

Lagrangian fillings and complicated Legendrian unknots

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ABSTRACT. An exact Lagrangian submanifold L in the symplectization of standard contact $(2n - 1)$ -space with Legendrian boundary Σ can be glued to itself along Σ . This gives a Legendrian embedding $\Lambda(L, L)$ of the double of L into contact $(2n + 1)$ -space. We show that the Legendrian isotopy class of $\Lambda(L, L)$ is determined by formal data: the manifold L together with a trivialization of its complexified tangent bundle. In particular, if L is a disk then $\Lambda(L, L)$ is the Legendrian unknot.

1. Introduction

If Σ is a closed Legendrian $(n - 1)$ -submanifold in standard contact $\mathbb{R}^{2n-1}$, and L_0 and L_1 are exact Lagrangian fillings of Σ in the symplectization $S\mathbb{R}^{2n-1} = \mathbb{R} \times \mathbb{R}^{2n-1}$, then one can glue compact versions of L_0 and L_1 along Σ to form a Legendrian submanifold $\Lambda(L_0, L_1)$ in standard contact $\mathbb{R}^{2n+1}$. The Legendrian $\Lambda(L_0, L_1)$ has the same Reeb chords as the original Legendrian Σ , but with Conley-Zehnder indices shifted up by 1. In [1], the second author used this construction for Legendrian spheres Σ with distinct disk fillings D_0 and D_1 and originally claimed that $\Lambda(D_0, D_1)$ were non-loose. These spheres were however later shown by Murphy to be loose. We present her proof in Section 4.

The Chekanov-Eliashberg algebra of a loose Legendrian is trivial. In [1], it was shown that the Chekanov-Eliashberg dg-algebra for the double $\Lambda(D_0, D_0)$ obtained by gluing a Lagrangian disk filling to itself is quasi-isomorphic to that of the standard Legendrian unknot, but the analysis there left open the question of whether $\Lambda(D_0, D_0)$ is actually Legendrian isotopic to the unknot. We prove here that this is indeed the case, see Corollary 1.2. We point out that the Reeb chord configuration of Σ and hence of the double $\Lambda(D_0, D_0)$ can be involved and that when this is the case, the Legendrian isotopy to the unknot must be rather complicated.

Disk fillings are special cases of Lagrangian fillings of more general topology where a similar result holds that we describe next. To state the result, recall that a Lagrangian embedding of a manifold L in $S\mathbb{R}^{2n-1}$ induces a trivialization of $TL \otimes \mathbb{C}$ that is well-defined up to homotopy.

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Theorem 1.1. *If $L \subset \mathbb{S}\mathbb{R}^{2n-1}$ is an embedded Lagrangian submanifold with Legendrian boundary, then the Legendrian isotopy class of $\Lambda(L, L) \subset \mathbb{R}^{2n+1}$ is determined by the induced trivialization of $TL \otimes \mathbb{C}$.*

Corollary 1.2. *If $D \subset \mathbb{S}\mathbb{R}^{2n-1}$ is an embedded Lagrangian disk with Legendrian boundary, then $\Lambda(D, D)$ is the Legendrian unknot.*

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2. Gluing two Lagrangian fillings

In this section we review the gluing construction from [1, Section 2.1] in notation convenient for our study here.

2.1. Basic notation for contact structures and exact Lagrangians

Let $n \geq 1$ and consider $\mathbb{R}^{2n-1}$ with coordinates $(q_1, p_1, \dots, q_{n-1}, p_{n-1}, \zeta)$ and the contact structure

$$\eta = \ker \left(\delta\zeta - \sum_{i=1}^{n-1} p_i \delta q_i \right),$$

and its symplectization $\mathbb{S}\mathbb{R}^{2n-1} = \mathbb{R} \times \mathbb{R}^{2n-1}$ with the canonical Liouville form

$$\lambda_{\text{can}} = e^t \left(\delta\zeta - \sum_{i=1}^{n-1} p_i \delta q_i \right),$$

where t is a standard coordinate on the additional $\mathbb{R}$ -factor in $\mathbb{S}\mathbb{R}^{2n-1}$. We write X_{can} for the canonical Liouville vector field on $\mathbb{S}\mathbb{R}^{2n-1}$, $X_{\text{can}} = \partial_t$, and ω_{can} for the canonical symplectic form, $\omega_{\text{can}} = \delta\lambda_{\text{can}}$.

Recall further that an *exact Lagrangian cobordism* in $\mathbb{S}\mathbb{R}^{2n-1}$ is a proper embedding $\phi: L \rightarrow \mathbb{S}\mathbb{R}^{2n-1}$ such that $\phi^*\lambda_{\text{can}} = \delta h$ for some compactly supported function $h: L \rightarrow \mathbb{R}$ (h is called a primitive of λ_{can}). This implies that for all sufficiently large $a > 0$, $\phi(L)$ coincides with the symplectization of a closed Legendrian submanifold Σ_- (resp. Σ_+) $(-\infty, -a) \times \Sigma_-$ in $(-\infty, -a) \times \mathbb{R}^{2n-1}$ (resp. $(a, \infty) \times \Sigma_+$ in $(a, \infty) \times \mathbb{R}^{2n-1}$). (In the case when Σ_- is empty, we say that L is an exact Lagrangian *filling* of Σ_+ .) It follows in particular that L is diffeomorphic to a compact cobordism $\bar{L}$ with cylindrical ends $(-\infty, -a] \times \Sigma_-$ and $[a, +\infty) \times \Sigma_+$ attached to the boundary $\partial\bar{L} = \Sigma_- \cup \Sigma_+$.

Remark 2.1. We will also use the notion *exact Lagrangian cobordism with Legendrian boundary* when the Lagrangian L and the ambient exact symplectic manifold W are compact manifolds with boundary. This then means that L is tangent to the Liouville vector field of W near the boundary ∂W . Let $L \subset W$ be a compact Lagrangian cobordism with Legendrian boundary consisting of $\Lambda_+ \subset \partial_+ W$ and $\Lambda_- \subset \partial_- W$, where $\partial_+ W$ (resp. $\partial_- W$)

denotes the convex (resp. concave) boundary of W where the Liouville vector fields points out of (resp. into) W . Then (W, L) can be canonically *completed* by attaching cylindrical ends $([0, \infty) \times \partial_+ W, [0, \infty) \times \Lambda_+)$ (resp. $((-\infty, 0] \times \partial_- W, (-\infty, 0] \times \Lambda_-)$) to the convex (resp. concave) boundary of (W, L) . In this way, the compact and non-compact cases are canonically related and we will use the same notation in both cases. For example, a Hamiltonian isotopy between compact Lagrangian cobordisms, means a Hamiltonian isotopy of contact type at infinity between their completions.

Consider next $\mathbb{R}^{2n+1}$ with coordinates $(x_1, y_1, \dots, x_n, y_n, z)$ and introduce polar coordinates on the last coordinate pair:

$$(x_n, y_n) = (r \cos \theta, r \sin \theta), \quad (r, \theta) \in [0, \infty) \times \mathbb{R}/2\pi\mathbb{Z}.$$

Using these coordinates, we equip $\mathbb{R}^{2n+1}$ with the contact form

$$\alpha = \delta z - \sum_{i=1}^{n-1} y_i \delta x_i + \frac{1}{2} r^2 \delta \theta,$$

and the corresponding contact structure $\xi = \ker \alpha$.

We will occasionally discuss front projections, and for that purpose we also consider the contact form

$$\alpha' = \delta z - \sum_{i=1}^{n-1} y_i \delta x_i - y_n \delta x_n,$$

with corresponding contact structure $\xi' = \ker \alpha'$. The diffeomorphism ψ of $\mathbb{R}^{2n+1}$ given by,

$$\psi(x_1, y_1, \dots, x_n, y_n, z) = \left(x_1, y_1, \dots, x_n, y_n, z + \frac{x_n y_n}{2} \right),$$

satisfies $\psi^* \alpha' = \alpha$ and is thus a contactomorphism intertwining the contact structures ξ and ξ' .

2.2. The gluing construction

We now recall the construction from [1, Section 2.1] (with slight notational changes). Pick a smooth function $\rho: [-1, 1] \rightarrow \mathbb{R}$ with the following properties:

- (1) ρ is even,
- (2) $\rho(u) = 1 - |u|$ near $u = \pm 1$,
- (3) $\rho > 0$ and its derivative ρ' satisfies $\rho' < 0$ on $(0, 1)$.

Use coordinates on $\mathbb{R}^{2n+1}$ as in Section 2.1 and consider the hypersurface W_ρ in $\mathbb{R}^{2n+1} \cap \{0 < x_n < 1\}$ given by,

$$W_\rho = \left\{ z = \left(\frac{\rho(x_n)}{\rho'(x_n)} - \frac{x_n}{2} \right) y_n \right\}.$$

The restriction of α to W_ρ is then

$$\lambda_\rho := \alpha|_{W_\rho} = \frac{\rho(x_n)}{\rho'(x_n)} \delta y_n - y_n \frac{\rho(x_n) \rho''(x_n)}{(\rho'(x_n))^2} \delta x_n - \sum_{i=1}^{n-1} y_i \delta x_i,$$

with corresponding Liouville vector field

$$X_\rho = \frac{\rho(x_n)}{\rho'(x_n)} \partial_{x_n} + y_n \frac{\rho(x_n) \rho''(x_n)}{(\rho'(x_n))^2} \partial_{y_n} + \sum_{i=1}^{n-1} y_i \partial_{y_i},$$

that is gradient-like for the function $-x_n$. Near $x_n = 1$, we have $\rho(x_n) = 1 - x_n$ and thus

$$\lambda_\rho = (x_n - 1) \delta y_n - \sum_{i=1}^{n-1} y_i \delta x_i.$$

Use coordinates on $\mathbb{R}^{2n-1}$ and $\mathbb{S}\mathbb{R}^{2n-1}$ as in Section 2.1 and consider the embedding germ $W_\rho \rightarrow \mathbb{S}\mathbb{R}^{2n-1}$ along $\{x_n = 1\}$ given by

$$\begin{aligned} t &= \log(1 - x_n), & \zeta &= -y_n, \\ q_j &= x_j, & p_j &= \frac{y_j}{1 - x_n}, \quad j = 1, \dots, n-1. \end{aligned}$$

The pull-back of λ_{can} under this germ equals λ_ρ . Since X_ρ is gradient like for $-x_n$, the germ extends to an embedding $\phi_\rho: W_\rho \rightarrow \mathbb{S}\mathbb{R}^{2n-1}$ with the property $\phi_\rho^* \lambda_{\text{can}} = \lambda_\rho$ and the extension is unique. A straightforward calculation shows that the image of the embedding ϕ_ρ is $\{t < \log \rho(0)\}$.

Consider an exact Lagrangian filling $L \subset \mathbb{S}\mathbb{R}^{2n-1}$ of $\Sigma \subset \mathbb{R}^{2n-1}$. We first push L along the Liouville flow for sufficiently large negative time to ensure that it is cylindrical in a neighborhood of the region $\{t \geq \ln \rho(0)\}$. Let h denote the compactly supported primitive of λ_{can} on L . Then push $\phi_\rho^{-1}(L)$ along the flow of ∂_z during the time $-h$. This gives a Legendrian submanifold $\Lambda_\rho^+(L) \subset \mathbb{R}^{2n+1}$.

The diffeomorphism $\tau: \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$,

$$\tau(x_1, y_1, \dots, x_{n-1}, y_{n-1}, x_n, y_n, z) = (x_1, y_1, \dots, x_{n-1}, y_{n-1}, -x_n, -y_n, z)$$

satisfies $\tau^* \alpha = \alpha$. Thus $\tau(\Lambda_\rho^+(L))$ is Legendrian and we define $\Lambda_\rho^-(L) := \tau(\Lambda_\rho^+(L))$.

Since ρ is even it follows that if L and L' are both exact Lagrangian fillings of Σ then the union $\Lambda_\rho^-(L) \cup \Sigma \cup \Lambda_\rho^+(L')$ is a smooth Legendrian submanifold diffeomorphic to the manifold resulting from gluing L and L' along Σ . We write

$$\Lambda_\rho(L, L') := \Lambda_\rho^-(L) \cup \Sigma \cup \Lambda_\rho^+(L').$$

If the function ρ is deformed in the convex set of functions satisfying conditions (1) – (3) above, then the Legendrian $\Lambda_\rho(L, L')$ varies by Legendrian isotopy. We therefore often drop the subscript ρ from the notation and write simply $\Lambda(L, L')$ when we discuss the Legendrian isotopy class.

Remark 2.2. If we pick the function ρ so that $\rho(u) = 1 - |u|$ in the region where L is cylindrical, then for each branch $\{\zeta = h(q_1, \dots, q_{n-1})\}$ of the front projection of Σ , the corresponding branch for $\psi(\Lambda_\rho^+(L))$ near $x_n = 0$ satisfies $z = \rho(x_n)h(x_1, \dots, x_{n-1})$. Hence the function ρ scales the front projection of Σ and the smoothing near $\{x_n = 0\}$ can alternatively be described in terms of front projections. In particular, the construction from [1, Section 2.1] is a special case of the one presented above.

3. Proofs of Theorem 1.1 and Corollary 1.2

We will use notation as in Sections 2.1 and 2.2. Fix a function ρ which satisfies (1)–(3) in Section 2.2 and choose it to be concave on $(0, 1)$.

3.1. Proof of Theorem 1.1

Step 1: A preliminary Legendrian isotopy.

Note that the vector field ∂_θ on $\mathbb{R}^{2n+1}$ is transverse to the hypersurface $W_\rho \subset \mathbb{R}^{2n+1}$. For $s \in [0, 1]$, consider the contact form on $\mathbb{R}^{2n+1}$ given by

$$\alpha_s = \frac{1}{1-s+sr^2/2} \left(\delta z + \frac{r^2}{2} \delta \theta \right) - \sum_{j=1}^{n-1} y_j \delta x_j,$$

with corresponding Reeb vector field

$$R_s = (1-s)\partial_z + s\partial_\theta.$$

Let $\lambda_s = \alpha_s|_{W_\rho}$ be the restriction of α_s to W_ρ . Flowing along the Liouville vector field $X_{\text{can}} = \partial_t$ of $\mathbb{S}\mathbb{R}^{2n-1}$ gives an isotopy of embeddings $\psi_s: W_\rho \rightarrow \mathbb{S}\mathbb{R}^{2n-1}$ with the following properties:

$$\psi_0 = \phi_\rho, \quad \psi_s^* \lambda_{\text{can}} = \lambda_s, \quad 0 \leq s \leq 1, \quad \text{and} \quad \psi_1 \text{ is surjective.}$$

Flowing $\psi_s(L)$ for time $-h$ along the Reeb vector field R_s produces a Legendrian isotopy $\Lambda_s^+(L)$, $0 \leq s \leq 1$, that starts at $\Lambda_0^+(L) = \Lambda_\rho^+(L)$. We have a directly analogous Legendrian isotopy that starts instead from $\Lambda_\rho^-(L)$. In order for these isotopies to glue to an isotopy of $\Lambda_\rho(L, L)$ we must make sure that the two halves do not intersect during the isotopy. It is easy to see that they do not intersect if the function h is sufficiently small since then $\Lambda_s^\pm(L)$ lies in $\{\pm x_n > 0\}$ for all $s \in [0, 1]$. If we flow L along the Liouville flow in negative time then the function h decreases exponentially. It follows that the desired isotopy exists.

Step 2: Construction of a pre-Lagrangian filling.

After this preliminary isotopy, we observe that $\Lambda(L, L) \subset \mathbb{R}^{2n+1}$ bounds a pre-Lagrangian submanifold (see [2] for this notion) Δ which is foliated by $\varphi_{\partial_\theta}^t(\Lambda^+(L))$ for $t \in [0, \pi]$, see Figure 1. The singular set of the characteristic foliation of Δ is precisely $\Sigma = \Delta \cap \{r = 0\}$.

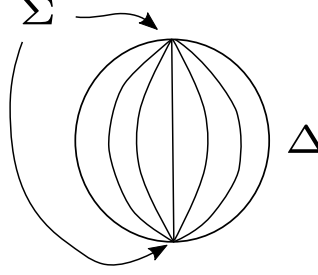


FIGURE 1. The pre-Lagrangian disk Δ with its singular Legendrian foliation. The singular set Σ is an equatorial sphere in $\partial\Delta$.

Step 3: Isotoping pre-Lagrangian fillings

Any two Lagrangian embeddings (with Legendrian boundary) of L lead to pre-Lagrangian fillings Δ with diffeomorphic Legendrian foliations. We show that they are contact isotopic.

Consider two such embeddings $f, g: \Delta \rightarrow \mathbb{R}^{2n+1}$. By assumption, the induced trivializations of $TL \otimes \mathbb{C}$ are homotopic and Gromov's h-principle for Legendrian immersions implies that the restrictions $f|_{L_0}$ and $g|_{L_0}$ to the central leaf L_0 of Δ are Legendrian regularly homotopic. By general position, the restriction of this regular homotopy to the codimension 1 skeleton of L_0 is an isotropic isotopy. Retracting L_0 to its codimension 1 skeleton, we then find that $f|_{L_0}$ and $g|_{L_0}$ are Legendrian isotopic. Extending this Legendrian isotopy to an ambient contact isotopy, we reduce to the case that $f|_{L_0} = g|_{L_0}$.

Assume then that $f|_{L_0} = g|_{L_0}$. We first deform g slightly near the singular set Σ of the Legendrian foliation of Δ so that f and g coincide there as follows. Pick a germ of Legendrian foliation that contains L_0 as a leaf and is transverse to all leaves of $f(\Delta)$ and of $g(\Delta)$ near Σ . The corresponding front projections of these leaves near Σ then consist of families of graphs of functions (all vanishing to first order on Σ) and a linear interpolation between these families give the desired isotopy near Σ .

We consider next a 1-jet space neighborhood of L_0 , here nearby leaves correspond to a family of 1-jet graphs of functions $h_t: L_0 \rightarrow \mathbb{R}$ which are increasing with respect to t and coincide near ∂L_0 . Linear interpolation connects any two such families of functions within the class of families that are increasing in t . Using the corresponding isotopies, we may then assume that f and g agree near the central leaf L_0 .

To finish the proof, identify $\Delta \setminus \Sigma$ with $L_0 \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ with the foliation given by $L_0 \times \{\theta\}$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, and consider the isotopy $\phi_s: \Delta \setminus \Sigma \rightarrow \Delta \setminus \Sigma$ given by

$$\phi_s(x, \theta) = (x, (1 - s + s\varepsilon)\theta).$$

For $\varepsilon > 0$ small enough, f and g agree on $\phi_1(\Delta \setminus \Sigma)$. The isotopy $f \circ \phi_s$ extends to a contact isotopy F_s since it has a fixed characteristic foliation. Similarly, $g \circ \phi_s$ extends to a contact isotopy G_s . Moreover, we may assume that $F_s = G_s$ near $f(\Sigma) = g(\Sigma)$ since f

and g agree there. Then $F_s^{-1} \circ G_s \circ g$ equals g for $s = 0$ and f for $s = 1$, and $F_s^{-1} \circ G_s$ extends smoothly to $f(\Sigma)$ by the identity. $\square$

3.2. Proof of Corollary 1.2

Since all trivializations of $TD \otimes \mathbb{C}$ are homotopic, Theorem 1.1 shows that the Legendrian isotopy class of $\Lambda(D, D)$ is independent of D . Consider the case when D is a trivial disk filling. Using the construction in [1, Section 2.1], see Remark 2.2, we represent D as half of the front of the standard unknot and then see directly that $\Lambda(D, D)$ is an unknot. $\square$

4. Loose Legendrian spheres

In this section we present Emmy Murphy's argument showing that $\Lambda(D_0, D_1)$ is loose for distinct disk fillings of the Legendrian knot in the topological isotopy class $\overline{9_{46}}$. The knot is shown in Figure 2. The two distinct disk fillings of the knot are obtained by pinching the two Reeb chords indicated in Figure 2.

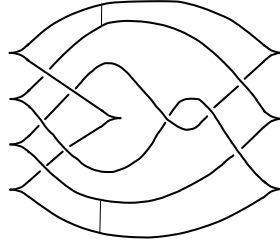


FIGURE 2. Front projection of the Legendrian $\overline{9_{46}}$. The two places to pinch to get the distinct disk fillings D_0 and D_1 are indicated as vertical lines.

Figure 3 shows a zig-zag path, embedded in the front. More precisely, the leftmost front diagram shows a slice in the front of $\Lambda(D_0, D_1)$ which lies in D_0 after the pinch. The path starts at the gray dot indicated there and moves up the left cusp to the cross. The path then continues as the cross in slices of D_0 approaching the central slice shown at the top of the figure, and down in the slices corresponding to D_1 until it passes the pinching slice, as indicated in the next front. Then the slices of D_1 undergo isotopy and ends in the lower right front. Here the cross goes up the right cusp completing the zig-zag. After deformation of the front along the zig-zag path we find an embedded zig-zag as shown in the lower left of the figure which then gives an explicit loose chart.

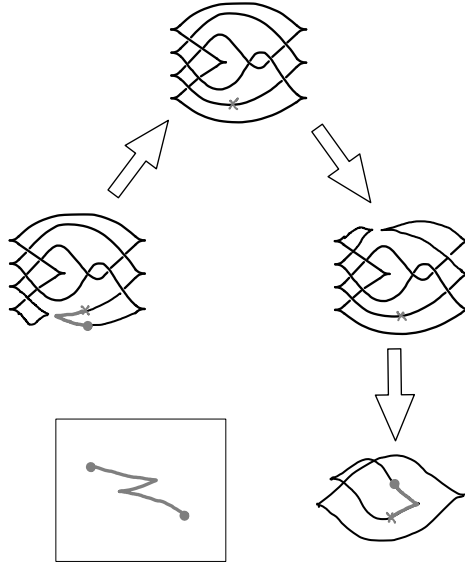


FIGURE 3. The embedded zig-zag in the front of $\Lambda(D_0, D_1)$.

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