

Moduli of Lagrangian immersions with formal deformations

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ABSTRACT. We introduce a joint project with Cheol-Hyun Cho on the construction of quantum-corrected moduli of Lagrangian immersions. The construction has important applications to mirror symmetry for pair-of-pants decompositions, SYZ and wall-crossing. The key ingredient is Floer-theoretical gluing between local moduli spaces of Lagrangians with different topologies.

1. Introduction

Moduli theory for vector bundles has been very well established. The groundbreaking discovery of Donaldson [Don85] and Uhlenbeck-Yau [UY86] built a deep connection between stability and canonical metric.

Theorem 1.1 ([Don85, UY86]). *A slope-semistable holomorphic vector bundle admits a Hermitian Yang-Mills metric.*

Stability conditions and GIT were essential to the construction. In general terms, Bridgeland [Bri07] developed a general mathematical theory of stability conditions for triangulated categories based on Douglas' work [Dou02] on Π -stability on D-branes in string theory. Foundational techniques have been developed in [Bri08, Tod09, BMT14] to construct stability conditions for derived categories of coherent sheaves. One remarkable feature of the theory is that, if we vary our choice of a stability condition, then the moduli spaces of stable objects undergo birational changes [Bri02], and they are related by Fourier-Mukai transforms.

We are working to develop a moduli theory for Lagrangians. According to homological mirror symmetry [Kon95], coherent sheaves on a complex manifold are mirror to Lagrangian submanifolds in its mirror symplectic manifold. Thus there should be a mirror moduli theory for Lagrangian submanifolds.

The following lists some ingredients that are new to the moduli theory for coherent sheaves.

- (1) **Complexification.** The classical moduli spaces are affine manifolds with singularities [Hit97, McL98]. To get better compactifications, we need to complexify the moduli spaces by considering flat connections or other boundary deformations on the Lagrangians. Technically we need to work over the Novikov ring

[FOOO09, FOOO10]

$$\begin{aligned}
\Lambda_+ &= \left\{ \sum_{i=0}^{\infty} a_i T^{A_i} \mid A_i > 0 \text{ increase to } +\infty, a_i \in \mathbb{C} \right\}, \\
\Lambda_0 &= \left\{ \sum_{i=0}^{\infty} a_i T^{A_i} \mid A_i \geq 0 \text{ increase to } +\infty, a_i \in \mathbb{C} \right\}, \\
\Lambda &= \left\{ \sum_{i=0}^{\infty} a_i T^{A_i} \mid A_i \text{ increase to } +\infty, a_i \in \mathbb{C} \right\}, \\
\Lambda_0^\times &= \mathbb{C}^\times \oplus \Lambda_+.
\end{aligned}$$

They are equipped with the valuation function

$$\text{val} : \sum_{i=0}^{\infty} a_i T^{A_i} \mapsto A_0 \text{ and } \text{val}(0) = +\infty.$$

The moduli are rigid analytic spaces.

- (2) **Quantum corrections.** The canonical complex structures need to be corrected using Lagrangian Floer theory [FOOO09]. For moduli of SYZ fibers, the combinatorial structure of quantum corrections was deeply studied by Kontsevich-Soibelman [KS06] and Gross-Siebert [GS11].
- (3) **Landau-Ginzburg models.** After compactification, the moduli are usually singular algebraic varieties. They can be described as critical loci of holomorphic functions which are known as Landau-Ginzburg models. Orlov [Orl09] found the theory of matrix factorizations for Landau-Ginzburg models. More generally we may encounter noncommutative Landau-Ginzburg models [Boc16, CHL].
- (4) **Singular Lagrangians.** Geometrically singular Lagrangians (or even isotropic skeletons) can appear as limits of smooth Lagrangians. They are still not yet very well understood. Kontsevich [Kon] proposed to study them using cosheaves of categories. Nadler [Nad17] is developing a study of arboreal singularities to classify them. We need to understand what objects these singular Lagrangians correspond to in the Fukaya category.

A particularly important class of examples is the moduli of fibers in a Lagrangian torus fibration. Strominger-Yau-Zaslow [SYZ96] conjectured that torus duality produces mirror geometries. Family Floer theory is the key ingredient to derive homological mirror symmetry in the SYZ setting. The theory was proposed by Fukaya [Fuk02] to study mirror symmetry by using Floer cohomologies of fibers of a Lagrangian torus fibration.

Tu [Tu14] took this approach to construct mirror spaces away from singular fibers. Abouzaid [Abo17] constructed family Floer functors for torus bundles and showed that the functor is fully faithful. The famous Fukaya trick plays an important role, which

relates Floer theories of different fibers by isotopies of A_∞ -algebras. However singular Lagrangian fibers place a major difficulty to understand SYZ and family Floer theory.

Lagrangian immersions are the best singular Lagrangians in the sense that they still have a well-defined Floer theory by Akaho-Joyce [AJ10]. In [CHL17, CHKL17, CHL], we constructed mirror geometries as the local moduli of Lagrangian immersions and derived homological mirror symmetry.

In the study of SYZ, the immersed two-sphere in (a) of Figure 1 is particularly interesting, since it (or the product with a torus) is the main source of wall-crossing phenomena.

Furthermore, it is hoped that any object in the derived Fukaya category can be given as a Lagrangian immersion; in particular any singular Lagrangian is isomorphic (as an object in the Fukaya category) to an immersion, so that we do not need to tackle singular Lagrangians other than immersions at all! For instance, Lagrangian immersions in the pair-of-pants were used by Seidel [Sei11] and Sheridan [She11, She15] to prove homological mirror symmetry for Fermat-type hypersurfaces (see (b) of Figure 1 for such an immersion in dimension 1).

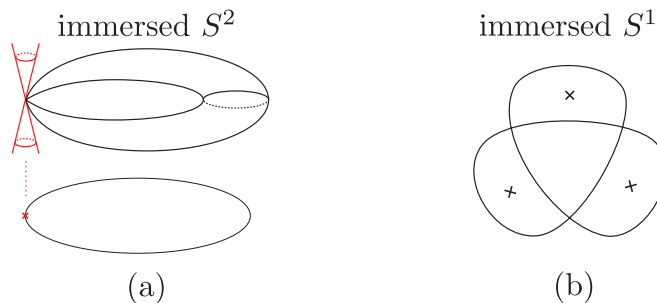


FIGURE 1. Immersed spheres

In this expository article, we introduce ongoing joint works with Cheol-Hyun Cho and with Yoosik Kim, in which we use Floer-theoretical gluing methods to construct moduli of Lagrangian immersions which are not necessarily tori. Detailed proofs will be given in a joint paper with Cho and that with Kim.

There are many occasions that non-tori play an important role, for instance in pair-of-pants decompositions or geometries related to knots. An interesting feature is that topologies of Lagrangians can jump within the same moduli space. We cannot use the Fukaya trick because the Lagrangians are no longer diffeomorphic to each other.

There is a technical advantage of having Lagrangian immersions in our moduli. Namely, immersed sectors provide bigger formal deformation spaces than flat connections. The deformation parameters associated to immersed sectors lie in Λ_+ (or Λ_0 in good cases), while flat connections are only $\mathbb{C}^\times$ -valued. (Lagrangian Floer theory does not allow deformations by $\Lambda^\times$ -connections since it would destroy the energy filtration. We will

consider them as ‘pseudo-deformations’.) Intuitively Λ_+ consists of infinitely many layers of $\mathbb{C}^\times$ in different energy levels. Thus the formal deformation space of an immersion typically covers a local family of Lagrangians (which can even be singular).

The main idea is to consider *pseudo-isomorphisms between Lagrangian immersions in the moduli and obtain gluing information from cocycle conditions on pseudo-isomorphisms*. This is explained in Section 3. We introduce two applications, namely mirror construction by pair-of-pants decompositions, and generic wall-crossing in SYZ.

We will present the main ideas and leave the details to separate papers. To begin with, let’s revisit the most basic example $\mathbb{P}^1$, which already contains some essential ideas used in our later developments. Fukaya-Oh-Ohta-Ono [FOOO10, FOOO11, FOOO16] studied in depth the Lagrangian Floer theory of toric manifolds and proved closed-string mirror symmetry.

Example: the two-sphere

Consider $\mathbb{P}^1$ equipped with the spherical area form (as a symplectic form) and the meromorphic top-form dz/z . It has an $\mathbb{S}^1$ -symmetry. The $\mathbb{S}^1$ -moment map $\mathbb{P}^1 \rightarrow [0, 1]$ gives a special Lagrangian fibration.

The classical moduli space of special Lagrangian fibers [McL98, Hit97] is given by $(0, 1)$. The induced affine structure on $(0, 1)$ is trivial. The classical moduli does not capture symplectic topologies of Lagrangians, for instance the non-displaceability of fibers.

We can complexify the moduli space by decorating the special Lagrangian fibers with flat $U(1)$ -connections. We have the set

$$\{(\text{fiber, flat } U(1)\text{-connection})\} = (0, 1) \times \mathbb{S}^1.$$

This is the well-known mirror space [HV, CL10].

Instead of taking the set of special Lagrangian fibers decorated by flat $U(1)$ -connections, we can take

$$\{(\text{stable Lagrangian in fiber class, flat } \Lambda_0^\times\text{-connection})\}/(\text{non-zero}) \text{ Quasi-isom.} \quad (1)$$

A ‘stable Lagrangian in fiber class’ is a Lagrangian which is quasi-isomorphic to a special Lagrangian fiber. Here we consider quasi-isomorphisms in the Fukaya category, see Section 3.1. Moreover we enlarge the deformation space for each Lagrangian by taking $\Lambda_0^\times = \mathbb{C}^\times \oplus \Lambda_+$. (Flat $\mathbb{C}^\times$ connections was introduced by [Cho08] to find non-displaceable Lagrangians.)

The above is simply $(0, 1) \times \Lambda_0^\times$ as a set. Different fibers do not intersect with each other and hence there is no morphism between them. This is not good enough for understanding the rigid-analytic structure of the moduli.

Consider the further enlargement

$$\{(\text{stable Lagrangian in fiber class, flat } \Lambda^\times\text{-connection})\}/(\text{non-zero}) \text{ Pseudo-isom.} \quad (2)$$

Note that we cannot allow flat $\Lambda^\times$ -connections in the Fukaya category, since it would destroy the energy filtration and Novikov convergence of the category. Lagrangians

equipped with flat $\Lambda^\times$ -connections are not really objects in the Fukaya category. However it is important to consider them for the rigid analytic structure of the moduli space. We call these to be ‘pseudo-deformations’. We define ‘pseudo-isomorphisms’ between them formally the same as quasi-isomorphisms, namely morphisms (linear combinations of transverse intersection points over Λ satisfying the cocycle condition) that have inverses up to homotopy. (See Definition 3.2 and 3.3.)

By allowing pseudo-deformations and pseudo-isomorphisms, there are relations between different fibers. For instance consider the fibers L_1 and L_2 as in Figure 2. Take a rotation of L_1 (which is a Hamiltonian perturbation) and get L'_1 . Since L'_1 is isomorphic to L_1 , it is stable.

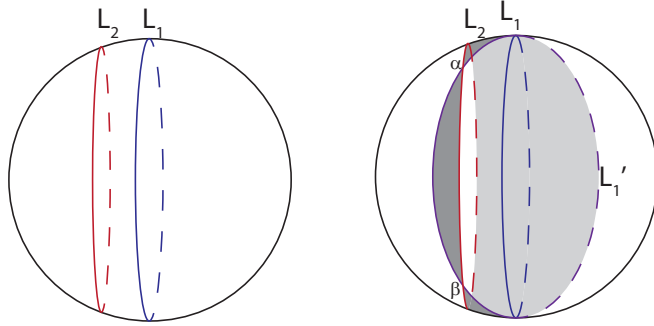


FIGURE 2. Stable Lagrangians in $\mathbb{P}^1$.

L'_1 intersects L_2 at two intersection points (α, β) . There are two strips bounded by L'_1 and L_2 . Note that these two strips have different areas. For a flat connection ∇_t on L'_1 , we have

$$m_1^{\nabla_t}(\alpha) = T^c(T^A - t)\beta$$

where c is a constant and $A > 0$ is the area of the cylinder bounded between L_1 and L_2 .

$T^A - t \neq 0$ if t is only allowed to be $\Lambda_0^\times$ -valued. Thus there is no morphism between (L'_1, ∇_t) and L_2 . This is as expected since L_2 (equipped with any decoration) is a trivial object in the Fukaya category, while L'_1 is not. (L'_1, ∇_t) and L_2 are different elements in Equation (1).

We can set $t = T^A$ if we allow it to be $\Lambda^\times$ -valued! Then (α, β) provides a pseudo-isomorphism between (L'_1, ∇_t) and L_2 . Thus (L'_1, ∇_t) (for $t = T^A \in \Lambda^\times$) and L_2 are the same element in Equation (2).

We see that any fibers are pseudo-isomorphic to (L_1, ∇_t) for some $t \in \Lambda^\times$ (even though they are not intersecting with each other). The expression (2) equals to $\Lambda^\times$ which has a

rigid analytic structure. Expression (1) is a subset of Expression (2) and hence equipped with the induced topology. Indeed it is $\Lambda_{0 < \text{val} < 1}^\times$ which equals to $(0, 1) \times \Lambda_0^\times$ as a set.

Expression (1) is not yet the moduli. We have a superpotential W coming from the m_0 -term of each element in (1). In this case $W = t + T^{A_S} t^{-1}$ where the two terms correspond to the left and the right discs bounded by a fiber (and A_S is the area of the sphere). The moduli is the Landau-Ginzburg model $(\Lambda_{0 < \text{val} < 1}^\times, W)$, whose critical locus consists of two points corresponding to two flat connections on the equator which is non-displaceable. This was deeply studied in [FOOO10].

2. Deformation space of a Lagrangian immersion

We review basic Lagrangian Floer theory, in particular the formal deformation of Lagrangian submanifolds following [FOOO09]. We focus on two specific types of formal deformation, one by flat line bundles over Lagrangian tori and the other by immersed generators for Lagrangian immersions. We will also briefly explain immersed Lagrangian Floer theory by Akaho-Joyce [AJ10].

2.1. Review on Lagrangian Floer theory: formal deformation

Let (X, ω) be a symplectic manifold (possibly noncompact). Consider a compact Lagrangian L in X .

By [FOOO09], we have a gapped filtered A_∞ -algebra $(CF(L, L), \{m_k\}_{k \geq 0})$ possibly with a nontrivial curvature m_0 . For a degree odd (or degree one when L is graded) element b , one can deform this A_∞ -structure as follows.

$$m_k^{b, \dots, b}(x_1, \dots, x_k) := \sum_{\substack{i_1, \dots, i_{k+1} \\ i_1 + \dots + i_{k+1} = l}} m_{k+l}(\overbrace{b, \dots, b}^{i_1}, x_1, b, \dots, b, x_k, \overbrace{b, \dots, b}^{i_{k+1}}). \quad (3)$$

It is easy to see that $\{m_k^b\}$ defines an A_∞ -algebra [FOOO09].

We are interested in b which satisfies the following condition so that (L, b) defines an object of the Fukaya category.

Definition 2.1. b is called a Maurer-Cartan element if

$$m_0^b = m_0(e^b) = m_0(1) + \sum_{k=1}^{\infty} m_k(b, \dots, b) = 0.$$

In this case (L, b) is said to be unobstructed.

It is called weak Maurer-Cartan if the right hand side is a (Novikov) constant multiple of $\mathbf{1}_L := PD[L]$, in which case (L, b) is said to be weakly unobstructed.

We denote the space of such b 's by $\mathcal{MC}(L)$ for Maurer-Cartan elements, and by $\mathcal{MC}_{weak}(L)$ for weak Maurer-Cartan elements. Note that deformations by different elements in $\mathcal{MC}(L)$ could lead to isomorphic objects. We shall identify them in the construction of moduli spaces. Isomorphisms are discussed in Section 3.

$\mathcal{MC}(L)$ can be interpreted as a (formal) deformation space of L . For instance, in the case of toric manifolds, $\mathcal{MC}_{weak}(L)$ agrees with the first cohomology of L which encodes geometric deformation of L (see [FOOO10]).

In case there is no danger of confusion, we will sometimes call $\mathcal{MC}(L)$ (or $\mathcal{MC}_{weak}(L)$) deformation space or moduli space, omitting the adjective “formal” in front of them.

2.2. Deformation by $\mathbb{C}^\times$ flat line bundles on L

Instead of using elements in $CF(L, L)$, one can vary the holonomy of a flat line bundle over a Lagrangian L to deform L (as an object of Fukaya category). More precisely, if we equip L with a line bundle E with a $\mathbb{C}^\times$ -flat connection ∇ , the A_∞ -operations on $CF(L, L)$ are deformed as follows:

$$m_k^{(L, \nabla)}(x_1, \dots, x_k) = \sum_{\beta \in \pi_2(M, L)} (hol_{\partial\beta} \nabla) m_{k, \beta}(x_1, \dots, x_k) T^{\omega(\beta)}$$

where $m_{k, \beta}$ is the contribution from holomorphic disks in class β to the original m_k -operator on $CF(L, L)$. By considering all possible holonomies, we obtain a (formal) moduli of objects in $\text{Fuk}(M)$ isomorphic to $(\mathbb{C}^\times)^{\dim_{\mathbb{R}} H_1(L; \mathbb{R})}$, which is nothing but the space of all $\mathbb{C}^\times$ flat line bundles on L modulo equivalence. One can slightly enlarge this space by considering $\Lambda^\times$ -line bundle, but the holonomy is still required to have valuation zero (i.e. it should start with nontrivial complex number without T) so we only have a slice of $(\Lambda^\times)^{\dim_{\mathbb{R}} H_1(L; \mathbb{R})}$.

In what follows, we will mainly consider Lagrangian tori equipped with a $\mathbb{C}^\times$ -flat line bundle. Furthermore, we will always choose a special representative of $\mathbb{C}^\times$ -line bundles whose connection behaves like a delta function. More precisely, for $L \cong \mathbb{R}^n/\mathbb{Z}^n$, we fix (oriented) hyper-tori $H_i = \epsilon_i + \mathbb{R}\langle e_i \rangle$ for $\epsilon_i \in \mathbb{R}/\mathbb{Z}$ so that the parallel transport over a path γ is given by multiplying $z_i^\pm$ whenever γ runs across H_i where the sign in the exponent is determined by the parity of the intersection $\gamma \cap H_i$. (See [CHL14] for more details.)

2.3. Immersed Floer theory

For an immersed Lagrangian L , we consider the boundary deformation induced by self-intersections. First, let us review Floer theory on L developed by [AJ10]. We assume that L has transversal intersection only. Let $\iota : L \rightarrow M$ be such a Lagrangian immersion. The Floer complex $CF(L, L)$ is generated by cochains on

$$L \times_\iota L = \{(x, y) \in L \times L \mid \iota(x) = \iota(y)\}.$$

Notice that each self-intersection point of ι gives two distinct generators, which are points in the off-diagonal component of $L \times_\iota L$. They have the following geometric meaning in Floer theory. Let A and B be two local branches of L meeting at a self-intersection point X of ι . Then the corresponding generators, say X and $\bar{X}$, of $CF(L, L)$ describe the jumps of the boundary of a holomorphic disk from A to B or from B to A , when we travel along the boundary in positive orientation.

If $X_1 \cdots, X_l$ denote odd-degree (degree one for L graded) generators from self-intersection points, we use the linear combination $b := \sum x_i X_i$ to formally deform the Lagrangian L as in (3) where x_i 's are a priori free variables. Namely x_i 's are taken from the free algebra generated by themselves. Here, we fix the convention for A_∞ -operations involving free variables by setting

$$m_k(x_{i_1} X_{i_1}, \dots, x_{i_k} X_{i_k}) := x_{i_k} \cdots x_{i_1} m_k(X_{i_1}, \dots, X_{i_k}),$$

i.e. we pull out the coefficients from the back.

Unlike $\mathbb{C}^\times$ -flat connections, not every such linear combination defines a legitimate deformation, as the Maurer-Cartan equation $\sum_k m_k(b, \dots, b)$ may impose nontrivial relations among free variables x_i 's. Therefore the corresponding deformation space is the Spec of the algebra generated by x_i 's modulo relations from the MC-equation. Note that this algebra is in general noncommutative, though commutative variables x_i solve MC-equation in most of examples we will consider in this article.

2.4. An example: pair-of-pants

Consider the Lagrangian immersion L from S^1 constructed by Seidel [Sei11] in a pair-of-pants. See Figure 3. There are three transverse immersed points, giving the immersed generators X, Y, Z in odd degree and $\bar{X}, \bar{Y}, \bar{Z}$ in even degree. The Floer complex is $\mathcal{F}(L, L) = \text{Span}\{\mathbf{1}, X, Y, Z, \bar{X}, \bar{Y}, \bar{Z}, \text{pt}\}$ as a vector space.

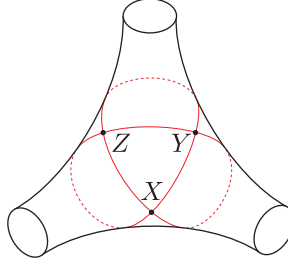


FIGURE 3. Seidel Lagrangian in a pair-of-pants

The following lemma is essential to make sense of the deformation theory (over $\mathbb{C}$).

Lemma 2.1 ([CHL17]). *The formal odd-degree deformations $b = xX + yY + zZ$ are weakly unobstructed for $x, y, z \in \mathbb{C}$. i.e. $\mathcal{MC}_{\text{weak}}(L) \cong \mathbb{C}^3$.*

Proof. Consider $m_0^b = \sum_{k>0} m_k(b, \dots, b)$ which has even degree. We need to argue that the coefficients of $\bar{X}, \bar{Y}, \bar{Z}$ are zero. Note that constant polygons do not contribute to these coefficients: for a constant polygon with an input corner to be X (and its output corner is one of $\bar{X}, \bar{Y}, \bar{Z}, \text{pt}$), it must have at least one input corner to be $\bar{X}$ in order to return to the same Lagrangian branch when we travel along its boundary once. But $\bar{X}$ does not appear in b and hence is not allowed to be an input.

The two triangles XYZ bounded by L contribute to the $\bar{Z}$ -output of $m_2(X, Y)$ and $m_2(Y, X)$, and other similar terms obtained by permuting X, Y, Z . By dimension reason only their disc classes contribute to m_0^b . From above there is no constant polygon component and so we have exactly one of these triangle components for each contribution. The areas of these two triangles bounded by L are the same. Following Seidel [Sei11], we take a non-trivial spin structure on L . Then the contributions of the two triangles cancel with each other. $\square$

We have $m_0^b = W \cdot \mathbf{1}_L$ where $W = xyz$ and $\mathbf{1}_L$ is the unit (namely the fundamental class). In conclusion the moduli space is given by the Landau-Ginzburg model $(\mathbb{C}^3, xyz)$. In Section 4 we consider a pair-of-pants decomposition and glue these local spaces together to form a global moduli.

Remark 2.1. The pair-of-pants can be compactified to an orbifold $\mathbb{P}_{a,b,c}^1$. In [CHL17, CHKL17] we used the Seidel Lagrangian L to construct and compute the mirror, and derived homological mirror symmetry. In an ongoing work with Amorim we are proving closed-string mirror symmetry along this line. In [LZ15] it was shown that the mirror superpotential W of elliptic orbifolds ($\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$) has interesting modular properties.

3. Gluing of formal deformation spaces

We will glue formal moduli spaces of Lagrangians making use of explicit isomorphisms in the Fukaya category. We first recall the notion of isomorphisms in an A_∞ -category, and explain how it can be used in our geometric setting in which we have two Lagrangians whose formal deformations are isomorphic over a certain subset of formal moduli spaces. At the end of the section, we describe a homotopy model for the B-side category for the glued formal moduli space following [BBB13].

3.1. Isomorphisms between objects of the Fukaya category

First, we introduce the notion of isomorphisms between two objects in an A_∞ -category.

Definition 3.1. Let L and L' be two objects in an A_∞ -category $\mathcal{C}$. $\alpha \in \text{Hom}_{\mathcal{C}}(L, L')$ is called a quasi-isomorphism if $m_1(\alpha) = 0$ and there exists $\beta \in \text{Hom}_{\mathcal{C}}(L', L)$ such that

$$m_2(\alpha, \beta) = \mathbf{1}_L + m_1(\gamma) \quad \text{and} \quad m_2(\beta, \alpha) = \mathbf{1}_{L'} + m_1(\gamma')$$

for some γ and γ' . If $m_1(\gamma) = m_1(\gamma') = 0$, then α (and β) is called a strict-isomorphism (or simply an isomorphism).

One can easily check that quasi-isomorphisms define an equivalence relation on the set of objects in $\mathcal{C}$. In particular, the composition of two quasi-isomorphisms (i.e. m_2 of the two) is also a quasi-isomorphism.

We next examine the gluing of formal moduli spaces for two different (possibly immersed) Lagrangians L and L' in a symplectic manifold M . For $b \in U \subset \mathcal{MC}(L)$ and $b' \in U' \subset \mathcal{MC}(L')$, suppose that there is a diffeomorphism $f : U \rightarrow U'$ and a quasi-isomorphism

$$\alpha : (L, b) \rightarrow (L', f(b))$$

between objects in $\text{Fuk}(M)$. In general α may depend on b as well, but here we assume that α is a fixed element in the vector space $\oplus_{x \in L \cap L'} \Lambda \langle x \rangle$ for simplicity.

When L and L' are weakly unobstructed with nontrivial potentials

$$W : \mathcal{MC}_{weak}(L) \rightarrow \Lambda, \quad W' : \mathcal{MC}_{weak}(L') \rightarrow \Lambda,$$

we further require that $W'(f(b)) = W(b)$ in order to glue two resulting Landau-Ginzburg models $(\mathcal{MC}_{weak}(L), W)$ and $(\mathcal{MC}_{weak}(L'), W')$. In fact, this condition is necessary to have a quasi-isomorphism $\alpha : (L, b) \rightarrow (L', f(b))$ since the Floer differential $m_1^{b, b'}$ on $CF((L, b), (L', b'))$ for $b' = f(b)$ squares to be zero only under such an assumption, since

$$\left(m_1^{b, b'}\right)^2 = W(b) - W'(f(b)).$$

3.2. Pseudo-deformations and Pseudo-isomorphisms

To understand the relation between two nearby Lagrangians in the same moduli space, we have to consider pseudo-deformations and pseudo-isomorphisms. They are not valid in the Fukaya category. However they provide a way to talk about Lagrangians close to L (but not isomorphic to L) by formal pseudo-deformations on L . Using this notion a neighborhood of L in the moduli can be identified as an open subset of Λ^n and is endowed with a rigid analytic structure.

Definition 3.2. A pseudo-deformation b of a Lagrangian immersion L is either a flat $\Lambda^\times$ -connection on L , or a linear combination of degree-one immersed generators of L over the Novikov field Λ , which satisfies that $m_k^{b, \dots, b}$ converge in Novikov sense for all $k \geq 0$.

Intuitively the Novikov convergence above means that the formal deformation is small enough so that we can talk about its image under the exponential map. Note that the above condition concerns L itself only but not any other Lagrangians. In the filtered Fukaya category we only allow formal deformations by flat $\Lambda_0^\times$ -connections or linear combinations of degree-one immersed generators with coefficients in $\Lambda_{>0}$ (or $\Lambda_{\geq 0}$ in better situations) which ensures Novikov convergence of the whole Fukaya category.

Definition 3.3. Let b, b' be weakly unobstructed pseudo-deformations on L, L' respectively. $\alpha \in \text{Hom}(L, L')$ is called a pseudo-isomorphism from (L, b) to (L', b') if the A_∞ -operations $m_k^{b_0, \dots, b_k}$ for all $k \geq 0$ (defined in the same way as in the case of formal deformations) converge in Novikov sense where b_i can be either b or b' , $m_1^{b, b'}(\alpha) = 0$ and there exists $\beta \in \text{Hom}(L', L)$ (with $m_1^{b', b}(\beta) = 0$) such that

$$m_2^{b, b', b}(\alpha, \beta) = \mathbf{1}_L + m_1^{b, b}(\gamma) \quad \text{and} \quad m_2^{b', b, b'}(\beta, \alpha) = \mathbf{1}_{L'} + m_1^{b', b'}(\gamma')$$

for some γ and γ' .

(L, b) and (L', b') are said to be pseudo-isomorphic to each other if there is a chain of pseudo-isomorphisms $\alpha_1 : (L, b) \rightarrow (L_1, b_1)$, $\alpha_2 : (L_1, b_1) \rightarrow (L_2, b_2)$, $\dots$, $\alpha_k : (L_{k-1}, b_{k-1}) \rightarrow (L', b')$.

Conceptually the Novikov convergence above means L, L' are close enough to each other in the moduli so that we can compare their pseudo-deformations.

Remark 3.1. As in the example of $\mathbb{P}^1$ in the Introduction, two pseudo-isomorphisms may not compose to a pseudo-isomorphism since the m_k operations involving the three Lagrangians with pseudo-deformations may not converge. Thus we need a chain of pseudo-isomorphisms in the above definition.

3.3. Construction of A_∞ -functor

Given two formal moduli spaces and their gluing data as in the previous section, we have a B-model category that encodes complex information of glued moduli whose construction goes as follows. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two dg-categories and suppose there are dg-functors

$$\phi_i : \mathcal{D}_i \rightarrow \mathcal{E} \quad i = 1, 2$$

that land on another dg-category $\mathcal{E}$. Given these data, we have the following new dg-category which models the gluing of $\mathcal{D}_1$ and $\mathcal{D}_2$ over $\mathcal{E}$.

Definition 3.4. The homotopy fiber product $\mathcal{D}_1 \times_{\mathcal{E}} \mathcal{D}_2$ is a dg-category whose objects are tuples of the form

$$(D_1, D_2, \phi_1(D_1) \xrightarrow{\sigma} \phi_2(D_2))$$

where σ is required to descend to an isomorphism on the cohomology level. A morphism between two objects is given as a tuple

$$\begin{aligned} & \text{Hom}_{\mathcal{D}_1 \times_{\mathcal{E}} \mathcal{D}_2}^i((D_1, D_2, \sigma), (D'_1, D'_2, \sigma')) \\ &= \text{Hom}_{\mathcal{D}_1}^i(D_1, D'_1) \oplus \text{Hom}_{\mathcal{D}_2}^i(D_2, D'_2) \oplus \text{Hom}_{\mathcal{E}}^{i-1}(\phi_1(D_1), \phi_2(D'_2)). \end{aligned} \quad (4)$$

A morphism (μ_1, μ_2, τ) in (4) is closed under the differential if and only if both μ_1 and μ_2 are closed morphisms and τ is a homotopy between $\sigma' \circ \phi_1(\mu_1)$ and $\phi_2(\mu_2) \circ \sigma$.

$$\begin{array}{ccc} \phi_1(D_1) & \xrightarrow{\sigma} & \phi_2(D_2) \\ \phi_1(\mu_1) \downarrow & \searrow \tau & \downarrow \phi_2(\mu_2) \\ \phi_1(D'_1) & \xrightarrow{\sigma'} & \phi_2(D'_2) \end{array}$$

We will not spell out the dg-structure on $\mathcal{D}_1 \times_{\mathcal{E}} \mathcal{D}_2$ in further details. See [BBB13] for the precise definition.

In our geometric situation, $\mathcal{D}_i$ is the category of coherent sheaves on $\mathcal{MC}(\mathbb{L}_i)$ for two fixed Lagrangians $\mathbb{L}_i$ $i = 1, 2$, or the matrix factorization category $\text{MF}(W_i)$ when we deal with weak Maurer-Cartan elements and the resulting Landau-Ginzburg model $(\mathcal{MC}_{\text{weak}}(\mathbb{L}_i), W_i)$. For an isomorphism

$$f : U_1 \subset \mathcal{MC}(\mathbb{L}_1) \rightarrow U_2 \subset \mathcal{MC}(\mathbb{L}_2)$$

(or between subsets of $\mathcal{MC}_{\text{weak}}(\mathbb{L}_i)$), which underlies a quasi-isomorphism

$$\alpha : (\mathbb{L}_1, b_1) \rightarrow (\mathbb{L}_2, f(b_1)), \quad (5)$$

$\mathcal{E}$ is either (dg-enhanced) $D^b\text{Coh}(U_1)$ or $\text{MF}(U_1, W|_{U_1})$ depending on the situation. The functors $\phi_i : \mathcal{D}_i \rightarrow \mathcal{E}$ are naturally induced by the restriction maps in this case.

Theorem 3.1. *We have a natural A_∞ -functor*

$$\text{Fuk}(M) \rightarrow \mathcal{D}_1 \times_{\mathcal{E}} \mathcal{D}_2$$

where $\mathcal{D}_i$ and $\mathcal{E}$ are given as in the explanation above.

Let us fix an inverse $\beta \in CF(\mathbb{L}_2, \mathbb{L}_1)$ of the quasi-isomorphism α given in equation (5). The functor sends a Lagrangian $L \in \text{Fuk}(M)$ to a tuple

$$(CF((\mathbb{L}_1, b_1), L), CF((\mathbb{L}_2, b_2), L), \sigma_\beta : CF((\mathbb{L}_1, b_1), L) \rightarrow f^*CF((\mathbb{L}_2, f(b_1)), L))$$

where b_1 and b_2 vary over the corresponding formal deformation spaces for the first two components, and over U and U' in the third component. Note that both $CF((\mathbb{L}_1, b_1), L)$ and $f^*CF((\mathbb{L}_2, f(b_1)), L)$ are bundles over U_1 . σ_β is a map (homotopy equivalence in A_∞ -language) defined by

$$p \in \mathbb{L}_1 \cap L \mapsto \sum_{i,j \geq 0} \pm m_{i+j+2}(\overbrace{b', \dots, b'}^i, \beta, \overbrace{b, \dots, b}^j, p).$$

Precise sign rules and the construction of higher components of the functor (as well as the proof of the theorem) will appear in a separate paper. Throughout the article, we will only study the (formal) moduli spaces of Lagrangians themselves, and will not discuss these functors further.

We remark that one can also consider gluing of more than two charts using the categorical model for such a situation as in, for e.g., [Wei16].

4. Application I: mirror construction for pair-of-pants decompositions

In this section we will use the gluing method given above to construct a moduli space of Lagrangian immersions in a pair-of-pants decomposition. It was briefly introduced in [Laua] and [Laub]. Here we make a more precise formulation. We shall focus only on punctured Riemann surfaces which already capture the essential ingredients. We will complete the detail in higher dimensions in future work.

Homological mirror symmetry was proved for punctured Riemann surfaces due to the works of Abouzaid-Auroux-Efimov-Katzarkov-Orlov [AAE⁺13], Bocklandt [Boc16], Pascaleff-Sibilla [PS16], Lee [Lee] and Lekili-Polishchuk [LP18]. In this article we put our focus on the construction of the mirror as a moduli space of Lagrangian immersions. Detailed proofs and the study of HMS will be given in a joint work with Cheol-Hyun Cho.

For simplicity let's consider the four-punctured sphere as shown below. The construction for more general pair-of-pants decompositions follows from the same method. (We will clarify a delicate point for general pair-of-pants decompositions in Section 4.4). Let's work over $\mathbb{C}$ at this stage, although later on we shall see that in general it is necessary to work over the Novikov ring Λ_0 (see Section 4.3).

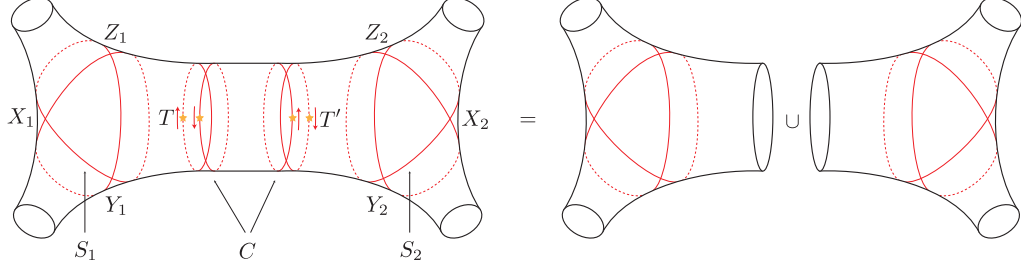


FIGURE 4. A pair-of-pants decomposition of the four-punctured sphere and immersed Lagrangians.

Recall from Section 2.4 that the deformation spaces of a Seidel Lagrangian is given by $(\mathbb{C}^3, W)$. We need to glue the deformation spaces $(\mathbb{C}^3, W)$ of the two Seidel Lagrangians S_1 and S_2 .

To change from S_1 to S_2 , there are two main processes: (partial) smoothing of S_1 (or S_2) to a double circle C , and gauge change of flat connections on C . (See Figure 4.) The gluing we need is

$$(\mathbb{C}^3, W^{S_1}) \xrightarrow{\text{smoothing}} (\mathbb{C}^\times \times \mathbb{C}^2, W^{(C,T)}) \xleftrightarrow{\text{gauge change}} (\mathbb{C}^\times \times \mathbb{C}^2, W^{(C,T')}) \xleftarrow{\text{smoothing}} (\mathbb{C}^3, W^{S_2}).$$

(The notation T and T' are explained below.)

4.1. Choices of gauge change

First we consider gauge change which is more standard. There is a vanishing sphere in the smoothing C corresponding to the immersed point X_1 of S_1 . In this dimension it is simply the union of two points.

Put a flat $\mathbb{C}^\times$ connection on C , which is acting by $t \in \mathbb{C}^\times$ when passing through the two points (in a chosen direction shown in Figure 4). The union of the two points with the prescribed normal orientation is called a gauge cycle, and it is denoted by T . Similarly C can be obtained by smoothing S_2 . Denote the corresponding gauge cycle by T' .

The gauge cycles T and T' are different. Changing the gauge cycles corresponds to gauge change of flat connections. Let's take a homotopy from T to T' by moving the gauge cycles along the Lagrangian immersion C (without jumping across branches). In the process the gauge cycles pass through immersed points Y_0 or Z_0 .

When a gauge cycle moves across an immersed point (say Y_0) of C , we can prove that the Floer theories before and after the move are isomorphic under a non-trivial change of coordinates.

Lemma 4.1. *Suppose a component in the gauge cycle T passes through the immersed point Y_0 (and denote the new gauge cycle by $\tilde{T}$). Denote the objects before and after the move by (C, ∇^{tT}) and $(C, \nabla^{i\tilde{T}})$ where ∇^{tT} and $\nabla^{i\tilde{T}}$ are flat connections with the*

corresponding gauge. Then the objects $(C, \nabla^{tT}, y_0 Y_0 + z_0 Z_0)$ and $(C, \nabla^{\tilde{t}\tilde{T}}, \tilde{y}_0 Y_0 + \tilde{z}_0 Z_0)$ are isomorphic via $\tilde{t} = t$, $\tilde{y}_0 = ty_0$ and $\tilde{z}_0 = z_0$.

Note that there are infinitely many different homotopies between T and T' . For instance we can move one of the gauge points through the immersed point Y , and the other through Z . Then the gluing is $t' = t^{-1}$, $y'_0 = ty_0$ and $z'_0 = tz_0$. (We have $t' = t^{-1}$ instead of $t' = t$, due to the difference of normal orientations of T and T' .)

In general

$$t' = t^{-1}, y'_0 = t^a y_0 \text{ and } z'_0 = t^b z_0$$

for fixed $a, b \in \mathbb{Z}$ with $a + b = 2$. Thus different choices result in different models for the moduli. Nevertheless the critical loci of the superpotentials are isomorphic and the Landau-Ginzburg models are equivalent.

4.2. Smoothing

Now consider the gluing between the deformation spaces of S_1 and C . They are given by $(x_1, y_1, z_1) \in \mathbb{C}^3$ and $(t, y_0, z_0) \in \mathbb{C}^\times \times \mathbb{C}^2$ respectively. (We only use $\mathbb{C}$ -valued deformations for the moment.)

The superpotentials are $x_1 y_1 z_1$ and $ty_0 z_0$ for C and S_1 respectively. Intuitively to match them, we should put $x_1 = t, y = y_0, z = z_0$. Below we justify this by Floer theory. In this example, the gluing map is very simple and we can obtain it by guess; the following gives a more systematic way to find the gluing map.

We claim that $(S_1, x_1 X_1 + y_1 Y_1 + z_1 Z_1)$ and $(C, tT + y_0 Y_0 + z_0 Z_0)$ are ‘pseudo’-isomorphic to each other under a suitable gluing map analogous to above. Note that S_1 and C are not intersecting with each other at all, and so they can never be isomorphic! We will come back to this point in Section 4.3.

The trick is to deform S_1 to S_1^x which intersects C at eight points a_i, b_i, c_i, d_i for $i = 1, 2$ as in the figure. We use cocycle conditions to deduce the gluing between S_1^x and C .

Consider $a_1 + b_1$ in $CF((C, \nabla^{tT}, y_0 Y_0 + z_0 Z_0), (S_1^x, x_1 X_1 + y_1 Y_1 + z_1 Z_1))$ and $c_2 + d_2$ in $CF((S_1^x, x_1 X_1 + y_1 Y_1 + z_1 Z_1), (C, \nabla^{tT}, y_0 Y_0 + z_0 Z_0))$. Consider cocycle conditions on $a_1 + b_1$ and $c_2 + d_2$. First we check that the cocycle conditions are satisfied. Then we show that $(a_1 + b_1, c_2 + d_2)$ gives isomorphisms between the two objects.

Proposition 4.2. *For a suitable choice of S_1^x , we have*

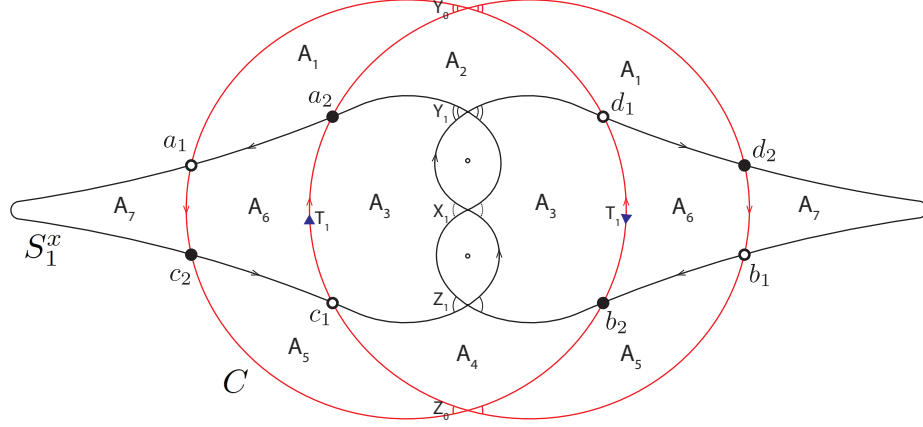
$$m_1^{((C, \nabla^{tT}, y_0 Y_0 + z_0 Z_0), (S_1^x, x_1 X_1 + y_1 Y_1 + z_1 Z_1))}(a_1 + b_1) = 0$$

and

$$m_1^{((S_1^x, x_1 X_1 + y_1 Y_1 + z_1 Z_1), (C, \nabla^{tT}, y_0 Y_0 + z_0 Z_0))}(c_2 + d_2) = 0$$

if and only if $x_1 = t, y_1 = y_0, z_1 = z_0$.

Each transverse Lagrangian intersection point corresponds two complementary Floer generators. Note that in the expressions in Proposition 4.2, for each intersection point at most one of the two complementary generators is taken to be input. In such a situation constant polygons cannot contribute for the same reason given in the proof of Lemma 2.1.


 FIGURE 5. The deformed Seidel Lagrangian S_1^x and the double-circle C .

The holomorphic polygons in Figure 6 contributes to the expressions in Proposition 4.2. The relations $x = t, y_1 = y_0, z_1 = z_0$ make sure their contributions cancel with each other. Any polygon that contributes must have the same Maslov index as these polygons. From the figure we can classify all such holomorphic polygons. In particular immersed polygons do not contribute since their Maslov indices are too high.¹

Similarly we can verify the following.

Proposition 4.3. *Under the same choice of S_1^x as in Prop. 4.2, $m_2(a_1 + b_1, c_2 + d_2) = \mathbf{1}_{S_1^x}$ and $m_2(c_2 + d_2, a_1 + b_1) = \mathbf{1}_C$ if $x_1 = t, y_1 = y_0, z_1 = z_0$.*

In the same way we have the gluing map between $(C, \nabla^{t'T'}, y'_0 Y'_0 + z'_0 Z'_0)$ and $(S_2^x, x_2 X_2 + y_2 Y_2 + z_2 Z_2)$.

¹For orbifolded Riemann surfaces like $\mathbb{P}^1_{a,b,c}$ [Sei11, CHL17], immersed polygons can contribute since the Maslov index receives fractional contributions from the interior orbifold marked points. Here we only consider punctured Riemann surfaces and such polygons do not contribute.

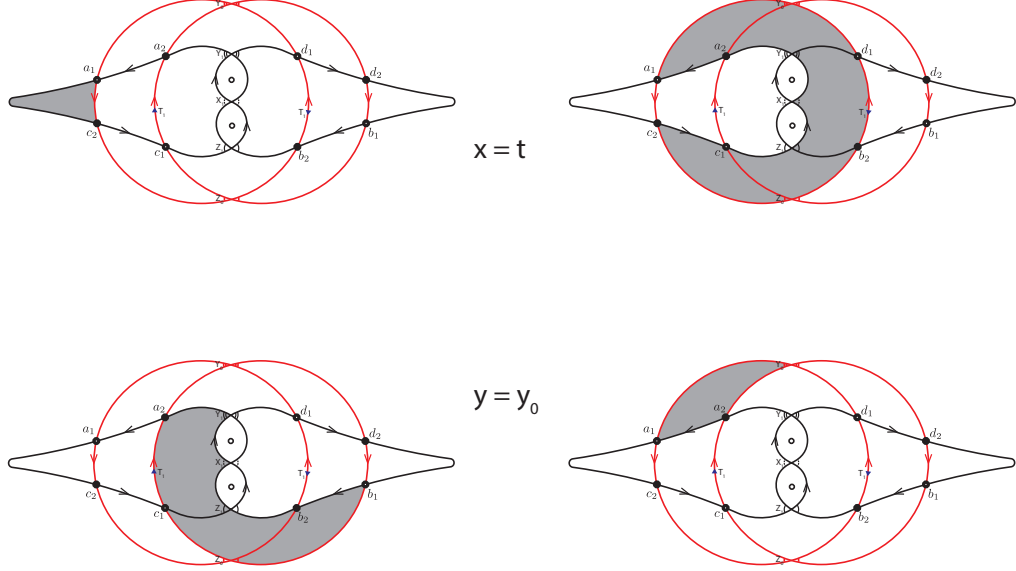


FIGURE 6. Holomorphic strips used in Proposition 4.2.

In conclusion, we have

$$(\mathbb{C}^3, W^{S_1}) \xrightarrow{\text{smoothing}} (\mathbb{C}^\times \times \mathbb{C}^2, W^{(C,T)}) \xrightarrow{\text{gauge change}} (\mathbb{C}^\times \times \mathbb{C}^2, W^{(C,T')}) \xleftarrow{\text{smoothing}} (\mathbb{C}^3, W^{S_2})$$

where the first map is $x_1 = t, y_1 = y_0, z_1 = z_0$, the second map is $t' = t^{-1}, y'_0 = t^a y_0$ and $z'_0 = t^b z_0$, the third map is $x_2 = t', y_2 = y'_0, z_2 = z'_0$.

The maps automatically preserve the superpotentials $W^{S_i} = x_i y_i z_i$, $W^{(C,T)} = t y_0 z_0$ and $W^{(C,T')} = t' y'_0 z'_0$. The resulting moduli space is

$$(\mathcal{O}(-a) \oplus \mathcal{O}(-b), W) \text{ with } a + b = 2$$

which is the Landau-Ginzburg mirror of the four-punctured sphere.

4.3. A paradox

We have shown in the previous section that $(S_1^x, x_1 X_1 + y_1 Y_1 + z_1 Z_1)$ is isomorphic to $(C, \nabla^{tT}, y_0 Y_0 + z_0 Z_0)$ by setting $x_1 = t, y_1 = y_0, z_1 = z_0$. On the other hand, it is not difficult to see that S_1 is isomorphic to S_1^x , see Figure 7 (where (α, β) serves as an isomorphism.)

Thus $(S_1, x_1 X_1 + y_1 Y_1 + z_1 Z_1)$ is isomorphic to $(C, \nabla^{tT}, y_0 Y_0 + z_0 Z_0)$. But S_1 and C never intersect and hence there is no morphism between them, a contradiction!

To resolve this, we need to take a closer look at areas of holomorphic discs, the relation between S_1 and S_1^x , and the relation between S_1^x and C . Following Fukaya-Oh-Ohta-Ono,

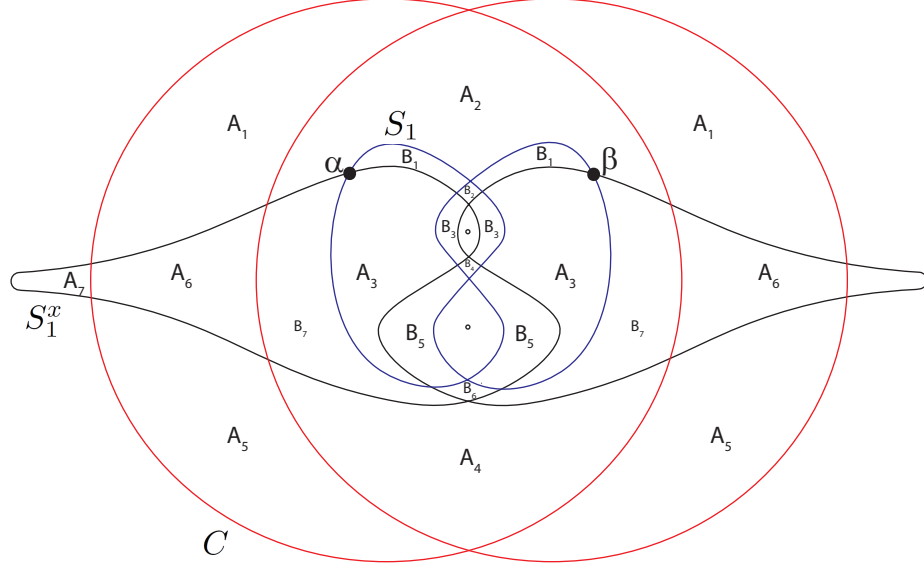


FIGURE 7. The immersions S_1 , S_1^x , and C . A_i denote the areas bounded by S_1^x and C , and B_i denote the areas bounded by S_1 and S_1^x . The figure is topological and does not represent the exact areas.

we use the Novikov ring

$$\Lambda_0 = \left\{ \sum_{i=0}^{\infty} a_i T^{A_i} : a_i \in \mathbb{C}, 0 \leq A_0 \leq A_1 \leq \dots \text{ increasing to } \infty \right\}$$

to filter deformations into different energy levels. Analogously one has the Novikov field Λ (where $A_0 \geq 0$ is not required) and Λ_+ (where $A_0 > 0$ is required).

First let's go back to the relation between $(S_1^x, x_1 X_1)$ and (C, ∇^{tT}) . (For simplicity we take $y_1 = z_1 = y_0 = z_0 = 0$.) The following is a more precise version of Proposition 4.2.

Proposition 4.4. *For a general choice of S_1^x , $(a_1 + b_1, c_2 + d_2)$ provides an isomorphism between (C, ∇^{tT}) and $(S_1^x, x_1 X_1)$ if and only if*

$$t = T^{A_1 + \dots + A_5 - A_7} x_1 \quad (6)$$

where A_i denote the areas bounded by S_1^x and C shown in Figure 7.

In Proposition 4.2 we take $A_1 + \dots + A_5 = A_7$.

Before we have restricted $x_1 \in \mathbb{C}$ for simplicity. Indeed we can take $x_1 \in \Lambda_0$, which makes the deformation space of S_1^x much bigger.

For the Lagrangian C , since ∇^{tT} is a flat connection, we can only allow $t \in \Lambda_{\text{val}=0} = \mathbb{C}^\times \oplus \Lambda_+$ in order to have the Fukaya category well-defined. Thus there is an overlapping region for Equation (6) if and only if

$$A_1 + \dots + A_5 \leq A_7.$$

Thus for arbitrary S_1^x the isomorphism may not exist. (For instance it does not exist for S_1 .) The following proposition gives a summary of the area criterion.

Proposition 4.5. *There exists $x \in \lambda_0$ and $t \in \Lambda_{\text{val}=0}$ such that $(S_1^x, x_1 X_1)$ is isomorphic to (C, ∇^{tT}) if and only if $A_1 + \dots + A_5 \leq A_7$.*

Now consider the relation between S_1 and S_1^x . We have the following.

Proposition 4.6. *(α, β) provides an isomorphism between $(S_1^x, x_1 X_1)$ and $(S_1, x'_1 X'_1)$ if and only if*

$$x_1 = T^{B_7+B_6+B_5-(B_3+B_2+B_1)} x'_1 \quad (7)$$

where B_i denote the areas bounded by S_1^x and S_1 shown in Figure 7.

Let's go back to the situation of Proposition 4.2 where S_1^x is chosen such that $A_1 + \dots + A_5 = A_7$ and hence $x_1 = t$. Note that $B_7 > A_7$ and $B_1 + B_2 + B_3 < A_3 + A_5$. Hence $B_7 + B_6 + B_5 - (B_3 + B_2 + B_1) > 0$. Then Equation (7) forces $\text{val}(x_1) > 0$. Thus $(S_1^x, x_1 X_1)$ is isomorphic to some $(S_1, x'_1 X'_1)$ only if $\text{val}(x_1) > 0$. On the other hand, $(S_1^x, x_1 X_1)$ is isomorphic to some (C, ∇^{tT}) only if $\text{val}(x_1) = 0$.

The two regions $\text{val}(x_1) = 0$ and $\text{val}(x_1) > 0$ are disjoint. As a result, there is no overlapping region where $(S_1, x'_1 X'_1) \cong (S_1^x, x_1 X_1) \cong (C, \nabla^{tT})$ simultaneously. This resolves the paradox.

4.4. Moduli over the Novikov field

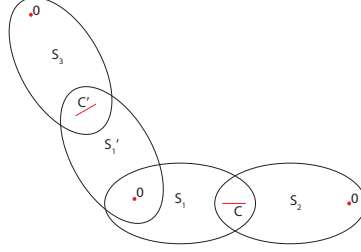
For the four-punctured sphere with two pair-of-pants in the decomposition, we can choose S_1^x and S_2^x as above in the two pair-of-pants such that the gluing $x_1 = t$ and $x_2 = t'$ does not involve the Novikov parameter T .

However suppose there is one more pair-of-pants in the decomposition. Then there are two 'necks' containing the exact double-circles C_1 and C_2 . In the pair-of-pants adjacent to both necks, we have a Seidel Lagrangian S_1^x which satisfies $(S_1^x, x_1 X_1) \cong (C_1, \nabla^{tT})$ under $x_1 = t \in \Lambda_{\text{val}=0}$, and another Seidel Lagrangian S_1^y which satisfies $(S_1^y, y'_1 Y'_1) \cong (C_2, \nabla^{t'T'})$ under $y'_1 = t' \in \Lambda_{\text{val}=0}$.

Unavoidably the gluing between deformation spaces of S_1 and S_1' involves the Novikov parameter as in Proposition 4.6. Thus we obtain a moduli space over the Novikov field Λ . Figure 8 gives a schematic picture of the moduli space.

4.5. Changing stability and flop

Let's briefly discuss stability conditions. Given a punctured Riemann surface, we can take different choices of pair-of-pants decompositions. They correspond to different


 FIGURE 8. The moduli space over Λ .

choices of quadratic differentials. Then we have different sets of stable Lagrangians; the corresponding moduli spaces are related by birational changes.

In this article we only consider geometric stability (rather than categorical stability) which is defined by a quadratic top-form (namely a section of $(\bigwedge^n T^*)^{\otimes 2}$). We define a special Lagrangian (of phase zero) as a smooth map $\iota : \hat{L}^n \rightarrow M^{2n}$ whose differential is injective everywhere except at zeros of the quadratic top-form, with $\iota^*\omega = 0$ and $\iota^*\text{Re}\Omega = 0$. By a domain-dependent Hamiltonian isotopy the map ι is made to avoid the zeros of the quadratic top-form such that its differential is everywhere injective, and it only self-intersects at transverse points. Then any Lagrangian which is quasi-isomorphic to such ι is said to be stable.

Take the four-punctured sphere as an example. Figure 9 depicts two different pair-of-pants decompositions and the corresponding stable Lagrangians. The corresponding moduli spaces are related by a flop.

We can even merge two simple zeros into a double zero of the quadratic differential. The corresponding flow trajectories and stable Lagrangians are depicted on the right of Figure 9. We have a union of two circles playing the role of the Seidel Lagrangian. It was studied in our previous work [CHL]. The resulting moduli space is the non-commutative resolution of the conifold corresponding to a quiver (together with a superpotential).

In [FHLY] we studied “flop” on a Lagrangian fibration for $T^*\mathbb{S}^3$, which results in the Atiyah flop on the mirror resolved conifold. If we take the union of two certain S^3 ’s in $T^*\mathbb{S}^3$ as a reference Lagrangian, then we produced the non-commutative resolution of the conifold. It is a three-dimensional analog of the above example.

Stability conditions for punctured Riemann surfaces were studied by Haiden-Katzarkov-Kontsevich [HKK]. We wish to understand the relation with the work of [HKK] later.

5. Application II: immersed two-sphere and wall-crossing

The two-dimensional immersed sphere given in (a) of Figure 1 leads to wall-crossing phenomena for SYZ Lagrangian fibrations. Fukaya studied this immersion and the relation with the Gross-Siebert program in his talks.

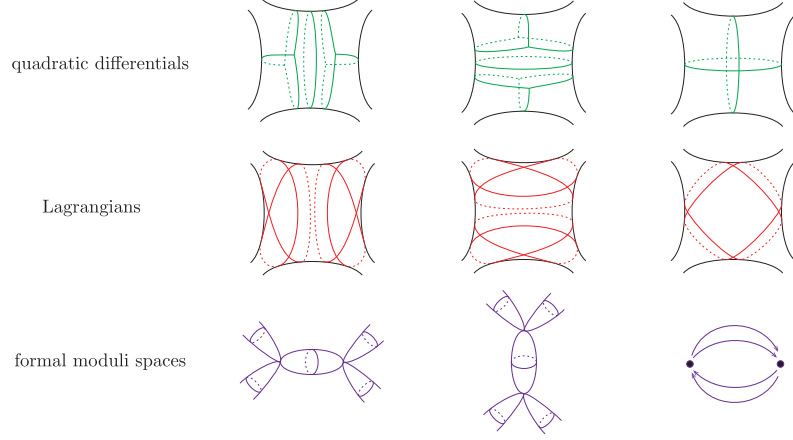


FIGURE 9. Different choices of quadratic differentials leads to different moduli related by flops.

Here we use the gluing method introduced before to understand the relations between the local deformations of a Chekanov torus, the immersed sphere, and a Clifford torus. The gluing between Chekanov and Clifford tori was computed in Seidel's lecture notes [Sei, Prop. 11.8] and is a special case of Pascaleff-Tonkonog [PT]. Thus we will focus on the gluing between the tori and the immersed sphere. More technical detail will be given in a joint work with Yoosik Kim.

One important thing is, we need to deform the Chekanov and Clifford fibers to make all three intersecting with each other appropriately. Otherwise the fibers (of a Lagrangian fibration) do not intersect with each other and there is no morphism between them.

The immersed sphere has exactly one transverse immersed point, which gives two degree-one generators U and V . U and V can be understood as jumpings from one branch to the other, see the arrows in Figure 10. We work with Morse model and pearl trajectories to compute the morphisms between the immersed sphere and the tori.

5.1. A local model

Consider the Hamiltonian S^1 -action on $\mathbb{C}^2$ defined by $(a, b) \mapsto (e^{i\theta}a, e^{-i\theta}b)$. The moment map for this S^1 -action is given by $(a, b) \mapsto |a|^2 - |b|^2$. The symplectic reduction map is $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ given by $f(a, b) = ab$.

We have the Lagrangian fibration $(|ab - \epsilon|, |a|^2 - |b|^2)$ on $\mathbb{C}^2 - \{ab = \epsilon\}$ for a fixed $\epsilon > 0$. It is special with respect to the holomorphic volume form $\Omega = \frac{da \wedge db}{ab - \epsilon}$. The fibers are denoted as $T_{r, \lambda}$ for $r > 0$ and $\lambda \in \mathbb{R}$. SYZ and the wall-crossing phenomenon for this example was studied by Auroux [Aur07], which serves as the basis of SYZ for more general cases [Aur09, CLL12, AAK16].

There is exactly one singular fiber $\mathbb{L} = T_{\epsilon,0}$, which is the two-dimensional immersed sphere given in (a) of Figure 1. We will focus on this local model in this section. The most generic wall-crossing in higher dimensions can be understood by considering the product of this Lagrangian fibration with the trivial torus bundle $(\mathbb{C}^\times)^k \rightarrow \mathbb{R}^k$.

5.2. Deformation spaces

Let $\mathbb{L}_1, \mathbb{L}_2 \subset \mathbb{C}^2 - \{ab = \epsilon\}$ be the Lagrangian tori which are the intersections of $\{|a|^2 = |b|^2\}$ and the f -preimages of the circles in Figure 10. They are Lagrangian isotopic to a Chekanov fiber $T_{r_1, \lambda=0}$ and a Clifford fiber $T_{r_2, \lambda=0}$ respectively where $r_1 < \epsilon < r_2$.

We deform $\mathbb{L}_i$ (for $i = 1, 2$) formally by flat $\Lambda_0^\times = \mathbb{C}^\times \oplus \Lambda_+$ connections. (We can allow flat $\Lambda_0^\times$ connections since it does not harm the convergence of m_k -operations. Their inverses live in the same space $\mathbb{C}^\times \oplus \Lambda_+$, and hence no negative power of T appears in m_k -operations.) The connections are parametrized by (x, y) and (x', y') for $\mathbb{L}_1, \mathbb{L}_2$ respectively.

We fix the gauge by choosing gauge hyper-tori as depicted in Figure 10. The parallel transport is given by multiplying y (for a path in $\mathbb{L}_1$) or y' (for a path in $\mathbb{L}_2$) when a path goes across the corresponding hyper-torus. For the fiber direction holonomy (x or x'), the associated hyper-tori are given in Figure 12 for x . (See [CHL14] for more details for gauge hypertori.)

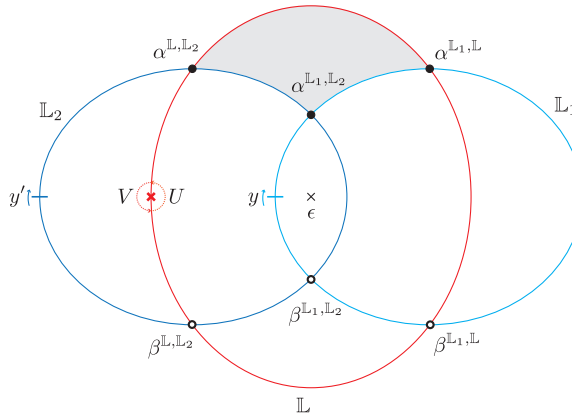


FIGURE 10. Base paths for $\mathbb{L}_1, \mathbb{L}_2$ and $\mathbb{L}$

Now consider the formal deformations $b = uU + vV$ for the immersed sphere $\mathbb{L}$, where U, V are the degree-one immersed generators. (See [CHL17] or [CHL] for more details for formal deformation spaces of Lagrangian immersions). $u, v \in \Lambda_+$ ensures Novikov convergence, namely there are only finitely many terms under every energy level. First we observe that we can allow $(u, v) \in \Lambda_0 \times \Lambda_+ \cup \Lambda_+ \times \Lambda_0$. It is because $\mathbb{L}$ only bounds

constant holomorphic polygons whose corners must be in alternating pattern $U, V, \dots$. Observe that uv still lies in Λ_+ even if we allow one of u, v in Λ_0 . Thus we have the following.

Lemma 5.1. *For $(u, v) \in (\Lambda_0 \times \Lambda_+) \cup (\Lambda_+ \times \Lambda_0)$, the A_∞ operations for $\{\mathbb{L}^{uU+vV}, \mathbb{L}_1, \mathbb{L}_2\}$ converge.*

Next, we observe that $\mathbb{L}$ is unobstructed, namely

$$m_0^b = m_0(1) + m_1(b) + m_2(b, b) + \dots \quad (8)$$

vanishes. m_0^b has degree 2 which must be proportional to the point class. Since there is no non-constant polygon bounded by $\mathbb{L}$, it is merely contributed by constant polygons supported at the self intersection point of $\mathbb{L}$ whose corners are $U, V, \dots, U, V$ or $V, U, \dots, V, U$. The constant polygons with corners $U, V, \dots, U, V$ cancel with that with $V, U, \dots, V, U$ and so $m_0^b = 0$. More detail about the cancellation will be given in a joint work with Yoosik Kim.

Lemma 5.2. *The formal deformation $b = uU + vV$ of $\mathbb{L}$ is unobstructed.*

For instance, the constant bigons contribute

$$m_2(uU, vV) = uv[\text{pt}], \quad m_2(vV, uU) = -uv[\text{pt}],$$

and hence they cancel with each other in (8).

5.3. Gluing deformation spaces

By analyzing cocycle conditions contributed from holomorphic strips bounded between two of $(\mathbb{L}_1, \nabla^{x,y})$, $(\mathbb{L}_2, \nabla^{x',y'})$ and $(\mathbb{L}, uU + vV)$, we have the following.

Proposition 5.3. *There are isomorphisms among $(\mathbb{L}_1, \nabla^{x,y})$, $(\mathbb{L}_2, \nabla^{x',y'})$ and $(\mathbb{L}, uU + vV)$ given as follows.*

- $(\mathbb{L}_1, \nabla^{x,y}) \cong (\mathbb{L}, uU + vV)$ if and only if

$$x = uv - 1, \quad y^{-1} = u$$

where $u, y \in \Lambda_0^\times$, $v \in \Lambda_+$, $x \in -1 + \Lambda_+$.

- $(\mathbb{L}_2, \nabla^{x',y'}) \cong (\mathbb{L}, uU + vV)$ if and only if

$$x' = uv - 1, \quad y' = v$$

where $v, y' \in \Lambda_0^\times$, $u \in \Lambda_+$, $x' \in -1 + \Lambda_+$.

- $(\mathbb{L}_1, \nabla^{x,y}) \cong (\mathbb{L}_2, \nabla^{x',y'})$ if and only if

$$x = x', \quad y' = y(x + 1)$$

where $y, y' \in \Lambda_0^\times$, $x, x' \in k + \Lambda_+$ for $k \in \mathbb{C}^\times - \{-1\}$.

Now we glue the three deformation spaces accordingly. Namely we take the disjoint union of $(\Lambda_0 \times \Lambda_+) \cup (\Lambda_+ \times \Lambda_0)$ (with coordinates (u, v)) and two copies of $\Lambda_0^\times \times \Lambda_0^\times$ (with coordinates (x, y) and (x', y') respectively), and take the quotient according to the above relations.

To better understand the topology, we should consider pseudo-deformations and pseudo-isomorphisms. Namely, we take the disjoint union of $\Lambda \times \Lambda \supset (\Lambda_0 \times \Lambda_+) \cup (\Lambda_+ \times \Lambda_0)$ and two copies of $\Lambda^\times \times \Lambda^\times \supset \Lambda_0^\times \times \Lambda_0^\times$, and take the quotient according to the same relations. The above moduli space is a subset of this. The advantage of extending by pseudo-deformations is that the gluing regions become open: they are $\Lambda^\times \times \Lambda^\times$ (in coordinates (x, y) or (x', y')) between $\mathbb{L}_i$ and $\mathbb{L}$, and $(\Lambda^\times - \{-1\}) \times \Lambda^\times$ between $\mathbb{L}_1$ and $\mathbb{L}_2$. We have the quotient topology on the bigger space which restricts to give a topology on the moduli space.

The gluing is illustrated in Figure 11. The upper component of the degenerate cone $\{uv = 0\}$ is a (partial) compactification of $\{x = -1\} \cong \mathbb{C}^\times$ by adding the point $\{x = -1, y = 0\}$, and similar for the lower component. The resulting moduli space is

$$\{(u, v) \in \Lambda_0 \times \Lambda_0 : uv \notin 1 + \Lambda_+\}.$$

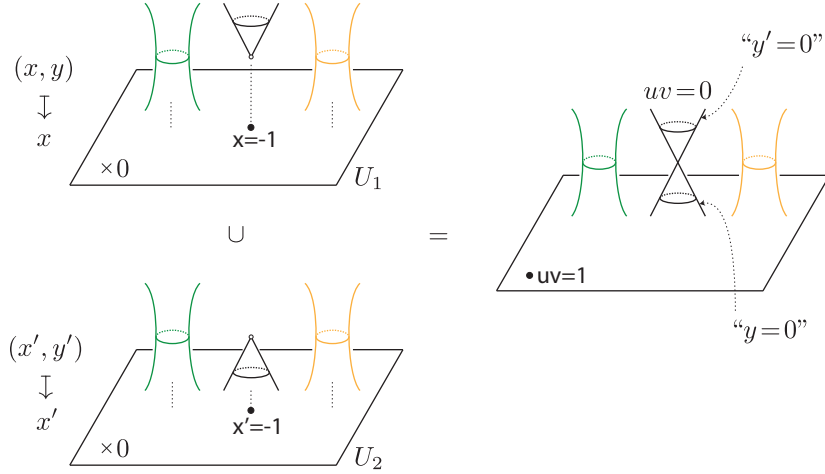


FIGURE 11. Gluing of two charts

In the following we sketch the proof of Proposition 5.3 by counting holomorphic strips that contribute to cocycle conditions.

5.4. Cocycle conditions and isomorphisms

The relation between $\mathbb{L}_1$ and $\mathbb{L}_2$ has been studied in Seidel's lecture notes [Sei] and also used by the work of Pascaleff-Tonkonog [PT]. Here we focus on the relation between $\mathbb{L}$ and $\mathbb{L}_1$ (and that between $\mathbb{L}$ and $\mathbb{L}_2$ is similar).

$\mathbb{L}_1$ and $\mathbb{L}$ intersect along two disjoint circles. See Figure 12. We perturb $\mathbb{L}_1$ by a Hamiltonian so that they intersect transversely at four points, which are denoted by

$$\alpha_0^{\mathbb{L}_1, \mathbb{L}}, \alpha_1^{\mathbb{L}_1, \mathbb{L}}, \beta_1^{\mathbb{L}_1, \mathbb{L}}, \beta_2^{\mathbb{L}_1, \mathbb{L}}.$$

The subscripts denote degrees of the morphisms.

We compute $d = m_1^{\nabla^{x,y}, b}$ of the unique degree-zero generator $\alpha_0^{\mathbb{L}_1, \mathbb{L}}$ in the Floer complex $CF((\mathbb{L}_1, \nabla^{x,y}), (\mathbb{L}, b = uU + vV))$ by counting pearl trajectories. There are two strips from $\alpha_0^{\mathbb{L}_1, \mathbb{L}}$ to $\beta_1^{\mathbb{L}_1, \mathbb{L}}$ whose projections under f are the shaded regions in (a) of Figure 12. We have chosen $\mathbb{L}_1$ such that the two strips have the same symplectic area Δ . We have

$$\langle d(\alpha_0^{\mathbb{L}_1, \mathbb{L}}), \beta_1^{\mathbb{L}_1, \mathbb{L}} \rangle = T^\Delta (1 - uy). \quad (9)$$

The strip on the right side gives T^Δ , while the strip on the left side gives $T^\Delta uy$ because it passes through the immersed point of $\mathbb{L}$ and the y -hypertorus.

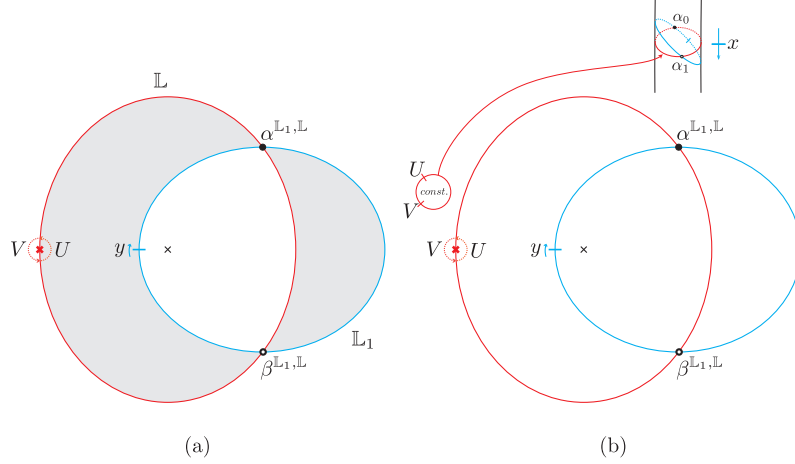


FIGURE 12. Pearl trajectories contributing to $d(\alpha^{\mathbb{L}_1, \mathbb{L}})$

For $\langle d(\alpha_0^{\mathbb{L}_1, \mathbb{L}}), \alpha_1^{\mathbb{L}_1, \mathbb{L}} \rangle$, we also have a pair of strips in $f^{-1}(\alpha^{\mathbb{L}_1, \mathbb{L}})$ contributing to it. One of the strips gives ± 1 , and the other gives $\pm x$ since it passes through the x -hypertorus. Additionally we have pearl trajectories consisting of a strip, Morse flow lines in (the

normalization of) $\mathbb{L}$, and constant polygons with corners $U, V, \dots, U, V$. The pearl trajectories form power series in uv . Hence

$$\langle d(\alpha_0^{\mathbb{L}_1, \mathbb{L}}), \alpha_1^{\mathbb{L}_1, \mathbb{L}} \rangle = xh(uv) + g(uv)$$

where g, h are power series with leading term ± 1 . In summary

$$d(\alpha_0^{\mathbb{L}_1, \mathbb{L}}) = T^\Delta (1 - uy) \beta_1^{\mathbb{L}_1, \mathbb{L}} + (xh(uv) + g(uv)) \alpha_1^{\mathbb{L}_1, \mathbb{L}}.$$

The cocycle condition gives

$$\begin{cases} x + H(uv) = 0 \\ y^{-1} = u \end{cases} \quad (10)$$

where $H = g/h$. One can check that the degree zero morphism in $CF(\mathbb{L}, \mathbb{L}_1)$ (which is the one dual to $\beta_2^{\mathbb{L}_1, \mathbb{L}}$) is also d -closed if and only if the same condition is satisfied. Moreover, $\alpha_0^{\mathbb{L}_1, \mathbb{L}}$ gives an isomorphism under the condition 10.

$H(uv)$ can be derived from the relations between $\mathbb{L}, \mathbb{L}_1, \mathbb{L}_2$.

Lemma 5.4. *We have $H(uv) = 1 - uv$ and hence $x = uv - 1$.*

Proof. We have chains of isomorphisms

$$(\mathbb{L}_1, (x, y)) \xrightarrow{\alpha_0^{\mathbb{L}_1, \mathbb{L}}} (\mathbb{L}, uU + vV) \xrightarrow{\alpha_0^{\mathbb{L}, \mathbb{L}_2}} (\mathbb{L}_2, (x', y')).$$

The composition of these two isomorphisms is given by

$$m_2(\alpha_0^{\mathbb{L}_1, \mathbb{L}}, \alpha_0^{\mathbb{L}, \mathbb{L}_2}) = T^{\Delta'} \alpha_0^{\mathbb{L}_1, \mathbb{L}_2}$$

where Δ' is the symplectic area of the triangle projecting down to the shaded region in Figure 10. This implies that $\alpha_0^{\mathbb{L}_1, \mathbb{L}_2}$ is also an isomorphism. Therefore, coordinate changes among three objects must be compatible (over the triple intersection). Using

$$x' = x, \quad y' = y(x + 1)$$

we get

$$1 - H(uv) = x + 1 = \frac{y'}{y} = \frac{v}{u^{-1}} = uv.$$

□

In conclusion, the moduli space of smooth Lagrangian tori has a puncture $u = v = 0$. We can glue in the deformation space of the immersed sphere to partially compactify and fill in this puncture. It also applies to generic singular fibers of a Lagrangian fibration in higher dimensions by taking a direct product with a torus factor.

In general situations, the deformation space of a singular Lagrangian is noncommutative, and our gluing technique will produce a noncommutative space. It is already manifested in the one-dimensional example in Figure 9. This is very interesting and we will further investigate it in future work.

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