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Infinitely many absolutely exotic manifolds

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Abstract. We show that the smooth 4-manifold M obtained by attaching a 2-handle to B^4 along a certain knot $K \subset \partial B^4$ admits infinitely many absolutely exotic copies M_n, $n = 0, 1, 2..$, such that each copy M_n is obtained by attaching 2-handle to a fixed compact smooth contractible manifold W along the iterates $f^n(c)$ of a knot $c \subset \partial W$ by a diffeomorphism $f : \partial W \to \partial W$. This generalizes the example in author's 1991 paper, which corresponds to $n = 1$ case in [A1].

Dedicated to the memory of Michael Atiyah

1. Construction

A relative exotic structure on a compact smooth 4-manifold M with boundary, is a self diffeomorphism $f : \partial M \to \partial M$, which extends to a self homeomorphism of M, but does not extend to a self diffeomorphism of M. If $F : M \to M$ a homeomorphism extending f, then the pull-back smooth structure M_F provides a relative exotic copy of M. We say that N is an absolutely exotic copy of M, if it is homeomorphic but not diffeomorphic to M (no condition on the boundary). The technique introduced in [AR] turns relative exotic structures to absolute exotic structures. This is done by choosing an invertible cobordism H with $\partial H = H_- \sqcup H_+$ and $H_- \approx \partial M$, and then gluing H it to the boundary of M in two different ways. Then the manifolds $M' = M \cup_{Id} H$ and $M'' = M \cup_f H$ become absolutely exotic copies of each other. Applying this construction to a cork W produces an absolutely exotic copy of W; and when W is an infinite order loose-cork ([A2]) then we get infinitely many absolutely exotic copies of W. This construction results a boundary H_+, which consists of hyperbolic manifolds glued along tori.

Ideally we want to produce small 4-manifolds with simple boundaries, admitting absolutely exotic copies (with the hope of capping boundaries to get small closed exotic manifolds). One such example is the cusp C ([A3]) with a Seifert fibered space boundary, which is obtained by attaching a 2-handle to B^4 along the trefoil knot with 0-framing. Performing knot surgeries $C \rightsquigarrow C_K$ along the torus inside (by using different knots K) provides infinitely many absolutely exotic copies of C. An interesting open problem here is to find corks inside of C, such that twisting C along them induce C_K of C.

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Here we produce another example M, similar to C, which also has a Seifert fibered space boundary, and is obtained by attaching a 2-handle to B^4 along a slice knot with -2 framing. But from this we can construct infinitely many different exotic copies of M, each obtained by cork-twisting along an infinite order loose-cork $W \subset M$ ([A2], [G]), rather than a knot surgery to M, as in the case of C above. Here unlike [AR] we are producing (infinitely many) absolutely exotic copies of M without altering its boundary.

Theorem 1.1. *The manifold M of Figure 1, which is obtained by attaching 2-handle to B^4 along the knot K of Figure 1 with -2 framing, admits infinitely many distinct absolutely exotic copies, and they can be detected twisting an infinite order loose-cork inside of M.*

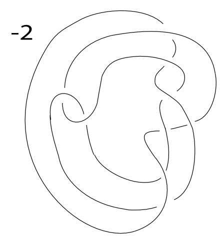

FIGURE 1. M

Proof. First of all by blowing up and down as in Figure 2, we see that ∂M can also be identified by $+2$ surgery to $(-4,2)$ twist knot (stevedores knot). ∂M is the small Seifert fibered space $M(-2;1/2,3/4,7/9)$ (e.g. [BW], [S], [T]), therefore its mapping class group is finite order.

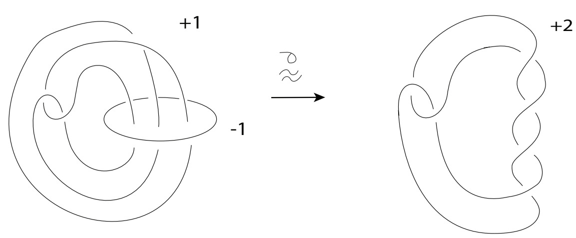

FIGURE 2. M

Recall the infinite order loose-cork W of [A2], which is shown in Figure 3 (where two of its alternative handlebody pictures are given). The order n diffeomorphism $f_n : \partial W \to \partial W$ is obtained by a delta move along the curve $\delta \subset \partial W$ (the dotted curve in the figure).

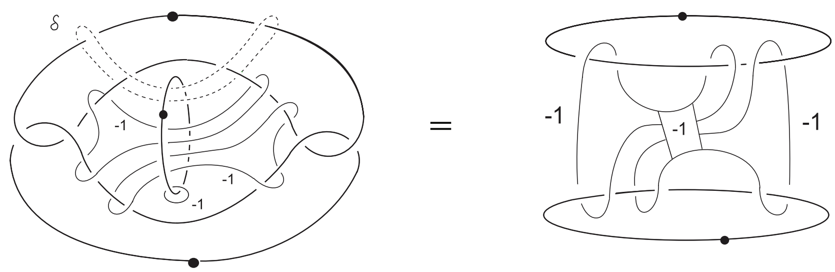

FIGURE 3. W

Now check that the handlebody pictures of Figure 4 describe the manifold M above (to see this cancel 1/2- handle pairs). The second picture of Figure 4 shows an imbedding $W \subset M$. That is, M is obtained from W by attaching a 2-handle along the knot c with 0-framing.

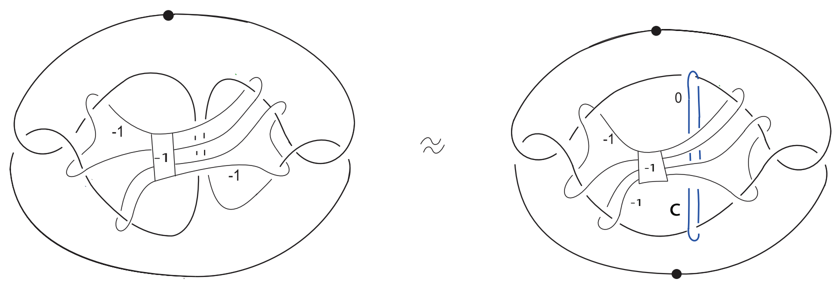

FIGURE 4. M

Now apply δ- moves to W inside M, n-times (where δ is chosen as in Figure 3), and call the resulting manifold M_n (which is the first picture of Figure 5). We claim $\{M_n\}$ are exotic copies of M rel boundary. To see this attach 2-handles to M_n along the knots a and b of the picture.

Call the manifold obtained from M_n by attaching 2-handles along a and b with -1 framings by $S_n = M_n + a^{-1} + b^{-1}$. Now we proceed as in [A2] by handle slides, to show that S_n is the manifold obtained from the Stein manifold S of Figure 7 by the knot surgery using the twist knot $(-2, -n)$. Furthermore we can compactify S into some closed symplectic manifold Z with $b_2^+(Z) > 1$ (by [LM], [AO], or [A4] p.108).

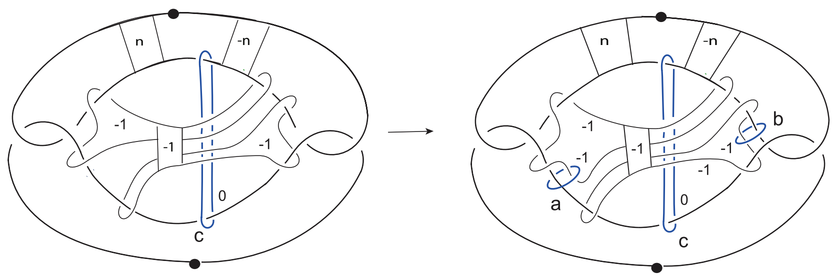

FIGURE 5. $M_n \rightsquigarrow S_n = M_n + a^{-1} + b^{-1}$

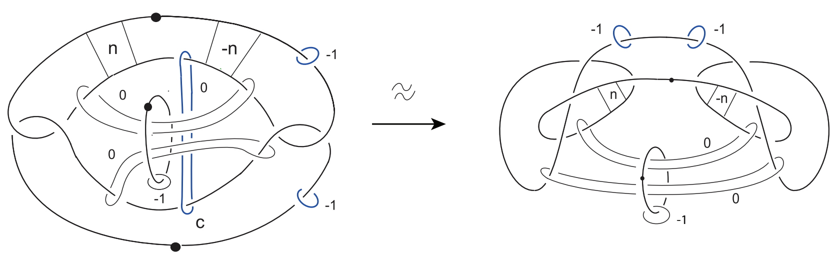

FIGURE 6. S_n

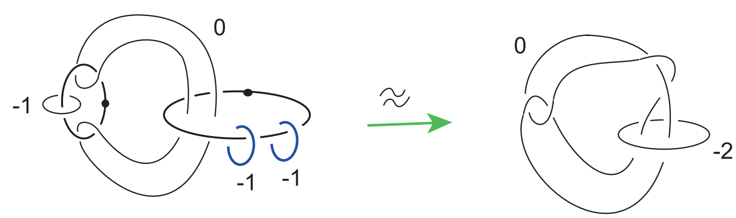

FIGURE 7. S

This shows that manifolds $\{M_n\}$ are exotic copies of M rel boundary, and they are obtained by iterating δ-moves to $f : \partial W \to \partial W$ inside $W \subset M$. Since the mapping class group of the Seifert fibered space $\partial M \approx M_n$ is finite [BO], by going to a subsequence we can assume that all $\{M_n\}$ are absolutely exotic copies of each other.

Now we analyze what an n-iterate of the δ- move does to M, well it turns it into M_n, and a close inspection shows that M_n is obtained from M by attaching 2-handle to W along the loop $f^n(c)$ with 0-framing, as shown in Figure 8. Recall also, performing the δ-move to W inside of M, has the affect of attaching a cancelling pair of 2/3-handles to M and performing the diffeomorphism described in [A1] resulting M_n.

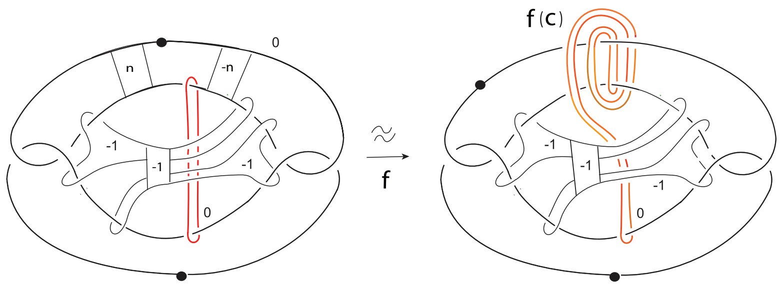

FIGURE 8. M_n

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