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Contact topology from the loose viewpoint

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Abstract. In this expository article, we describe a number of methods for studying high dimensional contact manifolds. We particularly focus on the concept of overtwistedness and looseness and their relation with geometric structures such as open books and surgery cobordisms. These notes are based on a lecture series given by the second author at the 2015 Gökova Geometry-Topology Conference, and built on the series of articles [8, 9, 10].

1. Introduction

1.1. Basics of contact geometry

A *contact structure* on an odd dimensional smooth manifold M^{2n+1} is a maximally non-integrable hyperplane field $\xi^{2n} \subseteq TM$. In this article we will always assume that the hyperplane field ξ is coorientable, i.e. we can write ξ as the kernel of a globally defined 1-form $\alpha \in \Omega^1(M)$. In this case, the condition of maximal non–integrability can be rephrased as the condition that $\alpha \wedge (d\alpha)^n$ is never zero, or equivalently the pair $(\xi, d\alpha)$ is a symplectic bundle. In particular the $(2n+1)$–form $\alpha \wedge (d\alpha)^n$ defines a volume form.

This is an open condition for the hyperplane field ξ and thus a C^1–small perturbation of a contact structure is still a contact structure. In other words, the rank of the 2–form $d\alpha$ has maximal rank when restricted to ξ , which is an open condition in the C^1–topology of the space of hyperplane fields of the tangent bundle TM. This is indeed the generic behaviour of a hyperplane field, in contrast to a situation in which we restrict the rank of $d\alpha$ to be non–maximal. An extreme instance of this latter case is the vanishing $d\alpha|_\xi = 0$, also known as the theory of foliations. See [22] for a discussion in the strongly interesting three–dimensional case.

Remark 1. Though the equality $\xi = \ker \alpha$ will be used systematically, we emphasize that contact structures ξ are a more natural geometric structure in comparison to the 1-form $\alpha \in \Omega^1(M)$ itself, which is only defined by ξ up to multiplication by positive functions (once the coorientation of ξ is fixed). Indeed, the contact condition $\alpha \wedge (d\alpha)^n \neq 0$ is invariant under the transformation $\alpha \rightsquigarrow e^f \alpha$ for any function $f \in C^\infty(M)$, and thus the resulting volume form is an artifact of the particular choice of α. The choice of a particular contact form $\xi = \ker \alpha$ fixes a particular vector field which preserves the contact

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structures (the Reeb field) and this gives a dynamical flavor to the theory [5].

We do note that the contact structure ξ together with its coorientation determine the sign of the volume form $\alpha \wedge (d\alpha)^n$ and therefore define an orientation on the smooth manifold M. Throughout the article, we will assume that all smooth manifolds M come a priori with an orientation, and by definition a cooriented contact structure is required to induce an orientation which agrees with the prescribed one.

The appearance of contact structures can be traced to the study of geometric optics and wave propagation [2, 6], and classical accounts on the subject can be found in [1, Appendix IV] and [4, Chapter IV]. Since a rotation in an odd dimensional vector space has a fixed axis, contact structures can only exist in odd dimensional smooth manifolds. The counterpart for even dimensional manifolds are symplectic structures, and contact geometry is the odd-dimensional sister of symplectic geometry. In particular, we have the following basic theorems:

Theorem 1.1 ([1]). Let (M, ξ) be a contact manifold, then it is locally equivalent to

$$\mathbb{R}^{2n+1}_{\text{std}} = (\mathbb{R}^{2n+1}, \xi = \ker \alpha_{\text{std}}), \quad \alpha_{\text{std}} = dz - \sum_{i=1}^{n} y_i dx_i.$$

In addition, the moduli of contact structures on a closed manifold is discrete: if ξ_t is a homotopy of contact structures on a closed manifold M, then there is an isotopy $f_t : M \longrightarrow M$ so that $(f_t)_*(\xi_0) = \xi_t$.

The first statement is referred to as Darboux's Theorem, and the second as Gray's stability theorem; these two theorems give contact geometry its marked topological flavor. For instance, Darboux's theorem implies that any contact manifold can alternatively be described by a contact atlas; that is, a smooth atlas where the transition functions are elements of the group of contact transformations

$$\text{Cont}(\mathbb{R}^{2n+1}_{\text{std}}) = \{\varphi \in \text{Diff}(\mathbb{R}^{2n+1}) \bigm| \varphi^* \alpha_{\text{std}} = e^f \alpha_{\text{std}} \text{ for some } f \in C^\infty M\}.$$

Since the contact condition is C^1–open, Gray's theorem implies that any two contact structures which are sufficiently C^1–close are isotopic, and in particular cut–and–paste operations and corner smoothings are well–defined up to contact isotopy.

1.2. An inadequate history of contact 3–manifolds

There is a long history of connections between 3–dimensional topology and contact structures on 3–manifolds [25]. Martinet established that every 3–manifold admits a contact structure [35], and soon after Lutz [34] showed that in fact a plane field on a 3–manifold is homotopic to a contact structure. In a beautiful application to low–dimensional topology, Eliashberg gave a new proof of the fact that every orientation preserving diffeomorphism of S^3 extends to D^4 using contact geometry [17, Section 6]. These are only three of a fantastic list of results in the establishment of contact geometry as a field [24, 26].

The interaction between the theory of knots and contact topology has also been strengthten these last two decades. For instance, an efficient method for getting lower bounds for the slice genus of smooth knots $K \subseteq \mathbb{R}^3$ is by representing them as knots everywhere transverse to the standard contact structure $\ker \alpha_{\text{std}}$ on $\mathbb{R}^3_{\text{std}}$. Such methods were first applied by Rudolph [41] using monopole Floer homology [30], and later by Plamenevskaya [40] using Heegaard-Floer homology. And the solution of Property P for smooth knots [33] also required a fair use of contact topology.

An important concept early in the development of the theory was the distinction between overtwisted and tight 3–manifolds [16]. We say that a 3–dimensional contact manifold (M, ξ) is *overtwisted* if there is an embedded $D^2 \subseteq (M, \xi)$ so that $TD^2|_{\partial D^2} = \xi|_{\partial D^2}$, and a contact manifold is called *tight* if it is not overtwisted. As an example of an overtwisted manifold, we have $\mathbb{R}^3_{\text{ot}} = (\mathbb{R}^3, \ker \alpha_{\text{ot}})$, where $\alpha_{\text{ot}} = \cos r dz + r \sin r d\theta$. The disk $D^2_{\text{ot}} = \{z = 0, r \leq \pi\}$ is an *overtwisted disk* demonstrating that $\mathbb{R}^3_{\text{ot}}$ is overtwisted. The role of overtwisted contact structures is defined by the following result [16, Theorem 1.6.1]:

Theorem 1.2 ([16]). Let ξ_1 and ξ_2 be two contact structures on a 3-manifold M, which are homotopic as (cooriented) plane fields and both overtwisted. Then, they are homotopic as contact structures, and therefore isotopic.

The power of this theorem is its local–to–global character: in order to show that a contact manifold is overtwisted, it suffices to exhibit a single overtwisted disk somewhere in the manifold. Once this has been found, the contact structure is totally classified by its homotopy class of the 2–plane field, which is readily understandable in terms of obstruction classes in algebraic topology. As an application, suppose ξ_1 and ξ_2 are two contact structures on M, which are homotopic as plane fields. They may or may not be isotopic, and detecting this is an extremely subtle question. But if (S^3, ξ_{ot}) is an overtwisted contact structure on the sphere, we see that $(M, \xi_1)\#(S^3, \xi_{\text{ot}}) \cong (M, \xi_2)\#(S^3, \xi_{\text{ot}})$.

In consequence, the classification of overtwisted contact structures is reduced to a strictly homotopy–theoretic problem which is a much coarser realm than the a prior differential geometric nature of contact structure. In contrast, tight contact structures are not classified only in terms of the homotopy classes of the underlying 2–plane fields and they exhibit a rich behaviour much more subtle than algebraic topology; we refer the reader to the articles [14, 23, 28, 29] for a substantial sample on the excitement they can provide.

1.3. Higher dimensions

Let us discuss the dichotomy between overtwisted and tight contact structure in higher dimensions. This is a salient instance of the flexible–rigid dichotomy that reigns in contact and symplectic topology [15, 21]. Let us first introduce the algebraic topological structure which underlies contact structures. The comparison to genuine contact structure serves to motivate the definition of overtwisted contact structures.

Definition 1. Let M^{2n+1} be a smooth oriented manifold, an *almost contact structure* is a pair $(\alpha, \omega) \in \Omega^1(M) \times \Omega^2(M)$ so that $\alpha \wedge \omega^n > 0$.

An almost contact structure (α, ω) on a smooth manifold M is a weaker notion than that of a contact structure $(M, \ker \alpha)$ since we can choose $\omega = d\alpha$; we thus think of an almost contact structure, up to homotopy through almost contact structures, as being a weakening of the defining property of a contact structure down to its purely algebro–topological invariants. This is captured in the following proposition:

Proposition 1.3. An almost contact structure is equivalent, up to homotopy, to either:

a. A stable almost complex structure, i.e. a bundle map $J : TM \oplus \mathbb{R} \to TM \oplus \mathbb{R}$ covering the identity so that $J^2 = -\operatorname{id}$, or
b. A lifting $\beta : M \to BU_n$ of the classifying map of the tangent bundle. That is, a map $\beta : M \to BU_n$ so that under the inclusion $\iota : U_n \to SO_{2n} \to SO_{2n+1}$, $B\iota \circ \beta : M \to BSO_{2n+1}$ classifies the tangent bundle.

Proof. Choose a metric on M, which is a contractible choice and thus unique up to homotopy. Since $U_{n+1} \subseteq GL_{n+1}\mathbb{C}$ is a homotopy equivalence we can homotope J to act orthogonally, which then defines a hyperplane field ξ on M by $\xi = J(\mathbb{R})^{\perp}$. The restriction $J|_\xi$ then presents $(\xi, J|_\xi)$ as a complex bundle, thus proving the equivalence of the two conditions in the proposition.

Let (α, ω) be an almost contact structure and note that, up to homotopy, the choice of α is the same as a choice of hyperplane field $\xi = \ker \alpha$. The almost contact condition $\alpha \wedge \omega^n > 0$ is tantamount to $\omega|_\xi$ being a non-degenerate skew form, which is a linear symplectic structure on ξ. Since $U_n \subseteq Sp(2n, \mathbb{R})$ is a homotopy equivalence, we see that this is also equivalent to the two given conditions. ☐

Remark 2. Since the inclusion $U(1) \subseteq SO(2)$ is a homotopy equivalence, a 3–dimensional almost contact structure is the same, up to homotopy, as a cooriented plane field.

Any contact structure is an almost contact structure, but only the former definition is interesting from a purely geometric perspective: $\alpha \wedge (d\alpha)^n > 0$ is a differential inequality, whereas the almost contact condition $\alpha \wedge \omega^n$ is a linear inequality and therefore can be treated with algebro–topological techniques. In the study of contact geometry, existence of contact structures on a given smooth manifold is quite a natural question: does every odd dimensional smooth manifold admit a contact structure?

This naive statement cannot be true because there are many smooth manifolds which do not even admit an almost contact structure. For example, the Steifel-Whitney class $w_2(M) \in H^2(M; \mathbb{Z}_2)$ must be the modulo 2 reduction of the class $c_1(\xi) \in H^2(M; \mathbb{Z})$, which implies that $\beta(w_2(M)) = 0 \in H^3(M, \mathbb{Z})$, where β is the Bockstein homomorphism.

Example 1. The smooth 5–fold $(S^1 \times \mathbb{CP}^2)/(\theta, [z_0, z_1, z_2]) \sim (-\theta, [\overline{z_0}, \overline{z_1}, \overline{z_2}])$ does not admit an almost contact structure.

Fantastically, the existence of an almost contact structure is the only obstruction for the existence of a contact structure:

Theorem 1.4 ([8]). Any almost contact structure on any manifold is homotopic to a genuine contact structure.

Just as in the 3–dimensional case, stated in Theorem 1.2, this existence theorem can be extended as a uniqueness theorem if we restrict to a particular class of contact structures: the overtwisted contact structures. Such uniqueness is essentially the defining property of overtwisted contact structures, but it must be first shown that they indeed exist. For that we now define the class of overtwisted contact structures in terms of containing a certain local model:

Definition 2. Let (M^{2n+1},ξ) be a contact manifold, then ξ is said to be *overtwisted* if, for every $R>0$ there is an isocontact embedding

$$i_R : (\mathbb{R}^3 \times B(R), \ker(\alpha_{\mathrm{ot}}+\lambda)) \longrightarrow (M,\xi).$$

Remark 3. Given any isocontact embedding $\mathbb{R}^3_{\mathrm{ot}} \subseteq (M^{2n+1},\xi)$, the Weinstein neighborhood theorem states that there exists a tubular neighborhood $\mathcal{O}p\,(\mathbb{R}^3_{\mathrm{ot}})$ and a positive radius $R\in\mathbb{R}^+$ such that

$$(\mathcal{O}p\,(\mathbb{R}^3_{\mathrm{ot}}),\xi) \cong (\mathbb{R}^3 \times B(R), \ker(\alpha_{\mathrm{ot}}+\lambda)).$$

By Gromov's h–principle for isocontact embeddings with positive codimension [27], any contact manifold of dimension at least 5 contains an abundance of such embeddings. For this reason, we emphasize that a central component of Definition 2 is that we need a contact embedding *for every* $R>0$. See also Section 6.3 below for a result on the existence of codimension–2 closed overtwisted submanifolds.

Definition 2 introduces a precise description of overtwisted contact structures, which can be shown to exist. Indeed, the proof of the existence result stated in Theorem 1.4 constructs overtwisted contact structures, and it can be said that Definition 2 was introduced with this purpose. But as in many h–principle arguments, the parametric uniqueness readily follows if the local model is good enough, and the right adaption of the argument for Theorem 1.4 yields the following uniqueness result:

Theorem 1.5 ([8]). Let ξ_1 and ξ_2 be two overtwisted contact structures on a smooth manifold M which are homotopic as almost contact structures. Then the ξ_1 and ξ_2 are isotopic contact structures.

Said differently, up to isotopy there is only one overtwisted contact structure in every almost contact homotopy class. Nevertheless, if we are given a contact manifold by some construction, how can we tell if it is the overtwisted one?

Note. In fact, the definition of an overtwisted contact structure does not actually require finding an embedding of $\mathbb{R}^3_{\mathrm{ot}} \times B_{\mathrm{std}}(R)$ for *all* radii $R\in\mathbb{R}^+$; there is a universal constant $R_0>0$, depending only on dimension, so that any contact manifold containing the contact ball $\mathbb{R}^3_{\mathrm{ot}} \times B_{\mathrm{std}}(R_0)$ is overtwisted. This might appear to be a much simpler condition, but the problem is the lack of explicit upper bounds for this constant R_0 and thus in practice we typically would need to find embeddings for all $R \in \mathbb{R}^+$ in order to satisfy either criterion.

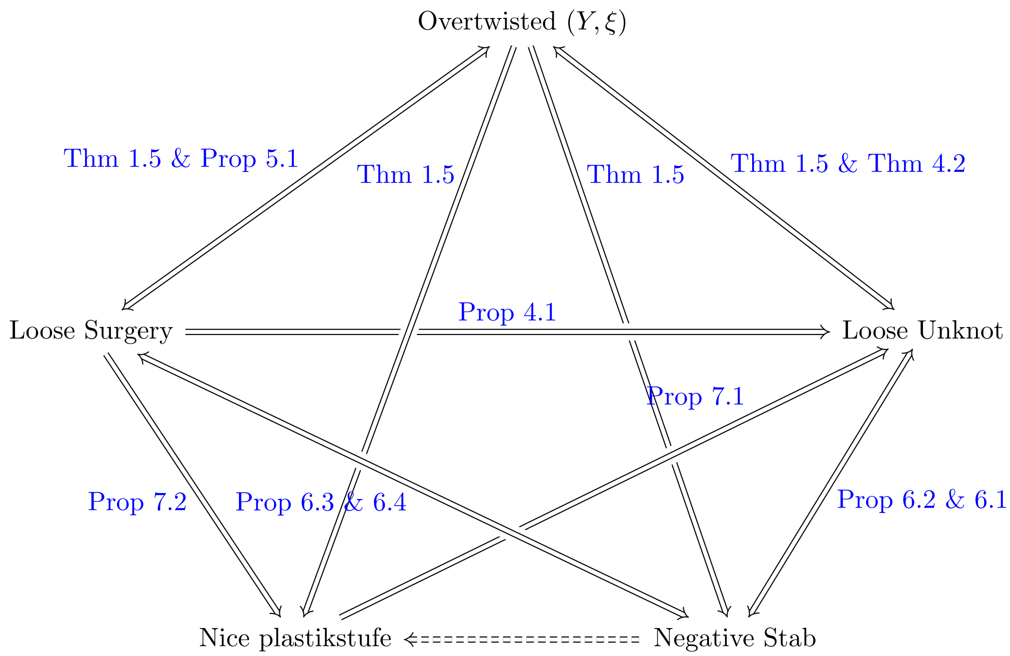
Assuming that we are working with a manifold which we know is overtwisted, we call the image of an embedding of $\mathbb{R}^3_{\mathrm{ot}} \times B_{\mathrm{std}}(R_0)$ an *overtwisted disk*. Since any overtwisted contact manifold contains an embedding of $\mathbb{R}^3_{\mathrm{ot}} \times B_{\mathrm{std}}(CR_0)$ for any $C > 0$, we note that an overtwisted contact manifold contains an unbounded number of disjoint overtwisted disks.

The problem of detecting whether a contact structure is overtwisted using Definition 2 is non–trivial. The main reason is that the definition itself is hard to use in practice: we would need to exhibit infinitely many contact embeddings, and even finding this embedding for a fixed radius $R \in \mathbb{R}^+$ may be through an extremely complicated coordinate choice. The central theme of this article is discovering criteria that can be useful for detecting overtwistedness in a variety of contexts (and of course, using this as an excuse, give an expository account of many structures in high dimensional contact geometry).

The following remark is in order. The original definition that appeared in the article [8] was much less tractable than the one given here, and requires a significant setup to even state. The equivalence of Definition 2 follows from criteria $(2a)$ in [10], or Theorem 10.2 in [8].

The article is organized as follows. Section 2 introduces the objects of interest in contact topology which will be used in order to characterize overtwisted contact structures. Section 3 discusses two essential operations, the Dehn twist and Legendrian handle slides, in terms of Legendrian front projections. Section 4 explains the relation between loose Legendrian knots and overtwisted contact structures. Then Section 5 discusses contact $(+1)$–surgery and Section 6 characterizes overtwisted contact structures in terms of open book decompositions, and Section 7 discusses the equivalence of nice plastikstufes and overtwistedness. We have included some constructions which might be of interest in higher–dimensional contact topology. These are just seeds for further development.

The equivalences between the different properties characterizing overtwistedness which are explained in this article are contained in the diagram below. There are arrows which we have not included, and the reader is invited to discover proofs for them. From a practical perspective, the arrows surrounding the plastikstufe are quite interesting, and a neat explicit description of them (including the dashed arrow) would improve our ability to detect overtwistedness. Similarly, it would be good to understand whether the hypothesis of niceness of the plastikstufe is genuinely required or non–contractible plastikstufes also provide overtwisted disks.



2. Contact geometry, from many viewpoints

2.1. The front projection

The most interesting submanifolds of a contact manifold $(Y^{2n+1}, \ker\alpha)$ are called *Legendrian submanifolds*; submanifolds $\Lambda \subseteq Y$ of dimension n satisfying $\alpha|_\Lambda = 0$. As stated in Theorem 1.1, every contact manifold is locally contactomorphic to $(\mathbb{R}^{2n+1}, \ker(dz - \sum y_i dx_i))$. We would like to be able to draw pictures of Legendrians in this standard local model, and an extremely useful tool for this is the *front projection*.

Let $\Lambda \subseteq \mathbb{R}^{2n+1}_{\text{std}}$ be a Legendrian manifold, and consider the projection $\pi : \mathbb{R}^{2n+1}_{\text{std}} \to \mathbb{R}^{n+1}$ given by $(x_i, y_i, z) \mapsto (x_i, z)$. After possibly C^∞ perturbing Λ through Legendrian embeddings, $\pi(\Lambda) \subseteq \mathbb{R}^{n+1}$ will be a singular hypersurface which is smooth outside of a codimension 1 subset. In fact $\pi(\Lambda)$ determines Λ completely.

To see this, first look at smooth points of $\pi(\Lambda)$. We know that $dz - \sum y_i dx_i|_\Lambda = 0$, which is to say $y_i = \frac{\partial z}{\partial x_i}$, so the y_i coordinates of Λ are determined by the front of Λ and the condition that Λ is Legendrian. Since the smooth points of $\pi(\Lambda)$ are dense in $\pi(\Lambda)$, this determines the y_i coordinates of Λ everywhere .

In general, the singularities of $\pi(\Lambda)$ cannot be classified. However, there is only one singularity that arises generically in codimension 1, all other singularities are in codimension 2 or more. This is the *cusp* singularity, and it is modelled on $(x, y, z) = (t^2, \frac{3}{2}t, t^3)$ (extended trivially in any remaining dimensions). Notice that this model defines a smoothly embedded Legendrian submanifold, but the front $(x, z) = (t^2, t^3)$ is a semi-cubic cusp.

Notice that, since $\frac{\partial z}{\partial x_i} = y_i \in \mathbb{R}$, fronts of Legendrians can never have vertical tangencies. Also, the front of a Legendrian can never be *self-tangent*: any two points in $\Lambda \subseteq \mathbb{R}^{2n+1}_{\mathrm{std}}$ with equal x_i and z coordinates must have distinct y_i coordinates. Conversely, a Legendrian front may self-intersect, and since the y_i coordinates are determined we do not need any additional data at multiple points (such as over/under crossing data) to determine the Legendrian embedding. See two instances of Legendrian fronts for the Legendrian unknot in Figure 1.

FIGURE 1. Two fronts representing the Legendrian unknot.

Front projections are extremely useful not only for visualization, but also for *construction*: a priori it may seem difficult to construct submanifolds satisfying $\alpha|_\Lambda = 0$. But if we draw any (possibly self-intersecting) hypersurface in $\mathbb{R}^{n+1}$ with no self-tangencies, vertical tangencies, and only cusp singularities, we can simply define $y_i = \frac{\partial z}{\partial x_i}$ to define a Legendrian manifold. In this way, Legendrians up to Legendrian isotopy are determined by singular hypersurfaces up to (some sort of) homotopy through nowhere self-tangent hypersurfaces, and are therefore topological objects, of a sort. Note, however, that the general classification problem of Legendrian spheres up to isotopy in $\mathbb{R}^{2n+1}_{\mathrm{std}}$ is known to be completely intractable, even though every n-dimensional sphere is smoothly unknotted (assuming $n > 1$).

2.2. Loose Legendrians

Let $B^3 \subseteq (\mathbb{R}^3, \xi_{\mathrm{std}})$ be a round ball in a contact Darboux chart and let $\Lambda_0 \subseteq (\mathbb{R}^3, \xi_{\mathrm{std}})$ be a stabilized Legendrian arc as seen in Figure 2. Consider a closed manifold Q and a neighborhood $\mathcal{O}p\,(Z) \subseteq T^*Q$ of the zero section $Z \subseteq T^*Q$. Then, the smooth submanifold $\Lambda_0 \times Z \subseteq (B^3 \times \mathcal{O}p\,(Z), \ker(\alpha_{\mathrm{std}} + \lambda_{\mathrm{std}}))$ is Legendrian submanifold.

Definition 3. Let Q be a closed manifold and $Z \subseteq T^*Q$ the zero section. The pair $(B^3 \times \mathcal{O}p\,(Z), \Lambda_0 \times Z)$ endowed with the contact structure $\ker(\alpha_{\mathrm{std}} + \lambda_{\mathrm{std}})$ is said to be a *loose chart*. Let $\Lambda \subseteq (Y, \xi)$ be a Legendrian in a contact manifold with $\dim(Y) \geq 5$. The Legendrian Λ is *loose* in (Y, ξ) if there is an open set $V \subseteq Y$ so that $(V, V \cap \Lambda)$ is contactomorphic to a loose chart.

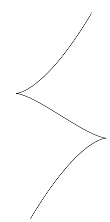


FIGURE 2. The front projection of a stabilized Legendrian arc.

Loose Legendrians were first defined in [37], where they were also classified up to Legendrian isotopy. We note that the definition presented above is slightly different than the definition introduced there, though it is equivalent (see [38, Section 4.2]). We refer to [37] for a detailed treatment, but we will attempt to give a very rough intuition; see Figure 3 for a Legendrian front diagram representing a loose Legendrian unknot.

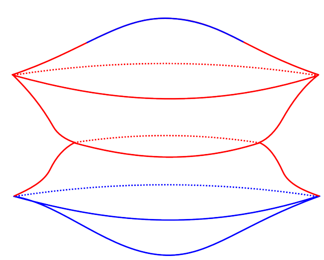


FIGURE 3. Loose Legendrian unknot, the loose chart is marked in red.

As stated above, the classification problem of Legendrians is intractable. But if we allow certain prescribed singularities[1] the Legendrian isotopy problem can be completely solved by topological methods. Loose Legendrians are smooth, but the local model which defines them is essentially a model which allows us to simulate any number of singularities in

[1]Of the Legendrians themselves! Not just singularities in the front projection

a smooth way. For us, the classification theorem is not so important as the following property, which is another kind of reduction from geometry to topology.

Theorem 2.1 ([37]). Let $\Lambda \subseteq (Y,\xi)$ be a loose Legendrian with loose chart $U \subseteq Y$, and let $f_t : \Lambda \longrightarrow Y$ be a smooth isotopy so that f_0 is the inclusion map, and $f_t|_U$ is the inclusion map of $\Lambda \cap U$ for all $t \in [0,1]$. Then, there exists a Legendrian isotopy $g_t : \Lambda \longrightarrow Y$ so that $g_t|_{Y\setminus U}$ is C^0–close to $f_t|_{Y\setminus U}$.

2.3. Weinstein manifolds

We review some basic definitions for Weinstein manifolds, see [15] for a thorough treatment of the topic. Weinstein manifolds should be thought of as symplectic manifolds which can be described by Morse theoretic methods. The level sets of Weinstein manifolds are contact manifolds, and therefore Weinstein handle attachments give us a way to construct new contact manifolds from old, by contact surgery.

Definition 4. A *Weinstein cobordism* is a triple (W,λ,f), where $(W,d\lambda)$ is a compact symplectic manifold with boundary, and $f : W \longrightarrow [0,1]$ is a Morse function such that $\partial W = \partial_- W \cup \partial_+ W = f^{-1}(0) \cup f^{-1}(1)$. We also require that the vector field V_λ symplectic dual to the Liouville form λ (defined by $V_\lambda \lrcorner\, d\lambda = \lambda$) is gradient-like for f.

From the definition it follows that $\lambda|_{f^{-1}(c)}$ is a contact form on the submanifold $f^{-1}(c)$ for any regular value $c \in [0,1]$. The descending manifold D^k_p associated to any critical point p of f satisfies $\lambda|_{D^k_p} = 0$. In particular, the submanifold D^k_p is isotropic and thus

$$\operatorname{ind}(p) = k \leq n+1 = \frac{1}{2}\dim W.$$

Critical points with index strictly less than $n+1$ are called *subcritical*, and a *subcritical* Weinstein cobordism (W,λ,f) is one where all critical points of f are subcritical.
In case $c \in [0,1]$ is a regular value, $\Lambda^c_p = D^k_p \cap f^{-1}(c)$ is an isotropic submanifold of the contact manifold $(f^{-1}(c), \ker\lambda)$. When $c \in [0,1]$ is a critical value with a unique critical point $p \in W$, the Weinstein cobordism $(f^{-1}([c-\varepsilon, c+\varepsilon]), \lambda, f)$ is determined, up to homotopy through Weinstein structures, by the contact manifold $(f^{-1}(c-\varepsilon), \ker\lambda)$ and the isotropic submanifold $\Lambda^{c-\varepsilon}_p$[2]. Hence the contact manifold $(f^{-1}(c+\varepsilon), \ker\lambda)$ is determined up to contactomorphism, and it is said to be obtained from $(f^{-1}(c-\varepsilon), \ker\lambda)$ by *contact surgery along* the isotropic sphere $\Lambda^{c-\varepsilon}_p$. Notice that $(f^{-1}(c-\varepsilon) \setminus \Lambda^{c-\varepsilon}_p, \ker\lambda)$ has a natural contact inclusion into $(f^{-1}(c+\varepsilon), \ker\lambda)$, defined by the flow of the gradient-like vector field V_λ. We refer to the monograph [15] for proofs of these statements and a more complete discussion of Weinstein handle attachments.
In case $c \in [0,1]$ is a critical value of f with a unique critical point p of index $n+1$ then $\Lambda^{c-\varepsilon}_p \subseteq (f^{-1}(c-\varepsilon), \ker\lambda)$ is a Legendrian submanifold. If this Legendrian is loose, we say that p is a *flexible* critical point.

[2]together with a framing of the symplectic normal bundle (which is necessarily trivial), and a smooth parametrization $S^{k-1} \cong \Lambda^{c-\varepsilon}_p$

Definition 5. A Weinstein cobordism (W, λ, f) is said to be *flexible* if every critical point of f is either subcritical or flexible.

In particular, every subcritical Weinstein cobordism is flexible. Owing to the flexibility properties of loose Legendrians, flexible Weinstein manifolds are completely classified, see [15, Chapter 14]. The following proposition is used to prove Theorem 4.2:

Proposition 2.2. Let (W, λ, f) be a flexible Weinstein cobordism so that $(\partial_- W, \ker \lambda)$ is an overtwisted contact manifold. Then, the contact manifold $(\partial_+ W, \ker \lambda)$ is overtwisted.

Proof. Split the cobordism (W, λ, f) into cobordisms with a single critical point

$$W = f^{-1}([0, c_1]) \cup \ldots \cup f^{-1}([c_s, 1]), \quad \text{for } 0 < c_1 < \ldots < c_s < 1.$$

The resulting attaching spheres $\Lambda_j \subseteq (f^{-1}(c_j), \ker \lambda)$ are either subcritical or loose Legendrians submanifolds. We show by induction that each contact manifold $(f^{-1}(c_j), \ker \lambda)$ is overtwisted. The $j = 0$ case follows from the fact that $(\partial_- W, \ker \lambda)$ is overtwisted, and the case $j = s$ case implies the result. The contact manifold $(f^{-1}(c_{j+1}), \ker \lambda)$ is obtained from $(f^{-1}(c_j), \ker \lambda)$ by a single Weinstein surgery along the isotropic sphere Λ_j, and any smooth isotopy of Λ_j can be C^0–approximated by a contact isotopy. Indeed, if Λ_j is subcritical, this follows from the h-principle for subcritical isotropic submanifolds [27], and if Λ_j is a loose Legendrian this is Theorem 2.1. In particular, we can find a contact isotopy which makes the attachig isotropic sphere Λ_j disjoint from any overtwisted disk in $(f^{-1}(c_j), \ker \lambda)$. □

Finally, we define a connected sum operation of Weinstein cobordisms. Let (W_1, λ_1, f_1), (W_2, λ_2, f_2) be two Weinstein cobordisms with non-empty negative boundary, and choose two points $p_i \in \partial_- W_i$ which are not in the descending manifold of any critical point. Let γ_i be the image curves of the points p_i by the flow of the gradient-like vector fields V_{λ_i}, so $\gamma_i \subseteq W_i$ are two curves which intersect transversely every level set of their corresponding ambient cobordisms exactly once. We define the connected sum cobordism

$$W_1 \overline{\#} W_2 = (W_1 \setminus \mathcal{O}p\,(\gamma_1)) \cup (W_2 \setminus \mathcal{O}p\,(\gamma_2)),$$

where the union glues a collar neighborhood of $\partial \mathcal{O}p\,(\gamma_1)$ to a collar neighborhood of $\partial \mathcal{O}p\,(\gamma_2)$ with a map that pulls back λ_2 to λ_1 and f_2 to f_1 (one can verify that such a map exists). The manifold $W_1 \overline{\#} W_2$ inherits a Weinstein structure $(W_1 \overline{\#} W_2, \lambda, f)$, the critical set of f is the union of critical sets of f_1 and f_2, and every regular level set $(f^{-1}(c), \ker \lambda)$ is contactomorphic to the contact connected sum $(f_1^{-1}(c), \ker \lambda_1) \# (f_2^{-1}(c), \ker \lambda_2)$. The Weinstein manifold $(W_1 \overline{\#} W_2, \lambda, f)$ is the *vertical connected sum* of (W_1, λ_1, f_1) and (W_2, λ_2, f_2). This operation is used in the proof of Theorem 4.2.

2.4. Open book decompositions

Weinstein manifolds give a good way to describe contact manifolds, by surgery. Another way to describe contact manifolds is by open book decompositions. Essentially an open book decomposition is a presentation of a closed manifold as a kind of relative mapping torus, and by requiring the fiber to be symplectic and the monodromy to be a symplectomorphism, this closed manifold will inherit a contact structure. Notice that Weinstein manifolds are a way to define contact manifolds using a symplectic manifold of one dimension larger, whereas open book decompositions use a symplectic manifold of one dimension smaller.

Let (W, λ) be a Weinstein domain, i.e. a Weinstein cobordism with $\partial_- W = \emptyset$, and let $\varphi : W \longrightarrow W$ be a compactly supported exact symplectomorphism, so that $\varphi^* \lambda = \lambda + dh$ for some compactly supported function $h \in C^\infty_c(W)$. The triple (W, λ, φ) is an (abstract) *open book decomposition*. An open book decomposition (W, λ, φ) canonically defines a contact manifold (Y, ξ) [13, 26], by considering

$$Y = W \times [0,1]/(x,1) \sim (\varphi(x), 0) \bigcup_{\partial W \times S^1} \partial W \times D^2$$

$$\xi = \ker\big((\lambda + Kd\theta + \theta dh) \cup (\lambda|_{\partial W} + Kr^2 d\theta)\big)$$

for a sufficiently large $K \in \mathbb{R}^+$. We write $(Y, \xi) = \mathrm{OB}(W, \lambda, \varphi)$ to denote this relationship, and say that (Y, ξ) is *compatible with* or *supported by* the open book (W, λ, φ). Notice that $\mathrm{OB}(W, \lambda, \varphi) = \mathrm{OB}(W, \lambda, \psi \circ \varphi \circ \psi^{-1})$ for any symplectomorphism ψ.

Open book decompositions are particularly useful in contact topology in light of E. Giroux's existence theorem [26].

Theorem 2.3 ([26]). Every contact manifold (Y, ξ) can be presented as $(Y, \xi) = \mathrm{OB}(W, \lambda, \varphi)$.

In order to construct open books, we would like to have a variety of compactly supported symplectomorphisms to choose from. Unfortunately, we really only know of one, the *Dehn-Seidel* twist [3, 42]. Let (W, λ) be a Weinstein manifold, and suppose it contains a (parametrized) Lagrangian sphere $L \subseteq (W, \lambda)$. A neighborhood of L is symplectomorphic to a neighborhood of the zero section in T^*S^n. Therefore, to define the Dehn twist around L, which we denote τ_L, it suffices to define it in this one model.

Fix the round metric on S^n, which identifies T^*S^n with TS^n, and let $(q, v) \in TS^n$ with $q \in S^n$, $v \in T_q S^n$. Define $\tau(q, v) = (q', v')$, where q' is obtained from q by geodesic exponentiation in the direction of v, for time $\pi - |v|$. v' is then defined as the parallel transport of v. When $|v| \geq \pi$, we define τ to be the identity. Notice that as $|v| \to 0$, q' limits to the antipodal point of q; and this is the essential property of the round metric: time π geodesic flow induces the antipodal map, independent of the choice of vector. It's easy to check that τ is a symplectomorphism, since it is the composition of two (non-compactly supported) symplectomorphisms: the contangent lift of the antipodal map on the sphere, and the Hamiltonian flow associated to $H = \max(\frac{1}{2}|v|^2, \frac{\pi^2}{2})$ (kinetic energy generates geodesic flow).

Since L is an exact Lagrangian, L defines a Legendrian Λ in the contact manifold $\mathrm{OB}(W, \lambda, \varphi)$ by integrating the exact form $\lambda|_L$. We denote this dependency by

$(Y, \xi, \Lambda) = \mathrm{OB}(W, \lambda, \varphi, L)$, and often times we say Λ *lies on the page* of the open book.[3] The conjugation invariance above now reads

$$\mathrm{OB}(W, \lambda, \varphi, L) = \mathrm{OB}(W, \lambda, \psi \circ \varphi \circ \psi^{-1}, \psi(L))$$

which can be verified by considering Λ as being near the page $\theta = 0$.

The following proposition relates Dehn twists to contact surgery:

Proposition 2.4. Suppose that $(Y, \xi, \Lambda) = \mathrm{OB}(W, \lambda, \varphi, L)$, then the contact manifold $\mathrm{OB}(W, \lambda, \varphi \circ \tau_L)$ is obtained from (Y, ξ) by contact surgery along Λ.

Consider a Lagrangian disk $D \subseteq (W, \lambda)$ with Legendrian boundary $\partial D \subseteq (\partial W, \ker \lambda)$ and attach a Weinstein handle to (W, λ) along the Legendrian sphere ∂D, obtaining a new Weinstein manifold $(W \cup H, \lambda')$. This manifold contains a Lagrangian sphere L, whose lower hemisphere is D and whose upper hemisphere is the core of the handle H. The open book $(W \cup H, \lambda', \varphi \circ \tau_L)$ is the *positive stabilization* of (W, λ, φ) along D, and $(W \cup H, \lambda', \varphi \circ \tau_L^{-1})$ is the *negative stabilization* of (W, λ, φ) along D.

Both the positive and the negative stabilization of an open book decomposition can be described as a contact connected sum. This description is the content of the following theorem.

Theorem 2.5 (E. Giroux). Let $(Y, \xi) = \mathrm{OB}(W, \lambda, \varphi)$ be a contact manifold, $D \subseteq (W, \lambda)$ any Lagrangian disk with Legendrian boundary $\partial D \subseteq (\partial W, \ker \lambda)$, and consider the contact structure $(S^{2n+1}, \xi_-) = \mathrm{OB}(T^*S^n, \tau^{-1})$. Then, the positive and negative stabilizations of (W, λ, φ) along D are diffeomorphic to Y. The positive stabilization is contactomorphic to (Y, ξ), and the negative stabilization is contactomorphic to the contact connected sum $(Y \# S^{2n+1}, \xi \# \xi_-)$.

This follows from a more general theorem. Its statement is known to experts, and we include a brief sketch of the proof here because to our knowledge no reference exists in the literature.

Proposition 2.6. (Murasugi Sum) Consider two $(2n+1)$–contact manifolds $(Y_i, \xi_i) = \mathrm{OB}(W_i, \lambda_i, \varphi_i)$ and two Lagrangian disks $D_i \subseteq W_i$ with Legendrian boundaries, for $i = 1, 2$. Let $U_i \subseteq W_i$ be a neighborhood of the disk D_i in W_i, identified[4] as

$$(U_i, D_i, \lambda_i|_{U_i}) \cong (D^n \times D^n, D^n \times \{0\}, \lambda = \sum_j^n x_j dy_j - y_j dx_j).$$

[3]Note that Λ does not literally lie on the page. However, because the Liouville flow of λ restricts to L as the action of a Hamiltonian isotopy, we can rescale λ_L by any exponential factor, and therefore assume that Λ is contained in a uniform neighborhood of the page.

[4]This identification can always be achieved after possibly adding an exact form to the Liouville form λ_i.

We form the new Liouville manifold (W, λ) by identifying the open neighborhoods U_1 and U_2 with the map $(x_j, y_j) \mapsto (y_j, -x_j)$; this is called the *plumbing* of (W_1, λ_1) and (W_2, λ_2). Since φ_1 and φ_2 have compact support, they each extend by the identity to symplectomorphisms of (W, λ) and we can consider the contact manifold $\mathrm{OB}(W, \lambda, \varphi_2 \circ \varphi_1)$; this is called the contact *Murasugi sum* of $\mathrm{OB}(W_1, \lambda_1, \varphi_1)$ and $\mathrm{OB}(W_2, \lambda_2, \varphi_2)$. Then, the contact *Murasugi sum* induces the connect sum of contact manifolds, that is

$$\mathrm{OB}(W, \lambda, \varphi_2 \circ \varphi_1) \cong (Y_1 \# Y_2, \xi_1 \# \xi_2).$$

Proof. Denote $Y = \mathrm{OB}(W, \lambda, \varphi_2 \circ \varphi_1)$ and let $B \cong \partial W \times D^2$ be the binding of Y. The mapping torus of the symplectomorphism $\varphi_2 \circ \varphi_1 \in \mathrm{Symp}_c(W, d\lambda)$ can be described by gluing two contactizations together:

$$Y \setminus B = (W \times [-2, -1] \cup [1, 2]) / \{(x, -1) \sim (\varphi_1(x), 1),\ (x, 2) \sim (\varphi_2(x), -2)\}.$$

Let $U \subseteq W$ be the identified copies of U_1 and U_2 in the plumbing, and divide its boundary as $\partial U = \partial_1 U \cup \partial_2 U$, where $\partial_i U = (\partial D_i) \times D^n$; in other words, $\partial_i U$ is the portion of the boundary of U which is contained in the contact boundary of the Liouville page W_i. In this division, the (extended) symplectic monodromy $\varphi_i \in \mathrm{Symp}_c(W)$ pointwise fixes the partial boundary $\partial_i U$. In order to identify the Murasugi sum $\mathrm{OB}(W, \lambda, \varphi_2 \circ \varphi_1)$ as a contact connected sum, it suffices to identify a separating sphere.

We define a separating sphere $S \cong S^{2n-1} \subseteq W$ as follows:

$$S = (U \times \{-2\}) \cup (U \times \{1\}) \cup (\partial_1 U \times [-2, -1]) \cup (\partial_2 U \times [1, 2]) \cup ((U \cap \partial W) \times \partial D^2)$$

Due to the above description, we have the identifications $U \times \{-2\} \cong \varphi_2^{-1}(U) \times \{2\}$ and $U \times \{1\} = \varphi_1^{-1}(U) \times \{-1\}$ for the first two factors in the union. The final piece $(U \cap \partial W) \times \partial D^2$ represents the portion of S contained in the binding, and notice that S is a closed manifold since each boundary component $\partial_i U$ is fixed by the respective monodromy φ_i. See Figure 4 for a diagramatic description of $S \subseteq Y$.

It remains to show that the complement $Y \setminus S$ is contactomorphic to the disjoint union of the two contact manifolds $Y_i \setminus B_i$, for $i = 1, 2$, where B_i denotes a standard Darboux ball. Given an interior point $p \in U$, the connected component of $Y \setminus S$ containing the point $(p, \frac{3}{2}) \in Y$ intersects the complement of the binding $Y \setminus B$ in the set

$$(W_2 \times (1, 2)) \cup (W \setminus W_1 \times [-2, -1]).$$

Since we have $W \setminus W_1 \cong W_2 \setminus U_2$ by construction, this intersection is contactomorphic to

$$(W_2 \times [0, 2]) / (x, 2) \sim (\varphi_2(x), 0) \setminus (U_2 \times [0, 1]).$$

This contactomorphism only uses a shift in the interval direction identifying $[-2, -1]$ to $[0, 1]$, and thus extends to the binding. By considering the contact Darboux ball $B_2 = U_2 \times [0, 1]$, we see that this connected component of $Y \setminus S$ is contactomorphic to

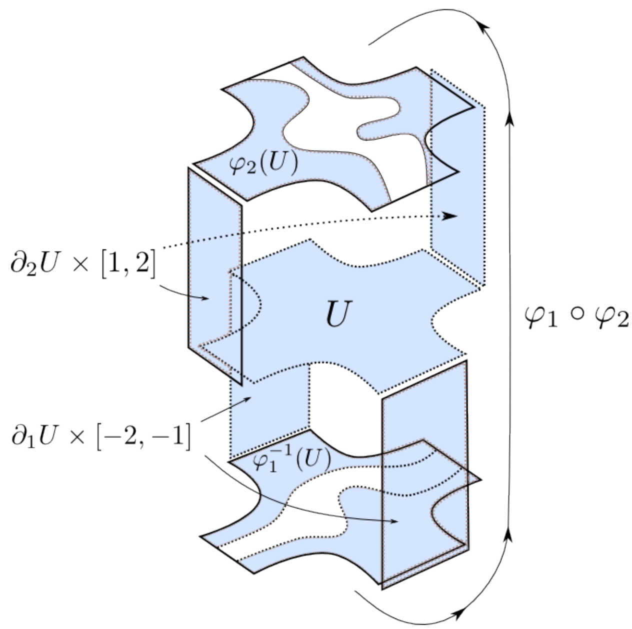


FIGURE 4. The separating sphere $S \subseteq Y$, depicted in blue.

$Y_2 \setminus B_2$. The same argument shows that the remaining component is contactomorphic to $Y_1 \setminus B_1$, which completes the proof. □

Note that the positive and negative stabilizations described previously can be interpreted in terms of a Murasugi sum: a positive (resp. negative) stabilization of $(Y, \xi) = \mathrm{OB}(W, \lambda, \varphi)$ is contactomorphic to the Murasugi sum of $\mathrm{OB}(W, \lambda, \varphi)$ and $(S^{2n+1}, \xi_{\mathrm{std}}) = \mathrm{OB}(T^*S^n, \lambda_{\mathrm{std}}, \tau)$ (resp. $(S^{2n+1}, \xi_-) = \mathrm{OB}(T^*S^n, \lambda_{\mathrm{std}}, \tau^{-1})$).

2.5. The plastikstufe

We provide details on the definition of the plastikstufe, first introduced in the article [39]. Let $\mathcal{O}p\,(\Delta^2_{\mathrm{ot}}) \subseteq (\mathbb{R}^3, \xi_{\mathrm{ot}})$ be a contact neighborhood of an overtwisted disk for any overtwisted contact structure $\xi_{\mathrm{ot}} = \ker \alpha_{\mathrm{ot}}$.

Definition 6. Let Q be a closed manifold and $\mathcal{O}p\,(Z) \subseteq T^*Q$ a neighborhood of the zero section. The contact manifold $(\mathcal{O}p\,(D^2_{\mathrm{ot}}) \times \mathcal{O}p\,(Z), \ker(\alpha_{\mathrm{ot}} + \lambda_{\mathrm{std}}))$ is said to be a *plastikstufe*. The submanifold $Q \subseteq \mathcal{O}p\,(D^2_{\mathrm{ot}}) \times \mathcal{O}p\,(Z))$ is the *core* of the plastikstufe.

Any contact manifold which contains a plastikstufe cannot be realized as the convex boundary of a symplectic manifold [39], in particular it cannot be realized by contact surgery on the standard sphere. We refer the reader to the paper for a complete proof, but the short version is that we look at a family of pseudo-holomorphic disks inside of a hypothetical symplectic filling, and derive a contradiction. The pseudo-holomorphic disks are those which have boundary contained in the plastikstufe, satisfying a marked point condition along an arc. We calculate that the moduli space of such disks is 1-dimensional. Gromov compactness tells us that this moduli space is compact. This moduli space has at least one boundary point (corresponding to the Bishop family of holomorphic disks converging to $0 \in D^2_{\mathrm{ot}}$), but we can also rule out any other boundary components (how difficult this is depends on what "symplectic filling" means, the easiest case is when the filling is exact with strongly convex boundary.) Since no 1-manifold exists with a single boundary point, we reach a contradiction.

For implications related to overtwistedness, we will need some additional hypotheses on our plastikstufe.

Definition 7. A plastikstufe $\mathcal{O}p\,(D^2_{\mathrm{ot}}) \times \mathcal{O}p\,(Z) \subsetneq (Y, \xi)$ is said to be *small* if it is contained in a smooth ball in Y. Let $\Lambda_0 \subseteq \mathcal{O}p\,(D^2_{\mathrm{ot}})$ be an open leaf of the characteristic foliation of the overtwisted disk. The plastikstufe $(\mathcal{O}p\,(D^2_{\mathrm{ot}}) \times \mathcal{O}p\,(Z), \ker(\alpha_{\mathrm{ot}} + \lambda_{\mathrm{std}}))$ has *trivial rotation* if the open Legendrian submanifold $\Lambda_0 \times Z$ has trivial rotation class.

Note that the rotation class of the Legendrian $\Lambda_0 \times Z$ is well defined since the hyperplane field ξ has a unique framing on the smooth ball up to homotopy. Observe that in the case $Q = S^{n-2}$, a plastikstufe $\mathcal{O}p\,(D^2_{\mathrm{ot}}) \times \mathcal{O}p\,(Z) \subseteq (Y, \xi)$ is both small and has trivial rotation if and only if $\Lambda_0 \times Z$, which is a Legendrian annulus $[0,1] \times S^{n-2}$, can be included into a Legendrian disk. Then this disk defines a smooth ball containing the plastikstufe, and since a Legendrian disk has a unique framing, it induces a trivial framing on its boundary collar.

3. The Front Gallery

In this section we describe Dehn twists and handle slides in terms of front projections [9, 10]. This is a useful tool, as it will already become apparent in these pages. Subsection 3.1 explains how to depict the front projection of the Legendrian lift of the image of an exact Lagrangian along a Dehn twist, and Subsection 3.2 details the front presentation of a Legendrian handle slide.

3.1. Legendrian Fronts of Dehn twists

The study of contact structures up to isotopy is equivalent to that of Liouville domains and their compactly supported symplectomorphisms. The symplectic Dehn twist [3, 43] is a salient instance of such symplectomorphisms and it provides the most thorough studied class of contact structures: those which are supported by an open book whose monodromy is a composition of Dehn twists (in particular any contact 3–fold).

Given a contact manifold $(Y, \xi) = \mathrm{ob}((W, \lambda), \varphi)$, an exact Lagrangian $S \subseteq (W, \lambda)$ and a fixed embedding of the Liouville page $(W, \lambda) \subseteq (Y, \xi)$, there exists a unique Legendrian lift of S up to isotopy. Suppose that S is a sphere and L is another exact Lagrangian, then $\tau_S(L) \subseteq (W, \lambda)$ is an exact Lagrangian; the purpose of Theorem 3.1 is to relate the Legendrian lift of $\tau_S(L)$ with the Legendrian lifts of S and L.

The statement focuses on the local situation where S and L intersect at a single point, and since the description is relative to the boundary, it extends to the rest of the manifold by the identity.

Theorem 3.1 (Dehn twist front [10]). Let $L, S \subseteq (W, \lambda)$ be two exact Lagrangians intersecting at a point $x = L \cap S$, and consider the contactization $(Y, \xi) = (W \times \mathbb{R}(z), dz - \lambda)$ of (W, λ). Then,

a. there exists a Darboux chart in (Y, ξ) around $x \in (W, \lambda)$ such that the front projection of the Legendrian lift of $\tau_S(L)$ is as depicted in Figure 5.

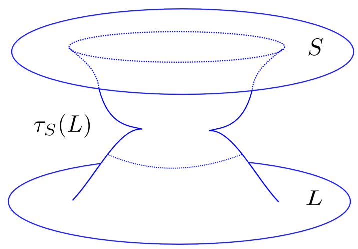


FIGURE 5. The Legendrian lift of $\tau_S(L)$.

b. there exists a Darboux chart in (Y, ξ) around $x \in (W, \lambda)$ such that the front projection of the Legendrian lift of $\tau_S^{-1}(L)$ is as depicted in Figure 6.

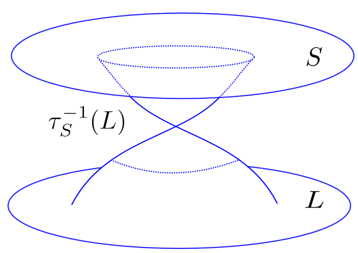


FIGURE 6. The Legendrian lift of $\tau_S^{-1}(L)$.

Proof. The proof of the general case is written in detail in [10], but the core geometry is already present in the 3–dimensional case. We now provide a proof assuming $\dim(L) = \dim(S) = 1$, and the reader is invited to generalize it to higher dimensions.

Consider a Darboux chart $(D^2(q,p) \times [-1,1], \alpha_{\text{std}}) \cong (\mathcal{O}p\,(x), \xi) \subseteq (Y, \xi)$ such that $(W \cap \mathcal{O}p\,(x), \lambda) \cong (D^2 \times \{0\}, \lambda_{\text{std}})$ and the two Lagrangians are

$$L = \{(q,p) \in D^2 \times \{0\} : q = p\}, \qquad S = \{(q,p) \in D^2 \times \{0\} : q = -p\}.$$

The Dehn twists of L along S are represented by the parametrized curves

$$\tau_S(L) = \{(\pm\cosh(t), \sinh(t)) : t \in \mathbb{R}\} \cap D^2 \times \{0\},$$
$$\tau_S^{-1}(L) = \{(-\sinh(t), \mp\cosh(t)) : t \in \mathbb{R}\} \cap D^2 \times \{0\},$$

depicted in Figure 7.

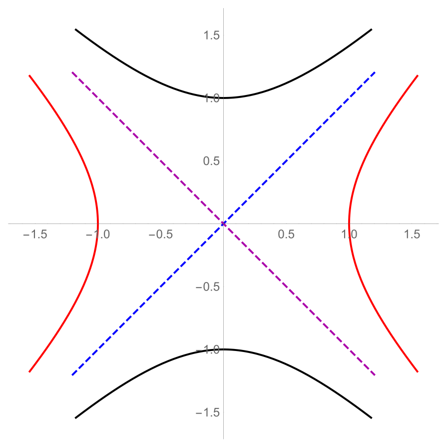


FIGURE 7. The Lagrangian L is the blue dashed positive slope line and the Lagrangian S is the purple dashed negative slope line. The curve $\tau_S(L)$ is represented by the red vertical hyperbola, whereas $\tau_S^{-1}(L)$ is the black horizontal hyperbola.

Near the origin $t \in \mathcal{O}p\,(0)$, the curves can be approximately parametrized by the curves

$$\tau_S(L) \simeq \{(\pm(1 + t^2/2), t) : t \in \mathbb{R}\} \cap D^2 \times \{0\},$$
$$\tau_S^{-1}(L) \simeq \{(-t, \mp(1 + t^2/2)) : t \in \mathbb{R}\} \cap D^2 \times \{0\}.$$

In order to lift these exact Lagrangians we need the action variables

$$z_+ = \int_{\tau_S(L)} pdq = \pm \int t^2 dt = \pm t^3/3,$$

$$z_- = \int_{\tau_S^{-1}(L)} pdq = \pm \int (1 + t^2/2) dt = \pm(t + t^3/6).$$

The front projection of the Legendrian lifts $\Lambda_\pm$ of $\tau_S^\pm(L)$ are thus

$$\Lambda_+ = \{(q, z_+) = (\pm(1 + t^2/2), \pm t^3/3)\},$$

$$\Lambda_- = \{(q, z_-) = (-t, \pm(t + t^3/6)\}.$$

The resulting two Legendrians $\Lambda_\pm$ are depicted in Figure 8; note that the intersection

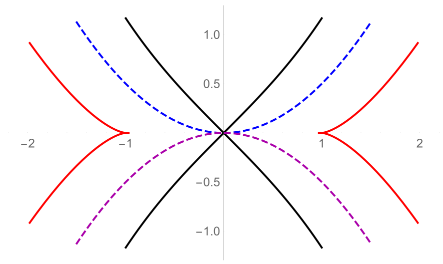


FIGURE 8. The Legendrian lift of the Lagrangian L is the blue dashed convex parabola and the lift of the Lagrangian S is the blue dashed concave parabola. The Legendrian lift Λ_+ of the curve $\tau_S(L)$ is represented by the red cusps, whereas the Legendrian lift Λ_- of the Lagrangian $\tau_S^{-1}(L)$ is the black cross.

point $x = L \cap S$ is sent to a Reeb chord between the Legendrian lifts of L and S, which in the picture has been set to have zero length. This concludes the proof in the 3–dimensional case. □

3.2. Front description of handle slides

In this subsection, we explain handle slides in contact topology; the smooth theory of handle slides is explained in the foundational work [36, 44] and the article [32] contains the diagramatic description in the 3–dimensional case, the so–called Kirby calculus.

Legendrian submanifolds $\Lambda \subseteq (Y, \xi)$ are canonically endowed with a framing by the contact structure ξ, thus a Legendrian sphere determines a Weinstein handle attachment on the symplectization of (Y, ξ) as long as we specify whether such attachment is convex or concave; we refer the reader to [15, 18, 45] for detailed accounts on Weinstein handle attachments and contact surgeries.

We aim to describe the contact handle slide of a Legendrian sphere Σ along a (± 1)–surgery Legendrian Λ in terms of the front projection on a neighborhood $(\mathcal{O}p\,(\Lambda), \xi) \cong (J^1\Lambda, \xi_{\text{std}})$ of the surgered Legendrian Λ. Theorem 3.2 explains how to diagramatically realize such contact handle slide of Σ along Λ in the front $J^1\Lambda$.

Theorem 3.2 (Legendrian handle slide [9]). Let (Y, ξ) be a contact manifold and $\Lambda \subseteq (Y, \xi)$ a Legendrian sphere. Then,

a. the Legendrians Σ and $h_\Lambda(\Sigma)$ presented in Figure 9 are Legendrian isotopic in the surgered manifold $(Y_\lambda(-1), \xi_\lambda(-1))$.

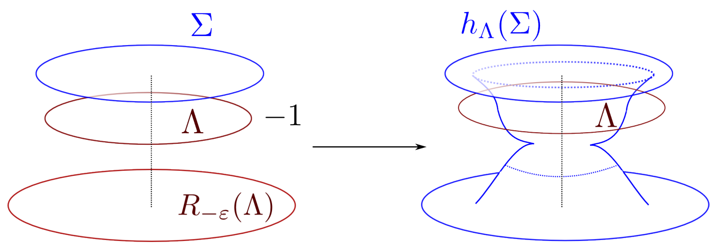


FIGURE 9. Handleslide of Σ along the (-1)–Legendrian Λ.

b. the Legendrians Σ and $h_\Lambda(\Sigma)$ presented in Figure 10 are Legendrian isotopic in the surgered manifold $(Y_\lambda(+1), \xi_\lambda(+1))$.

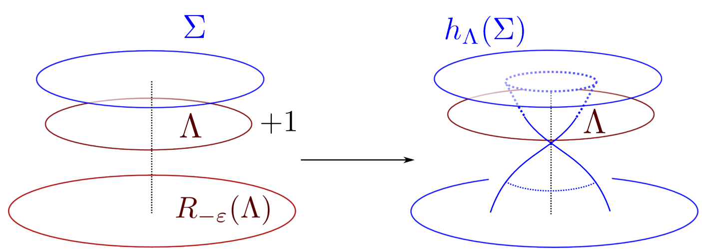


FIGURE 10. Handleslide of Σ along the $(+1)$–Legendrian Λ.

Proof. The statement follows if we realize the Legendrian handle slide as a Dehn twist along a Lagrangian sphere, and then use Theorem 3.1. Consider the unit bidisk $h = \{(q,p) \in \mathbb{R}^n \times \mathbb{R}^n : |q| \leq 1, |p| \leq 1\} \subseteq \mathbb{R}^{2n}$ and the function

$$f : \mathbb{R}^{2n} \longrightarrow \mathbb{R}, \quad (q,p) \longmapsto q^2 - p^2.$$

This Morse function has a critical point p at the origin $0 \in h$ of index n and the pair (h, f) is an index–n elementary cobordism between the level sets $\{f = -1\}$ and $\{f = 1\}$. The harmonic function f is the real part $\Re(F)$ of the holomorphic function

$$F : \mathbb{C}^n = \mathbb{R}^n(q) \oplus i\mathbb{R}^n(p) \longrightarrow \mathbb{C}, \quad z \longmapsto z^2$$

and thus its imaginary part $\Im(F) = q \cdot p$ provides a piece of an open book projection on each level set of $\Re(F)$. By endowing (h, f) with its standard index–n Weinstein structure [45], the regular level sets are contactomorphic to $(J^1 S^{n-1}, \xi_{\text{std}})$ where the zero section is either the attaching or the belt Legendrian sphere depending on the sign of the regular value. The open book projection given by $\Im(F)$ restricted to a level set is compatible with this contact structure because the Reeb field is transverse to the pages, and the diagramatic description in Figure 11 is both a smooth and Weinstein depiction of the situation: Figure 11 proves that the Legendrian handle is indeed a Dehn twist, as we

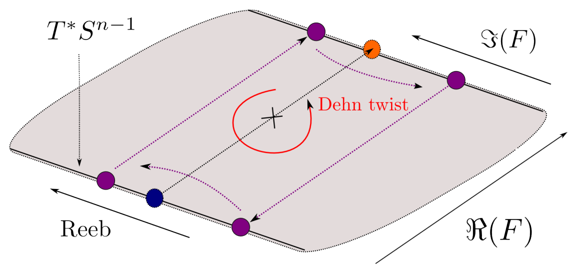


Figure 11. The elementary cobordism in Picard–Lefschetz theory.

now explain. The zero section of the fiber T^*S^{n-1} over the blue point is the attaching sphere and the zero section of the fiber over the orange point is the belt sphere. The handle slide takes place on the boundary $\{\Re(F) = 1\}$, which is the contact manifold $\{\Re(F) = -1\} \cong J^1S^{n-1}$ surgered along the Legendrian zero section. The monodromy at the initial level $\{\Re(F) = -1\}$ is given by the Reeb flow and it is thus the identity on the pages $(T^*S^{n-1}, \lambda_{\mathrm{std}})$; since the monodromy around the isolated critical point of F is a symplectic Dehn twist along the zero section of its Milnor fibre [3], the monodromy of the open book in $\{\Re(F) = 1\}$ is a Dehn twist along the Lagrangian zero section of the pages T^*S^{n-1}.

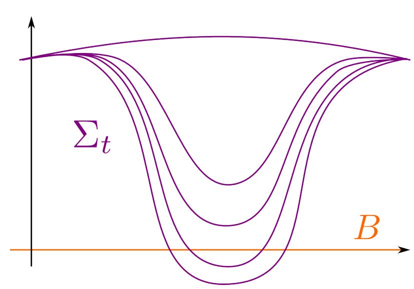


Figure 12. The fronts of the family realizing the handle slide.

Choose a Darboux chart $\mathcal{O}p\,(B) \cong (J^1B, \xi_{\mathrm{std}})$ around the belt sphere $B \subseteq \{\Re(F) = 1\}$ and the family Σ_t of Legendrian disks C^∞–identical in their boundaries depicted in Figure 12. By definition, a handle slide is the effect in the initial level set of crossing the belt sphere at a point; since the family Σ_t intersects B at exactly one point (because the crossing is tangential only once), we only need to compare the fronts of Σ_0 and Σ_1 flown backwards to the initial level set. These will give the Legendrians Σ and $h_\Lambda(\Sigma)$ in the statement of the theorem. Note that the trivialization in Figure 11 had a Dehn twist monodromy in the surgered level set and the identity in the initial one, but in the coordinates provided by $(J^1B, \xi_{\mathrm{std}})$ we have the identity in the surgered level set. In consequence, the monodromy in initial level set acts as a Dehn twist and the difference between the backwards flow Σ of Σ_0 and that of Σ_1 is precisely a Dehn twist along the attaching sphere $\Lambda \subseteq \{\Re(F) = 1\}$. Hence, $h_\Lambda(\Sigma)$ is a Dehn twist of Σ along Λ.

This proves part (a) of the statement, and part (b) follows by inverting the cobordism: in this case, we would use the anti–holomorphic $F(z) = \bar{z}^2$ map, which realizes a negative Dehn twist instead. $\square$

Remark 4. Note that the height in the Legendrian front of Figure 9 is important. Thus, in the case of a convex attachment one has to consider the Reeb push–off in the opposite direction to which the family Σ_t is approaching. In the concave case, this is a posteriori irrelevant since a cone vertically crossing a plane is a Reidemeister move.

We have given a conceptual proof of Theorem 3.2 but the reader is also invited to prove the statement with a direct computation in the Weinstein handle. The details of this second approach and further discussions will appear in the article [9].

Example 2. Let $H \subseteq (S^{2n+1}, \xi_{\mathrm{std}})$ be the Hopf link presented as the linked union of a Legendrian unknot and a small Reeb push–off of itself. Perform $(+1)$–surgery along H to obtain a contact structure $\xi_H(+1)$ on the same smooth sphere S^{2n+1}; then the Legendrian unknot in this surgered contact $(S^{2n+1}, \xi_H(+1))$ manifold is loose. Indeed, a possible isotopy exhibiting a loose chart is depicted in Figure 13 where the Legendrian unknot S is slid along each component of the surgery link once following the description in Theorem 3.2.

4. Loose Legendrians

In this section, we discuss the relation between loose Legendrian submanifolds, contact $(+1)$–surgery and overtwistedness. The results from the Front Gallery above readily imply that performing a contact $(+1)$–surgery along a loose Legendrian sphere yields a contact structure in which the Legendrian unknot is also a loose Legendrian. This is the content of the following statement:

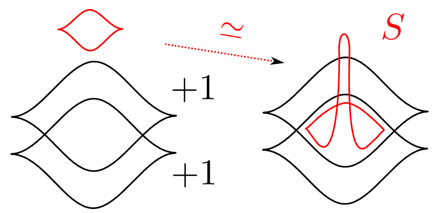


FIGURE 13. The Legendrian unknot S (red) is slid twice along the (+1)–surgery link H (in black) one per each component, until a loose chart appears on the complement of $H \subseteq (S^{2n+1}, \xi_{\mathrm{std}})$.

Proposition 4.1. Let (Y, ξ) be a contact manifold and $\Lambda \subseteq (Y, \xi)$ a loose Legendrian knot. The Legendrian unknot in the (+1)–contact surgery $(Y_\Lambda(+1), \xi_\Lambda(+1))$ along Λ is a loose Legendrian.

Proof. ([9]) Consider a loose chart (U, ξ_{std}) for $\Lambda \subseteq (Y, \xi)$ and the Legendrian unknot $\Lambda_0 \subseteq (U, \xi_{\mathrm{std}})$; in this chart handle–slide Λ_0 along the surgery (+1)–curve Λ. Performing two Reidemeister moves of types 1 and 2 creates a loose chart for the handle-slid Λ_0, see Figures 14 and 15 for a diagramatic description of such moves. □

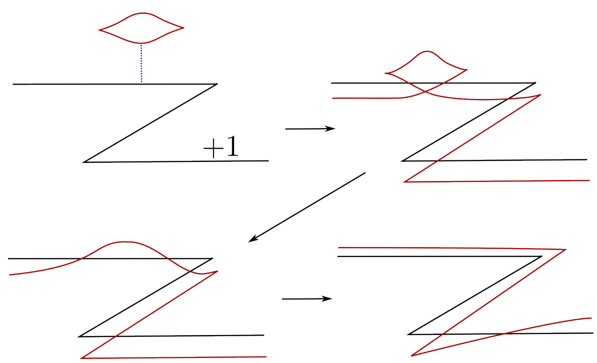


FIGURE 14. Proof of Proposition 4.1 by exhibiting an explicit loose chart for the Legendrian unknot.

The h–principle satisfied by overtwisted contact structures [8] implies that a Legendrian submanifold which is disjoint from an overtwisted ball has a loose chart. The converse is false since loose Legendrian submanifolds exist in abundance already in a contact Darboux ball. However, a more subtle beautiful converse holds:

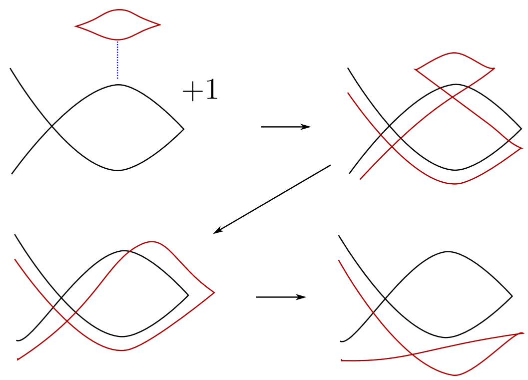


FIGURE 15. Proof of Proposition 4.1 in the fish–tailed loose chart.

Theorem 4.2. Let (Y, ξ) be a contact manifold and $\Lambda_0 \subseteq (Y, \xi)$ the Legendrian unknot. Suppose that Λ_0 is a loose Legendrian, then (Y, ξ) is overtwisted.

We prove Theorem 4.2 by a cobordism argument; this method of proof actually yields a stronger statement and we will use the technique again in Proposition 6.1. It would be of strong interest to be able to study whether it is possible to weaken the hypothesis on having a loose unknot to the existence of other types of loose Legendrian submanifolds, for instance Legendrian tori.

Proof of Theorem 4.2. The argument has two steps. First, we consider a Weinstein cobordism $\mathbb{W} = (W, \lambda, \varphi)$ with contact boundaries

$$\mathbb{W} : (S^{2n+1}, \xi_k) = \operatorname{ob}\left(\S_{i=0}^{k} T^*S_i^n, \prod_{i=0}^{k} \tau_{S_i}^{-1}\right) \longrightarrow \operatorname{ob}\left(\S_{i=0}^{k} T^*S_i^n, \prod_{i=0}^{k} \tau_{S_i}\right) \cong (S^{2n+1}, \xi_0),$$

where $\S_{i=0}^{k} T^*S_i^n$ denotes any plumbing of $k+1$ cotangent bundles of spheres S_i along their cotangent fibres and zero sections. The cobordism is realized by attaching $2(k+1)$ critical handles, two handles along each of the zero sections S_i, for $0 \leq i \leq k$. A Weinstein handle attachment along a Legendrian sphere which is the Legendrian lift of an exact Lagrangian in the page induces a change of monodromy by composition with a Dehn twist along this Lagrangian sphere. The crucial fact about this cobordism is that its concave boundary $\partial_- \mathbb{W} = (S^{2n+1}, \xi_k)$ is an overtwisted contact manifold for any $k \in \mathbb{N}$; nevertheless, it will suffice for us to prove such a statement for k large enough:

Assertion: (S^{2n+1}, ξ_k) is overtwisted for k large enough.

The proof of the assertion will be given right after this argument, so let us suppose that the assertion holds. The second step in order to prove Theorem 4.2 is noticing that looseness of the Legendrian unknot in (Y, ξ) implies that the connected sum Weinstein cobordism

$\mathbb{W}\overline{\#}SY$ between $\mathbb{W}$ and the symplectization $S(Y, \xi)$ is a flexible Weinstein cobordism. Indeed, the loose chart of the unknot $\Lambda_0 \subseteq (Y, \xi)$ is disjoint from the Legendrian attaching spheres S_i and thus, by connected summing with the unknot, these attaching spheres also inherit a loose chart. This proves Theorem 4.2 since a flexible Weinstein cobordism with overtwisted concave end, in this case $\partial_-\mathbb{W} \cong (S^{2n+1}, \xi_k)\#(Y, \xi)$, must have an overtwisted convex end, in this case $\partial_+\mathbb{W} \cong (S^{2n+1}, \xi_0)\#(Y, \xi) \cong (Y, \xi)$, by the h–principle satisfied by loose Legendrians [15]. □

Proof of Assertion. By Definition 2, a contact structure is overtwisted if it contains an overtwisted three dimensional ball with a (sufficiently) large neighborhood. This is equivalent to the existence of a closed overtwisted 3–dimensional submanifold with a large neighborhood. The assertion can be proven by induction on the dimension n, the base case being the fact that the contact 3–fold

$$(S^3, \xi_k) = \mathrm{ob}\left(\S_{i=0}^{k} T^*S_i^1, \prod_{i=0}^{k} \tau_{S_i}^{-1}\right)$$

is overtwisted. This follows from the fact that (S^3, ξ_k) is contactomorphic to the connected sum of $(S^3, \xi_1)\#(S^3, \xi_{k-1})$ and (S^3, ξ_1) is overtwisted; for instance, because the zero section of the page in $(S^3, \xi_1) = \mathrm{ob}(T^*S^1, \tau_{S^1}^{-1})$ is a $tb = 0$ Legendrian bounding a 2–disk, which violates Thurston–Bennequin inequality [6].

Let us now show that (S^5, ξ_k) is overtwisted for k large enough. Note that (S^5, ξ_1) contains the overtwisted submanifold (S^3, ξ_1), but a priori its neighborhood may not be large enough: as we will explain later, the standard sphere (S^5, ξ_0) does contain many overtwisted 3–spheres as contact submanifolds, and thus having an overtwisted submanifold cannot be a sufficient condition for overtwistedness.

The strategy becomes apparent: we want to increase the neighborhood of the overtwisted contact submanifold $(S^3, \xi_1) \subseteq (S^5, \xi_1)$. The normal form for a contact form in a neighborhood of the contact submanifold $(S^3, \xi_1 = \ker \alpha_1)$ reads $\alpha_1 + r^2 d\theta$, where $(r, \theta) \in D^2(R)$ are polar coordinates; the size of the neighborhood is precisely $R = \max r$. The crucial observation is the rescaling equality $(\sqrt{k}r)^2 d\theta = r^2 d(k\theta)$ for any $k \in \mathbb{N}$. Geometrically, the angle $d\theta$ is well–defined away from the origin $\{r = 0\}$ and multiplication by k corresponds to a regular k–covering: the local model extending to the origin is given by the map $z \longmapsto z^k$, i.e. a branched cover with branch locus $\{r = 0\}$. In consequence, the contact branched k–cover of a contact manifold (Y, ξ) along a contact submanifold $(M, \xi|_M)$ is a contact manifold (Y_k, ξ_k) which also contains the contact submanifold $(M, \xi|_M) \cong (M, \xi_k|_M)$, as the branch set, but the contact neighborhood of $(M, \xi_k|M)$ inside (Y_k, ξ_k) has k times the size of the contact neighborhood of $(M, \xi|_M) \subseteq (Y, \xi)$.

Let us then perform a branched k–cover of the contact manifold (S^5, ξ_1) along its overtwisted contact submanifold (S^3, ξ_1). By the above rescaling equality, the resulting contact manifold (Y_k, ξ'_k) is overtwisted if $k \in \mathbb{N}$ is large enough. In order to conclude the proof of the assertion, it suffices to show that $(Y_k, \xi'_k) \cong (S^5, \xi_k)$. First, notice that the the inclusion $S^3 \subseteq S^5$ is smoothly an unknotted sphere and since the branched cover of S^5 along an unknotted S^3 is diffeomorphic to S^5, we conclude that $Y_k \cong S^5$ as smooth manifolds. Second, observe that

$$(S^5, \xi_0) \cong \text{ob}\left(\S_{i=0}^{k} T^* S_i^2, \prod_{i=0}^{k} \tau_{S_i}\right),$$

with *positive* Dehn twists, is the open book decomposition for S^5 induced by the isolated A_{k+1}–singularity

$$\{z^{k+1} + w_1^2 + w_2^2 = 0\} \subseteq \mathbb{C}^3,$$

which realizes S^5 as a $(k+1)$–branched cover of itself (using the variable $z \in \mathbb{C}$) whose branched locus is precisely our 3–sphere $S^3 \cong \{z = 0\} = \{w_1^2 + w_2^2 = 0\} \cap S^5 \subseteq S^5$. The only difference between this situation and ours is the positivity of the Dehn twists, but this is readily solved by noting that conjugating in $\mathbb{C}^3$ gives the same geometric picture but with monodromy being a composition of negative Dehn twists. This concludes that $(Y_k, \xi'_k) \cong (S^5, \xi_k)$ and thus the induction step for $n = 2$; the reader can adapt now the argument for a general induction step. □

We have thus proven that the existence of a loose chart for the Legendrian unknot in a contact manifold implies that the contact manifold is overtwisted. In short, loose unknot implies overtwisted.

5. (+1)–contact surgery

In this section, we discuss the relation between (+1)–contact surgery and overtwistedness. The Thurston–Bennequin inequality [6] provides a direct criterion for overtwistedness in contact 3–folds, we shall use the split nature of Definition 2 to prove the following result:

Proposition 5.1. Let (Y, ξ) be a contact manifold and $\Lambda \subseteq (Y, \xi)$ a loose Legendrian knot. The (+1)–contact surgery $(Y_\Lambda(+1), \xi_\Lambda(+1))$ along Λ is an overtwisted contact manifold.

Proof. Let us proceed by induction on the dimension of the contact manifold, the base case being $\dim(Y) = 3$. In the 3–dimensional case we can argue using the Thurston–Bennequin inequality; indeed, the destabilizing Legendrian $\widetilde{\Lambda}$ bounds a disk in $Y_\Lambda(+1)$ because the framing of the surgery coincides with $\text{lk}(\Lambda, \widetilde{\Lambda})$. By construction, the contact framing along $\widetilde{\Lambda}$ coincides with this Seifert framing, so $\widetilde{\Lambda}$ is a $\text{tb} = 0$ unknot and $(Y_\Lambda(+1), \xi_\Lambda(+1))$ is overtwisted.

We now perform the induction step for $\dim(Y) = 5$, the general case being analogous. First, note that there exists an isocontact embedding

$$(J^1S^1, \xi_{\mathrm{std}}) \longrightarrow (J^1S^2, \xi_{\mathrm{std}}) \cong (\{(q,p;z) \in T^*\mathbb{R}^3 \times \mathbb{R} : q \cdot p = 0, |q| = 1\}, dz - \lambda_{\mathrm{std}})$$

induced by the inclusion of the equator $S^1 \subseteq S^2$ and the choice of a metric. The contact submanifold $(J^1S^1, dz - \lambda_{\mathrm{std}})$ has an infinite neighborhood inside $(J^1S^2, dz - \lambda_{\mathrm{std}})$ due to the fact that $p_2 \in \mathbb{R}$ is unconstrained. In general, the submanifold

$$(\{(q,p;z) \in T^*\mathbb{R}^3 \times \mathbb{R} : q \cdot p = 0, |q| = 1, |p_1| \leq 1, q_2 = p_2 = 0\}, dz - \lambda_{\mathrm{std}})$$

inside the contact manifold

$$(\{(q,p;z) \in T^*\mathbb{R}^3 \times \mathbb{R} : q \cdot p = 0, |q| = 1, |p_1| \leq 1, |p_2| \leq C\}, dz - \lambda_{\mathrm{std}})$$

has a contact neighborhood of size proportional to C. Note that these can be taken to be the convex boundaries of a critical Weinstein attachment.

Second, the $(+1)$–surgery is realized by an inverse Weinstein cobordism and the structure of the 6–dimensional handle D^6 can be split in $D^4 \times D^2$ such that it induces a 4–dimensional handle attachment in the 3–dimensional submanifold $J^1S^1 = \partial_+ D^4$ in $D^4 \times \{0\}$ of $J^1S^2 = \partial_+ D^6$. In particular, the surgered 5–fold will contained the surgered 3–fold as a contact submanifold.

Finally, consider a loose chart $U \times D^2(R) \subseteq (\mathbb{R}^3, \xi_{\mathrm{std}}) \times \mathbb{R}^2$ for Λ with $R \in \mathbb{R}^+$ large enough. Then perform the $(+1)$–surgery on Λ such that the stabilized arc in U coincides with an arc of the equator $S^1 \subseteq S^2$ and use a handle with the coordinate p_2 bounded by $|p_2| \leq 2R$. The resulting surgered 5–fold now contains a neighborhood of an overtwisted 2–disk as a 3–dimensional submanifold, by the base case, and with a neighborhood proportional to $2R$. □

Proposition 5.1 is explained in a more explicit manner in [20, Corollary 2.4].

6. Open book decompositions

The Giroux correspondence [26] establishes contact open book decompositions as a central tool in contact topology. In this section, we relate looseness to the existence of a negatively stabilized supporting open book; in particular, this characterizes higher–dimensional tightness in terms of open book decompositions. Note that such characterization is known for contact 3–folds, where even a criterion in terms of the monodromy is available [31].

In Subsection 6.1 and 6.2, we will focus on loose Legendrians and negative stabilizations; each subsection can be deduced from the other, but we present independent arguments. Subsection 6.3 provides a construction of codimension–2 overtwisted embeddings by using negatively stabilized open books.

6.1. Loose unknots and Negative Stabilizations

We now establish the equivalence between looseness of the Legendrian unknot and the existence of a supporting open book which is a negative stabilization. Proposition 6.1 and Proposition 6.2 provide the two directions: the former uses the cobordism argument and the latter the description of Dehn twists in the front projection.

Proposition 6.1. Let (Y, ξ) be a contact structure such that the Legendrian unknot is loose. Then, there exists a supporting negatively stabilized open book.

Proof. Consider the cobordism

$$\mathbb{W} : (S^{2n+1}, \xi_{\mathrm{ot}}) = \mathrm{ob}\left(\S_{i=0}^{k} T^*S_i^n, \prod_{i=0}^{k} \tau_{S_i}^{-1} \right) \longrightarrow \mathrm{ob}\left(\S_{i=0}^{k} T^*S_i^n, \prod_{i=0}^{k} \tau_{S_i} \right),$$

which features in the proof of Theorem 4.2 and note that the source contact manifold is a negative stabilization. In particular, there exists a Darboux ball B, an open ball $U \subseteq (S^{2n+1}, \xi_{\mathrm{ot}})$ and an isocontact proper embedding

$$i : \mathrm{ob}(T^*S^n, \tau^{-1}) \setminus B \longrightarrow (U, \xi_{\mathrm{ot}}).$$

Now we use the cobordism argument from Section 4 to isotope the Legendrian attaching spheres in $\mathbb{W} \# \operatorname{Symp}(Y, \xi)$ away from the open ball

$$U \subseteq \partial_-(\mathbb{W} \# \operatorname{Symp}(Y, \xi)) \cong (S^{2n+1}, \xi_{\mathrm{ot}}) \# (Y, \xi)$$

such that a contactomorphic copy of U is present in the convex end

$$\partial_+(\mathbb{W} \# \operatorname{Symp}(Y, \xi)) \cong (S^{2n+1}, \xi_{\mathrm{std}}) \# (Y, \xi) = (Y, \xi).$$

In consequence, the isocontact embedding i allows us to express our initial contact manifold (Y, ξ) as the contact connected sum

$$(Y, \xi) \cong (Y', \xi') \# \, \mathrm{ob}(T^*S^n, \tau^{-1})$$

for some contact manifold (Y', ξ'). By Giroux's existence theorem, there exists an open book decomposition $(Y', \xi') = \mathrm{ob}(W, \varphi)$ and by Murasugi sum [10], we conclude that (Y, ξ) is a negative stabilization. □

We have thus established that the existence of a loose Legendrian unknot implies the existence of a negatively stabilized open book supporting the contact structure. The following statement provides the converse implication:

Proposition 6.2 ([10]). Let (Y, ξ) be a contact structure supported by a negatively stabilized open book, then the Legendrian unknot is loose.

Proof. First note that the zero section S of a Weinstein page T^*S^n in the open book $(S^{2n+1}, \xi_{\mathrm{std}}) = \mathrm{ob}(T^*S^n, \tau_S)$ is the Legendrian unknot. This can be seen by observing that S is the vanishing cycle of the standard Lefschetz fibration

$$z^2 : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$$

and thus, bounds a Lagrangian disk isotopic to the standard flat Lagrangian plane $\mathbb{R}^n \oplus 0 \subseteq \mathbb{R}^n \oplus i\mathbb{R}^n = \mathbb{C}^n$. Alternatively, one can proceed by induction on dimension. The 3–dimensional case holds since $r = 0$ and $tb = -1$, for it is the linking of the negative Hopf band $\partial T^*S^1 \subseteq S^3$. The induction step is based on the observation that given the zero section S^{n-1}, the next higher–dimensional zero section S^n is constructed by suspending S^{n-1}, which yields inductively the front description of the Legendrian unknot.

Taking into account the contact connected sum decomposition

$$(Y, \xi) = \mathrm{ob}(W \S T^*S^n, \varphi \circ \tau^{-1}) \cong \mathrm{ob}(W, \varphi) \# \,\mathrm{ob}(T^*S^n, \tau^{-1}),$$

it suffices to show that the Legendrian unknot in the contact sphere $\mathrm{ob}(T^*S^n, \tau^{-1})$ is loose. We will now prove that this is indeed the case by showing that the Legendrian unknot is a D^n–stabilization of another Legendrian sphere. Let $L \subseteq \mathrm{ob}(T^*L, \tau_L^{-1})$ and $S \subseteq \mathrm{ob}(T^*S, \tau_S)$ be the Legendrian spheres given by the zero section in any of their Weinstein pages $(T^*S^n, \lambda_{\mathrm{std}})$. On the one hand, observe that

$$S \subseteq \mathrm{ob}(T^*L \S T^*S, \tau_L^{-1} \circ \tau_S)$$

is the Legendrian unknot. On the other, the Legendrian S is isotopic to the Legendrian lift of the exact Lagrangian sphere $\tau_L^{-1}(S)$, which by Theorem 3.1 is isotopic to a D^n–stabilization of L. Indeed, the resulting Legendrian $\tau_L^{-1}(S) \simeq \tau_S(L)$ has a front as depicted in Figure 5 which contains a spherically symmetric cusp dividing the Legendrian in two disjoint and unlinked components, one of them being the Legendrian unknot S minus a small Legendrian disk; such a Legendrian is isotopic to a D^n–stabilization of L and thus loose. Alternatively the existence of the spherically symmetric cusp already provides a loose chart if one again observes that no other part of the Legendrian intersects such a chart. In either case, the Legendrian unknot S has a loose chart. □

Remark 5. The fact that the Legendrian unknot S in the contact manifold $\mathrm{ob}(T^*\Lambda, \tau_\Lambda^{-1})$ is a loose Legendrian submanifold, which is the main ingredient in Proposition 6.2, also follows Example 2. Alternatively, one can combine Proposition 4.1 and the fact that the zero section in $\mathrm{ob}(T^*S^{n-1}, id)$ is loose, which is explained in the proof of Proposition 6.3.

6.2. Loose surgery and Negative Stabilizations

We now describe the relation between being a (+1)–surgery along a loose Legendrian and admitting a negatively stabilized supporting open book. First we notice that the existence of such open book implies being a loose surgery:

Proposition 6.3. Let (Y, ξ) be a contact structure supported by a negatively stabilized open book. Then, there exists (Y', ξ') and a loose Legendrian $\Lambda \subseteq (Y', \xi')$ such that (Y, ξ) is contactomorphic to $(Y'_\Lambda(+1), \xi'_\Lambda(+1))$.

Proof. Consider the contact manifold $(S^{n-1} \times S^n, \xi_{\mathrm{std}}) = \mathrm{ob}(T^*S^{n-1}, id)$, first note that the zero section Λ in any of its Weinstein pages $(T^*S^{n-1}, \lambda_{\mathrm{std}})$ is a loose Legendrian inside $(S^{n-1} \times S^n, \xi_{\mathrm{std}})$. This follows either from Figure 16, which uses Theorem 3.2, or from the observation that $(S^{n-1} \times S^n, \xi_{\text{std}})$ is the contact boundary of the subcritical Stein manifold $(S^{n-1} \times D^{n+1}, \xi_{\text{std}})$ obtained by attaching a $(n-1)$–handle to $(D^{2n}, \lambda_{\text{std}})$.

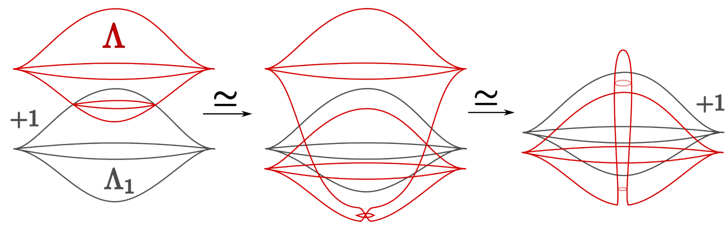


FIGURE 16. Proof of Proposition 6.3.

By hypothesis we have the negatively stabilized open book decomposition

$$(Y, \xi) = \text{ob}(W \S T^* S^n, \varphi \circ \tau^{-1}).$$

Then we can set $(Y', \xi') := \text{ob}(W \S T^* S^n, \varphi)$ and for the Legendrian sphere

$$\Lambda := S^n \subseteq \text{ob}(T^* S^n, id) \setminus D^{2n+1}(\varepsilon) \subseteq \text{ob}(T^* S^n, id) \# \, \text{ob}(W, \varphi) = (Y', \xi').$$

As we explained above, Λ is loose and the proposition now follows from the fact that a $(+1)$–surgery along a Legendrian, which sits as an exact Lagrangian in the page of a supporting open book, is supported by the contact open book whose monodromy is obtained by composing the given monodromy with a negative Dehn twist along this Lagrangian. □

Conversely, the existence of a loose chart for the Legendrian unknot implies the existence of a negatively stabilized supporting open book:

Proposition 6.4. Let (Y, ξ) be a contact manifold and $\Lambda \subseteq (Y, \xi)$ a loose Legendrian knot. The $(+1)$–contact surgery $(Y_\Lambda(+1), \xi_\Lambda(+1))$ along Λ is supported by a negatively stabilized open book.

The reader is invited to prove Proposition 6.4 by using the cobordism argument; we have preferred to provide an alternative approach by using the following result:

Fact 1. Let (Y, ξ) be a contact manifold, Ł $\subseteq Y$ a Legendrian submanifold and D a Legendrian disk which intersects Ł transversely at a single point. Then there exists a supporting open book $(Y, \xi) = \text{ob}(W, \varphi)$ such that Ł $\cup D \subseteq W$ is an exact Lagrangian submanifold and $\partial D \subseteq \partial W$.

This result is a consequence of the special position theorem for open book decompositions in Weinstein cobordisms [11].

Proof of Proposition 6.4. Consider the Legendrian $\widetilde{\Lambda}$ whose D^n–stabilization is the given loose Legendrian $s(\widetilde{\Lambda}) = \Lambda$, and the supporting open book $(Y, \xi) = \text{ob}(W, \varphi)$ provided by Fact 1 applied to $\text{Ł} = \widetilde{\Lambda}$ and D a cotangent fiber of $T^*\widetilde{\Lambda}$. By performing a positive stabilization along the cotangent fiber of Λ, we obtain the decomposition

$$(Y_\Lambda(+1), \xi_\Lambda(+1)) = \text{ob}(W\S T^*S, \varphi \circ \tau_\Lambda^{-1} \circ \tau_S).$$

If the Legendrian $\widetilde{\Lambda}$ is seen as a Lagrangian in the page, Theorem 3.2 allows us to identify $\Lambda = \tau_S(\widetilde{\Lambda})$. The iterated use of the braid relation $\tau_S \tau_{\widetilde{\Lambda}} \tau_S = \tau_{\widetilde{\Lambda}} \tau_S \tau_L$ and conjugation invariance yield the following isotopies for the monodromy:

$$\tau_{\widetilde{\Lambda}}^{-1} \tau_S \varphi = \tau^{-1}_{\tau_S^{-1}(\widetilde{\Lambda})} \tau_S \varphi = \tau_S^{-1} \tau_{\widetilde{\Lambda}}^{-1} \tau_S \tau_S \varphi = \tau_S^{-1} \tau_{\widetilde{\Lambda}}^{-1} \tau_S^{-1} \tau_S \varphi \tau_{\widetilde{\Lambda}}^2 = \tau_S^{-1} \tau_{\widetilde{\Lambda}}^{-1} \varphi \tau_{\widetilde{\Lambda}}^2$$

Hence, the contact manifold $(Y_\Lambda(+1), \xi_\Lambda(+1))$ is the negative stabilization of the contact manifold $\text{ob}(W, \tau_{\widetilde{\Lambda}}^{-1} \varphi \tau_{\widetilde{\Lambda}}^2)$. □

6.3. An application: overtwisted spheres in (S^{2n+1}, ξ_0)

Let us construct isocontact embeddings of overtwisted contact $(2n-1)$–spheres into the standard contact sphere (S^{2n+1}, ξ_0), and hence into any $(2n+1)$–dimensional contact manifold; note that this codimension–2 closed case is not covered by Gromov's h–principle [19, Theorem 12.3.1].

Consider the smooth projective quadric $Q_{n-2} \subseteq \mathbb{CP}^{n-1}$ and the total space of the holomorphic line bundle $\pi : \Omega(1) \longrightarrow \mathbb{CP}^{n-1}$. The contact boundaries

$$B_0 = \partial(\pi^{-1}(Q_{n-2})) \cong \partial(T^*S^{n-1}, \lambda_0), \qquad B_1 = \partial\Omega(1) \cong (S^{2n-1}, \xi_0)$$

are both endowed with the S^1–action induced by the standard circle action on $\mathbb{C}$; the holomorphic base inclusion induces a contact S^1–equivariant inclusion $B_0 \subseteq B_1$. Now consider a Liouville embedding of the smooth quadric (T^*S^{n-1}, λ_0) into the unit ball $(D^{2n}(1), \lambda_0) \subseteq \mathbb{C}^n$ preserving such S^1–actions on their boundaries. Alternatively, one can consider the affine complex $(n-1)$–dimensional singular quadric

$$\{z_1^2 + z_2^2 + \ldots + z_n^2 = 0\} \subseteq \mathbb{C}^n \cap D^{2n}(1),$$

where the Hopf action on the boundary ∂D^{2n} induces an S^1–action on ∂T^*S^{n-1}: symplectically smoothing the singularity relative to the contact boundary also gives such an embedding.

Denote by $T \in \text{Symp}_c(D^{2n}, \lambda_0)$ the negative fibered Dehn twisted induced by the Hopf S^1–action on $\partial(D^{2n}, \lambda_0) \cong (S^{2n-1}, \xi_0)$; we refer the reader to the article [12] for a definition of a fibered Dehn twist, and note that a fibered Dehn twist on T^*S^{n-1} is the square of a Dehn twist. Then the discussion above gives an isocontact embedding

$$(S^{2n-1}, \xi_T) \cong \text{ob}(T^*S^{n-1}, T|_{T^*S^{n-1}}) \subseteq \text{ob}(D^{2n+1}, T) \cong \text{ob}(D^{2n+1}, \text{id}) \cong (S^{2n+1}, \xi_0).$$

Note that the isomorphism $\text{ob}(D^{2n+1}, T) \cong \text{ob}(D^{2n+1}, \text{id})$ comes from the fact that in a disk the fibered Dehn twist can be unravelled.

Theorem 6.5 ([9]). The contact spheres (S^{2n-1}, ξ_{T^k}) are overtwisted for $k \geq 1$.

The theorem follows by generalizing Proposition 6.3. This yields isocontact embeddings of the overtwisted contact structures (S^{2n-1}, ξ_T^k) in any contact $(2n+1)$–fold. Note though that this series of contact structure are (often) not in the same almost contact homotopy class [8, Lemma 10.5].

Also, the above discussion explicitly proves the existence of a small neighborhood of an overtwisted disk in any contact manifold without invoking Gromov's h–principle, see Remark 3 above:

Corollary 1. *For any contact manifold* $(Y, \ker \alpha)$*,* $\exists \varepsilon(\alpha) \in \mathbb{R}^+$ *and an isocontact embedding* $\mathbb{R}^3_{ot} \times B(\varepsilon(\alpha)) \subseteq (Y, \ker \alpha)$*.*

7. The plastikstufe

The plastikstufe was first introduced by [39], preceeding the definition of the higher–dimensional overtwisted disk [8, 10]. Even after the geometric criterion for overtwistedness [10], it still remains a central geometric object of interest in the study of higher–dimensional contact structures. For now, it is unknown whether the existence of a general plastikstufe is tantamount to overtwistedness. However, under the right set of hypotheses this is indeed the case.

Following the pentagon of equivalences in Figure 1.3, we relate nice plastikstufes to loose Legendrians. The first implication is a consequence of the result [38][Theorem 1.1], whose statement we can particularize as:

Proposition 7.1 ([38]). Let (Y, ξ) be a contact manifold containing a nice plastikstufe. Then, the Legendrian unknot is a loose Legendrian.

Proof. The proof is detailed in [38], but let us briefly highlight its two steps. First, notice that the statements holds in the 3–dimensional case from the fact that a Legendrian knot K disjoint from an overtwisted 2–disk, which is a nice plastikstufe in dim(3), can be destabilized by considering the Legendrian connected sum with the Legendrian boundary ∂D^2_{ot} of the overtwisted disk. Indeed, $tb(\partial D^2_{\text{ot}}) = 0$ and thus $tb(K \# \partial D^2_{\text{ot}}) = tb(K) + 1$, which in the case of the Legendrian unknot $K = \Lambda_0$ gives a smooth unknot with tb equal to 0; by the h–principle the stabilization of $K \# \partial D^2_{\text{ot}}$ is Legendrian isotopic to K if the latter is disjoint from the overtwisted disk, a property satisfied in the case $K = \Lambda_0$. Now, the second step in the proof is a careful parametric version of the 3–dimensional argument applied along the family of overtwisted 2–disks parametrized by the core of the plastikstufe. $\square$

Remark 6. In order for this proof to work, it is crucial that the plastikstufe is contained in a ball, and in particular has a contractible core. This is the strongest of the three hypotheses required in a nice plastikstufe, whereas the remaining two (spherical core and trivial rotation class) can likely be weakened or even removed.

We can explicitly exhibit a nice plastikstufe in the (+1)–contact surgery on a loose Legendrian knot, thus proving:

Proposition 7.2 ([10]). Let (Y, ξ) be a contact manifold and $\Lambda \subseteq (Y, \xi)$ a loose Legendrian knot. The (+1)–contact surgery $(Y_\Lambda(+1), \xi_\Lambda(+1))$ along Λ contains a nice plastikstufe.

Proof. The argument is again based on the fact that the statement is known for contact 3–folds. In that case, (+1)–surgery along a stabilized Legendrian creates an overtwisted disk as explained in the proof of Proposition 5.1. The spherical plastikstufe appears in higher dimensions by symmetrically rotating this picture, as we now explain. Let $\widetilde{\Lambda} \subseteq (Y, \xi)$ be a Legendrian sphere whose spherical stabilization is isotopic to the given Legendrian Λ. By identifying $(\mathcal{O}p\,(\widetilde{\Lambda}), \Lambda; \xi_{\text{std}}) \cong (J^1 S^{n-1}, S^{n-1}; \xi_{\text{std}})$, the Legendrian Λ is represented by zero section stabilized over the equator $S^{n-2} \subseteq S^{n-1}$.

In order to find the 3–dimensional model inside this local description, we consider the unique meridian $S^1_x \subseteq S^{n-1}$ passing through a fixed point $x \in S^{n-2}$ in the equator and both the north and south poles; in particular, the submanifold $J^1(S^1_x) \subseteq T^*S^{n-1} \times \mathbb{R}$ is a 3–dimensional contact submanifold and under this contactomorphism the intersection $\Lambda \cap \mathcal{J}^1(S^1_x)$ is given as the stabilization of the zero section.

Consider the description of (+1)–contact surgery as drilling the surgery sphere and quotienting the boundaries T^*S^{n-1} by a Dehn twist; since a Dehn twist can be modelled in terms of the geodesic flow, we can assume that it induces a surface Dehn twist on the symplectic submanifold $T^*S^1_x \subseteq T^*S^{n-1}$ and thus the induced (+1)–surgery creates an 3–dimensional overtwisted submanifold (M_x, ξ_{ot}). Note that the construction can be made parametrically on the points of the equator $x \in S^{n-2}$, and each of the resulting 3–folds is overtwisted because the base locus $\mathcal{J}^1(S^1_x) \cap \mathcal{J}^1(S^1_y) \cong S^0 \times \mathbb{R}$ is disjoint from the overtwisted disk. In consequence, we obtain a plastikstufe P with spherical core which can be verified to be nice. □

Let us present a simple application of (1)=(2).

Corollary 2. *Let (Y, ξ) be a contact 5–fold and $(M^3, \xi|_M) \subseteq (Y, \xi)$ an overtwisted contact divisor. Then, the contact fiber connected sum*

$$(Y, \widetilde{\xi}_M) = (Y, \xi)\#_{(M, \xi|_M)}(\S(M), \xi_{std})$$

along the contact divisor $(M, \xi|_M)$ is an overtwisted 5–fold.

Note that the space of contact elements $(\S(M), \xi_{\text{std}})$ is a Stein fillable contact manifold and the 5–fold (Y, ξ) might as well be. In general, there is a smooth cobordism between the connected sum $Y\#\S(M)$ to the fiber connected sum Y, but it does contain smooth handles of index 4 and it cannot be deformed to a Stein cobordism for topological reasons. In the case of (Y,ξ) being Liouville fillable, the cobordism cannot even be made Liouville for symplectic fillability reasons.

The fiber sum operation in Corollary 2 is the 5–dimensional analogue of a Lutz twist. Indeed, performing a 3–dimensional Lutz twist along a transverse knot $K \subseteq (Y^3,\xi)$ yields an overtwisted contact manifold isotopic to the fiber sum

$$(Y,\widetilde{\xi}_K) = (Y,\xi)\#_{S^1}(S^1\times S^2,\xi_{\mathrm{std}}),$$

where S^1 is embedded as tranverse knots $K \subseteq (Y,\xi)$ and $S^1\times\{*\} \subseteq (S^1\times S^2,\xi_{\mathrm{std}})$.

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