mathcal{L}\mathcal{L}oc(T^{van}_{C\ell}) \simeq \mathcal{L}oc(T^{van}_{C\ell} \times S^1) \simeq \mathcal{L}oc(T_{C\ell}) \tag{129}$$

Remark 4.2. Under the equivalence $\mathcal{P}_A \simeq \mathcal{P}_B$, the framing given by the coordinate z_n corresponds to the homogenous coordinate section $x_n : \mathcal{O} \to \mathcal{O}(1)$.

More fundamentally, the choice of the coordinate z_n provides an extension

$$\begin{tikzcd} \overline{W} : \mathbb{C}^{n-1} \times \mathbb{CP}^1 \arrow[r] & \mathbb{CP}^1 \end{tikzcd} \qquad \overline{W} = [W, z] \tag{130}$$

where we equip $\mathbb{C}^{n-1}$ with coordinates $z_1, \ldots, z_{n-1}$, and $\mathbb{CP}^1$ with coordinates $[z_n, z]$.

The circles $\gamma_{\mathcal{C}h}, \gamma_{C\ell} \subset \mathbb{C}$, with their counterclockwise orientations, are naturally isotopic when regarded within $\mathbb{CP}^1 \setminus \{0\} \simeq \mathbb{C}$, but with opposite orientations. We also obtain an identification $T^{van}_{\mathcal{C}h} \simeq T^{van}_{C\ell}$ by parallel transporting each within $\mathbb{C}^{n-1} \times \mathbb{CP}^1$ above the respective intervals $[\epsilon - \rho_{\mathcal{C}h}, \infty]$, $[-\infty, \epsilon - \rho_{C\ell}]$ to the fiber at $\infty \in \mathbb{CP}^1$.

Thus altogether, we have a canonical identification $T_{\mathcal{C}h} \simeq T_{C\ell}$ compatible with the above splittings, but notably with the inverse map on the factor S^1. Now with the above identifications, one can trace through the definitions to conclude the following.

Theorem 4.2. *For the perverse schober $\mathcal{P}_A$, the Chekanov and Clifford functors factor into the compositions*

$$F_{\mathcal{C}h} : \mathcal{LD}_{\spadesuit} \hookrightarrow \mathcal{L}\mathcal{L}oc(T^{van}_{\mathcal{C}h}) \simeq \mathcal{L}oc(T_{\mathcal{C}h}) \hookrightarrow \mu Sh_{L_{\mathcal{C}h}}(M) \simeq \Gamma(Cyl, \mathcal{P}_A|_{Cyl}) \tag{131}$$

$$F_{C\ell} : \mathcal{LD}_{\spadesuit} \hookrightarrow \mathcal{L}\mathcal{L}oc(T^{van}_{C\ell}) \simeq \mathcal{L}oc(T_{C\ell}) \hookrightarrow \mu Sh_{L_{C\ell}}(M) \simeq \Gamma(Cyl, \mathcal{P}_A|_{Cyl}) \tag{132}$$

Corollary 4.3. *The wall-crossing formula of Corollary 3.5 is the birational map on moduli of objects for the partially defined functor $\mathcal{L}oc(T_{\mathcal{C}h}) \to \mathcal{L}oc(T_{C\ell})$ given by comparing clean local systems on the Chekanov and Clifford tori as objects in $\Gamma(Cyl, \mathcal{P}_A)$ with coordinates related by the framing τ.*

Acknowledgements: I thank Vivek Shende for asking whether one could deduce the wall-crossing formula for toric mutations from the structure of the Landau-Ginzburg A-model of $\mathbb{C}^n$ with superpotential $W = z_1 \cdots z_n$. I thank AIM for hosting the recent workshop [2] at which Vivek first posed this question. I also thank Dmitry Tonkonog for explaining various aspects of his work and an anonymous referee for comments on the exposition.

I am grateful for the support of NSF grant DMS-1502178.

References

[1] R. Anno, and T. Logvinenko, *Spherical DG-functors*, arXiv:1309.5035.

[2] *Arborealization of singularities of Lagrangian skeleta*, AIM, March 26–30, 2018.

[3] D. Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol., **1**:51–91, 2007.

[4] M. Kapranov and V. Schechtman, *Perverse Schobers*, arXiv:1411.2772.

[5] D. Nadler, *A combinatorial calculation of the Landau-Ginzburg model $M = \mathbb{C}^3, W = z_1z_2z_3$*, arXiv:1507.08735.

[6] D. Nadler, *Mirror symmetry for the Landau-Ginzburg A-model $M = \mathbb{C}^n, W = z_1 \cdots z_n$*, arXiv:1601.02977.

[7] D. Nadler, *Wrapped microlocal sheaves on pairs of pants*, arXiv:1604.00114.

[8] J. Pascaleff and D. Tonkonog, *The wall-crossing formula and Lagrangian mutations*, arXiv:1711.03209.

[9] P. Seidel, *Lectures on categorical dynamics*, available at author's website, 2013.

Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720-3840
Email address: `nadler@math.berkeley.edu`