

Fredholm topology and enumerative geometry: reflections on some words of Michael Atiyah

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ABSTRACT. The first part of this article is a short and selective survey of developments in differential and algebraic geometry from the 1980's involving enumerative questions and nonlinear elliptic partial differential equations. In the second part we discuss possible extensions of these ideas to structures on 4-manifolds, in particular to complex structures on surfaces of general type.

Dedicated to the memory of Sir Michael Atiyah

1. Introduction

“What we have here is a nonlinear way to use the index theorem”.

It was my great good fortune to have many mathematical discussions with Michael Atiyah, and I learnt an immense amount from him. Much of what he said to me I only remember in general shape but in some cases I have more vivid memories and the quote above, uttered by Atiyah around 1982, is such an example. (There is another example at the beginning of Section 4 below). The first part of this article is a general discussion motivated by this quote, and survey of some developments over the past four decades. In the second part I will outline some new strands in these developments, involving moduli spaces of self-dual and complex structures. In addition to these brief reminiscences the influence of Atiyah's ideas, work and general approach to mathematics runs through the whole article.

Index theory can be seen as an extension of linear algebra to infinite dimensions. Thus if $T : H_1 \rightarrow H_2$ is a linear map between finite dimensional vector spaces we have

$$\dim \ker T - \dim \operatorname{coker} T = \dim H_1 - \dim H_2.$$

If H_1, H_2 are infinite dimensional Banach spaces the right hand side is not defined but if T is a *Fredholm* operator the kernel and cokernel are finite dimensional (by definition) and the left hand side is defined: the index of T . It is often useful to think, formally, of the index as the regularised version of the difference of the two infinite dimensions.

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In particular it is a deformation invariant on the space of Fredholm operators. This general theory is not difficult: the significance of the *Atiyah-Singer Index Theorem* is to connect with geometry; finding the index for elliptic differential operators over manifolds.

Differential topology is based on linear algebra. For example, if we have submanifolds P^p, Q^q of a manifold M^n in general position their intersection is a manifold of dimension $n - (p + q)$, just as happens in the linear case. Putting the two strands together, yields a “Fredholm differential topology” involving infinite dimensional manifolds and nonlinear maps between them with Fredholm derivatives. In the situations we consider the general picture is that we have a Banach manifold $\mathcal{B}$, a vector bundle with Banach space fibres $\mathcal{E} \rightarrow \mathcal{B}$ and a section σ which is locally represented by a nonlinear map with Fredholm derivative. (That is, around a point $b \in \mathcal{B}$ we choose a local coordinate chart and a local trivialisation of the bundle $\mathcal{E}$ so a section is represented by a map from this chart to the fibre $\mathcal{E}_b$ and we require that this be a smooth map with Fredholm derivative.) The index of the derivative is independent of the local representation and is formally the difference of the dimension of $\mathcal{B}$ and the fibre dimension of $\mathcal{E}$. If the section σ is transverse to the zero section then the zero set $Z_0 = \sigma^{-1}(0)$ is a manifold of dimension equal to the index. The crucial issue is the compactness of Z_0 . This is far from automatic since the ambient space $\mathcal{B}$ is not compact and, as we will see will typically fail in a straightforward sense in cases of interest. But if for the moment we assume the relevant compactness, then Z_0 carries a fundamental homology class $\zeta = [Z_0] \in H_\mu(\mathcal{B})$ where μ is the index and this class is a deformation invariant, independent of the choice of transverse section σ (for deformations through Fredholm sections, preserving compactness). We assume that ζ can be defined in homology with rational coefficients although that requires a discussion of orientations. The proof of deformation invariance is the same as in finite dimensions: the standard differential topological construction of the Poincaré dual of the Euler class of a vector bundle. So the upshot is that under suitable hypotheses there is a way to define what is formally the (homology) Euler class of the infinite dimensional bundle $\mathcal{E} \rightarrow \mathcal{B}$.

In some cases of interest the homology groups of the infinite dimensional ambient space $\mathcal{B}$ can be computed. In other cases one at least knows certain cohomology classes which can be paired with the homology class ζ to produce numerical invariants. That is, one has a graded ring R with a homomorphism $R \rightarrow H^*(\mathcal{B})$ and the pairing gives an element of degree μ in $\tilde{R} = \text{Hom}_{\mathbf{Q}}(R, \mathbf{Q})$.

These ideas, in the abstract, can be traced back a long way, certainly to the 1965 paper [32] of Smale and are related to the older Leray-Schauder degree theory. The developments which are our focus here, starting around 1980, involve the application of these ideas to nonlinear differential equations arising in geometry. Most of these developments fall into two broad lines.

- (1) Equations involving gauge fields, particularly over 4-manifolds. These include the Yang-Mills instanton equations (which were the context for Atiyah’s remark above) and the Seiberg-Witten equations. In the latter case the theory has been extended to more sophisticated topological constructions such as the Bauer-Furuta invariants in stable homotopy [5]. There are other equations such

as the Vafa-Witten equations which fit into the Fredholm framework but where compactness is only partially understood (with recent developments in work of Taubes [34]). Similarly for various analogues over manifolds of higher dimension, as discussed in [14]: the main case where compactness difficulties are resolved occurs in the “DT invariants” for Calabi-Yau 3-folds and at present the resolution has to pass through algebraic geometry [35].

- (2) Pseudoholomorphic curves in symplectic manifolds. This goes back to Gromov’s 1986 paper and has become an enormous field. There are again some analogous equations for calibrated submanifolds in higher dimensions where compactness is not adequately understood.

The situations in which the ideas sketched above can be usefully applied (even making optimistic conjectures regarding compactness) are comparatively rare and linked to “low dimensional phenomena”. There are many nonlinear equations in differential geometry with Fredholm linearisations—very often the linearisation is a variant of the Laplace operator. For one example we can take closed geodesics $\gamma : S^1 \rightarrow M$. But in most such cases there is no hope of achieving compactness of the solution set: there could be arbitrarily long geodesics. On the other hand there are many situations where some form of compactness holds but which involve *overdetermined* equations.

To illustrate this, consider deformations of a compact complex submanifold P in a fixed complex manifold M . Let $N_P = TM|_P/TP$ be the normal bundle— a holomorphic bundle over P . The linearised equation is given by the $\bar{\partial}$ -operator on sections of N and the solutions, which give the infinitesimal deformations of P , are the holomorphic sections $H^0(P; N_P)$. But it may not be possible to extend these to genuine deformations; there are potential obstructions in $H^1(P; N)$. In the case of curves, when $\dim_{\mathbb{C}} P = 1$, this fits perfectly into our Fredholm picture (as in item (2) above). The cohomology groups $H^0(P, N_P), H^1(P, N_P)$ are the kernel and cokernel of the linearised operator and the (real) index is twice the Euler characteristic $\mu = 2(\dim H^0 - \dim H^1)$. If we are in a transverse situation then H^1 vanishes and the space Z_0 of holomorphic curves near to P is a (real) manifold of dimension μ (and in fact a complex manifold). But it might happen that we are not in a transverse situation and Z_0 could have some very different structure: it could be singular or a manifold of dimension greater than μ . Then we can follow two paths, one differential geometric and one algebro-geometric. (One expects these to reach the same endpoint, although technically this may be highly non-trivial, in general.)

- Differential geometrically, we can perturb our equations in some way, for example by perturbing the complex structure on X to an almost-complex structure, so that the perturbed equation is transverse and we get a solution set Z of the perturbed equation which is a manifold of dimension μ .
- Algebro-geometrically, one can add additional structure to the original set Z_0 which enables one to define a *virtual fundamental class* in $H_{\mu}(Z_0)$. In the simplest case, when the potential obstructions do not actually occur and Z_0 is a manifold of real dimension $2\dim H^0(N)$, one considers the vector bundle over Z_0 formed by

the cohomology groups $H^1(P; N_P)$. The virtual fundamental class is the Poincaré dual of the Euler class of this bundle, which lies in $H_\mu(Z_0)$.

The picture is fundamentally different when P has dimension greater than 1. From the differential geometric point of view, the equations defining a complex submanifold become overdetermined and if we perturb the complex structure on M to a generic almost-complex structure we expect there to be no solutions. From an algebro-geometric point of view we still have a deformation theory, with infinitesimal deformations in $H^0(P, N_P)$ and potential obstructions in $H^1(P, N_P)$. There is a “Kuranishi model” of a neighbourhood in the space of submanifolds as the solutions of h_1 equations in h_0 complex variables where $h_i = \dim H^i$, so it might seem reasonable to think of the “expected” complex dimension of this neighbourhood as $h_0 - h_1$, as before. The difference is that there are now higher cohomology groups $H^i(P; N_P)$ for $i \geq 2$. The Euler characteristic $\sum (-1)^i h_i$ is a deformation invariant but, without some control of the higher cohomology, the difference $h_0 - h_1$ is not and the expected dimension could be different at different points of Z_0 .

For another example, consider the space V of conjugacy classes of irreducible representations in $SU(r)$ of the fundamental group $\pi = \pi_1(M)$ of a compact oriented manifold M . This set V has the structure of a real algebraic variety and clearly only depends on the group π . When M has dimension 3 there is a sense in which the “expected dimension” of V is 0, even though the actual dimension of V could be very different. This is the idea behind the Casson invariant (for homology 3-spheres) which “counts” the points in V . To fit this into our general framework (as was done by Taubes [33]), we consider V as the set of isomorphism irreducible flat connections on an $SU(r)$ bundle $E \rightarrow M$. The infinite dimensional space $\mathcal{B}$ is the quotient of the space of all connections by the gauge group $\text{Aut}(E)$. The curvature $F(A)$ of a connection A can be viewed as a cotangent vector in the space of connections and we get a Fredholm section σ of the cotangent bundle $T^*\mathcal{B} \rightarrow \mathcal{B}$ whose zero set is identified with V . Then the Casson invariant is one half the homology class we discussed above: i.e. half the number of zeros (counted with suitable signs) of a generic perturbation of σ . From another point of view, the deformation theory of a flat connection A over a manifold M can be discussed through the de Rham complex:

$$\Omega^0(\text{ad } E) \xrightarrow{d_A} \Omega^1(\text{ad } E) \xrightarrow{d_A} \Omega^2(\text{ad } E) \dots \quad (1)$$

Here $\text{ad } E$ is the bundle of Lie algebras associated to the adjoint representation and d_A is the coupled exterior derivative defined by A . The cohomology groups H_A^i of the complex are the cohomology groups of the manifold M in the local coefficient system $\text{ad } E$. If, as we are assuming, the connection is irreducible the group H_A^0 vanishes. The group H_A^1 corresponds to infinitesimal deformations of A and there are potential obstructions in H_A^2 . Poincaré duality gives a dual pairing between H_A^i and H_A^{n-i} so when M is a 3-manifold there are just two non-zero cohomology groups H_A^1, H_A^2 which are dual, and this fits in with the fact that the expected dimension of V is zero. For a higher dimensional manifold we are in the same position as in the previous example: there are higher dimensional cohomology groups $H_A^3, \dots$ and we don’t have a way to define an expected dimension of V .

There is a very similar discussion for holomorphic vector bundles over compact complex manifolds. The deformation theory of a holomorphic bundle $E \rightarrow X$ (with fixed determinant) is discussed through the cohomology groups $H^i(X, \text{End}_0 E)$, where $\text{End}_0 E$ is the bundle of trace-free endomorphisms. The group H^0 is related to the “stability” of the bundle and we can assume it is zero for this discussion. The groups H^1, H^2 give infinitesimal deformations and obstructions, as before. To have a virtual dimension we need control of the higher cohomology groups H^i which would typically come through vanishing: $H^i = 0$ for $i \geq 3$. This is automatic if X is a complex surface (and then the holomorphic bundles are intimately related to Yang-Mills instantons). If X is a Calabi-Yau 3-fold then Serre duality gives the vanishing of H^3 and the picture is closely analogous to the previous discussion of the Casson invariant: this is the starting point for the definition of “DT invariants” counting holomorphic bundles (or more generally sheaves) over X [35]. More generally, we get vanishing of H^3 for any 3-fold X with a nonzero section of K_X^{-1} . So, for example, moduli spaces of bundles over $X = \mathbf{CP}^3$ have an “expected dimension (which is also related to Yang-Mills instantons on S^4 , through twistor theory). But there is no obvious way to define an expected dimension for moduli spaces of bundles over $\mathbf{CP}^n$ for $n \geq 4$.

These examples illustrate the special and low-dimensional nature of these Fredholm and virtual fundamental class techniques. There are some situations in which these ideas can be applied in less direct ways.

- Furuta and Ohta define a Casson-like invariant through representations of the fundamental group of 4-manifolds by interpreting flat connections as Yang-Mills instantons [16].
- Borisov and Joyce [9] and Cao and Leung [10] define enumerative invariants for sheaves over Calab-Yau 4-folds by, from a differential geometer’s point of view, interpreting these within $\text{Spin}(7)$ -geometry.

2. Towards enumerative theories for structures on manifolds

In this section we discuss two other cases where these Fredholm/virtual fundamental class techniques might be applicable; different in nature from those considered above. We begin with some general background in Subsection 2.1 and then review the geometric setting for these two examples in Subsection 2.2.

2.1. Miller-Mumford-Morita classes

A general picture is that we want to consider some kind of structure on a compact oriented n -manifold M , which we can take to be given by a tensor field τ (for example a Riemannian metric). So we have an infinite dimensional space $\mathcal{X}$ of these tensors which is acted on by the group Diff of orientation preserving diffeomorphisms of M and we form the quotient space $\mathcal{B} = \mathcal{X}/\text{Diff}$. Suppose initially that the action is free, so that $\mathcal{B}$ is an infinite dimensional manifold. The space $\mathcal{X}$ is a principal Diff bundle over $\mathcal{B}$ and we have

a classifying map

$$g : \mathcal{B} \rightarrow B\text{Diff},$$

and $g^* : H^*(B\text{Diff}) \rightarrow H^*(\mathcal{B})$. In examples the assumption that Diff acts freely is too restrictive and we should weaken it to the assumption that stabilisers are finite. Then $\mathcal{B}$ is an infinite dimensional orbifold and if we use rational coefficients we still have a map g^* .

The cohomology of $B\text{Diff}$ might not be accessible but there is a standard way to produce classes in it. Let γ be a $p + n$ -dimensional rational characteristic class of $SL(n; \mathbf{R})$ (or equivalently $SO(n)$) with $p \geq n$. There is a universal bundle $\mathcal{U} \rightarrow B\text{Diff}$ with fibre M and a tangent bundle along the fibres $T_V \rightarrow \mathcal{U}$, so we get $\gamma(T_V) \in H^{p+n}(\mathcal{U})$ and integration over the fibre of $\mathcal{U} \rightarrow B\text{Diff}$ gives us a class in $H^p(B\text{Diff})$. Finally we pull-back by g to get $I(\gamma) \in H^p(\mathcal{B})$. We could define $I(\gamma)$ without going through $B\text{Diff}$ and the universal bundle by considering directly an orbifold bundle with fibre M over $\mathcal{B}$.

The classical example is when $n = 2$, for Riemannian metrics on an oriented surface. The characteristic classes in question are just polynomials in the Euler class, or first Chern class, $c_1 \in H^2$ and we get *Miller-Mumford-Morita classes* $I(c_1^{p+1}) \in H^{2p}(\mathcal{B})$. In other words, if we form a graded ring

$$\mathcal{R} = \mathbf{Q}[\sigma_1, \sigma_2, \dots]$$

freely generated by objects σ_p in degree $2p - 2$ then we have a homomorphism

$$\Gamma : \mathcal{R} \rightarrow H^*(\mathcal{B}),$$

taking σ_p to $I(c_1^{p+1})$. This extends in the obvious way to define a ring $\mathcal{R}_{SO(n)}$ in each dimension n taking characteristic classes which are polynomials in the Pontryagin classes and, for even n , the Euler class. There have been many recent advances in geometric topology, in the understanding of these generalised Miller-Mumford-Morita (MMM) classes and $H^*(B\text{Diff})$, see the survey [17] for example. (These classes are also called “tautological classes” in the literature.)

There are parallel constructions using K -theory. Suppose that n is even, the manifold M is a spin manifold and we have a spin structure on the vertical tangent bundle T_v . Take a Euclidean metric on T_v so the fibres of $\mathcal{U} \rightarrow B\text{Diff}$ become Riemannian spin manifolds. Let ρ be a representation of $\text{Spin}(n)$. We get an associated bundle over the fibres and a coupled Dirac operator D_ρ . The index of the family defines a class $\text{ind} D_\rho$ in $K(B\text{Diff})$ and the Atiyah-Singer theory gives a formula for the Chern character $\text{ch}(\text{ind} D_\rho) \in H^*(B\text{Diff})$ in terms of the MMM classes. Conversely we can express all the MMM classes in terms of the $\text{ind} D_\rho$. (Note: For questions involving rational cohomology the spin assumption is not important because the D_ρ exist for at least half of the representations ρ , which suffice to generate all the MMM classes. Also in the construction above we should strictly restrict to compact subsets of $B\text{Diff}$, but this also will not be important for us.)

There is an extension of this discussion in the case when the tensors τ we are considering give a reduction of the structure group of TM to some subgroup $G \subset SL(n, \mathbf{R})$ which is not homotopy equivalent to $SO(n)$. Then we can start with other characteristic classes

of G to get a ring $\mathcal{R}_G$ which maps to $H^*(\mathcal{B})$. Another variant is when we have as part of our structure a symplectic form ω on the manifold M which we take as fixed and reduce the symmetry group to the symplectic diffeomorphisms $\text{SDiff}(M, \omega) \subset \text{Diff}$.

With this background in place we come to the main point. Suppose that we have a suitable differential equation for structures τ . That is, we have some vector bundle $\mathcal{E} \rightarrow \mathcal{B}$, a Fredholm section σ of index μ , as considered in the previous section, and our equation is $\sigma(\tau) = 0$. Then, modulo compactness and orientation questions, we can fit into that general framework and define an element of degree μ in the dual $\tilde{\mathcal{R}}_{SO(n)}$ of the ring $\mathcal{R}_{SO(n)}$ (or $\tilde{\mathcal{R}}_G$ for a structure group G).

In the classical case, when τ is a Riemannian metric on a surface of genus $g \geq 2$ we take $\sigma(\tau) = K_\tau + 1$ where K_τ is the Gauss curvature. So the solutions of our equation are metrics of curvature -1 , the index is $6g - 6$ and the zero set $Z \subset \mathcal{B}$ is the moduli space $\mathcal{M}_g$. By the uniformisation theorem we achieve the same end by taking τ to be an almost-complex structure, with no equation, and in that set-up we just have $\mathcal{B} = \mathcal{M}_g$, so $\mathcal{B}$ is finite-dimensional. Since $\mathcal{X}$ is contractible the map g^* is an isomorphism on rational cohomology. There is a huge literature on $H^*(\mathcal{M}_g)$, the MMM classes and the integrals of MMM classes over $\mathcal{M}_g$. On the K -theory side, the irreducible representations of $Spin(2) = S^1$ are labelled by an integer r . Taking for simplicity $r = 1 - 2q$ for integers $q \geq 1$, the corresponding virtual bundle over $\mathcal{M}_g$ is the vector bundle which assigns to each complex structure the vector space $H^0(M, K_M^q)$.

2.2. Two structures in real dimension 4

Before going further let us emphasise that, in the context of possible enumerative theories, the material we are discussing below is largely speculative and on a very different footing from the well-established theories we reviewed in Section 1.

- Our first example is *self-dual conformal structures*, for which a basic reference is the paper [4] of Atiyah, Hitchin and Singer. Recall that if (M, g) is an oriented Riemannian 4-manifold the Weyl tensor decomposes as $W = W_+ \oplus W_-$ where $W_\pm \in s_0^2(\Lambda_\pm^2)$, the trace-free symmetric 2-tensors on the 3-dimensional $\Lambda_\pm^2$. The Weyl curvature is conformally invariant and a conformal structure with $W_- = 0$ is called self-dual. In our set-up we take $\mathcal{X}$ to be the space of conformal structures τ and the section to be that induced by W_- , so the zero set Z_0 is the moduli space of self-dual conformal structures modulo diffeomorphism. This is a Fredholm section of index

$$\text{index} = \frac{1}{2}(29\sigma(M) - 15e(M)), \quad (2)$$

where $\sigma(M), e(M)$ are the signature and Euler characteristic. The deformation theory is worked out in [21]. There is an elliptic complex

$$\Gamma(TM) \rightarrow \Gamma(s_0^2 T^* M) \rightarrow \Gamma(s_0^2 \Lambda_-^2) \quad (3)$$

where the first term consists of vector fields on M (the Lie algebra of Diff), the second consists of deformations of a conformal structure (the tangent space of $\mathcal{X}$) and the third is the space where W_- lives *a priori* (the fibre of the bundle $\mathcal{E}$).

The MMM classes in this situation lie in dimensions 0 modulo 4 so for a manifold M with $e = 3\sigma \pmod{8}$ one might hope to define pairings with a virtual fundamental class.

- The second example is *complex surfaces*. Ignoring for the moment some fundamental difficulties, we consider a compact oriented 4-manifold M and the space $\mathcal{X}$ of almost-complex structures τ . The section σ of a bundle over $\mathcal{B}$ is induced by the Nijenhuis tensor $N(\tau)$ of an almost complex structure, which lies in $\Lambda^{0,2}(T)$ and this is a Fredholm section of (real) index

$$\mu = 2\chi(TM) = 2(10\chi(S) - 2c_1^2(S)). \quad (4)$$

Here we are writing $\chi(S)$ for the holomorphic Euler characteristic which is equal, by the Riemann-Roch formula, to $(c_1^2 + c_2)/12$. Using standard formulae we can also write $\mu = -(3e + 7\sigma)$.

As a set, the zeros of our section correspond to equivalence classes of complex structures on M , which we would like to call the “moduli space” of complex structures. The fundamental difficulty we encounter is the well-known one that this is not in general a good object: the natural topology on the set may not be Hausdorff. We will come back to this in the next section.

For a general complex manifold X the Kodaira-Spencer-Kuranishi theory describes the “versal deformation” in terms of the cohomology groups $H^p(TX)$. As in the examples from the previous section, the special feature of complex dimensions 1, 2 are reflected in the fact that there are no higher cohomology groups H^i for $i \geq 3$.

In this situation the structure group is $GL(2, \mathbf{C})$ and the characteristic classes are generated by c_1, c_2 . The relevant ring is

$$\mathcal{R}_{U(2)} = \mathbf{Q}[\sigma_{pq}],$$

with generators σ_{pq} of degree $2(p+2q-2)$ corresponding to $c_1^p c_2^q$. One might—in suitable situations—hope to define pairings of $\mathcal{R}_{U(2)}$ with a virtual fundamental class. (Even for moderate values of the index μ this would give a large collection of numbers. For example, the dimension of the degree 8 part of $\mathcal{R}_{U(2)}$ is 30.)

It has to be said that there is not the same clearcut motivation for developing such enumerative theories as there is in other situations, with the construction of 4-manifold and symplectic topology invariants. It is also not so easy to find natural deformations of the equations (although we will encounter something on those lines in the next section). From the point of view of this article we could say that the motivation is that, as we have explained, the situations where this kind of nonlinear Fredholm theory is possibly relevant are comparatively rare and special, hence precious, and one wants to understand them as far as possible.

3. Compactification

“You need to be careful compactifying moduli spaces: people spend their lives doing that”.

This is our second quote from Michael Atiyah. The context was that we were discussing the “Uhlenbeck compactification” of moduli spaces of instantons. For the immediate purposes then, given the fundamental analytical results of Uhlenbeck, this was quite straightforward to define. But Atiyah’s remark holds true in the sense that understanding in detail the structure of the Uhlenbeck compactification is crucial in establishing deep properties of the instanton invariants of 4-manifolds such as Witten’s conjecture on the relation with Seiberg-Witten invariants, as in the work of Feehan and Leness [15], and this is something which is still not fully understood.

In the context of this article Atiyah’s remark points to the core of the matter. The zero set Z_0 of our Fredholm section will usually not be compact so does not carry a homology class and the evaluation of cohomology classes has no clear meaning. What we would like to do is to compactify the zero set in such a way that the relevant cohomology classes and the deformation theory extend over the compactification.

There is not much known about compactification of moduli spaces of self-dual conformal structures. If we have a sequence of such structures τ_i on a fixed smooth 4-manifold M we would like to identify some kind of geometric limit of a subsequence. If we suppose that within each conformal class there are metrics g_i with constant scalar curvature and bounded Sobolev constant then there are results of Tian and Viaclovsky [36]. But one can say that the construction of a rigorous general theory seems, at best, very far off. There are some explicit examples known of connected components of moduli spaces and we mention two.

Example 1

For the manifold $M = \mathbf{CP}^2 \sharp \mathbf{CP}^2$ there is a component of the moduli space constructed by Poon [30] which is an open interval $(0, 1)$. (The index here is -1 but the structures have 2-torus symmetry which modifies the “expected dimension” discussion, so in this context the dimension is the expected one.) The natural compactification is the closed interval $\overline{Z} = [0, 1]$. The geometric meaning is that the point $0 \in \overline{Z}$ can be thought of as corresponding to a wedge $\mathbf{CP}^2 \vee \mathbf{CP}^2$ of two copies of $\mathbf{CP}^2$ with its Fubini-Study metric, joined at a point. The convergence to this singular limit is realised by a sequence of conformal structures on the connected sum with the “neck” shrinking to zero size. The other end point 1 is similar. There is a well-known Eguchi-Hanson metric on the tangent bundle of the 2-sphere, which is self-dual and asymptotically locally Euclidean.

The conformal 1-point compactification of this gives a compact self dual orbifold T with one singular point and the end-point 1 corresponds to the wedge $T \vee T$ of two copies glued at their singular points. The convergence is realised by shrinking the neck of a “connected sum” of these orbifolds. (See the discussion in [13].)

Example 2

Let M be the $K3$ manifold with the opposite of its standard orientation. The Calabi-Yau metrics on M define self-dual conformal structures and the Torelli theorem for $K3$ surfaces shows that a connected component of the moduli space has an explicit description U/Γ . Here U is a certain dense open set in the negative Grassmannian $Gr^-(19, 3)$ of negative 3-dimensional subspaces in the indefinite space $\mathbf{R}^{19,3}$ and Γ is a subgroup of the isometry group of the $K3$ lattice $\Lambda_{K3} \subset \mathbf{R}^{3,19}$. The complement of U in the Grassmannian is a union of explicit codimension-3 sets and these correspond to structures with mild orbifold singularities. As a first step towards a compactification we can add these points to get a moduli space $Z_0 = Gr^-(19, 3)/\Gamma$.

The index formula (2) gives a virtual dimension 52 whereas the dimension of $Gr^-(19, 3)$ is $3 \cdot 19 = 57$. The difference is accounted for by the fact that the linearised operator has a cokernel: i.e. the cohomology H^2 of the deformation complex (3) is nonzero. The Calabi-Yau metric induces a flat connection on the bundle $s_0^2 \Lambda_-^2$ and the space H^2 can be identified with the 5-dimensional space of parallel sections. This gives an explicit description of the obstruction bundle over Z_0 as the quotient by Γ of the rank 5 bundle $s_0^2(V)$, where V is the tautological $\mathbf{R}^3$ bundle over the Grassmannian. So a generic perturbation of the moduli space corresponds to the zero set $Z \subset Z_0$ of a generic section of $s_0^2(V)$. To develop this further we would have to consider a suitable compactification of the moduli space Z_0 , which is an important topic for other purposes, but there are some other interesting points that arise.

Recall that the indefinite orthogonal group $O(19, 3)$ has four connected components, corresponding to the action on the orientations of positive and negative subspaces. So we have two homomorphisms $o_+, o_- : O(19, 3) \rightarrow \{\pm 1\}$. It is known that the group Γ is the intersection of $O(\Lambda_{K3})$ with the kernel of o_- : the diffeomorphisms of M preserve the orientation of the negative subspace and this is a fundamental phenomenon in smooth 4-manifold theory [12]. On the other hand the homomorphism o_+ is non-trivial on Γ : the reflections by classes of square 2 are realised by generalised Dehn twists. These observations show that the bundle $s_0^2(V)$ over Z_0 is orientable but the space Z_0 is not, so we cannot define a virtual fundamental class in rational homology even ignoring compactness. But we will suggest a variant of the set-up which gets around this.

In both the $SU(2)$ -instanton and Seiberg-Witten theories the orientation of moduli spaces is governed by the orientation of the positive and negative parts of the second cohomology of the underlying 4-manifold. Consider pairs (τ, A) consisting of a self-dual conformal structure and an anti-self-dual connection A of Chern class k .

The discussion above shows that, at least in this case of the K3 manifold, the moduli space of these pairs *is* orientable. It has virtual dimension $8k - 60 + 52 = 8k - 8$ (which is divisible by 4). One can also deform to an interesting coupled system. We assume for simplicity that we have chosen some way to fix a metric within each conformal class and consider the equations for pairs (τ, A) :

$$F_A^+ = 0 \quad W^-(\tau) = \epsilon F_A^- * F_A^-,$$

where ϵ is a real parameter and $*$ denotes the quadratic map combining the Killing form on the Lie algebra with $\Lambda_-^2 \otimes \Lambda_-^2 \rightarrow s_0^2(\Lambda_-^2)$. It seems likely that for generic ϵ the moduli space of solutions of this coupled system is an orbifold of dimension $8k - 8$. There is a similar discussion for the Seiberg-Witten case. The constructions are related to the Seiberg-Witten invariants for families [28], [26]. It would be interesting to pursue a general study of these orientation questions for self-dual structures.

Another obvious issue in this K3 discussion is that the rank of the obstruction bundle $s_0^2 V$ is odd and in a standard situation the Euler class of a bundle of odd rank vanishes in rational cohomology. However it is possible that the right treatment of the compactification could allow non-trivial pairings.

4. Surfaces of general type

We now turn to the main technical topic of this article, considering complex structures on 4-manifolds defining surfaces of “general type”, which are the analogues of complex curves of genus two or more. There is a huge literature about these and in particular there is a well-developed Kollár, Shepherd-Barron, Alexeev (KSBA) theory of moduli space compactification.

To set up the basic picture differential geometrically we can start with a compact symplectic 4-manifold (X, ω) . The symplectic structure defines a first Chern class $c_1 \in H^2(X)$ and we assume that $c_1 = -[\omega]$. Consider the space $\mathcal{X}$ of almost-complex structures on X , algebraically compatible with ω , and let $\mathcal{B}$ be the quotient by the symplectic diffeomorphism group. A compatible almost-complex structure defines a Riemannian metric $g(J, \omega)$ on X and this means that there is no difficulty in forming the quotient space. One can define a “Hermitian scalar curvature” $S(J, \omega)$ which reduces to the ordinary scalar curvature when J is integrable and which has the property that

$$\int_X (S(J, \omega) + 4) \omega^2 = 0.$$

We consider the section σ of a bundle over $\mathcal{X}$ which corresponds to the pair of tensors $(N(J), S(J, \omega) + 4)$, where $N(J)$ is the Nijenhuis tensor. Thus a zero of σ gives a Kähler structure on X in the class $-c_1$ with scalar curvature -4 and a standard integral identity then shows that this metric is in fact a *Kähler-Einstein* metric, with Ricci curvature $-g(J, \omega)$. Thus the space $Z_0 \subset \mathcal{B}$ is the moduli space of Kähler-Einstein structures. This is a Fredholm section of index μ where μ is given by the same formula (4).

If, for simplicity, we assume that $H^1(X, \mathbf{R}) = 0$ the relevant deformation complex is

$$C_0^\infty \rightarrow \Gamma(s_{\mathbf{C}}^2 T) \rightarrow \Gamma(\Lambda^{0,2} \otimes T) \oplus C_0^\infty, \quad (5)$$

where C_0^∞ denotes real valued functions of integral zero and $s_{\mathbf{C}}^2 T$ is the symmetric square of the tangent bundle, regarded as a complex vector bundle using J .

A complex surface which admits a “negative” Kähler-Einstein metric, $\text{Ricci} = -g$, is of general type and the converse is almost true. Let Y be a smooth complex surface of general type. It might be that the canonical bundle K_Y is ample and in that case Y admits a unique Kähler-Einstein metric by the theorem of Aubin and Yau so we can take $X = Y$ above. But in the theory it is best to consider moduli spaces of all structures with fixed numerical invariants (c_1^2, χ) , so there could be *different* underlying symplectic 4-manifolds (X, ω) (although no example is known). If K_Y is not ample there is a *canonical model* obtained by contracting all -1 and -2 curves. This is an orbifold with ADE singularities and carries a unique orbifold Kähler-Einstein metric by an extension of the Aubin-Yau theorem [22]. In short, before compactification, we should strictly run the differential geometric discussion above with possibly a finite number of different symplectic 4-manifolds or orbifolds, but we will not go into this further.

The KSBA theory produces a compactified moduli space $\overline{\mathcal{M}}_{a,b}$ of surfaces of general type with $c_1^2 = a, \chi = b$. (The original references are [1], [23] and there is a helpful exposition in [18].) It is analogous to the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ of curves of genus $g \geq 2$. Berman and Guenancia show that the singular surfaces represented in the compactification are precisely those which admit Kähler-Einstein metrics, with a suitable technical definition of what that means in the singular case [8]. This is analogous to the hyperbolic geometry description of the Deligne-Mumford space, but the metrics are not necessarily complete and the picture is much more complicated. There are many interesting questions about the asymptotics of these metrics. In any case, while in the future it may be possible to proceed in a more differential geometrical fashion, we will now switch to a purely algebro-geometric point of view. Our discussion is based on the following foundational premise.

PREMISE

- (1) *The virtual fundamental class theory of Behrend and Fantechi [7] and Li and Tian [25], or some variant of that, can be used to define a class $\zeta \in H_\mu(\overline{\mathcal{M}}_{a,b}, \mathbf{Q})$ where $\mu = \mu(a, b) = 20b - 4a$.*
- (2) *The MMM classes extend to $H^*(\overline{\mathcal{M}}_{a,b}, \mathbf{Q})$.*

The author does not have the expert knowledge required to write a useful discussion of this premise and is certainly not suggesting that it must be true: the main point here is to raise the questions. The author’s impression is that this statements should be true at least for moduli spaces in which the surfaces involved are not too badly singular, as in the example we study in Section 5 below. In any event, assuming—for now—the premise, we immediately get the existence of elements $\rho_{a,b} \in \check{\mathcal{R}}_{U(2)}$ for each (a, b) such

that $\mu(a, b) \geq 0$, which are the main point of this article. (Note added in proof. Yunfeng Jiang informs the author that he has some results in this direction.)

Remarks

- (1) One does not have to go far to encounter cases where the virtual moduli space dimension is different from the actual dimension. For example, for a smooth surface S of degree $d \geq 5$ in $\mathbf{CP}^3 = \mathbf{P}(U)$ standard exact sequences show that $H^2(TS)$ is isomorphic to the cokernel of the natural map

$$s^{d-4}(U) \rightarrow s^{d-5}(U) \otimes U$$

which has dimension $(d-2)(d-3)(d-5)/2$ and is non-trivial if $d \geq 6$. The actual dimension of the moduli space is

$$\frac{(d+1)(d+2)(d+3)}{6} - 16,$$

while the virtual dimension is less by the dimension of $H^2(TS)$. Catanese [11] and Manetti [27] have shown that the moduli spaces $\mathcal{M}_{a,b}$ can have a large number of components of different dimensions.

- (2) There is a well-known “geography” of surfaces of general type, with a region $\mathcal{S}$ in the (a, b) -plane outside which the moduli spaces $\mathcal{M}_{a,b}$ are empty. (See [6], VII.9, for example.) The line $\mu(a, b) = 0$ cuts through the middle of this region $\mathcal{S}$. For many interesting surfaces $\mu(a, b) < 0$ and the virtual class theory evaporates. From the differential geometric point of view one can ask the question, when does a given symplectic 4-manifold admit a Kähler structure? One might try to develop some analytical scheme to prove existence. But it is harder to imagine how such a scheme could find these surfaces with $\mu < 0$, since at the index level the solutions “ought not to exist”.
- (3) The obstruction space for deformations of a compact complex manifold V of any dimension is always $H^2(TV)$. A special feature of complex dimension 2 is that this space has a direct geometric interpretation: the Serre dual is then $H^0(T^*V \otimes K_V)$ and elements of this give singular holomorphic foliations of the surface V . This is analogous to the description, in complex dimension 1, of the cotangent bundle of $\mathcal{M}_g$ in terms of quadratic differentials.
- (4) In regard to the second item in the “Premise” the two-dimensional MMM classes exist in $H^2(\overline{\mathcal{M}}_{a,b}; \mathbf{Q})$ and are studied in the literature.

In terms of line bundles, there are Knudsen-Mumford line bundles $\mathcal{L}_0, \mathcal{L}_2 \rightarrow \overline{\mathcal{M}}_{a,b}$ such that:

$$\det \pi_*(K^p) = \mathcal{L}_2^{N_2(p)} \otimes \mathcal{L}_0^{N_0(p)} \quad (6)$$

where, writing $\underline{p} = p - 1/2$,

$$N_2(p) = \frac{p^3}{3} - \frac{p}{12} \quad , \quad N_0(p) = -2\underline{p} \quad (7)$$

The left hand side in (6) refers to the relative canonical bundle of the family $\mathcal{U} \rightarrow \overline{\mathcal{M}}_{a,b}$. Note that for a general ample line bundle L over a surface the Knudsen-Mumford theory gives four lines $\mathcal{L}_3, \mathcal{L}_2, \mathcal{L}_1, \mathcal{L}_0$ such that

$$\det \pi_*(L^p) = \mathcal{L}_3^{n_3(p)} \mathcal{L}_2^{n_2(p)} \mathcal{L}_1^{n_1(p)} \mathcal{L}_0^{n_0(p)},$$

where $n_i(p) = \binom{p}{i}$, but for the canonical bundle Serre duality implies that $\mathcal{L}_3 = \mathcal{L}_2^2, \mathcal{L}_1 = \mathcal{L}_0^{-2}$ and we get the expressions (6),(7).

To be more precise, these are all orbifold or $\mathbf{Q}$ -line bundles, due to the possible presence of finite automorphism groups. The Grothendieck-Riemann-Roch theorem shows that

$$c_1(\mathcal{L}_2) = -\frac{1}{2}I(c_1^3) \quad , \quad c_1(\mathcal{L}_0) = 24I(c_1c_2) \quad (8)$$

The line bundle $\mathcal{L}_3 = \mathcal{L}_2^2$ is known as the CM line bundle. Patakfalvi and Xu [29] show that it is an ample line bundle over $\overline{\mathcal{M}}_{a,b}$, and it follows the MMM class $I(c_1^3)$ is non-zero in $H^2(\overline{\mathcal{M}}_{a,b}; \mathbf{Q})$.

In the opposite direction, Randall-Williams shows that on a moduli space of *smooth* hypersurfaces in projective space all MMM classes are trivial in rational cohomology [31]. That is, we need to go to a compactification to see any interesting topology, from this point of view.

- (5) For sufficiently large p , the direct images $\pi_*(K^p)$ are vector bundles over the moduli space, not just virtual bundles. As in other moduli problems (see [3] p. 582 for example), this gives restrictions on their Chern characters and should lead to relations between the MMM classes.
- (6) Let S be a smooth surface of general type and Π be a 2-dimensional subspace of $H^0(K_S^q)$. If q is sufficiently large and Π is generic this defines a Lefschetz pencil on S : the curves in the linear system have at worst ordinary double points. This linear system gives a map

$$\Gamma : \mathbf{CP}^1 = \mathbf{P}(\Pi) \rightarrow \overline{\mathcal{M}}_g,$$

where $2g - 2 = q(q + 1)c_1^2(S)$. It seems possible that this construction could be developed to give some connections between the putative enumerative theory of surfaces and curve-counting invariants in the Deligne-Mumford spaces $\overline{\mathcal{M}}_g$. (See the discussion in 5.4 below.)

- (7) To gain some insight into the obstruction spaces for surface deformations we consider smoothings of normal crossings. Let S_1, S_2 be surfaces containing curves C_1, C_2 with normal bundles N_1, N_2 . If C_1 and C_2 are isomorphic, say $C_1 = C_2 = C$, we form a singular surface $\Sigma = S_1 \cup_C S_2$ and study the smoothings of this. The infinitesimal deformations of the singularity correspond to sections of the line bundle $N_1 \otimes N_2$ over C . We focus on the case when S_1, S_2 are cubic surfaces in $\mathbf{P}^3$ and C_i are in the linear system $|\mathcal{O}(3)|$ on S_i . In the end the smoothings we construct will be sextic surfaces and from another point of view

what we are studying is the degeneration of smooth sextic surfaces to a union of two cubics. The general picture follows the same pattern as the “gluing” techniques which have been employed in all the Fredholm theories discussed in Section 1 (and there are important recent developments in understanding the behaviour of Kähler-Einstein metrics in such situations [19]). The moduli space of cubic surfaces has dimension 4 so the moduli space of pairs (S, C) of a cubic surface and curve in $|\mathcal{O}(3)|$ has dimension 22. We need to study the matching problem for two pairs $(S_1, C_1), (S_2, C_2)$ to have isomorphic curves C_1, C_2 . The curves C_i have genus 10 and the moduli space $\mathcal{M}_{10}$ has dimension 27 so our first guess is that the space of matching pairs has dimension $22 + 22 - 27 = 17$. This will be true if the natural maps between the various moduli spaces have suitable transversality properties. The normal bundles N_i have degree 27 and it follows from Riemann-Roch that, given a matching pair, the dimension of $H^0(N_1 \otimes N_2)$ is 45. So our first guess is that the moduli space of the smoothed surface has dimension $45 + 17 = 62$ and this is indeed the *virtual* dimension of the moduli space of sextics, as in item (1) above, but not the actual dimension which is 68.

The explanation for this deviation from the first-guess dimension count is that the space of genus 10 curves which enter in the discussion is a lower-dimensional subset $\mathcal{M}'_{10}$ of $\mathcal{M}_{10}$: the general curve of genus 10 cannot be embedded as the intersection of two cubic surfaces in $\mathbf{P}^3$. In fact if $C = S_1 \cap S_2$ where S_i are cubic surfaces then $K_C = \mathcal{O}(2)$ in other words $\mathcal{O}(1) = K_C^{1/2}$ is a spin structure on C . At this point we can refer to the article [2] of Atiyah which explains that the complex geometry of spin structures on complex curves can be understood through the theory of skew-adjoint Fredholm operators. (An operator T on a complex Hilbert space is skew adjoint if $\langle x, Ty \rangle$ is a skew symmetric complex bilinear form.) The $\bar{\partial}$ -operator on $K_C^{1/2}$ becomes skew-adjoint with respect to standard Hermitian structures. There are 2^{20} spin structures on a curve of genus 10 but for the purposes of this discussion, which only involves small deformations, we can suppose that we have a distinguished one. Then the condition for a curve C to lie in $\mathcal{M}'_{10}$ is that $H^0(K_C^{1/2})$ has dimension 4, while for generic curves it will have dimension 0. Consider the abstract situation of the space $SFred$ of skew-adjoint Fredholm operators on a Hilbert space and the subset $SFred_4 \subset SFred$ of operators with kernel of dimension 4. For $T \in SFred_4$ let K_T be the 4-dimensional kernel. Then one finds that normal bundle of $SFred_4$ in $SFred$ has a 6-dimensional fibre $\Lambda^2 K_T$ at T . (This is a straightforward extension of the obvious case when the Hilbert space has dimension 4.) In this way one finds that $\mathcal{M}'_{10}$ has codimension 6 in $\mathcal{M}_{10}$ and the normal bundle is identified with $\Lambda^2 H^0(K_C^{1/2})$.

The deformation theory of the singular surface Σ yields a space $\mathbf{T}^1(\Sigma)$ of infinitesimal deformations and an obstruction space $\mathbf{T}^2(\Sigma)$. There is an exact sequence

$$\dots H^0(N_1 \otimes N_2) \rightarrow \mathbf{T}^1(\Sigma) \rightarrow \mathbf{T}^1(S_1, C_1) \oplus \mathbf{T}^1(S_2, C_2) \rightarrow H^1(C; TC) \rightarrow \mathbf{T}^2(\Sigma) \dots$$

where $\mathbf{T}^1(S_i, C_i)$ is the space of infinitesimal deformations of the pair (S_i, C_i) . The term $H^1(C; TC)$ is the tangent space of $\mathcal{M}_{10}$ at C but the incoming map in the sequence maps to the codimension 6 subspace $T\mathcal{M}'_{10}$ and from the discussion above we get a map from $\Lambda^2 \mathbf{C}^4$ to $\mathbf{T}^2(\Sigma)$ which is in fact an isomorphism, fitting in with what we saw in item (1) above. If we ran the whole discussion with S_1 a cubic surface but S_2 a quadric the corresponding curve C has genus 4 and $K_C = \mathcal{O}(1)$. The canonical system of a general curve of genus 4 embeds the curve in $\mathbf{P}^3$ as the intersection of a cubic and a quadric so our first-guess dimension count is correct in this case, fitting in with the fact that $H^2(TS)$ vanishes for a smooth quintic surface.

We see from this that the appearance of the obstruction spaces for surface deformations is bound up with the special properties of curves on surfaces stemming from the fact that curves are also divisors. This is well-known in the enumerative geometry of curves on surfaces. For a surface with $p_g > 0$ (*i.e.* $b^+ > 1$) “most” curves appear in families of the wrong dimension and do not contribute to the Gromov invariants. Our situation is different because a cubic surface has $p_g = 0$. Start with our 4-dimensional family of complex structures on the smooth 4-manifold underlying a cubic surface and perturb this slightly to a 4-dimensional family of almost-complex structures. Then we still have a 22-dimensional family of pairs (S, C) but now we expect that the matching problem will behave in a generic way and that the space of matching pairs $(S_1, C_1), (S_2, C_2)$ will have dimension 17, rather than 23 as occurs in the integrable case. (Strictly we should pass to real dimensions here, since the moduli spaces will not have complex structures.)

5. Study of an example

For most surfaces that one can construct easily the moduli spaces have very large dimension. Imposing symmetry by a finite group allows us to cut down the dimension to get manageable spaces but still exhibiting some essential features of the situation. In this Section we will discuss one example of this kind and compute the enumerative invariants (modulo some foundational assumptions).

Remark The significance of the numerical invariants $a = c_1^2, b = \chi$ of a surface is that they define the Hilbert polynomial for $\dim H^0(K_S^p)$. Fix a finite group Γ and consider surfaces of general type with a Γ action. The spaces $H^0(K_S^p)$ are then representations of Γ so the Hilbert polynomial extends to a “ Γ -Hilbert function”, taking values in the representation ring of Γ which is the generalisation of the pair (a, b) .

5.1. A family of sextic surfaces

Take coordinates (x_1, y_1, x_2, y_2) on $\mathbf{C}^4$ and let ζ be a primitive sixth root of unity. Let G be the subgroup of $GL(4, \mathbf{C})$ generated by:

$$\begin{aligned} (x_1, y_1, x_2, y_2) &\mapsto (\zeta x_1, \zeta^{-1} y_1, x_2, y_2) \\ (x_1, y_1, x_2, y_2) &\mapsto (x_1, y_1, \zeta x_2, \zeta^{-1} y_2) \\ (x_1, y_1, x_2, y_2) &\mapsto (x_2, y_2, x_1, y_1). \end{aligned}$$

Let V be the vector space of polynomials of degree 6 invariant under G . These have the form

$$\alpha x_1^6 + \beta y_1^6 + \alpha x_2^6 + \beta y_2^6 + A Q_+^3 + B Q_+ Q_-^2, \quad (9)$$

where $Q_{\pm} = x_1 y_1 \pm x_2 y_2$. The $\mathbf{C}^*$ -action on $\mathbf{C}^4$

$$(x_1, y_1, x_2, y_2) \mapsto (\lambda x_1, \lambda^{-1} y_1, \lambda x_2, \lambda^{-1} y_2),$$

induces an action on V :

$$(\alpha, \beta, A, B) \mapsto (\lambda^6 \alpha, \lambda^{-6} \beta, A, B).$$

The stable points in V for the torus action are those where α and β are non-zero so each stable orbit in V contains a representative

$$\alpha(x_1^6 + y_1^6 + x_2^6 + y_2^6) + A Q_+^3 + B Q_+ Q_-^2,$$

which is unique up to change in the sign of α . Thus we get a moduli space M of “GIT stable” sextic surfaces with this G -action which is the quotient of $\mathbf{C}^2$ by ± 1 , where a point (A, B) in $\mathbf{C}^2$ corresponds to the surface

$$S_{AB} = \{x_1^6 + y_1^6 + x_2^6 + y_2^6 + A Q_+^3 + B Q_+ Q_-^2 = 0\} \subset \mathbf{CP}^3.$$

The locus $\Delta \subset \mathbf{C}^2$ of points (A, B) where S_{AB} is singular has four components. Write $F = (3A - B)$ and for $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ write $G_1 = 3A + 3B + 5\epsilon_1$, $G_2 = 3A + 3B + 5\epsilon_2$. Then for each of these 4 choices of signs there is a component of Δ with equation

$$(F^2 - G_1 G_2)^2 = 4F^2(G_1 - F)(G_2 - F).$$

The singularities that arise are mild and will not enter further in our discussion.

The reason for choosing this example is that the virtual dimension of M is not equal to the actual dimension. In general if we have a surface S with the action of a finite group Γ the deformation theory, for surfaces with Γ -action, works in the obvious way. The group Γ acts on $H^i(TS)$, infinitesimal deformations are given by the invariant part $H^1(TS)^\Gamma$ and obstructions in $H^2(TS)^\Gamma$. We go back to amplify the description of the obstruction spaces for hypersurfaces we mentioned in Section 3, for the case of sextic surfaces. On $\mathbf{P}^3 = \mathbf{P}(U)$ we have the dual Euler sequence

$$T^* \mathbf{P}^3(1) \rightarrow \underline{U}^* \rightarrow \mathcal{O}(1),$$

and taking tensor product with $\mathcal{O}(1)$ gives

$$T^* \mathbf{P}^3(2) \rightarrow \underline{U}^*(1) \rightarrow \mathcal{O}(2).$$

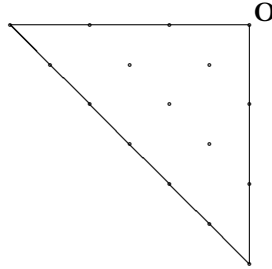
Taking sections we get

$$H^0(T^*\mathbf{P}^3(2)) \rightarrow U^* \otimes U^* \rightarrow s^2(U^*),$$

which shows that $H^0(T^*\mathbf{P}^3(2))$ is canonically isomorphic to $\Lambda^2 U^*$. Now for a smooth sextic surface $S \subset \mathbf{P}^3$ cut out by a section s of $\mathcal{O}(6)$ we have a restriction map $T^*\mathbf{P}^3|_S \rightarrow T^*S$ and an isomorphism of line bundles $K_S = \mathcal{O}(2)$. The second isomorphism depends, up to a factor, on the choice of s and a volume form $\Lambda^4 U = \mathbf{C}$. Combining these ingredients, we get a map $r : \Lambda^2 U^* \rightarrow H^0(T^*S \otimes K_S)$ which one readily sees is an isomorphism. Now suppose that a group Γ acts on V , preserving S . If Γ acts with determinant 1 on V and if Γ preserves the section s cutting out S then it follows that r is a Γ -equivariant map, for the standard actions on source and target. This holds in our situation, with $\Gamma = G$, and we conclude that the dual of the obstruction space for our surfaces with G -action is the G -invariant part of $\Lambda^2 U^*$. One finds that this is 1-dimensional, spanned by $\omega = dx_1 dy_1 + dx_2 dy_2$. So the virtual (complex) dimension of M is 1.

There is an obvious “naive” compactification $\overline{M}_{GIT}$ of M , which is to take $\mathbf{C}^2 \subset \mathbf{CP}^2$ and the quotient of $\mathbf{CP}^2$ by ± 1 . This is the Geometric Invariant theory compactification obtained by adding “polystable” points. The points at infinity in $\overline{M}_{GIT}$ correspond to solutions of the equation $AQ_+^3 + BQ_+Q_-^2 = 0$. If $A, B \neq 0$ these form a union of three quadrics meeting in four lines; if $B = 0$ we get a quadric $\{Q_+ = 0\}$ with multiplicity 3 and if $A = 0$ a union of one quadric $\{Q_+ = 0\}$ and another $\{Q_- = 0\}$ with multiplicity 2. None of these objects is allowed in the KSBA compactification.

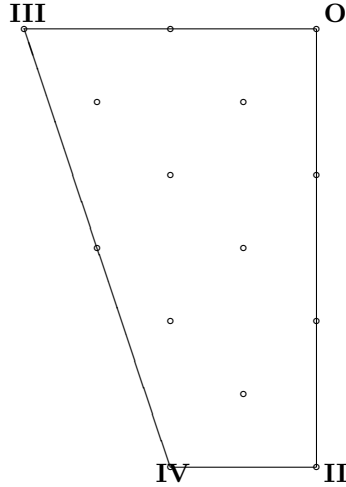
The naive compactification $\overline{M}_{GIT}$ is a toric surface and has a toric description (P, Λ) where $\Lambda \subset \mathbf{Z}^2$ is the lattice of pairs $(n, m) \in \mathbf{Z}^2$ with $n + m$ even and P is the triangle with vertices $(0, 0), (-6, 0), (0, -6)$. (Of course, the complex structure only determines the “fan” of normals to the edges of P and we could scale P by a factor.) This toric structure is partly accidental—the torus does not act on the family of surfaces parametrised by the moduli space.



5.2. The KSBA compactification

In this subsection we will describe another compactification $\overline{M}$ of the moduli space M . Before going to the general picture we begin by discussing the 1-parameter family of surfaces $S_{A,0}$ which is more straightforward. The basic point is that the correct limit when $A \rightarrow \infty$ is a triple branched cover of the quadric $Q_+ = 0$. Let $\pi : Y \rightarrow \mathbf{CP}^3$ be the cyclic triple cover of $\mathbf{CP}^3$ branched over the smooth surface $S_{0,0}$. If s is the section of $\mathcal{O}(6)$ over $\mathbf{CP}^3$ cutting out $S_{0,0}$ there is by construction a section η of $\pi^*(\mathcal{O}(2))$ over Z with $\eta^3 = s$. We also have another section Q_+ of $\pi^*(\mathcal{O}(2))$. Let W be the hypersurface in $Z \times \mathbf{P}^1$ defined by the equation written $\eta = \lambda Q_+$ in terms of an affine coordinate λ on $\mathbf{P}^1$. So we have a projection $W \rightarrow \mathbf{P}^1$ and the fibre over a finite point λ can be identified with $S_{A,0}$ where $A = \lambda^3$. The fibre over infinity is a smooth surface S_{III} : the triple cover of the quadric $Q_+ = 0$ branched over the intersection with $S_{0,0}$. The fact that we have to take the cube root $\lambda = A^{1/3}$ to construct the family is the usual orbifold phenomenon: the $\mathbf{Z}/3$ -action of the triple cover means that S_{III} has a larger automorphism group than the generic surface $S_{A,0}$.

Our compactification $\overline{M}$ is also a toric variety. Let Π be the quadrilateral with vertices $0 = (0, 0), II = (0, -6), III = (-4, 0), IV = (-2, -6)$ and $\Lambda \subset \mathbf{Z}^2$ be the same lattice as above. Then we define $\overline{M}$ to be polarised toric variety corresponding to (Π, Λ) (we discuss the polarisation later). To see this as a moduli space we want to take a toric chart for each vertex, construct a corresponding family of surfaces and glue these together over the intersections of the charts. For brevity we will do this in full for just two charts corresponding to the vertices $0, IV$. The first we already have: it is the family of surfaces $S_{A,B}$ parametrised by $\mathbf{C}^2/\pm 1$ and the centre of the chart corresponds to the surface $S_O = S_{0,0}$.



Before going on to the vertex IV we describe the surfaces corresponding to the rest of the boundary of the quadrilateral. Recall that a smooth quadric in $\mathbf{CP}^3$ can be identified with $\mathbf{P}^1 \times \mathbf{P}^1$ and take standard affine co-ordinates (s, t) on the latter, so we have four lines $s = 0, s = \infty, t = 0, t = \infty$. This configuration of four lines will appear often in what follows and we will denote it by Λ . Let $C_\mu \subset \Sigma$ be the curve in the linear system $\mathcal{O}(6, 6)$ over $\mathbf{P}^1 \times \mathbf{P}^1$ with affine equation

$$1 + s^6 + t^6 + s^6 t^6 + \mu s^3 t^3 = 0 \quad (10)$$

- The segment from O to III corresponds to the surfaces S_{A0} , and the segment from O to II to the S_{0B} .
- The open segment from III to IV corresponds to triple covers of $\mathbf{P}^1 \times \mathbf{P}^1$ with simple branching over the curves $C_\mu, C_{-\mu}$, for $\mu \in \mathbf{C} \setminus \{0\}$. The limit as $\mu \rightarrow 0$ is the surface S_{III} we discussed above: a $\mathbf{Z}/3$ -cyclic cover branched over C_0 .
- The segment from II to IV , including the end-point II but not IV , corresponds to the following family of surfaces. For $\mu \in \mathbf{C}$ take the double cover of $\mathbf{P}^1 \times \mathbf{P}^1$ branched over the divisor in $\mathcal{O}(8, 8)$ given by the union of C_μ and the four lines Λ . So Λ lifts to the double cover. Each line meets C_μ in 6 points and these give ordinary double points in the branched cover. Blow up these 24 points and then collapse the proper transform of Λ to a point. Taking $\mu = 0$ gives the surface S_{II} corresponding to the vertex II .

To study the vertex IV we let $\mathbf{P}_w^5$ be the weighted projective space with homogeneous coordinates $(x_1, y_1, x_2, y_2, h_+, h_-)$ and weights 1 on the first four coordinates and 2 on the last two. We consider $\mathbf{P}^3$ as embedded in $\mathbf{P}_w^5$ in the obvious way and write $\mathbf{P}_h^1$ for the line $\{(0, 0, 0, 0, h_+, h_-)\}$ in $\mathbf{P}_w^5$. For parameters $\alpha, \beta \in \mathbf{C}$, let $S^{\alpha, \beta}$ be the complete intersection in $\mathbf{P}_w^5$ defined by the equations:

$$x_1^6 + y_1^6 + x_2^6 + y_2^6 + h_+^3 + h_+ h_-^2 = 0,$$

$$x_1 y_1 = \alpha h_+ + \beta h_-,$$

$$x_2 y_2 = \alpha h_+ - \beta h_-.$$

If α, β are both nonzero we can write $h_+ = (x_1 y_1 + x_2 y_2)/2\alpha$ and $h_- = (x_1 y_1 - x_2 y_2)/2\beta$ and, substituting into the first equation, we get

$$8(x_1^6 + y_1^6 + x_2^6 + y_2^6) + \alpha^{-3}(x_1 y_1 + x_2 y_2)^3 + \alpha^{-1} \beta^{-2}(x_1 y_1 + x_2 y_2)(x_1 y_1 - x_2 y_2)^2 = 0.$$

The surface $S^{\alpha, \beta}$ does not meet the line $\mathbf{P}_h^1$ in $\mathbf{P}_w^5$ and projection from this line maps $S^{\alpha, \beta}$ to $S_{A, B}$ in $\mathbf{CP}^3$ where

$$8A = \alpha^{-3} \quad 8B = \alpha^{-1} \beta^{-2} \quad (11)$$

We define S_{IV} to be the surface $S^{0, 0}$. Let $\mathbf{P}_w^4$ be the weighted projective space with co-ordinates (x_i, y_j, h_-) . The projection from S_{IV} to $\mathbf{P}_w^4$ is well defined and the image

is a cone $\text{Cone}(\Lambda)$ over the configuration Λ of 4-lines in $\mathbf{P}^3$. The projection exhibits S_{IV} as a triple branched cover of $\text{Cone}(\Lambda)$. The surface $S^{0,0}$ meets $\mathbf{P}_h^1$ in three points $h_+ = 0, h_+ = h_-, h_+ = -h_-$ and the triple cover maps these three points to the vertex of the cone.

Next consider the case when α is zero but β is not. Then the equations give $x_1y_1 + x_2y_2 = 0$ and the projection from $S^{0,\beta}$ to $\mathbf{P}^3$ has image this quadric surface. Take affine co-ordinates $x_1 = s, y_1 = t, x_2 = st, y_2 = 1$ on this quadric, as before. We have $h_- = \beta^{-1}x_1y_1 = \beta^{-1}st$ and our surface is defined by the equation

$$(1 + s^6 + t^6 + s^6t^6) + h_+^3 + \beta^{-2}s^2t^2h_+ = 0.$$

The projection to the quadric exhibits $S^{0,\beta}$ as a triple cover with branch locus

$$(1 + s^6 + t^6 + s^6t^6) \pm \frac{2}{3\beta^3\sqrt{-3}}s^3t^3 = 0,$$

which agrees with our previous discussion of the boundary segment from *III* to *IV*.

The case $\beta = 0$ is similar but a little more complicated. The equations give $x_1y_1 - x_2y_2 = 0$ defining another smooth quadric in $\mathbf{CP}^3$ containing the same line configuration Λ . We parametrise this quadric surface by $x_1 = s, y_1 = t, x_2 = st, y_2 = -1$. The surface $S^{\alpha,0}$ meets the line $\mathbf{P}_h^1$ in the point $h_+ = 0$ and the projection from $S^{\alpha,0}$ to $\mathbf{P}^3$ is not defined there. Blowing up this point we get a surface $\tilde{S}^{\alpha,0}$ which maps to $\mathbf{P}^3$ with image the quadric. Writing $h_+ = \alpha^{-1}x_1y_1$ this blown up surface $\tilde{S}^{\alpha,0}$ is defined by the equation

$$(1 + s^6 + t^6 + s^6t^6) + \alpha^{-3}s^3t^3 + \alpha^{-1}sth_-^2 = 0$$

If $s = 0$ and $t^6 = -1$ the co-ordinate h_- is unconstrained and we get a line in $\tilde{S}^{\alpha,0}$; similarly for $s = \infty, t = 0, \infty$. Collapsing these 24 lines in $\tilde{S}^{\alpha,0}$ gives the double cover of the quadric branched over the $\mathcal{O}(8,8)$ divisor

$$st((1 + s^6 + t^6 + s^6t^6) + \alpha^{-3}s^3t^3) = 0.$$

which agrees with the previous discussion for the boundary segment from *II* to *IV*.

From another point of view, there is a well-known toric degeneration of the quadric $\mathbf{P}^1 \times \mathbf{P}^1$ to the cone $\text{Cone}(\Lambda)$. We start with S_{IV} , the triple cover of the cone, and deform this to triple covers of $\mathbf{P}^1 \times \mathbf{P}^1$ to get the surfaces $S^{0,\beta}$, which are smooth for small β . Similarly for the $S^{\alpha,0}$ but with the extra complication due to the point $h_+ = 0$ in $\mathbf{P}_h^1$ which is not smoothed by the deformation and remains a singular point in $S^{\alpha,0}$.

We can make similar constructions in toric charts corresponding to the vertices *II*, *III* but the key formula for constructing the toric moduli space $\overline{M}$ is (11). First, we see that only β^2 appears, so we write $\beta^2 = \gamma$ and we have $A^{-1} = \alpha^3, B^{-1} = \alpha\gamma$. Recall that $(-A, -B)$ defines the same point in the moduli space as (A, B) . Thus changing the sign of α does not change the surface $S^{\alpha,\beta}$. Consider a monomial $A^{-p}B^{-q}$ on $(\mathbf{C}^*)^2$ which is equal to $\alpha^{p+3q}\gamma^q$. For this to descend to a well-defined function on the moduli space (i.e. invariant under change of sign of α) we need $p+3q$ to be even, so $p+q$ is even. For the monomial to extend holomorphically over $\alpha = \gamma = 0$ we need $3p+q \geq 0, q \geq 0$.

So the holomorphic functions on a neighbourhood of the point corresponding to S_{IV} in the moduli space have a basis given by the intersection of our lattice $\Lambda \subset \mathbf{Z}^2$ with the convex set $\{(p, q) \in \mathbf{R}^2 : 3p + q \geq 0, q \geq 0\}$. Standard toric theory shows that the compact space $\overline{M}$ is defined by the polytope given by intersecting further with a set $\{(p, q) : p \leq C_1, q \leq C_2\}$ for any fixed $C_1, C_2 > 0$. We have taken $C_1 = 2, C_2 = 6$ and then translated the quadrilateral so that the origin is at the vertex O .

The author has a strong feeling that this moduli space $\overline{M}$ is the KSBA moduli space (for these surfaces with G -action) but he is not qualified to certify that as a definite fact. In any case we will proceed with our study based on that assumption.

Remarks

1. We can make the same constructions in $\mathbf{P}^9$ using the canonical embeddings of our surfaces $S_{A,B}$, but then we are in high codimension with many equations and variables which do not play any real role. The advantage of the weighted projective space is that it allows us to bring in just the two sections of K_S which are really relevant. It seems likely to the author that the moduli space we are constructing is the Chow stable moduli space, under the canonical embedding.

2. Starting with the GIT compactification $\overline{M}_{GIT}$ we can get the compactification $\overline{M}$ by performing weighted blow-ups at the two points corresponding to the vertices $(-6, 0), (0, -6)$ and contracting the proper transform of the line at infinity. For GIT moduli spaces there are techniques of Jeffrey and Kirwan [20] which, in favourable cases, can be used to calculate pairings of the kind we are concerned with, so the comparison of the different compactifications is a relevant topic. Laza investigates this comparison, for another moduli problem, in [24].

5.3. Calculations in cohomology

The moduli space $\overline{M}$ has virtual complex dimension 1 so there is a virtual fundamental class $\zeta \in H_2(\overline{M})$. We have two MMM classes, associated to the characteristic classes c_1^3 and $c_1 c_2$. The goal of this subsection is to calculate the pairing of ζ with these two classes. By standard toric theory $H^2(\overline{M}; \mathbf{Q})$ is two dimensional. Each edge of the quadrilateral Π corresponds to a 2-sphere in $\overline{M}$ and so defines a homology class. Let D_{II}, D_{III} be the 2-spheres corresponding to the edges from O to III and O to II respectively and use the same symbols for their homology classes. These give a basis for $H_2(\overline{M}, \mathbf{Q})$ in which we will do our calculations. We begin by calculating the 2-dimensional MMM classes.

Consider first a general situation. Let Y be a Calabi-Yau 3-fold and $L \rightarrow Y$ an ample line bundle. Serre duality implies that the Hilbert polynomial of Y is odd, say:

$$\dim H^0(L^p) = H(p) = \alpha p^3 + \beta p, \quad (12)$$

for sufficiently large p (in fact $p \geq 1$). Suppose that s_0, s_1 are sections of L defining a Lefschetz pencil on Y . Then we have a subvariety $W \subset Y \times \mathbf{P}^1$ cut out by the section of $L \otimes \mathcal{O}(1)$ written in an affine coordinate on $\mathbf{P}^1$ as $s_0 - \lambda s_1$. Write $\pi : W \rightarrow \mathbf{P}^1$ for the projection and K_v for the relative canonical bundle. The adjunction formula gives

$K_v = L \otimes \mathcal{O}(1)$. For large enough p we have a vector bundle $V_p = \pi_*(K_v^p)$ over $\mathbf{P}^1$ of rank r_p and degree d_p . We want to find d_p in terms of the data α, β . We could apply the Riemann-Roch theorem for families but in this situation there is a more elementary direct argument. If V is a vector bundle over $\mathbf{P}^1$ of rank r and degree d then for large enough q we have

$$\dim H^0(\mathbf{P}^1, V \otimes \mathcal{O}(q)) = qr + (d + r).$$

Applied to V_p we get

$$\dim H^0(W, L^p \otimes \mathcal{O}(p + q)) = qr_p + (d_p + r_p),$$

for large enough p, q . The restriction sequence for $W \subset M \times \mathbf{P}^1$ gives

$$\dim H^0(W, L^p \otimes \mathcal{O}(p + q)) = H(p)(p + q + 1) - H(p - 1)(p + q),$$

and, comparing the two formulae, we get

$$d_p = pH(p) - (p - 1)H(p - 1).$$

Writing $\underline{p} = p - 1/2$ as before, this is

$$d_p = \alpha(4\underline{p}^3 + \underline{p}) + 2\beta\underline{p}. \quad (13)$$

Comparing with the formulae (6),(7) for the Knudsen-Mumford line bundles $\mathcal{L}_0, \mathcal{L}_2$ we see that

$$\langle c_1(\mathcal{L}_0), [\mathbf{P}^1] \rangle = -(\alpha + \beta), \quad \langle c_1(\mathcal{L}_2), [\mathbf{P}^1] \rangle = 12\alpha. \quad (14)$$

To apply this in our situation we begin with the 2-sphere $D_{III} \subset \overline{M}$. The family that we discussed at the beginning of subsection 6.2 above can be embedded in weighted projective space $\mathbf{P}_w^4$. In our co-ordinates $(x_1, y_1, x_2, y_2, h_+)$ we let Y be the degree 6 hypersurface defined by the equation

$$x_1^6 + y_1^6 + x_2^6 + y_2^6 + h_+^3 = 0.$$

The adjunction formula in $\mathbf{P}_w^4$ shows that Y is a Calabi-Yau 3-fold. We take the line bundle $\mathcal{O}(2)$ over Y and consider the pencil $h_+ - \lambda(x_1y_1 + x_2y_2)$, so for finite λ the fibre of $\pi : W \rightarrow \mathbf{P}^1$ is the surface $S_{A,0}$ with $A = \lambda^3$ and the fibre over ∞ is S_{III} . To find the Hilbert polynomial $H(p)$ for this pair (Y, L) let n_p be the dimension of the space of homogeneous polynomials of degree $2p$ on $\mathbf{C}^4$, so

$$n_p = \frac{1}{6}(2p + 1)(2p + 2)(2p + 3).$$

Then if we write $N_p = \dim H^0(\mathbf{P}_w^4, \mathcal{O}(2p))$ we have

$$N_p = n_p + n_{p-1} + \dots + n_0,$$

whereas the restriction sequence for $Y \subset \mathbf{P}_w^4$ gives, for large enough p , $H(p) = N(p) - N(p - 3)$. So we conclude that $H(p) = n_p + n_{p-1} + n_{p-2}$ and this gives

$$H(p) = 4p^3 + 7p. \quad (15)$$

Since the true parameter on D_{III} is $A^2 = \lambda^6$ we get a factor of $1/6$ in the formulae, and we arrive at

$$\langle c_1(\mathcal{L}_0), [D_{III}] \rangle = -11/6 \quad , \quad \langle c_1(\mathcal{L}_2), [D_{III}] \rangle = 8. \quad (16)$$

The discussion for D_{II} is very similar. This time we define a degree 6 hypersurface $Y' \subset \mathbf{P}_w^4$ by the equation

$$x_1^6 + y_1^6 + x_2^6 + y_2^6 + (x_1 y_1 + x_2 y_2) h_-^2 = 0,$$

and consider the pencil $(x_1 y_1 - x_2 y_2) - \lambda h_-$. The only difference is that Y' is singular, with a singular point at $P_\infty = (0, 0, 0, 0, 1)$ but the singularity does not affect the calculations. The true parameter on D_{II} is $B^2 = \lambda^4$ so we get

$$\langle c_1(\mathcal{L}_0), [D_{II}] \rangle = -11/4 \quad , \quad \langle c_1(\mathcal{L}_2), [D_{II}] \rangle = 12. \quad (17)$$

and we see that $c_1(\mathcal{L}_0) = -(11/48)c_1(\mathcal{L}_2)$ in $H^2(\overline{M}, \mathbf{Q})$.

The quadrilateral Π has been chosen to correspond to the polarisation $\Omega = c_1(\mathcal{L}_2)/4 \in H^2(\overline{M})$. By general toric theory

$$\langle \Omega^2, [\overline{M}] \rangle = 2 \text{Area}_\Lambda(\Pi),$$

where Area_Λ means the area relative to the lattice, which is half the standard area. So we see that $\langle \Omega^2, [\overline{M}] \rangle = 18$ and hence

$$\langle c_1(\mathcal{L}_2)^2, [\overline{M}] \rangle = 16.18 = 288. \quad (18)$$

We now consider the virtual fundamental class $\zeta \in H_2(\overline{M}; \mathbf{Q})$. Due to the foundational gap expressed in our “Premise” we do not have an official definition of this, but we calculate in what seems the appropriate way. Recall that the standard symplectic form ω on $\mathbf{C}^4$ defines a section s_ω of $T^*\mathbf{P}^3(2)$. This section has no zeros so defines a rank-2 sub-bundle $E \subset T^*\mathbf{P}^3$. (This is a well-known object: it is the “null correlation bundle” which corresponds via twistor theory to the standard Yang-Mills instanton on S^4 .) The sub-bundle E is a holomorphic contact structure on $\mathbf{CP}^3$ so there is no surface $\Sigma \subset \mathbf{CP}^3$ (even locally) such that the restriction of s_ω to $T^*\Sigma(2)$ vanishes. We lift s_ω by the projection $\mathbf{P}_w^4 \setminus P_\infty \rightarrow \mathbf{P}^3$ to define a section s'_ω of $T^*\mathbf{P}_w^4(2)$ away from the point P_∞ . Now consider the degree 6 hypersurface $Y \subset \mathbf{P}_w^4$ and pencil in $\mathcal{O}(2)$ defining a family $\pi : W \rightarrow \mathbf{P}^1$ as above, with $W \subset Y' \times \mathbf{P}^1 \subset \mathbf{P}_w^4 \times \mathbf{P}^1$. Lifting the section s'_ω to the product and restricting to W we finally get a section s''_ω of $T^*W \otimes K_v \otimes \mathcal{O}(-1)$. Let $\Delta \subset \mathbf{C} \subset \mathbf{P}^1$ be the finite set corresponding to singular surfaces in the pencil. We have a line bundle $\mathcal{L} \rightarrow \mathbf{C} \setminus \Delta$ with fibres the G -invariant part of the obstruction spaces $H^2(TS)$ and a dual line bundle $\mathcal{L}^*$ with fibres the G -invariant part of $H^0(T^*S \otimes K_S)$. The section s''_ω defines by restriction to fibres a non-vanishing section of $\mathcal{L}^*(-1)$. On the smooth part of a singular fibre or on the fibre over $\lambda = \infty$ the section restricts to a finite and non-vanishing section of $T^*S \otimes K_S$, so the natural extension of $\mathcal{L}^*$ to $\mathbf{P}^1$ is isomorphic to $\mathcal{O}(1)$. The same discussion applies to the family corresponding to D_{II} : we just restrict

to the smooth part. Taking account again of the coverings we get

$$\zeta.D_{III} = -1/6 \quad \zeta.D_{II} = -1/4. \quad (19)$$

We make a short digression to consider further the $\mathbf{Z}/3$ -cyclic cover S_{III} which gives insight into the denominators in these formulae. Let V_+, V_- be 2-dimensional complex vector spaces with fixed isomorphisms $\Lambda^2 V_\pm = \mathbf{C}$ and write $\mathbf{P}^1 \times \mathbf{P}^1 = \mathbf{P}(V_+) \times \mathbf{P}(V_-)$. Let $S \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$ be a $\mathbf{Z}/3$ -cyclic cover branched over a smooth curve of bi-degree $(6, 6)$. Using standard theory one finds that

$$H^0(T^*S \otimes K_S) = s^2(V_+^*) \oplus s^2(V_-^*) \quad (20)$$

Now let U be the 4-dimensional space $V_1 \otimes V_2$. The trivialisations of $\Lambda^2 V_\pm$ define a non-degenerate quadratic form on U and $\mathbf{P}^1 \times \mathbf{P}^1$ is embedded in $\mathbf{P}(U)$ as the corresponding quadric surface. This is the usual description of 4-dimensional (complex) oriented Euclidean geometry in terms of spin spaces $V_\pm$. We have the usual splitting of the 2-forms $\Lambda^2 U^* = \Lambda_+^2 \oplus \Lambda_-^2$ and one has the spinor description $\Lambda_\pm^2 = s^2(V_\pm^*)$. What we essentially see from this is that the obstruction spaces behave in a simple way for a family of smooth sextic surfaces converging to a triple cover of a quadric. In the case at hand the group G acts on $V_\pm$ and G -invariant piece corresponding to the self-dual form ω matches up with a 1-dimensional invariant subspace in $s^2(V_+^*)$. However the isomorphism (20) depends on an isomorphism $K_S = \mathcal{O}(2, 2)$ and the $\mathbf{Z}/3$ covering group acts non-trivially on this. So on our $\mathbf{P}^1$ covering D_{III} , with parameter λ , we have a line bundle $\mathcal{L} = \mathcal{O}(-1)$ but the additional $\mathbf{Z}/3$ symmetry group of S_{III} acts non-trivially on the fibre over ∞ .

Similarly there is an additional $\mathbf{Z}/2$ symmetry group of the surface S_O which works the equivalence $S_{A,B} = S_{-A,-B}$. This is given by $x_i \mapsto -x_i, y_j \mapsto y_j$ and so takes ω to $-\omega$. So for both S_O and S_{III} if we take account of their full symmetry groups the corresponding invariant part of the obstruction space vanishes. This means that under a generic perturbation both S_O and S_{III} “persist”: more precisely, slightly perturbed versions of them, with the additional symmetries, persist. The picture is that the solutions Z of the perturbed problem can be viewed (approximately) as a subset of $\overline{M}$ but the symmetries of the situation force this subset to contain the points in $\overline{M}$ corresponding to vertices of the quadrilateral. The moduli space $\overline{M}$ is only a rational homology manifold at these points and the intersection pairing on integral homology takes rational values.

To sum up: we have three classes in $H^2(\overline{M}, \mathbf{Q})$ given by $c_1(\mathcal{L}_0), c_1(\mathcal{L}_2)$ and the Poincaré dual of ζ and these are all proportional:

$$PD[\zeta] = (1/48)c_1(\mathcal{L}_2) = -(1/11)c_1(\mathcal{L}_0).$$

Using (18) we find the pairings

$$\langle \zeta, c_1(\mathcal{L}_2) \rangle = -6 \quad , \quad \langle \zeta, c_1(\mathcal{L}_0) \rangle = 11/8. \quad (21)$$

Using the formulae (8) we see that the pairings with $I(c_1^3), I(c_1 c_2)$ are 12 and $-11/192$ respectively.

5.4. Curves

Let η be a 12th root of unity and take the action of $\mathbf{Z}/12$ on $\mathbf{C}^2$ generated by $(z_1, z_2) \mapsto (\eta z_1, \eta^{-1} z_2)$. Let $H \subset \mathbf{Z}/12 \times \mathbf{Z}/12$ be the subgroup of pairs (a, b) with $a = b \pmod{2}$ and take the obvious action of H on $\mathbf{C}^2 \times \mathbf{C}^2$. We consider the space of polynomials of bidegree $(6, 6)$ invariant under H . A basis for this space, written in our usual affine coordinates, is $1, s^6, t^6, s^6 t^6, s^3 t^3$ and there is an action of a 2-torus on the space, generated by $s \rightarrow \mu s, t \rightarrow \nu t$. We want to consider the corresponding curves in $\mathbf{P}^1 \times \mathbf{P}^1$. There is a similar discussion to the case of surfaces in subsection 6.1: if any of the coefficients of the first four monomials vanish we get an unstable point for the torus action, which we omit. Then we can use the torus action to put the equation in the form

$$P(s^6 + t^6) + Q(1 + s^6 t^6) + R s^3 t^3 = 0. \quad (22)$$

So we have a family of smooth curves $C_{P,Q,R}$ parametrised by an open subset of $\mathbf{CP}^2$. There is some residual equivalence: taking s to $-s$ gives $C_{P,Q,R} = C_{P,Q,-R}$ and taking s to s^{-1} gives $C_{P,Q,R} = C_{Q,P,R}$. We write

$$\mathcal{W}^H = \mathbf{CP}^2 / (\mathbf{Z}/2 \times \mathbf{Z}/2)$$

for the quotient by this action of $\mathbf{Z}/2 \times \mathbf{Z}/2$ and then the moduli space of these smooth curves is a subset of $\mathcal{W}^H$.

Now go back to our G -invariant sextic surfaces S_{AB} . The group action gives a distinguished pencil of curves in the canonical system $\mathcal{O}(2)$, defined by $x_2 y_2 - \lambda x_1 y_1$. For $\lambda \neq 0, \infty$ the curve is the intersection of S_{AB} with a smooth quadric which we parametrise by $x_1 = s, y_1 = t, x_2 = st, y_2 = \lambda$. Then the curve has equation

$$s^6 + t^6 + s^6 t^6 + \lambda^6 + f_{AB}(\lambda) s^3 t^3 = 0,$$

where

$$f_{AB}(\lambda) = (1 + \lambda) (A(1 + \lambda)^2 + B(1 - \lambda)^2).$$

Replacing s, t by $\lambda^{1/2} s, \lambda^{1/2} t$ we put this curve into our standard form

$$\lambda^3 (s^6 + t^6) + (1 + s^6 t^6) + f_{AB}(\lambda) s^3 t^3 = 0,$$

so $P = \lambda^3, Q = 1, R = f_{AB}(\lambda)$. We view this as a degree 3 map $\Gamma_{A,B} : \mathbf{CP}^1 \rightarrow \mathbf{CP}^2$. Let τ be the involution of $\mathbf{CP}^2$ defined by interchanging P, Q . It is easy to check that the Γ_{AB} are exactly the degree 3 maps Γ such that

- (1) $\Gamma(\lambda^{-1}) = \tau\Gamma(\lambda)$;
- (2) $\Gamma(0)$ lies in the line $\{Q = 0\} \subset \mathbf{CP}^2$ and Γ has second order contact with the line at that point (i.e. $Q \circ \Gamma = O(\lambda^3)$ as $\lambda \rightarrow 0$);
- (3) $\Gamma(\infty)$ lies in the line $\{P = 0\} \subset \mathbf{CP}^2$ and Γ has second order contact with the line at that point.
- (4) Γ does not pass through the point $P = Q = 0$.
- (5) $\Gamma(-1)$ lies in the line $\{R = 0\}$

Of course the third item is a consequence of the first two.

Thus we see that, roughly speaking, our moduli space of sextic surfaces S_{AB} can be interpreted as a space of maps Γ to a moduli space of curves and we would like to investigate how this interacts with moduli space compactifications.

If P and Q are non-zero the curve defined by the equation (22) has at worst ordinary double points so the question is how to extend the family when P or Q vanish. We use the same procedure as before. For parameters α, β we consider the complete intersection $C^{\alpha, \beta}$ defined by the equations

$$x_1 y_1 = \alpha h \quad , \quad x_2 y_2 = \beta h \quad , \quad x_1^6 + y_1^6 + x_2^6 + y_2^6 + h^3 = 0$$

in $\mathbf{P}_w^4$. If α, β are both nonzero then we view this as a curve in the smooth quadric $x_1 y_1 = (\alpha/\beta) x_2 y_2$ and one finds that it is equivalent to $C_{P, Q, R}$ with $R = 1, P = \alpha^3 Q = \beta^3$. When $\alpha = \beta = 0$ we get a curve C_0 in the cone $\text{Cone}(\Lambda)$ over the line configuration Λ . Let Σ be the $\mathbf{Z}/3$ -cyclic cover of $\mathbf{P}^1$ branched over the six roots of $z^6 + 1 = 0$, so Σ has genus 2 and there are three points in Σ lying over $z = 0$ and three lying over $z = \infty$. The component of C_0 in each component of $\text{Cone}(\Lambda)$ is a copy of Σ so we can obtain C_0 by taking 4 copies of Σ and identifying 24 points (6 in each copy) in pairs. So C_0 is a stable curve in the sense of Deligne-Mumford, with 12 ordinary double points. When just one of α, β is zero we are considering the degeneration of the quadric to a pair of planes and we get a 1-parameter family of stable curves which deforms C_0 by smoothing 6 of the double points.

The conclusion is that in this case the naive compactification is the right thing to consider. For each point (P, Q, R) in $\mathbf{CP}^2$ we have a stable curve $C_{P, Q, R}$ and we obtain a moduli space $\mathcal{W}^H = \mathbf{CP}^2/(\mathbf{Z}/2 \times \mathbf{Z}/2)$ of stable curves with this symmetry group, contained in the full Deligne-Mumford compactification $\overline{\mathcal{M}}_{25}$ of curves of genus 25. Our moduli space $M = \mathbf{C}^2/\pm 1$ of surfaces S_{AB} parametrises a family of curves $\underline{\Gamma}_{AB} : \mathbf{CP}^1 \rightarrow \mathcal{W}^H$. That is, we use the equivariance property (1) above to factor

$$\mathbf{CP}^1 \rightarrow \mathbf{CP}^2 \rightarrow \mathcal{W}^H$$

through the quotient $\mathbf{CP}^1 \rightarrow \mathbf{CP}^1$ mapping λ to $\lambda + \lambda^{-1}$, which identifies λ with λ^{-1} . But it is easier to compute with the lifted maps Γ_{AB} .

On thing which is clarified by our family $C^{\alpha, \beta}$ is the second order contact condition (2),(3) above. The curves $C^{0, \beta}$ have an additional $\mathbf{Z}/3$ -symmetry which means that this contact condition is the condition that there is genuine family of curves $\Gamma(\lambda)$ for small λ : this is just the fact that $P = \alpha^3$. We should really view $\mathbf{CP}^2$ as an orbifold with orbifold model at the origin given by the quotient $\mathbf{C}^2/(\mathbf{Z}/3 \times \mathbf{Z}/3)$ and this orbifold structure encodes the contact conditions (2),(3).

There is now another compactification $\overline{M}_{\text{Maps}}$ of M defined by the theory of stable maps $\underline{\Gamma} : \mathbf{CP}^1 \rightarrow \mathcal{W}^H$. We will not attempt to work this out in full here but one can observe some phenomena.

- For a surface on the boundary component from *III* to *IV* in $\overline{M}$ defined by an equation

$$h^3 + cs^2t^2h + (1 + s^6 + t^6 + s^6t^6) = 0$$

we take the pencil $st - \lambda h$ which gives the curves

$$(\lambda^{-3} + c\lambda^{-1})s^3t^3 + (1 + s^6 + t^6 + s^6t^6) = 0$$

in other words

$$P = Q = \frac{\lambda^3}{1 + c\lambda^2} \quad R = 1;$$

a degree 3 map to the line $P = Q$ in $\mathbf{CP}^2$.

- For a surface on the boundary component from *II* to *IV* in $\overline{M}$ defined by an equation

$$sth^2 + cs^3t^3 + (1 + s^6 + t^6 + s^6t^6) = 0$$

we get

$$P = Q = \frac{\lambda^2}{1 + c\lambda^2}, \quad R = 1;$$

a degree 2 map to the line $P = Q$.

- For the surface S_{IV} we have a pencil defined by $h^+ - \lambda h^-$ but for all but a finite number of λ the curves are isomorphic to C_0 .
- For finite A, B with $A + B \neq 0$ the image of the map Γ_{AB} meets the line $R = 0$ at the points $P/Q = -1, \lambda_+^3, \lambda_-^3$ where

$$\lambda_{\pm} = \frac{(B - A) \pm \sqrt{-4AB}}{A + B}.$$

This expression is homogeneous in A, B , so the 1-parameter family of maps $\Gamma_{tA, tB}$ meet the line at infinity in the same three points. These points are recorded in the stable maps limit of the $\Gamma_{tA, tB}$ as $t \rightarrow \infty$.

The first two items suggest that the compactification $\overline{M}_{\text{Maps}}$ is not the same as $\overline{M}_{GIT}$ and the last shows that it is not the same as $\overline{M}$. It seems likely that $\overline{M}_{\text{Maps}}$ is a toric blow-up of each of these, with collapsing maps

$$\overline{M}_{GIT} \leftarrow \overline{M}_{\text{Maps}} \rightarrow \overline{M}.$$

Of course the fact that $\overline{M}_{\text{Maps}}$ is different from $\overline{M}$ does not rule out the possibility of relating our enumerative theory to curve-counting theories.

Counting parameters shows that the virtual (complex) dimension of the space of maps Γ satisfying the constraints (1)-(5) is 2, the same as the actual dimension, and so not the same as the virtual dimension of $\overline{M}$. The explanation for this is similar to what we saw in Section 4, Remark (7). The subset $\mathcal{W}^H \subset \overline{\mathcal{M}}_{25}$ lies inside a larger family $\overline{\mathcal{M}}_{25}^H \subset \overline{\mathcal{M}}_{25}$ of curves with an H -action. More generally there is a moduli space $\mathcal{W}$ of curves in the linear system $\mathcal{O}(6, 6)$ on $\mathbf{P}^1 \times \mathbf{P}^1$ divided by $PSL(2, \mathbf{C}) \times PSL(2, \mathbf{C})$ which has dimension $49 - 1 - 6 = 42$, while the full moduli space $\mathcal{M}_{25}$ of curves of genus 25 has dimension

$3 \cdot 25 - 3 = 72$. So for such a curve C there is a 30-dimensional family of deformations which do not embed in $\mathbf{P}^1 \times \mathbf{P}^1$. To see this explicitly, we take the exact sequence of bundles on C

$$0 \rightarrow TC \rightarrow \mathcal{O}(2, 0) \oplus \mathcal{O}(0, 2) \rightarrow \mathcal{O}(6, 6) \rightarrow 0$$

which gives

$$H^0(C; \mathcal{O}(6, 6)) \rightarrow H^1(TC) \rightarrow H^1(C, \mathcal{O}(2, 0) \oplus \mathcal{O}(0, 2)) \rightarrow H^1(C; \mathcal{O}(6, 6)).$$

The last term vanishes and the first term represents the deformations of C within $\mathbf{P}^1 \times \mathbf{P}^1$. In terms of our 2-dimensional spaces $V^\pm$, as considered in subsection 6.3, one finds that

$$H^1(C; \mathcal{O}(2, 0) \oplus \mathcal{O}(0, 2)) = s^4(V_+) \otimes s^2(V_-) \oplus s^2(V_-) \otimes s^4(V_+)$$

which indeed has dimension $5 \cdot 3 + 3 \cdot 5 = 30$.

In our context we want to consider an H -invariant curve C and the H -invariant subspace of $s^4(V_+) \otimes s^2(V_-) \oplus s^4(V_+) \otimes s^2(V_-)$. It is easy to check that this space is 2-dimensional so $\mathcal{W}^H$ has codimension-2 in $\overline{\mathcal{M}}_{25}^H$. To relate the deformation theories it is clearest to work in the general case of $\mathcal{W} \subset \overline{\mathcal{M}}_{25}$ and a curve $\Gamma : \mathbf{P}^1 \rightarrow \mathcal{W}$ defined by intersecting a sextic surface S with a pencil of quadrics in $\mathbf{P}^3$. Without loss of generality we consider a point p_0 of $\mathbf{P}^1$ corresponding to our standard quadric $\mathbf{P}^1 \times \mathbf{P}^1$ and a non-zero tangent vector $v \in T\mathbf{P}_p^1$. The pencil involves another quadric and the choice of v gives an element

$$E_v \in H^0(\mathbf{P}^1 \times \mathbf{P}^1; \mathcal{O}(2, 2)) = s^2(V_+^*) \otimes s^2(V_-^*).$$

Now suppose that we have an element $\Omega \in \Lambda^2 U^*$. As we recalled in subsection 5.3 we can regard Ω as lying in $s^2(V_+^*) \oplus s^2(V_-^*)$. Then we have a product

$$\Omega \cdot E_v \in s^4(V_+^*) \otimes s^2(V_-^*) \oplus s^2(V_-^*) \otimes s^4(V_+^*),$$

which is the dual of $H^1(C; \mathcal{O}(2, 0) \oplus \mathcal{O}(0, 2))$. Now, as we have explained above, this last space can be regarded as the fibre of the normal bundle $N_{\mathcal{W}}$ of $\mathcal{W}$ in $\overline{\mathcal{M}}_{25}$. The upshot is that we get map

$$\Lambda^2 U^* \rightarrow H^0(\mathbf{P}^1, T^* \mathbf{P}^1 \otimes \Gamma^*(N_{\mathcal{W}}^*)).$$

Using Serre duality on $\mathbf{P}^1$ the transpose is a map

$$H^1(\mathbf{P}^1, \Gamma^*(N_{\mathcal{W}})) \rightarrow \Lambda^2 U.$$

Composing with the map induced by projection of $T\overline{\mathcal{M}}_{25}$ to the normal bundle we get a map from $H^1(\mathbf{P}^1, \Gamma^* T\overline{\mathcal{M}}_{25})$ to $\Lambda^2 U$ which relates the obstruction spaces in the two theories.

As we wrote at the beginning of this section, the main motivation for studying this example with finite group action is to gain insight into the larger questions, such as for general sextic surfaces. The dimension of the space of pairs consisting of a sextic surface S and a pencil in $|K_S|$ is $68 + 16 = 84$ and for each such pair we get (roughly speaking) a rational curve Γ in the 42-dimensional space $\mathcal{W}$ inside the 72-dimensional $\overline{\mathcal{M}}_{25}$. Since $84 = 2 \cdot 42$ we expect that for typical points $C_1, C_2 \in \mathcal{W}$ there is a 1-dimensional space

of curves Γ through C_1 and C_2 . But if we consider the space $\mathcal{W}$ as a subset of $\overline{\mathcal{M}}_{25}$ we expect that the virtual dimensions reduce by 6. It seems interesting to study both the detailed geometry and the enumerative geometry of this situation.

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