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Thoughts about a good classification of manifolds

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Abstract. "Good" is a matter of taste. But there are mathematical concepts which most mathematicians agree on that they are good. For example if one wants to classify complex vector bundles it seems to be a good idea to look at isomorphism classes of vector bundles modulo addition of trivial vector bundles. The resulting set of equivalence classes is denoted by $\tilde{K}^0(X)$, and for compact spaces this is a group under the operation of Whitney sum. Another reason why it is good, is that $\tilde{K}^0(X)$ is the degree 0 subgroup of a generalized cohomology theory which allows an attack by the standard tools of algebraic topology like exact sequences or spectral sequences.

In this note we take this as a model for the classification of closed connected manifolds. In analogy to the vector bundles we consider diffemorphism classes of smooth manifolds modulo connected sum with a "trivial" manifold T. Whereas we don't see a good candidate for T for odd dimensional manifolds, we take $T = S^n \times S^n$ in even dimensions and pass to what we call the reduced stable diffeomorphism classes of manifolds. In contrast to vector bundles the reduced stable diffeomorphism classes of smooth manifolds don't form a group. But we will see that they decompose as quotients of groups by a linear action of another group. Most of the results in this note are not new, they are all based on the results of my papers [13], [14]. But we add a perspective which readers might find good.

1. The model K-theory

Let X be a compact topological space. We denote the set of isomorphism classes of finite dimensional complex vector bundles over X by $Vect(X)$. This is a monoid under the Whitney sum. Computations of $Vect(X)$ are at present not accessible, even in the case of spheres, where it is equivalent to the computation of (unstable) homotopy groups of the orthogonal groups $O(n)$. But if $k > n$, the set of isomorphism classes of k-dimensional vector bundles over S^n can be computed. The set is independent of $k > n$ and was computed by Bott with his periodicity theorem: It is trivial if n is odd, and $\mathbb{Z}$ if n is even.

More generally, if one considers for a compact space X the equivalence relation **reduced stable equivalence**, which means that two complex vector bundles E and E' are reduced stably equivalent, if there are integers k and l, such that $E \oplus \mathbb{C}^k \cong E' \oplus \mathbb{C}^l$, the reduced stable equivalence classes form a group denoted by $\tilde{K}^0(X)$. The point here is that all finite dimensional vector bundles can be embedded into a trivial vector bundle, which implies that if the addition on the reduced stable equivalence classes is induced by

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the Whitney sum, all vector bundles have an inverse in $\tilde{K}^0(X)$. Using Bott periodicity Atiyah and Hirzebruch [2], [1] extend this to a reduced generalized cohomology theory $\tilde{K}^*(X)$.

It would be nice to define a similar relation on the set of diffeomorphism classes of closed smooth connected manifolds of a fixed dimension n, which makes the set of equivalence classes a group with a geometrically defined group structure. Let's look at the case of surfaces. There the obvious candidate for an addition is the connected sum which should play the role of the Whitney sum for vector bundles. The next step would be to find a replacement for the trivial bundle, a manifold T: the "trivial manifold" replacing the trivial vector bundle. Choosing such a T one can call surfaces F and F' **reduced T-stably isomorphic** if there are integers k and l such that $F\sharp kT$ is diffeomorphic to $F'\sharp lT$. Now one can look for a T, such that reduced T-stable isomorphism classes form a group. We would need a neutral element N which means that $F\sharp N\sharp kT$ is diffeomorphic to $F\sharp lT$ for some k and l, and an inverse, which means that for each F there is an F' such that $F\sharp F'\sharp kT$ is diffeomorphic to $N\sharp lT$ for some k and l. If we could find such a T which is orientable, then also N must be orientable and then a non-orientable manifold has no inverse. Thus one has to take T non-orientable, for example $T = \mathbb{R}P^2$. Then it works, but the result is disappointing since the resulting group is the trivial group.

But this is the only dimension, where this can work. In dimension $n \geq 3$ such a T cannot exist. If so, there would be a manifold N representing the neutral element. If we would have a group structure on the reduced stable isomorphism classes for each closed connected n-manifold M there is a closed n-manifold M' - the inverse - such that $M\sharp M'\sharp rT \cong N\sharp lT$ for some k and l, which alone from the point of view of homology groups is impossible. Namely, $tors\ H_1(T)$ and $tors\ H_1(N)$ are finite and so there is a prime p, such that $H_1(T)$ and $H_1(N)$ have no p-torsion, and so a manifold M with non-trivial p-torsion has no inverse.

2. A weakening of the model of vector bundles

In odd dimensions I don't have any good idea what to do. But the following result by Wall about 1-connected oriented 4-manifolds is a guide to a modification. In his case $T = S^2 \times S^2$. Instead of "reduced $S^2 \times S^2$-stably equivalent" we just write "reduced stably diffeomorphic".

Theorem 2.1. [22] *Two closed smooth 1-connected oriented 4-manifolds M and M' are reduced stably diffeomorphic if and only*
- sign(M) = sign(M')
- both are Spin or both do not admit a Spin-structure.

This suggests to consider two disjoint subsets of the reduced stable diffeomorphism classes of closed simply connected 4-manifolds: $\mathcal{M}(1)$, the reduced stable diffeomorphism classes of 1-connected Spin manifolds and $\mathcal{M}(2)$, the reduced stable diffeomorphism classes of 1-connected manifolds admitting no Spin structure. Then Wall's theorem implies that both sets are actually groups under the operation given by connected sum, and both are isomorphic to $\mathbb{Z}$ under the signature in the non-Spin case and under the signature divided by 16 in the Spin case. In the Spin case the neutral element is represented by the empty manifold, in the non-Spin case by $\mathbb{C}P^2\sharp\bar{\mathbb{C}P}^2$. If we return for a moment to surfaces and take $T = S^1 \times S^1$ we have an analogous result. The reduced stable diffeomorphism classes are the disjoint union of two subsets, the orientable and the non-orientable surfaces, and both of these subsets are groups: In the orientable case the trivial group, in the non-orientable case the group is $\mathbb{Z}/2$ detected by the Euler characteristic mod 2. There the neutral element is $\mathbb{R}P^2\sharp\mathbb{R}P^2$.

This raises the following question. To make things precise we consider pointed connected smooth manifolds together with an orientation at the base point. Diffeomorphisms have to preserve the base point and the local orientation. We only consider even-dimensional connected manifolds and say that closed $2n$-dimensional connected manifolds M and M' are **reduced stably diffeomorphic** if there are integers k and l such that $M\sharp k(S^n \times S^n)$ is diffeomorphic to $M'\sharp l(S^n \times S^n)$.

Question: *Can we divide the set of reduced stable diffeomorphism classes of closed connected $2n$-manifolds into subsets which are groups?*

The answer to this question is in a very unsatisfying way: Yes. Namely, there are only countably many reduced stable diffeomorphism classes of closed connected $2n$-manifolds. Thus we can identify them with the integers by choosing an enumeration. But this is clearly not a good answer. We are looking for an answer, which fulfills something like the following properties:

- There should be a "in principle" way to decide, which closed connected $2n$-manifold sits in which of the disjoint subsets.
- The group structure on the different subsets should be geometric, whatever that means, for example like the connected sum.
- There should be a "in principle" way to determine the isomorphism classes of the groups given by the disjoint subsets.

We don't see a way to achieve this, but we will later modify the question again in a way which allows a positive answer.

3. The surgery exact sequence and Sullivan's classification up to finite ambiguity

Before we come to this modification we look at the classification of closed manifolds within a given homotopy type. Here one divides the set of actual diffeomorphism classes into disjoint subsets, namely the subsets are parametrized by the different homotopy types. With other words we fix a finite CW-complex X and consider the set $\mathcal{M}(X)$ of closed manifolds homotopy equivalent to X. This gives a decomposition of all smooth closed manifolds into disjoint subsets $\mathcal{M}(X)$. The aim of classical surgery is to determine the sets $\mathcal{M}(X)$. But this is not what one really does. One considers instead the set of homotopy smoothings $S_h(X)$ of X, the diffeomorphism classes of pairs (M, f), where

$f : M \to X$ is a homotopy equivalence. This set can be attacked by classical surgery as developed by Browder, Novikov, Wall and Sullivan, leading to the surgery exact sequence [23] of sets

$$L_{n+1}(\pi_1(X), w_1) \to S_h(X) \to [X, G/O] \to L_n(\pi_1(X), w_1),$$

where $n \geq 5$ and we assume that $S_h(X)$ is non-empty. It is natural to ask the question we raised above, is there a group structure on $S_h(X)$, ideally such that the exact sequence above is an exact sequence of groups? In the smooth case this is not known, but if we pass to topological manifolds this was positively answered by Quinn [17]. But the group structure on the topological structure set is not geometric, at least a geometric description in the spirit of the connected sum is not known and probably does not exist. And the action of the group of homotopy classes of self equivalences of X on this group is not linear. Thus one might look for another algebraic picture.

The idea of such an algebraic picture of smooth closed manifolds was intensively studied by Sullivan in the seventies and led to his important paper [18]. The philosophy in this paper is in a certain sense a model of what we will later describe. To obtain a very smooth answer one has to pay a price: One needs a restriction of the fundamental group, for example simply connected manifolds, and one considers a classification up to finite ambiguity. Then Sullivan gives - roughly speaking - the picture, that "this mathematical object behaves up to "finite ambiguity" like a finite dimensional real vector space with additional structure provided by tensors, lattices, and canonical elements. This algebraic model is derived directly from the differential forms on the manifold M." More precisely this is an algebraic picture (up to finite ambiguity and for simply connected manifolds) of the structure set $S_h(X)$. Then Sullivan studies the group of homotopy classes of self equivalences of X and proves a very remarkable theorem. Namely he proves that this group is commensurable with an **arithmetic group**, where commensurable means that there is a finite sequences of homomorphisms starting with the original group and ending with the other group, which have finite kernels and cokernels. This arithmetic group acts algebrically on the algebraic data mentioned before and the quotient is up to finite ambiguity the set of diffeomorphism classes of manifolds within the rational homotopy type given by X.

Thus if one is willing to pay the price of a classification up to finite ambiguity and to restrict the fundamental group one can divide the manifolds of dimension ≥ 5 into subsets which are the orbit space of algebraic actions of groups commensurable to an arithmetic group acting on certain algebraic and - at least in principle - computable data. We will soon see a similar picture for stable diffeomorphism classes of closed even dimensional manifolds, although the data are less computable since we don't consider a classification up to finite ambiguity and consider arbitrary fundamental groups.

4. A concept of a good classification of closed connected even-dimensional manifolds up to reduced stable diffeomorphisms

After this excursion to the classification of manifolds within a given homotopy type we return to the reduced stable diffeomorphism classes and modify our original question as follows. We consider closed, connected, pointed at the base point oriented smooth manifolds. By a diffeomorphism we mean a base point and the local orientation preserving diffeomorphism.

Question: *Can we divide the reduced stable diffeomorphism classes of closed, connected, pointed and at the base point oriented 2n-dimensional manifolds into disjoint subsets $\mathcal{M}(i)$, such that*
- there is a "in principle way" to decide in which $\mathcal{M}(i)$ a closed connected manifold sits
- for each i there are groups with a geometric addition $\hat{M}(i)$ and groups $G(i)$ acting linearly on $\hat{M}(i)$, such

$$\hat{M}(i)/_{G(i)} \cong \mathcal{M}(i)$$

- there is a "in principle way" to determine the groups $\hat{M}(i)$, $G(i)$ and the action of $G(i)$ on $\hat{M}(i)$.

In this paper we concentrate on smooth manifolds. The same techniques apply to PL- or topological manifolds with appropriate modifications by passing from the normal bundle to the PL-normal bundle or the topological normal bundle. Some of the applications we later mention are for topological manifolds.

5. An answer in dimension 4

We begin with an answer to the question raised above in the special case of closed, connected, oriented 4-manifolds, where the answer is more explicit than in higher dimensions. This is based on Peter Teichner's thesis [19], [20]. We begin with the definition of the index set over which we form the disjoint union. Let π be a finitely presentable group and w either ∞ or $w \in H^2(\pi;\mathbb{Z}/2)$. Two such pairs (π, w) and (π', w') are called isomorphic if there is a group isomorphisms $\varphi : \pi \to \pi'$ and either $w = w' = \infty$ or $\varphi^*(w') = w$. Given a pointed closed connected 4-manifold (M, x_0) we associate to it the pair $(\pi_1(M, x_0), w(M) := \infty)$, if the universal covering $\tilde{M}$ is not a Spin manifold, and the pair $(\pi_1(M, x_0), w(M))$, if $\tilde{M}$ is Spin and $w(M) \in H^2(\pi_1(M, x_0), \mathbb{Z}/2)$, such that $u^*(w(M)) = w_2(M)$, where $u : M \to K((\pi_1(M, x_0), 1), y_0)$ is a base point preserving classifying map of the universal covering. The choice of $w(M)$ depends on the choice of u, but u^* is up to composition with an automorphism of $\pi_1(M, x_0)$ unique. Thus the equivalence class of $(\pi_1(M, x_0), w(M))$ is a well defined invariant of (M, x_0). If $f : (M, x_0) \to (M', x_0')$ is a diffeomorphism then f_* induces an isomorphism of the corresponding pairs.

Given this we can define the set $\mathcal{M}([\pi, w)])$ as the set of reduced stable diffeomorphism classes of pointed closed connected 4-manifolds such that $[(\pi_1(M, x_0), w(M))] = [\pi, w]$. Obviously the set of all reduced stable diffeomorphism classes of closed, connected, pointed, oriented 4-manifolds M is the disjoint union of the sets $\mathcal{M}([\pi, w])$, where $[\pi, w]$ runs over the isomorphism classes of pairs with π a finitely presentable group.

Next we define the sets $\hat{M}([\pi, w])$. We fix for each π a model of $K(\pi, 1)$ and equip it with a base point. Then we define $\hat{M}([\pi, w])$ as the set of reduced stable diffeomorphism classes of pairs (M, u), where M is a closed, connected, oriented 4-manifold and $u : M \to K(\pi, 1)$ is a pointed map inducing an isomorphism on the fundamental group, if $w = \infty$. If $w \neq \infty$ we chose a fibration $p : K(\pi, 1) \to K(\mathbb{Z}/2, 2)$ representing the class w. Now we also represent the universal Stiefel-Whitney class $w_2 \in H^2(BSO; \mathbb{Z}/2)$ by a map g and consider the pull back fibration under g and denote this fibration over BSO (depending on the choices, which we make once and for ever) by $B(\pi, w)$. Then $\hat{M}([\pi, w])$ is defined as reduced stable diffeomorphism classes of pairs $(M, \bar{\nu}_M)$, where $\bar{\nu}_M$ is a homotopy class of lifts of the oriented normal Gauss map $\nu_M : M \to BSO$. We further require, that $\bar{\nu}_M$ is a **normal 1-smoothing**, that means that $\bar{\nu}_M$ is a 2-equivalence, an isomorphism on π_1 and surjective on π_2. Here two such pairs $(M, \bar{\nu}_M)$ and $(M', \bar{\nu}_{M'})$ are called **reduced stably diffeomorphic** if there is a base point and orientation preserving diffeomorphism $f : M \sharp r(S^2 \times S^2) \to M' \sharp q(S^2 \times S^2)$ commuting up to fibre homotopy with $\bar{\nu}_{M \sharp r(S^2 \times S^2)}$ and with $\bar{\nu}_{M' \sharp q(S^2 \times S^2)}$. Here we choose the normal 1-smoothings on the stabilized manifolds in such a way that the manifolds together with the maps $\bar{\nu}$ to $B(\pi, w)$ are bordant to $(M, \bar{\nu}_M)$ resp. $(M', \bar{\nu}_{M'})$. We call the set $\hat{M}(\pi, w)$ the **stabilized normal 1-structure set**.

The group $G(\pi, w)$ of homotopy classes of fibre homotopy self equivalences acts on $\hat{M}(\pi, w)$ by composition and the orbit space is isomorphic to $M(\pi, w)$, as desired.

Before we define the geometric group structure on $\hat{M}(\pi, w)$ we answer the third part of the question and give a "in principle way" to compute this set. For this we recall the definition of bordism groups of manifolds with B-structure. Let $B \to BO$ be a fibration. Then a B-manifold is a pair $(M, \bar{\nu}_M)$, where as above $\bar{\nu}_M$ is a lift of the normal Gauss map, but we don't require, that it is a k-equivalence for some k. The lift $\bar{\nu}_M$ is called a normal B-structure on M. The set of bordism classes of normal B-structures on closed n-dimensional manifolds M, where a zero bordism of $(M, \bar{\nu}_M)$ is a compact manifold W with normal B structure whose boundary is $(M, \bar{\nu}_M)$, is denoted by $\Omega_n(B)$. These bordism classes are a group under disjoint union. We note that by construction the pair $(M, \bar{\nu}_M)$ is bordant to $(M, \bar{\nu}_{M \sharp r(S^2 \times S^2)})$ in $\Omega_4(B(\pi, w))$. Returning to $\hat{M}(\pi, w)$ we consider the forgetful map (forgetting that $\bar{\nu}_M$ is a 2-equivalence) $\hat{M}(\pi, w) \to \Omega_4(B(\pi, w))$. In [13] we proved that this map is a bijection:

Theorem 5.1. *The forgetful map*

$$\hat{M}(\pi, w) \to \Omega_4(B(\pi, w))$$

is a bijection.

Next we define the geometric group structure on $\hat{M}(\pi, w)$. For this we chose a set of generators $\alpha_1, ..., \alpha_r$ of π and represent $(u_*)^{-1}(\alpha_i)$ by disjoint smooth embedded circles S^1_i in M (for this we thicken the base point to a disc allowing that the elements of the fundamental group can be represented by disjoint embeddings). Since M is oriented, the normal bundle of S^1_i is trivial. We choose a trivialization and via this a tubular neighborhood. Then we join these tubular neighborhoods via embedded thickened arcs with the base point, to obtain a thickening of a wedge of circles. We do the same for M' and circles representing $(u'_*)^{-1}(\alpha_i)$. Then we chose an orientation reversing diffeomorphism between these thickenings, remove the interior and identify the boundaries. We call the result a **connected sum along a 1-skeleton**. The reason for this notation is that if M and M' are simply connected, then the result is the ordinary connected sum. This construction depends on choices. If $w = \infty$ the reduced stable diffeomorphism class is unique. This follows from the fact that by construction it has fundamental group isomorphic to π with isomorphism induced from either of the pieces (using the Seifert van Kampen Theorem) and thus gives an element in $\hat{M}(\pi, \infty)$ which by construction is bordant in $\Omega_4(K(\pi_1, 1))$ to the disjoint union, and so by our theorem the result follows.

If $w \neq \infty$ we have to chose a preferred framing of the embedded circles. We will see that the normal B-structure determines a framing or - in this case - equivalently a Spin structure. The restriction of $\bar{\nu}_M$ to S^1_i determines a Spin structure, since it is equivalent to a lift to the fibre of $BSO \to BO$, which is $BSpin$. Thus in both cases we obtain an induced framing. We use this framing on M and the opposite framing obtained by composing with a reflection to identify the tubular neighborhoods. Then we extend the tubular neighborhoods to a thickening of the wedge of circles, remove them from M and M' and identify the boundaries using the identification above. The compatible choice of the identification allows to construct a bordism between the disjoint union of M and M' and the resulting manifold by taking the cylinder over both manifolds and identifying the thickenings via the identification above. The normal 1-smoothings of M and M' extend to this bordism and the restriction to the part of the boundary given by the cut and paste construction above gives us an element in $\hat{M}(\pi, w)$, which, since it is bordant to the sum in the bordism group, is well defined.

Thus in all cases we have a geometrically defined addition, called the **connected sum along the 1-skeleton**, such that the bijection $\hat{M}(\pi, w)) \to \Omega_4(B(\pi, w)$ is an isomorphism. This is the last step to fulfill the last condition of a good reduced stable diffeomorphism classification in dimension 4 in the spirit of the reduced stable classification of vector bundles in the case of K-theory:

Theorem 5.2. *The connected sum along the* 1*-skeleton defines a group structure such that the forgetful map*

$$\hat{M}(\pi, w) \to \Omega_4(B(\pi, w))$$

is an isomorphism of groups.

We would like to discuss the three properties of a “good classification” listed above. The first two properties are obvious, the third amounts to a computation of $\Omega_4(B(\pi, w))$. If $w = \infty$ this group is (using the Atiyah-Hirzebruch spectral sequence) isomorphic to $\mathbb{Z} \oplus H_4(\pi; \mathbb{Z})$, where the map is given by the signature of M and the image of the fundamental class under u_*. If $w \neq \infty$, the group can be attacked by the James spectral

spectral sequence [20]. The E_2-diagram is given by $H_p(\pi;\Omega_q^{Spin})$ and the differential is also determined in [20] in terms of the Steenrod operations and w.

Even in the case, where $w=\infty$, an explicit computation of the group is not finished by identifying it with $\mathbb{Z}\oplus H_4(\pi;\mathbb{Z})$. It would be desirable to find more explicit invariants, which detect the element in $H_4(\pi;\mathbb{Z})$. For finite fundamental groups a lot was done in that direction (not only for $w=\infty$), in particular the quadratic 2-type was introduced consisting of the isometry class of the quadruple given by the fundamental group, the second homotopy group as module over the fundamental group, the first k-invariant and the intersection form with values in the group ring on the second homotopy group. For a survey about some aspects of this see [6] and the literature cited there. An explicit computation in the general case is not known, but recently in [11] the group was determined if π is the fundamental group of a 3-manifold.

The constructions and statements can be generalized to non-orientable 4-manifolds. Then one has to add the first Stiefel-Whitney class to the data, so instead of (π,w) one considers triples (π,w_1,w), where $w_1\in H^1(\pi;Z/2)$ and considers manifolds with $u:M\to K(\pi1,1)$ such that $u^*(w_1)=w_1(M)$. For details see [19].

6. An answer in dimension >4

The discussion in dimension 4 is an explicit version of the general answer in all even dimensions. Let's fix an integer $n>2$ and study the set of reduced stable diffeomorphism classes of closed, connected, pointed, locally oriented $2n$-manifolds. We begin with the definition of the index set which parametrizes the disjoint subsets. A Serre fibration $p:B\to X$ of path connected pointed spaces is called n-coconnected if the homotopy groups of the homotopy fibre vanish in degree $\geq n$. For example the fibrations $B(\pi,w)\to BSO$ considered above are 2-coconnected. We consider the set of fibre homotopy equivalence classes of pointed n-coconnected fibrations $B\to BO$ such that B is homotopy equivalent to a CW-complex with finite n-skeleton. This is our index set for $2n$-manifolds. Given such a fibration $B\to BO$ we call a pair $(M,\bar\nu_M)$ a normal $(n-1)$-smoothing in B if $\bar\nu_M$ is a lift of the normal Gauss map $\nu_M:M\to BO$, which is a n-equivalence. We can stabilize $(M,\bar\nu_M)$ by connected sum with $S^n\times S^n$ is such a way that $M\sharp(S^n\times S^n)$ also admits a normal $(n-1)$-smoothing in B. Thus we can consider the set of stable diffeomorphism classes of closed, connected, pointed, locally oriented $2n$-manifolds admitting a normal $(n-1)$-smoothing in B. We denote this set by $\mathcal{M}(B)$.

Proposition 6.1. *The set of reduced stable diffeomorphism classes of closed, connected, pointed, locally oriented $2n$-manifolds is the disjoint union of the subsets $\mathcal{M}(B)$, where B runs over all fibre homotopy equivalence classes of pointed n-coconnected fibrations $B\to BO$ such that B is homotopy equivalent to a CW-complex with finite n-skeleton.*

Proof. We have to show that for each closed, connected, pointed, locally oriented $2n$-manifold there is a pointed n-coconnected fibration $B\to BO$ such that B is homotopy equivalent to a CW-complex with finite n-skeleton and the reduced stable diffeomorphism class of M sits in $\mathcal{M}(B)$. For this we consider the normal Gauss map $\nu_M:M\to BO$.

Then we consider the Moore-Postnikov decomposition of ν_M. This is a tower of fibrations $B_k \to B_{k-1} .... \to BO$, such that $B_k \to BO$ is k-coconnected and there is a lift by a k-equivalence $\bar{\nu}_M : M \to B_k$ of ν_M for all k. Then $B_n \to BO$ fulfills all requirements.

To see that the subsets are disjoint, one notes that if two manifolds are diffeomorphic, then the Moore-Postnikov towers are fibre homotopy equivalent. □

The fibre homotopy equivalence class of the n-th stage $B_n \to BO$ of the Moore-Postnikov decomposition of $\nu_M \to BO$ for a $2n$-dimensional manifold M, is an invariant of the normal homotopy type of M and it is called the **normal** $(n-1)$**-type**.

Next we define the sets $\hat{M}(B)$ for a pointed n-coconnected fibration $B \to BO$ such that B is homotopy equivalent to a CW-complex with finite n-skeleton. This consists of reduced stable diffeomorphism classes of normal $(n-1)$-smoothings $(M, \bar{\nu}_M)$ in B. Here we stabilize $(M, \bar{\nu}_M)$ by connected sum with $S^n \times S^n$ to obtain $(M\sharp(S^n \times S^n), \bar{\nu}_{M\sharp(S^n\times S^n)})$ such that we obtain a normal $(n-1)$-smoothing which considered as a lift to B is bordant to $(M, \bar{\nu}_M)$. The stable classification result below will show that the reduced stable diffeomorphism class is independent on the choice of the normal $(n-1)$-smoothing $\bar{\nu}_{M\sharp(S^n\times S^n)}$. In the case of 4-manifolds, where $B = B(\pi, w)$, the definition of $\hat{M}(\pi, w)$ agrees with the present of $\hat{M}(B(\pi, w))$.

As in dimension 4 the relation between $\hat{M}(B)$ and $\mathcal{M}(B)$ is the following. The group of fibre homotopy self equivalences $Aut(B)$ acts on $\hat{M}(B)$ by composition. The forgetful map (forgetting $\bar{\nu}_M$) induces a bijection $\hat{M}(B)/Aut(B) \cong \mathcal{M}(B)$.

As in dimension 4 we consider the bordism group $\Omega_{2n}(B)$ of pairs $(M, \bar{\nu}_M)$, where here $\bar{\nu}_M$ is just a lift of the normal Gauss map to B, not an n-equivalence. We have a forgetful map $\hat{M}(B) \to \Omega_{2n}(B)$ and we have the following generalization of Theorem 3:

Theorem 6.2. *Let B be a pointed n-coconnected fibrations $B \to BO$ such that B is homotopy equivalent to a CW-complex with finite n-skeleton. The forgetful map*

$$\hat{M}(B) \to \Omega_{2n}(B)$$

is a bijection.

This follows from [13] Proposition 4, for surjectivity and Theorem 2 for injectivity. As Diarmuid Crowley pointed out, the proof of Proposition 4 is not correct. A correct proof is contained in [14], chapter 2.

As in dimension 4 we want to define a geometric addition on $\hat{M}(B)$ such that the forgetful map becomes a homomorphism. The idea is the same but we distinguish between dimension 4 and higher dimensions since we want apply results by Wall about thickenings which he only proved for $n > 2$ [24]. Let $(M, \bar{\nu}_M)$ be a normal (n - 1)-smoothing. We choose (once and for ever) a finite $(n-1)$-skeleton K of B. Since $\bar{\nu}_M$ is a n-equivalence there is a unique up to homotopy lift $f : K \to M$ of $\bar{\nu}_M$. By Wall there is a thickening N_M, a codimension 0 smooth compact submanifold with boundary, such that f factors through N_M by a simple homotopy equivalence [24], page 76. Since $n-1 < 2n$, we are in the stable range and so the thickenings are classified up to concordance by $[K, BO]$ [24], page

80. Since $\bar{\nu}_M$ is a n-equivalence, the different thickenings are classified by the choice of an orientation. Now we consider another normal $(n-1)$-smoothing $(M', \bar{\nu}'_M)$ and thickenings N_M on M compatible with the local orientation of M and $N_{M'}$ for M' compatible with the opposite local orientation and identify them. We remove the interior of N_M and $N_{M'}$ from M resp. M' and identify the corresponding boundaries. By construction there is a normal $(n-1)$-smoothing on this manifold which on the complement of the thickenings agrees with $\bar{\nu}_M$ resp. $\bar{\nu}_{M'}$. Thus we obtain a new normal $(n-1)$-smoothing, which we denote by $(M\sharp_K M', \bar{\nu}_{M\sharp_K M'})$. There is a B-bordism between $(M, \bar{\nu}_M) + (M', \bar{\nu}_{M'})$ and $(M\sharp_K M', \bar{\nu}_{M\sharp_K M'})$, obtained by taking the cylinder over $M + M'$ and identifying N_M with $N_{M'}$. Thus we have defined a geometric addition $\sharp_K$ on $\hat{M}(B)$ such that the forgetful map is a homomorphism. Theorem 5 implies that the addition $\sharp_K$ is well defined and makes $\hat{M}(B)$ a group. It also implies that this addition is independent on the choice of K. We call $\sharp_K$ the **connected sum along a $(n-1)$-skeleton**.

Thus we have proved:

Theorem 6.3. *The connected sum along a $(n-1)$-skeleton defines a group structure on $\hat{M}(B)$, the set of reduced stable diffeomorphisn classes of 2n-dimensional normal $(n-1)$-smoothings, such that the forgetful map is an isomorphism*

$$\hat{M}(B) \to \Omega_{2n}(B).$$

In this section we assumed $n > 2$, since we used thickenings to define the group structure on $\hat{M}(B)$. But we have the same statement in dimension 2 and 4, where the fibrations B are more explicitly described.

Now we discuss in how far Proposition 4 and Theorem 6 gives an answer to the question posed at the end of section 3. We consider manifolds of dimension $2n$. Our index set is the set of fibre homotopy equivalence classes of pointed n-coconnected fibrations $B \to BO$ such that B is homotopy equivalent to a CW-complex with finite n-skeleton. For each such B we have defined subsets $\mathcal{M}(B)$, which are pairwise disjoint. Given M we can read off from the normal homotopy type in which $\mathcal{M}(B)$ the reduced stable diffeomorphism class is contained. Thus one can use methods of unstable homotopy theory to attack this problem, which we consider a "in principle way" to decide the problem.

Then we defined groups $\hat{M}(B)$ with the geometrically defined addition connected sum along a $(n-1)$-skeleton generalizing the ordinary connected sum in the case where this skeleton consists of a point, and the group of fibre homotopy self equivalences $Aut(B)$ acts linearly by composition on $\hat{M}(B)$ such that the orbit space is isomorphic to $\mathcal{M}(B)$. Finally there is an isomorphism $\hat{M}(B) \cong \Omega_{2n}(B)$. This group is by the Pontrjagin Thom construction isomorphic to the stable homotopy group $\pi_{2n}(TB)$, where $T(B)$ is the pull back to B of the Thom spectrum over BO. Thus one can use methods of stable homotopy theory to attack $\hat{M}(B)$, whereas one can use obstruction theory to attack $Aut(B)$ and in favorable cases determine the action of $Aut(B)$ on $\hat{M}(B)$. This is what we mean by a "in principle" computation of $\hat{M}(B)$, $Aut(B)$ and the action of $Aut(B)$ on $\hat{M}(B)$.

7. From stable to unstable classification

Like in our model case of vector bundles the group structure and the "in principle" computability had it's price: We had to pass from the actual classification to the reduced stable classification. In the case of vector bundles one can - under appropriate conditions - go backwards, from the reduced stable classification to the actual classification. For example, if X is a finite k-dimensional CW-complex, and if E and E' are bundles of real dimension $> k$, then E and E' are actually isomorphic if and only if $dim\, E = dim\, E'$ and the bundles are reduced stably isomorphic. This is a cancellation result. A similar cancellation result for manifolds would be desirable.

The first observation is that the Euler characteristic plays the role of the dimension of vector bundles and has to be controlled. We don't know a general cancellation result in the spirit of the cancellation of trivial bundles in the case of vector bundles. But one can define an invariant (besides the Euler characteristic) which "in principle" decides the question for $n > 2$.

Suppose that M and M' are closed, connected, pointed, locally oriented $2n$-manifolds with equal Euler characteristic $e(M) = e(M')$ such that M and M' are stably diffeomorphic. Then M and M' have the same normal $(n-1)$-type B and there are normal $(n-1)$-smoothings $(M, \bar{\nu}_M)$ and $(M', \bar{\nu}_{M'})$, such that they are bordant in $\Omega_{2n}(B)$. Let $(W, \bar{\nu}_W)$ be a bordism between $(M, \bar{\nu}_M)$ and $(M', \bar{\nu}_{M'})$. Then there is an obstruction $\theta(W, \bar{\nu}_W)$ in a monoid $l_{2n+1}(\pi_1(B), w_1(B))$ (where $w_1(B)$ is the pull back to B of the first universal universal Stiefel-Whitney class of BO) such that $\theta(W, \bar{\nu}_W)$ is a so called elementary element if and only if $(W, \bar{\nu}_W)$ is bordant rel boundary to an s-cobordism and so by the s-cobordism theorem M is diffeomorphic to M', if $n > 2$. This is a long story and for details we refer to [13]. The monoids $l_{2n+1}(\pi_1(B), w_1(B))$ are very complicated, but under some conditions one can show that the obstructions are elementary leading to cancellation results. We quote some cancellation results which indicate what can happen ([13], Theorem 5):

Theorem 7.1. *Let M and M' be reduced stably diffeomorphic $2n$-manifolds with equal Euler characteristic and $n > 2$. Then:*
- If n is odd and M is simply connected, M is diffeomorphic to M'.
- If n is even and $M = M_0 \sharp (S^n \times S^n)$ and M is simply connected, M is diffeomorphic to M'.
- If $\pi_1(M)$ is finite and $M = M_0 \sharp 2(S^n \times S^n)$, then M is diffeomorphic to M'.

Remark 7.1. In dimension 4 Ian Hambleton and the author have proved a better result in the topological category, namely under the same conditions as in the previous theorem, if $\pi_1(M)$ is finite and $M = M_0 \sharp (S^2 \times S^2)$, then M is homeomorphic to M' [7]. This proof was actually the model for the proof of the theorem above.

Remark 7.2. There is no cancellation result known in the smooth case in dimension 4. Donaldson (and later many more examples of the same type were constructed) proved that for example $\mathbb{C}P^2 \sharp 9 \mathbb{C}P^2$ has infinitely many smooth structures [4] and so we obtain by Wall's result above infinitely many stably diffeomorphic simply connected 4-manifolds which are pairwise non-diffeomorphic. Thus cancellation down to zero is not possible. But this is essentially all we know about cancellation in the smooth category in dimension 4.

This leads to the following challenging questions:

Questions in dimension 4:

- *Is there any closed, smooth, connected* 4*-manifold and integer k and $M = M_0 \sharp k(S^2 \times S^2)$ such that if M' is stably diffeomorphic to M, then M is diffeomorphic to M'?*
- *Is it true that if Σ is stably diffeomorphic to S^4, then Σ is diffeomorphic to S^4? This is equivalent to the smooth* 4*-dimensional Poincaré conjecture.*

8. Comparison with the classification via classical surgery and "best" classification results

The question we stressed - modeled on the definition of reduced K-theory as reduced stable isomorphism classes of complex vector bundles - was to decompose manifolds into subsets which are quotients of groups by linear actions is one way to attack manifolds. Classical surgery is another way which looks at the classification of manifolds within a given homotopy type X instead of a n-coconnected fibration $B \to BO$. Both approaches have a main difficulty right at the beginning. If one wants to decide, whether two $2n$-dimensional manifolds M and M' are diffeomorphic one has to decide in the classical case, whether they are homotopy equivalent, and in the modified case, whether they have the same normal $(n-1)$-type. Both is in general out of range, even though for both one has "in principle" methods to attack the problem by looking at the Postnikov towers. But only a few cases are worked out, where this functions well. For example the homotopy classification of closed 4-manifolds does not have a simple answer.

For both approaches there are special cases where one of them works well and the other not so well. The most important application of classical surgery for classifying manifolds are examples where the homotopy type is easy to determine. The first example was exotic spheres, which by Kervaire and Milnor were classified in dimension > 4 in terms of an exact sequence involving the cyclic subgroup bP , the stable homotopy groups of spheres and the Kervaire invariant [12].

The second example is the Borel conjecture which states that if f is a homotopy equivalence between two closed topological manifolds M and M' which are aspherical, then f is homotopic to a homeomorphism. In the language of the structure set this just means that $S_h(M)$ is trivial for aspherical manifolds. One could call this a "best" classification results, since a single invariant determines the manifolds. The first step in classical surgery is here for free, the homotopy type is determined by the fundamental group. This is of course only the starting point. One has to analyze the maps in the surgery exact sequence, which by a conjecture of Farrell and Jones about certain assembly maps are isomorphisms (even in a more general setting than fundamental groups of $K(\pi_1, 1)$-manifolds). In the last years there has been dramatic progress in this direction. For a survey we refer to the article by Wolfgang Lück [16] and more recent work by Bartels, Farrell and Lück and the literature cited there [3]. There are a few "best" classification results for non-aspherical manifolds by Wolfgang Lück and the author based on solutions of the Farrell-Jones conjecture, in part based on classical surgery or on modified surgery [15].

For modified surgery there are many results in low dimension (between 4 and 8) where one makes use of the fact that the normal $(n-1)$-type is roughly determined by the skeleton of half the dimension of the manifold and the bordism groups are tractable. An example of a "best" classification result is the homeomorphism classification of closed 4-manifold with cyclic fundamental groups [8]. For a survey of some other results we refer to [6], for more recent results to [10], [8].

In higher dimensions the advantage of control up to half the dimension doesn't play any role and there are very few applications. The most attractive one concerns the diffeomorphism classification of n-dimensional complex complete intersections (the real dimension is $2n$). By the Lefschetz hyperplane theorem the skeleton of dimension n is roughly $\mathbb{C}P^{[n/2]}$ and in fact the normal $(n-1)$-type is easy to determine. Since there is a bit stability, one can apply the cancellation result above and as a consequence the diffeomorphism type is determined by the bordism class in the normal $(n-1)$-type plus the Euler characteristic. Sullivan has studied the classification of complete intersections and a conjecture attributed to him says that the obvious invariants: total degree, Euler characteristic plus Pontrjagin classes determine the diffeomorphism type of a complete intersection. This is wrong in complex dimension 2 [5] using Donaldson invariants but open in higher dimensions. This would be another example of a "best" classification result. A step towards this fascinating conjecture is, that it one adds the bordism class in the normal $(n-1)$-type, the conjecture is true. Claudia Traving [21] has actually proved that the bordism class is determined by the Pontrjagin classes if enough small primes (compared to the dimension) divide the total degree. For details about all this we refer to [13].

Thus both the classical approach and my modified approach lead to explicit classification results in special situations, and sometimes even "best" classification results. The point of this article was that one can construct a general picture for the world of even dimensional manifolds in the spirit of the K-theory classification of vector bundles.

9. Odd dimensional manifolds

The main thing we have to say here is that we have nothing useful to say. The question, whether there is a $(2n+1)$-dimensional manifold T, such that, if we consider reduced stable diffeomorphism classes of closed connected $(2n+1)$-manifolds modulo connected sum with copies of T, this set can be decomposed into subsets as above, remains completely open. Dimension 3 might play a special role and so the fact that the geometrization theorem indicates that nothing in that direction looks promising, might not exclude that in higher dimensions something is possible.

Of course, this doesn't mean, that there are no classification results in odd dimensions (including some "best" classification results). Both classical and modified surgery can be applied and there are numerous concrete classification results. Only the general picture given here for even-dimensional manifolds has at present no analogy in odd dimensions. Thus for the moment we have to leave this as a challenge.

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