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Non semi-simple TQFTs from unrolled quantum $sl(2)$

Christian Blanchet, Francesco Costantino, Nathan Geer, and Bertrand Patureau-Mirand

Abstract. Invariants of 3-manifolds from a non semi-simple category of modules over a version of quantum $\mathfrak{sl}(2)$ were obtained by the last three authors in [4]. They are invariants of 3-manifolds together with a cohomology class which can be interpreted as a line bundle with flat connection. In [1] we have extended those invariants to graded TQFTs on suitable cobordism categories. Here we give an overview of constructions and results, and describe the TQFT vector spaces. Then we provide a new, algebraic, approach to the computation of these vector spaces.

Introduction

New quantum invariants of 3-manifolds equipped with 1-dimensional cohomology classes over $\mathbb{C}/2\mathbb{Z}$, or equivalently $\mathbb{C}^*$ flat connections, have been constructed in [4] from a variant of quantum $\mathfrak{sl}(2)$. This family of invariants is indexed by integers $r \geq 2$, $r \not\equiv 0$ mod 4, which gives the order of the quantum parameter. In the case $r \equiv 0$ mod 4, we have obtained in [2] invariants of 3-manifolds equipped with generalised spin structures corresponding to certain flat connections on the oriented framed bundle. These invariants are built from surgery presentations and have common flavor with the famous Witten-Reshetikhin-Turaev quantum invariants, but are indeed very different. First, they are defined for 3-manifolds equipped with cohomology classes, and second they use a stronger version of quantum $sl(2)$ which in particular avoids the semi-simplification procedure required for producing modular categories. To emphasize the power of these new invariants we quote that for the smallest root of unity, $r = 2$, the multivariable Alexander polynomial and Reidemeister torsion are recovered, which allows us to reproduce the classification of lens spaces, see [1].

The TQFT extension of these invariants has been carried out in [1]. The main achievement is a functor on a category of *decorated cobordisms* with values in finite dimensional graded vector spaces. An object in this category is a surface equipped with the following data: a base point on each connected component, a possibly empty set of colored points, a 1-dimensional cohomology class over $\mathbb{C}/2\mathbb{Z}$ and a Lagrangian subspace. A morphism is a cobordism with: a colored ribbon graph, a cohomology class and a signature defect which all satisfy certain admissibility conditions. A description of these TQFT vector spaces split into two cases, depending on whether the cohomology class is integral or not. In the non-integral case, it can be done using colored trivalent graphs with a pattern similar to the Witten-Reshetikhin-Turaev case. In the integral case we are able to prove finite dimensionality in general and to provide a Verlinde formula for their graded dimension under the further assumption that the surface contains a point with a projective color. A new result in this paper is an Hochschild homology description of the TQFT vector spaces. In the integral case, our statement is proved under the previous assumption.

1. Unrolled quantum $sl(2)$ and modified trace invariant

In this section we recall the unrolled quantum $sl(2)$ at a root of unity. In the whole paper, $r \geq 2$ is an integer which is non zero modulo 4, $r' = r$ if r is odd and $r' = \frac{r}{2}$ else.

Let $q = \mathrm{e}^{\frac{i\pi}{r}}$, $q^x = \mathrm{e}^{\frac{ix\pi}{r}}$ for $x \in \mathbb{C}$. Recall (see [5]) the $\mathbb{C}$-algebra $\overline{U}_q^H\mathfrak{sl}(2)$ given by generators E, F, K, K^{-1}, H and relations:

$$KEK^{-1} = q^2E, \qquad KFK^{-1} = q^{-2}F, \qquad [E,F] = \frac{K-K^{-1}}{q-q^{-1}}, \qquad E^r = 0,$$
$$HK = KH, \qquad [H,E] = 2E, \qquad [H,F] = -2F, \qquad F^r = 0.$$

The algebra $\overline{U}_q^H\mathfrak{sl}(2)$ is a Hopf algebra where the coproduct, counit and antipode are defined in [5]. A *weight module* is a finite dimensional module which splits as a direct sum of H-weight spaces and is such that K acts as q^H.

The category $\mathscr{C}$ of weight modules is $\mathbb{C}/2\mathbb{Z}$-graded (by the weights modulo $2\mathbb{Z}$) that is $\mathscr{C} = \bigoplus_{\overline{\alpha}\in\mathbb{C}/2\mathbb{Z}} \mathscr{C}_{\overline{\alpha}}$ and $\otimes : \mathscr{C}_{\overline{\alpha}} \times \mathscr{C}_{\overline{\beta}} \to \mathscr{C}_{\overline{\alpha}+\overline{\beta}}$. The category $\mathscr{C}$ is a ribbon category and we have the usual Reshetikhin-Turaev functor from $\mathscr{C}$-colored ribbon graphs to $\mathscr{C}$ (which is given by Penrose graphical calculus).

The simple modules in $\mathscr{C}$ are highest weight modules with any complex number as highest weight. The generic simple modules are those which are projective. They are indexed by the set

$$\ddot{\mathbb{C}} = (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}.$$

For $\alpha \in \ddot{\mathbb{C}}$, the r-dimensional module $V_\alpha \in \mathscr{C}_{\overline{\alpha+r-1}}$ is the irreducible module with highest weight $\alpha + r - 1$.

The group of invertible modules is generated by the one dimensional vector space denoted by $\varepsilon = \mathbb{C}_r$ with H-weight equal to r and degree equal to r modulo 2. The subgroup of invertible objects with trivial degree is generated by $\sigma = \mathbb{C}_{2r'}$, the one dimensional vector space with H-weight equal to $2r'$ (if r is even then $\sigma = \varepsilon$). For each integer j, $0 \leq j \leq r-1$ the simple module with highest weight j is $j+1$ dimensional. For $0 \leq j < r-1$ it is not projective, but has a $2r$-dimensional projective cover P_j. The non simple indecomposable projective modules are the $P_j \otimes \mathbb{C}_r^{\otimes k}$, $0 \leq j < r-1$, $k \in \mathbb{Z}$.

The link invariant underlying our construction is the re-normalized link invariant ([8]) that we recall briefly. The modified dimension is the function defined on $\{V_\alpha\}_{\alpha\in\ddot{\mathbb{C}}}$ by

$$\mathsf{d}(\alpha) = (-1)^{r-1}\frac{r\{\alpha\}}{\{r\alpha\}},$$

where $\{\alpha\} = 2i\sin\frac{\pi\alpha}{r}$. Let L be a $\mathscr{C}$-colored oriented framed link in S^3 with at least one component colored by an element of $\{V_\alpha : \alpha \in \ddot{\mathbb{C}}\}$. Opening such a component of L gives a 1-1-tangle T whose open strand is colored by some $\alpha \in \ddot{\mathbb{C}}$ (here and after we identify $\ddot{\mathbb{C}}$ with the set of coloring modules $\{V_\alpha\}$). The Reshetikhin-Turaev functor associates an endomorphism of V_α to this tangle. As V_α is simple, this endomorphism is a scalar $\langle T\rangle \in \mathbb{C}$. The modified invariant is $F'(L) = \mathsf{d}(\alpha)\langle T\rangle$.

Theorem 1.1 ([8])**.** *The assignment $L \mapsto F'(L) = \mathsf{d}(\alpha)\langle T\rangle$ described above is an isotopy invariant of the colored framed oriented link L.*

The modified invariant F' extends to $\mathscr{C}$-colored ribbon graphs provided the given colored graph contains at least one edge with projective color. In the case where the selected projective color is simple, then the above description holds. The general case is obtained by using the fact that any indecomposable projective module is a submodule in the tensor product of two simple projective ones, see [5].

2. Invariants of closed 3-manifolds

We will consider here (compact and oriented) closed 3-manifolds with decorations and admissibility conditions. A decoration in a closed 3-manifold M is a $\mathscr{C}$-colored ribbon graph T together with a cohomology class $\omega \in H^1(M \setminus T, \mathbb{C}/2\mathbb{Z})$ such that the coloring of T is compatible with ω, i.e. each oriented edge e of T is colored by an object in $\mathscr{C}_{\omega(m_e)}$ where m_e is the oriented meridian of e.

Definition 2.1. A connected decorated manifold (M, T, ω) is admissible if either there exists a loop γ such that $\omega(\gamma) \notin \mathbb{Z}/2\mathbb{Z}$, or the colored ribbon graph T contains at least one edge whose color is a projective module.

Definition 2.2. A surgery presentation of a connected decorated 3-manifold is a triple (L, T, c) where

(1) L is an oriented framed link in S^3.
(2) T is a $\mathscr{C}$-colored ribbon graph in $S^3 \setminus L$,
(3) c is a cohomology class in $H^1(S^3 \setminus (L \cup T), \mathbb{C}/2\mathbb{Z})$ which is compatible with the coloring of T and vanishes on the preferred parallels of components of L.

The surgery presentation is computable if either L is empty and (S^3, T, c) is admissible, or for all meridians m_j of components of L, one has $c(m_j) \notin \mathbb{Z}/2\mathbb{Z}$.

The decorated 3-manifold represented by (L, T, c) is $(S^3(L), T, \omega)$ where $S^3(L)$ is the manifold obtained by doing surgery on L and ω is the unique extension of c. The following proposition allows us to use computable surgery presentations in order to define invariants of admissible decorated 3-manifolds. A weak version of this proposition is proved in [4]. In [1] the admissibility condition was slightly extended to the case of integral cohomology classes and ribbon graphs containing at least one projective color. It is shown that up to local skein equivalence (relations on ribbon graphs living in a ball) we may suppose that the cohomology class is non integral, see [1].

Proposition 2.1. *Any connected admissible decorated 3-manifold is skein equivalent to a decorated 3-manifold which admits a computable surgery presentation.*

Let $H_r = \{1-r, 3-r, \dots, r-1\}$. For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ we define the Kirby color Ω_α as the formal linear combination

$$\Omega_\alpha = \sum_{k \in H_r} \mathsf{d}(\alpha + k) V_{\alpha+k}.$$

Its degree is $\overline{\alpha} \in \mathbb{C}/2\mathbb{Z}$. Although the Kirby color depends on a complex number, its contribution in the formula for the 3-manifold invariant depends only on its degree.

Theorem 2.2 ([4, 1])**.** *Let (M, T, ω) be a connected admissible decorated 3-manifold. If (L, T, c) is a computable surgery presentation of (M, T, ω) then*

$$\mathsf{N}(M, T, \omega) = \frac{F'(L \cup T)}{\Delta_+^p \ \Delta_-^s}$$

is a well defined topological invariant (i.e. depends only on the orientation preserving diffeomorphism class of the triple (M, T, ω)), where $\Delta_+, \ \Delta_-$ are given in Equation (2) below, (p, s) is the signature of the linking matrix of the surgery link L and for each i the component L_i with meridian m_i is colored by a Kirby color of degree $c(m_i)$.

The TQFT construction uses a renormalized version of N which is defined for possibly disconnected admissible decorated 3-manifolds with extended structures. For a connected 3-manifold, the extended structure is an integer, interpreted as a signature defect, which fixes the so called framing anomaly.

The renormalised invariant Z is multiplicative for disjoint unions, and in the connected case is defined by

$$\mathsf{Z}(M, T, \omega, n) = \eta \lambda^{b_1(M)} \delta^n \mathsf{N}(M, T, \omega) \tag{1}$$

where n is the extended structure and $b_1(M)$ is the first Betti number. The scalars λ, η and δ are given by

$$\lambda = \frac{\sqrt{r'}}{r^2}, \qquad \eta = \frac{1}{r\sqrt{r'}} \qquad \text{and} \qquad \delta = \lambda \Delta_+ = (\lambda \Delta_-)^{-1} = q^{-\frac{3}{2}} \mathrm{e}^{-i(s+1)\pi/4} \tag{2}$$

where s is in $\{1, 2, 3\}$ with $s \equiv r \bmod 4$. Another expression is the following:

$$\mathsf{Z}(M, T, \omega, n) = \eta \lambda^m \delta^{-\sigma+n} F'(L \cup T) \tag{3}$$

where the components of L are colored by Kirby colors as in Theorem 2.2, $m \in \mathbb{N}$ is the number of components of L, $\sigma \in \mathbb{Z}$ is the signature of the linking matrix $\mathrm{lk}(L)$.

3. The category of decorated cobordisms

In this section we define the category of decorated cobordisms. This category is a $2+1$-dimensional cobordism category. The general idea is to be able to cut admissible decorated extended 3-manifolds along surfaces and get gluing formulas for the invariant Z. Note that the definition of a decorated cobordism contains an admissibility condition.

Definition 3.1 (Objects)**.** A *decorated surface* is a 4-tuple $\mathbf{\Sigma} = (\Sigma, \{p_i\}, \omega, \mathscr{L})$ where:

- Σ is a closed, oriented surface which is an ordered disjoint union of connected surfaces each having a distinguished base point $*$;
- $\{p_i\}$ is a finite (possibly empty) set of homogeneous $\mathscr{C}$-colored oriented framed points in Σ distinct from the base points, i.e. each $p_i \in \Sigma$ is equipped with a sign, a non-zero tangent vector and a color which is an object of $\mathscr{C}_{\overline{\alpha_i}}$ for some $\overline{\alpha_i} \in \mathbb{C}/2\mathbb{Z}$;
- $\omega \in H^1(\Sigma \setminus \{p_1, \dots, p_k\}, *; \mathbb{C}/2\mathbb{Z})$ is a cohomology class such that $\omega(m_i) = \overline{\alpha_i}$ where m_i is an oriented meridian around p_i;
- $\mathscr{L}$ is a Lagrangian subspace of $H_1(\Sigma; \mathbb{R})$.

The *opposite or negative* of $\mathbf{\Sigma} = (\Sigma, \{p_1, \dots p_k\}, \omega, \mathscr{L})$ is defined as

$$\overline{\mathbf{\Sigma}} = (\overline{\Sigma}, \{\overline{p}_1, \dots \overline{p}_k\}, \omega, \mathscr{L})$$

where $\overline{\Sigma}$ is Σ with the opposite orientation, $\overline{p}_i$ is p_i with the same vector and color, but opposite sign.

We orient the boundaries of manifolds using the "outward vector first" convention. Here, we also use this convention for the boundary of a graph. Reshetikhin-Turaev ribbon functor F is adapted to this convention, so that an upward arc colored by V represents the identity of V.

Definition 3.2 (Cobordisms)**.** Let $\mathbf{\Sigma}_\pm = (\Sigma_\pm, \{p_i^\pm\}, \omega_\pm, \mathscr{L}_\pm)$ be decorated surfaces. A *decorated cobordism* from $\mathbf{\Sigma}_-$ to $\mathbf{\Sigma}_+$ is a 5-tuple $\boldsymbol{M} = (M, T, f, \omega, n)$ where:

- M is an oriented 3-manifold with boundary ∂M;
- $f : \overline{\Sigma_-} \sqcup \Sigma_+ \to \partial M$ is a diffeomorphism that preserves the orientation; denote the image under f of the base points of $\overline{\Sigma_-} \sqcup \Sigma_+$ by $*$;
- T is a $\mathscr{C}$-colored ribbon graph in M such that $\partial T = \{\overline{f(p_i^-)}\} \cup \{f(p_i^+)\}$ and the color of the edge of T containing $f(p_i^\pm)$ equals the color of $p_i^\pm$;
- $\omega \in H^1(M \setminus T, *; \mathbb{C}/2\mathbb{Z})$ is a cohomology class relative to the base points on ∂M, such that the restriction of ω to $(\partial M \setminus \partial T) \cap \Sigma_\pm$ is $(f^{-1})^*(\omega_\pm)$;
- the coloring of T is compatible with ω, i.e. each oriented edge e of T is colored by an object in $\mathscr{C}_{\omega(m_e)}$ where m_e is the oriented meridian of e;
- n is an arbitrary integer called the *signature-defect* of $\boldsymbol{M}$;
- each connected component of M disjoint from $f(\overline{\Sigma_-})$ is admissible.[1]

We consider decorated cobordisms from $\mathbf{\Sigma}_-$ to $\mathbf{\Sigma}_+$ up to diffeomorphism: a *diffeomorphism*

$$g : (M, T, f, \omega, n) \to (M', T', f', \omega', n')$$

is an orientation preserving diffeomorphism of the underlying manifolds M and M', still denoted by g, such that $g(T) = T'$, $g \circ f = f'$, $\omega = g^*(\omega')$ and $n = n'$. Remark that up to diffeomorphism, $\boldsymbol{M} = (M, T, f, \omega, n)$ only depends on f up to isotopy.

[1]Here we extend Definition 2.1 to the case $\partial M \neq \emptyset$.

Notice that the last condition in Definition 3.2 ensures that all connected decorated cobordisms from $\emptyset$ to $\mathbf{\Sigma}_2$ are asked to be admissible.

Remark 3.1. Two different cohomology classes that are compatible with the same pair (M,T) differ by an element of $H^1(M,\partial M;\mathbb{C}/2\mathbb{Z})$. When this group is zero the compatible cohomology class is unique.

Given a decorated cobordism $\boldsymbol{M}=(M,T,f,\omega,n)$, from $\mathbf{\Sigma}_-$ to $\mathbf{\Sigma}_+$, let

$$f^-=f_{|\overline{\Sigma_-}}:\overline{\Sigma_-}\to M \text{ and } f^+=f_{|\Sigma_+}:\Sigma_+\to M$$

be the components of f. If $\mathscr{L}_-$ and $\mathscr{L}_+$ are the Lagrangians of $\mathbf{\Sigma}_-$ and $\mathbf{\Sigma}_+$, respectively, then we define two other Lagrangians

$$M_*(\mathscr{L}_-)=(f^+)_*^{-1}\big((f^-\circ\overline{\mathrm{Id}_{\Sigma_-}})_*(\mathscr{L}_-)\big)\subset H_1(\mathbf{\Sigma}_+;\mathbb{R})$$

and

$$M^*(\mathscr{L}_+)=(f^-\circ\overline{\mathrm{Id}_{\Sigma_-}})_*^{-1}\big((f^+)_*(\mathscr{L}_+)\big)\subset H_1(\mathbf{\Sigma}_-;\mathbb{R})$$

where $\overline{\mathrm{Id}_{\Sigma_-}}:\Sigma_-\to\overline{\Sigma_-}$ is the identity. The fact that $M_*(\mathscr{L}_-)\subset H_1(\Sigma_+,\mathbb{R})$ and $M^*(\mathscr{L}_+)\subset H_1(\Sigma_-,\mathbb{R})$ are indeed Lagrangians follows from the analysis of the Lagrangian relations induced by f_+ and f_- (see [9, Section IV.3.4]).

Definition 3.3. For $j=1,2$, let $\boldsymbol{M}_j=(M_j,T_j,f_j,\omega_j,n_j)$ be a decorated cobordism such that $\mathbf{\Sigma}=(\Sigma,\{p_i\},\omega_\Sigma,\mathscr{L})=\partial_+\boldsymbol{M}_1=\overline{\partial_-\boldsymbol{M}_2}$. Let $\mathscr{L}_-$ and $\mathscr{L}_+$ be the Lagrangian subspaces of the decorated surfaces $\overline{\partial_-\boldsymbol{M}_1}$ and $\partial_+\boldsymbol{M}_2$, respectively. The composition $\boldsymbol{M}_2\circ\boldsymbol{M}_1$ is the decorated cobordism (M,T,f,ω,n) from $\overline{\partial_-M_1}$ to ∂_+M_2 defined as follows:

- $M=M_1\cup_\Sigma M_2$ is the manifold obtained by gluing $\partial_+\boldsymbol{M}_1$ and $\overline{\partial_-\boldsymbol{M}_2}$ through the map $f_2^-\circ(f_1^+)^{-1}$,
- T is the $\mathscr{C}$-colored ribbon graph $T_1\cup_{\{p_i\}}T_2$ in M,
- $f=f_1^-\sqcup f_2^+$,
- ω is the cohomology class obtained using the Mayer-Vietoris map and forgetting the base points in the gluing surface,
- $n=n_1+n_2-\mu(M_{1*}(\mathscr{L}_-),\mathscr{L},M_2{}^*(\mathscr{L}_+))$ where μ is the Maslov index of the Lagrangian subspaces of $H_1(\Sigma,\mathbb{R})$.

As we now explain, decorated surfaces and cobordisms form a tensor category $\mathcal{C}ob$. The objects of $\mathcal{C}ob$ are decorated surfaces and morphisms are diffeomorphism classes of decorated cobordisms. The composition is given in Definition 3.3. The tensor product of $\mathcal{C}ob$ is given by the disjoint union: if $\mathbf{\Sigma}_j=(\Sigma_j,\{p_i^j\},\omega_j,\mathscr{L}_j)$, for $j=1,2$, are two decorated surfaces then define $\mathbf{\Sigma}_1\otimes\mathbf{\Sigma}_2$ as $(\Sigma_1\sqcup\Sigma_2,\{p_i^1\}\sqcup\{p_i^2\},\omega_1\oplus\omega_2,\mathscr{L}_1\oplus\mathscr{L}_2)$. Here the ordering of the components of $\Sigma_1\sqcup\Sigma_2$ is obtained by putting those of Σ_1 first then those of Σ_2. In particular, $\mathbf{\Sigma}_1\sqcup\mathbf{\Sigma}_2\neq\mathbf{\Sigma}_2\sqcup\mathbf{\Sigma}_1$ but they are isomorphic. Similarly, if $\boldsymbol{M}_j=(M_j,T_j,f_j,\omega_j,n_j)$, for $j=1,2$, are two cobordisms then define $\boldsymbol{M}_1\otimes\boldsymbol{M}_2$ as $(M_1\sqcup M_2,T_1\sqcup T_2,f_1\sqcup f_2,\omega_1\oplus\omega_2,n_1+n_2)$. Often, for the sake of clarity we will write $\boldsymbol{M}_1\sqcup\boldsymbol{M}_2$ instead of $\boldsymbol{M}_1\otimes\boldsymbol{M}_2$. The composition of cobordisms is given in Definition 3.3.

4. Graded TQFT functor

The main achievement in [1] is the following theorem.

Theorem 4.1. *For each $r \geq 2$, $r \not\equiv 0$ mod. 4, there exists a monoidal graded TQFT functor $\mathbb{V}$ from the category of decorated cobordisms to the category of finite dimensional $\mathbb{Z}$-graded vector spaces, which extends the renormalised invariant Z.*

The construction uses a variant of a universal construction initiated in [3], which we will now briefly discuss. First, given a decorated surface $\mathbf{\Sigma}$, the generators of $\mathbb{V}(\mathbf{\Sigma})$ are all decorated cobordisms from $\emptyset$ to $\mathbf{\Sigma}$. The relations are defined as the kernel of the pairing with decorated cobordisms from $\mathbf{\Sigma}$ to $\emptyset$. This construction produces finite dimensional vector spaces, but fails to be monoidal. In order to produce the full monoidal functor $\mathbb{V}$, we introduce colored spheres $\widehat{S}_k$, $k \in \mathbb{Z}$. Recall that V_0 is the simple projective highest module of weight $r-1$ and, for $k \in \mathbb{Z}$, $\sigma^k = \mathbb{C}_{2kr'}$ is the one dimensional module with weight $2kr'$. Let $\widehat{S}_k$ be the sphere with colored points U, where

$$U = \begin{cases} ((V_0, +1), (\sigma^k, +1), (V_0, -1)), & \text{if } k \neq 0 \\ ((V_0, +1), (V_0, -1)), & \text{if } k = 0. \end{cases}$$

The cohomology class and the Lagrangian in the object $\widehat{S}_k$ are trivial. Using the objects $\widehat{S}_k$ we now give a description of the graded vector space $\mathbb{V}(\mathbf{\Sigma}) = \oplus_k \mathbb{V}_k(\mathbf{\Sigma})$ associated with a decorated surface $\mathbf{\Sigma}$ equivalent to the definition given in [3]. The vector space $\mathbb{V}_k(\mathbf{\Sigma})$ is generated by all decorated cobordisms from $\widehat{S}_k$ to $\mathbf{\Sigma}$. Relations are defined as the kernel of the pairing with decorated cobordisms from $\mathbf{\Sigma}$ to $\widehat{S}_k$.

The monoidality property uses natural cobordisms which we call "pants" $\mathsf{P}_{k,l}^{k+l}$ from $\widehat{S}_k \sqcup \widehat{S}_l$ to $\widehat{S}_{k+l}$ and similar morphisms "upside down" $\mathsf{P}^{k,l}_{k+l}$ [1, Section 4]. The category of decorated cobordisms has a symmetry extending the standard flip of two connected components. We have shown in [3, Section 5] that the monoidal functor $\mathbb{V}$ is compatible with standard symmetry on vector spaces when r is odd, and supersymmetry on graded vector spaces when r is even.

In genus zero the TQFT vector spaces can be described from the category $\mathscr{C}$. Indeed the degree zero part for a sphere with one positive point colored with a projective object is the space of invariants [1, Section 6.1]:

$$\mathbb{V}_0(S^2, (V, +1)) = \mathrm{Hom}(\mathbb{C}, V). \tag{4}$$

If there are several points with $\pm$ sign and at least one projective color, we reduce to this case via duality and tensor product. It would be convenient to have such a formula for the whole graded vector space $\mathbb{V}(S^2, (+, V))$. This is done by introducing graded morphisms. The category $\mathfrak{C}$ has the same objects as in $\mathscr{C}$, but morphisms are $\mathbb{Z}$-graded vector spaces, graded by k, defined by

$$\mathbb{H}\mathrm{om}(V, W) \cong \oplus_k \mathrm{Hom}(V \otimes \sigma^k, W). \tag{5}$$

The composition of $f \in \mathrm{Hom}(U \otimes \sigma^k, V)$ with $g \in \mathrm{Hom}(V \otimes \sigma^l, W)$ is $g \circ (f \otimes \mathrm{Id}_{\sigma^l})$ in $\mathrm{Hom}(U \otimes \sigma^{l+k}, W)$. We then have:

$$\mathbb{V}(S^2, (V, +1)) = \mathbb{H}\mathrm{om}(\mathbb{C}, V). \tag{6}$$

In the next section, a general description of the TQFT vector spaces will be obtained from elementary bricks consisting of spheres with two or three points colored with indecomposable projective modules.

For simple projective modules, the list of bricks is the following [5, Theorem 5.2]

$$\mathbb{V}(S^2, (V_\alpha, -), (V_\beta, +)) = \mathbb{H}\mathrm{om}(V_\alpha, V_\beta) = \begin{cases} \mathbb{C} \text{ if } \beta - \alpha = 2kr' \text{ with } k \in \mathbb{Z} \text{ giving degree,} \\ 0 \text{ else.} \end{cases} \tag{7}$$

Recall that $H_r = \{-r+1, -r+3, \ldots, r-1\}$ and that ε, σ^k are the invertible modules of weight r and $2kr'$ respectively.

$$\mathbb{V}_k(S^2, (V_\alpha, +), (V_\beta, +), (V_\gamma, +)) = \mathrm{Hom}(\sigma^k, V_\alpha \otimes V_\beta \otimes V_\gamma) = \begin{cases} \mathbb{C} \text{ if } \alpha + \beta + \gamma \in H_r + 2r'k, \\ 0 \text{ else.} \end{cases} \tag{8}$$

In the first case above we will say that the triple (α, β, γ) is r-admissible for degree k. In other words, a triple (α, β, γ) is r-admissible (for some degree $k \in \mathbb{Z}$) if and only if $\alpha + \beta + \gamma + r - 1 \equiv 0 \bmod 2\mathbb{Z}$; we remark that such k is uniquely determined if r is odd and that there are two possible values of k (differing by 1) if r is even. So, given a trivalent graph G with oriented edges, an r-admissible coloring is a map $c : \mathrm{Edges}(G) \to \mathbb{C}$ such that the colors incoming to each vertex (if an edge is outgoing we consider it as an incoming edge with opposite color) form an r-admissible triple and if r is even the datum of a specification of the degree of the triple around each vertex. The total degree of an r-admissible coloring on G is the sum of the degrees of the triples at the vertices. We get a basis for the TQFT vector space using r-admissible colorings in generic case as follows :

Theorem 4.2. *Let G be an oriented uni-trivalent planar graph and $\Sigma = \partial H_G$ be the boundary of a regular neighborhood of G. Suppose Σ is equipped with a cohomology class which is non integral on all meridians of edges of G. Then a basis of $\mathbb{V}(\Sigma)$ is indexed by r-admissible colorings of G, where colors are compatible with the evaluation on meridians and stay in the range $]-r, r]$ if r is odd and $[0, r[$ if r is even. Moreover, the degree is the sum of degrees at the vertices.*

The theorem above is an improved version of [1, Section 6.3] and can be proved using surgery formulas and skein methods. It can also be deduced from the general splitting theorem 5.1 in the next section.

Remark 4.1. Here for the skein description of the basis we need to add a germ of incoming edge at each trivalent vertex corresponding to the incoming $\widehat{S}_k$. For r even, due to supersymmetry property, we also need to fix an ordering of trivalent vertices up to an even permutation. We give an example in Figure 1. Here, an admissible coloring is given

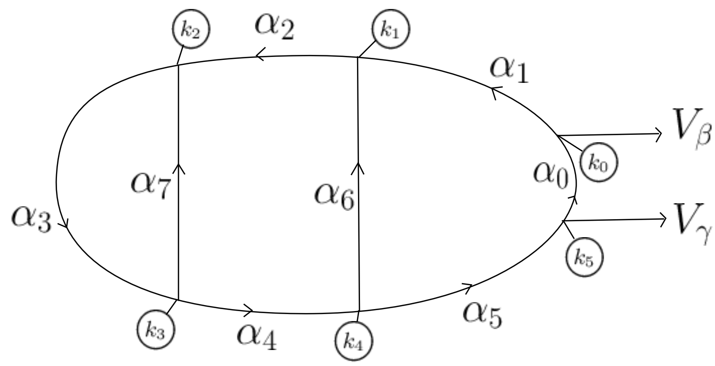

FIGURE 1. Colored graph in generic case

by the complex values α_i in the fundamental band and integers k_j satisfying the above r-admissibility conditions.

Corollary 4.3. *The dimension for the TQFT vector space of a genus g surface with non integral cohomology class is r^{3g-3} if r is odd, and $\frac{r^{3g-3}}{2^{g-1}}$ if r is even.*

Verlinde formula and the graded dimensions of $\mathbb{V}(\Sigma)$

Definition 4.1. A decorated surface $\mathbf{\Sigma} = (\Sigma, \{p_i\}, \omega, \mathscr{L})$ is called admissible if either there exists a loop γ such that $\omega(\gamma) \notin \mathbb{Z}/2\mathbb{Z}$, or at least one of the p_i has projective color.

In this section, we state the Verlinde formula giving the graded dimension of the TQFT vector spaces for admissible surfaces.

The proof is given in [1, Theorem 5.9] and uses the pivotal structure which does not fully exist on the decorated cobordism category because the standard coevaluation cobordism is admissible only for admissible surfaces. We do not know the Verlinde formula for non admissible surfaces.

Let $\mathbf{\Sigma} = (\Sigma, \{p_i\}, \omega, \mathscr{L})$ be a connected admissible decorated surface with base point $*$. For each $\overline{\beta} \in \mathbb{C}/2\mathbb{Z}$ let $\mathrm{Id}_\Sigma^\varphi : \Sigma \to \Sigma$ be the product $[0,1] \times (\Sigma, \{p_i\})$ with cohomology class φ which restricts to ω on each boundary component and whose evaluation on the relative cycle $[0,1] \times *$ is equal to $\overline{\beta}$. The mapping torus of $\mathrm{Id}_\Sigma^\varphi$, obtained by gluing the two ends is denoted by $S^1_{\overline{\beta}} \times \mathbf{\Sigma}$.

Theorem 4.4 (Verlinde formula for graded dimensions)**.** *Let $\mathbf{\Sigma} = (\Sigma, \{p_1, \dots p_n\}, \omega, \mathscr{L})$ be a connected admissible surface of genus g. Define the graded (super)-dimension of $\mathbb{V}(\mathbf{\Sigma})$ by*

$$\dim_t(\mathbb{V}(\mathbf{\Sigma})) = \begin{cases} \sum_{k\in\mathbb{Z}} \dim(\mathbb{V}_k(\Sigma))t^k & \text{if } r \text{ is odd,} \\ \sum_{k\in\mathbb{Z}} (-1)^k \dim(\mathbb{V}_k(\Sigma))t^k & \text{if } r \text{ is even.} \end{cases}$$

$$\textit{Then} \qquad \dim_{q^{2r'\beta}}(\mathbb{V}(\Sigma)) = \mathsf{Z}(S^1_\beta \times \mathbf{\Sigma}).$$

If p is empty, then

$$\mathsf{Z}(S^1_\beta \times \mathbf{\Sigma}) = \frac{1}{r}(r')^g \sum_{k \in H_r} \left(\frac{\{r\beta\}}{\{\beta + k\}} \right)^{2g-2}$$

where $\{\gamma\} = q^\gamma - q^{-\gamma}$.
If p_i is colored by V_{c_i} for $c_1, \ldots, c_n \in \ddot{\mathbb{C}}$ then

$$\mathsf{Z}(S^1_\beta \times \mathbf{\Sigma}) = \frac{(-1)^{n(r-1)}}{r}(r')^g q^{c\beta} \sum_{k \in H_r} q^{ck} \left(\frac{\{r\beta\}}{\{\beta + k\}} \right)^{2g-2+n}$$

where $c = \sum_i c_i$.

5. Description of graded TQFT vector spaces for admissible surfaces

This section contains new results. Our aim is to describe the graded TQFT vector spaces for all admissible surfaces (Definition 4.1). This includes the case of a surface with vanishing cohomology class and one point colored with a projective object of degree 0, e.g P_0 or V_0 in case of odd r.

Projective objects and multiplicity modules

For each grading $\alpha \in \mathbb{C}/2\mathbb{Z}$ we define a projective object $\mathbf{P}_\alpha$ which is a direct sum of r' indecomposable projective ones representing orbits under the action of degree $\overline{0} \in \mathbb{C}/2\mathbb{Z}$ invertible objects.

For r odd and $\alpha \in \mathbb{C}/2\mathbb{Z}$ we define $\mathbf{P}_\alpha$ as follows:

$$\mathbf{P}_\alpha := \begin{cases} P_0 \oplus P_2 \oplus \cdots \oplus P_{r-3} \oplus V_0 \oplus (\varepsilon \otimes (P_1 \oplus P_3 \oplus \cdots \oplus P_{r-2})) & \text{if } \alpha = \overline{0} \in \mathbb{C}/2\mathbb{Z} \\ P_1 \oplus P_3 \oplus \cdots \oplus P_{r-2} \oplus (\varepsilon \otimes (P_0 \oplus P_2 \oplus \cdots \oplus P_{r-3} \oplus V_0)) & \text{if } \alpha = \overline{1} \in \mathbb{C}/2\mathbb{Z} \\ \bigoplus_{\beta \equiv \alpha, \Re(\beta) \in]-r,r]} V_\beta & \text{else.} \end{cases}$$

Here $\Re(\beta)$ denotes the real part of β.

For r even and $\alpha \in \mathbb{C}/2\mathbb{Z}$ we define $\mathbf{P}_\alpha$ as follows:

$$\mathbf{P}_\alpha := \begin{cases} P_0 \oplus P_2 \oplus \cdots \oplus P_{r-2} & \text{if } \alpha = \overline{0} \in \mathbb{C}/2\mathbb{Z} \\ P_1 \oplus P_3 \oplus \cdots \oplus P_{r-3} \oplus V_0 & \text{if } \alpha = \overline{1} \in \mathbb{C}/2\mathbb{Z} \\ \bigoplus_{\beta \equiv \alpha, \Re(\beta) \in [0,r[} V_\beta & \text{else.} \end{cases}$$

In all cases we set $\mathbb{A}_\alpha = \mathbb{E}\mathrm{nd}(\mathbf{P}_\alpha)$, and call it the degree α basic algebra. For generic α it is semisimple. For degree 0 and 1, it is described in [5]; we quote that it contains nilpotent elements of order 2 as well as non diagonal morphisms.

For α, β, γ in $\mathbb{C}/2\mathbb{Z}$, we define the multiplicity module

$$\mathbb{H}(\alpha, \beta, \gamma) = \mathbb{H}\mathrm{om}(\mathbb{C}, \mathbf{P}_\alpha \otimes \mathbf{P}_\beta \otimes \mathbf{P}_\gamma)$$

which support a right action of $\mathbb{A}_\alpha \otimes \mathbb{A}_\beta \otimes \mathbb{A}_\gamma$.

We remark that $\mathbb{H}(\alpha, \beta, \gamma)$ is isomorphic to $\mathbb{V}\left(S^2, (\mathbf{P}_\alpha, +), (\mathbf{P}_\beta, +), (\mathbf{P}_\gamma, +)\right)$. If we represent the pair of pants by a planar trivalent vertex, then we need to add a germ of incoming edge corresponding to the incoming $\widehat{S}_k$ in order to fix the isomorphism, and use a braiding with this colored edge to write the isomorphism corresponding to a cyclic permutation.

Given a trivalent vertex v with ordered oriented edges labelled with degrees in $\mathbb{C}/2\mathbb{Z}$, then we have an associated multiplicity module $\mathbb{H}(v)$ defined using color $\mathbf{P}$ if the orientation is outgoing and the dual if the orientation is ingoing. For an outgoing edge, we have a right action of the algebra corresponding to the grading, and for ingoing edge we have left action. If some edge has fixed color, then the multiplicity module is adapted using this color.

TQFT vector space as a degree 0 Hochschild homology

Now we are able to describe the TQFT vector spaces in the admissible case as follows. Suppose that $G = (\mathcal{V}, \mathcal{E})$ is an oriented uni-trivalent planar graph and $\Sigma = \partial H_G$ is the boundary of a tubular neighborhood of G. Let $(\alpha_e)_{e\in\mathcal{E}} \in (\mathbb{C}/2\mathbb{Z})^{\mathcal{E}}$ be a 1-cycle coloring the edges of G and assume that each external edge e (adjacent to a univalent vertex) is equipped with an object of $\mathscr{C}_{\alpha_e}$. We identify the univalent vertices of G with marked points on Σ. Let $\mathbf{\Sigma} = (\Sigma, \{p_i\}, \omega, \mathscr{L})$ be the admissible decorated surface such that $\mathscr{L}$ is defined by H_G and the cohomology class ω, whose value on the meridian of an edge $e \in \mathcal{E}$ is α_e, vanishes on all longitudes (curves on the surface which are the preferred parallels of the cycles of the graph G using the blackboard framing). Suppose that $\mathbf{\Sigma}$ is admissible (so either one of the colors α_e is non integral or one of the colors of the 1-valent vertices of G is projective). If $\mathbf{A}$ is an algebra and $\mathbf{M}$ is an $\mathbf{A}$-bimodule, then the degree 0 Hochschild homology module $HH_0(\mathbf{A}, \mathbf{M})$ is defined as the quotient of the module $\mathbf{M}$ by the relations $uv - vu$ for all $(u, v) \in \mathbf{A} \times \mathbf{M}$.

Then the following holds:

Theorem 5.1. *Let $\mathbf{\Sigma}$ be an admissible decorated surface and G be a graph for $\mathbf{\Sigma}$ as above.*
a) There exists a surjective graded homomorphism

$$\mathbb{H}_G := \otimes_{v\in\mathcal{V}} \mathbb{H}(v) \to \mathbb{V}(\mathbf{\Sigma}).$$

b) Let $\mathcal{E}_{\text{int}}$ denote the set of internal edges, then the tensor product $\mathbb{H}_G = \otimes_{v\in\mathcal{V}} \mathbb{H}(v)$ is a bimodule over $\mathbb{A}_G = \otimes_{e\in\mathcal{E}_{\text{int}}} \mathbb{A}_{\alpha_e}$ and we get an isomorphism between degree zero Hochschild homology and the TQFT vector space:

$$HH_0(\mathbb{A}_G, \mathbb{H}_G) \cong \mathbb{V}(\mathbf{\Sigma}).$$

Remark 5.1. Any admissible decorated surface can be represented in this way. The vanishing hypothesis on longitudes can be removed if we use twisted action of the basic algebras.

This theorem is the new part of this paper. We expect that a similar statement is true for the non-admissible case, but we leave this for further investigation.

Proof of Theorem 5.1

In order to prove the theorem, we extend our TQFT to decorated surfaces with boundary, and follow a method from [3]. This is part of an extended TQFT functor which will be fully developed in [7]. Let us first recall some definitions from Appendix A of [3]:

Definition 5.1.
- An algebroid is a $\mathbb{C}$-linear category Δ; if $a, b \in Ob(\Delta)$ one denotes ${}_a\Delta_b = \mathrm{Hom}_\Delta(a,b)$ and the composition of $y \in {}_b\Delta_c$ and $x \in {}_a\Delta_b$ is also denoted $y \circ x = xy$.
- A left Δ-module is a functor $F : \Delta \to \mathbf{Vect}$; similarly, a right Δ-module is a functor $\Delta^{op} \to \mathbf{Vect}$. If M is a left (resp. right) Δ-module and $a \in Ob(\Delta)$, we denote $M(a) \in \mathbf{Vect}$ also by ${}_aM$ (resp. M_a).
- If Δ' is another algebroid, a $(\Delta \times \Delta')$-bimodule is a functor $F : \Delta \times (\Delta')^{op} \to \mathbf{Vect}$.
- If Δ is an algebroid, M a right Δ-module and N a left Δ-module, then their *tensor product* $M \otimes_\Delta N$ is the quotient of the vector space

$$\bigoplus_a M_a \otimes_{\mathbb{C}} {}_aN$$

(where a ranges over all the objects of Δ or if Δ is not small in a small skeleton of Δ) by the subvector space generated by the relations $u\alpha \otimes v - u \otimes \alpha v$ where $u \in M_a, v \in_b N, \alpha \in_a \Delta_b$.
- If Δ is an algebroid and M is a Δ-bimodule, the Hochschild homology module $HH_0(\Delta, M)$ is defined as the quotient of the module $\bigoplus_a {}_aM_a$ by the relations $uv - vu$ for all $u \in {}_a\Delta_b, v \in {}_b\Delta_a$.

Example 5.2. If Δ is a k-linear category, then Δ is a left and right Δ-module. The left module structure is given by the functor $a \to \mathrm{Hom}(\cdot, a)$; the right module structure by $a \to \mathrm{Hom}(a, \cdot)$.

Let γ be an oriented closed (possibly non connected) curve with one base point $*_i$ per component and $\omega_\gamma \in H^1(\gamma, \cup_{\pi_0\gamma} *_i; \mathbb{C}/2\mathbb{Z})$, then we define the algebroid $\Delta(\gamma, \omega_\gamma)$ where objects are admissible decorated surfaces $\mathbf{\Sigma} = (\Sigma, p, \omega, \mathscr{L})$ with boundary (γ, ω_γ). Here we ask that $\cup *_i$ is the set of base points in Σ and the restriction of ω to $\partial\Sigma$ is ω_γ; furthermore the Lagrangian subspace $\mathscr{L}$ of a surface with boundary Σ is by definition a maximal isotropic subspace of $H_1(\Sigma; \mathbb{R})$. The space of morphisms from $\mathbf{\Sigma}$ to $\mathbf{\Sigma}'$ is the graded TQFT vector space $\mathbb{V}(\overline{\mathbf{\Sigma}} \cup_\gamma \mathbf{\Sigma}')$. (Here the Lagrangian subspace of the glueing of two surfaces $\overline{\mathbf{\Sigma}} \cup_\gamma \mathbf{\Sigma}'$ is the image in $H_1(\overline{\mathbf{\Sigma}} \cup_\gamma \mathbf{\Sigma}'; \mathbb{R})$ of $\mathscr{L}_1 \oplus \mathscr{L}_2$ via Mayer-Vietoris map.) Composition is induced by gluing along the intermediate decorated surface, using also the pair of pants map $\mathsf{P}_{k,l}^{k+l}$ from $\widehat{S}_k \sqcup \widehat{S}_l$ to $\widehat{S}_{k+l}$. For a (possibly non admissible) decorated surface $\mathbf{\Sigma}$ with boundary (γ, ω_γ), which we consider as a cobordism from $\emptyset$ to (γ, ω_γ), we define the TQFT module $\mathbb{V}(\mathbf{\Sigma})_{\_}$ which is a right module over $\Delta(\gamma, \omega_\gamma)$ by $\mathbb{V}(\mathbf{\Sigma})_{\Sigma'} = \mathbb{V}(\mathbf{\Sigma} \cup_\gamma \overline{\mathbf{\Sigma}'})$, and similarly if $\mathbf{\Sigma}''$ is a decorated cobordism from (γ, ω_γ) to $\emptyset$ we define the TQFT module ${}_{\_}\mathbb{V}(\mathbf{\Sigma})$ which is a left module over $\Delta(\gamma, \omega_\gamma)$ by ${}_{\Sigma'}\mathbb{V}(\mathbf{\Sigma}'') = \mathbb{V}(\mathbf{\Sigma}' \cup_\gamma \mathbf{\Sigma}'')$.

We have the following general splitting theorem.

Theorem 5.2 (Splitting theorem)**.** *If $\mathbf{\Sigma} = \mathbf{\Sigma}_1 \cup_\gamma \mathbf{\Sigma}_2$ is a closed decorated surface which splits along the multicurve γ as a cobordism $\mathbf{\Sigma}_1$ from $\emptyset$ to (γ, ω_γ) followed by a cobordism $\mathbf{\Sigma}_2$ from (γ, ω_γ) to $\emptyset$, and suppose that $\mathbf{\Sigma}_1$ is admissible then we have an isomorphism*

$$\mathbb{V}((\mathbf{\Sigma}_1))_{\_} \otimes_{\Delta(\gamma,\omega_\gamma)} {}_{\_}\mathbb{V}((\mathbf{\Sigma}_2)) \cong \mathbb{V}(\mathbf{\Sigma}) \ .$$

Proof. The map from left to right is induced by gluing and the pair of pants $\mathsf{P}^{k,l}_{k+l}$ (recall indeed that in Section 4 we defined $\mathbb{V}(\Sigma) = \bigoplus_k \mathbb{V}_k(\Sigma)$ where $\mathbb{V}_k(\Sigma)$ is generated by the morphisms from $\widehat{S}_k$ to Σ . We have to build the inverse map. Since $\mathbf{\Sigma}_1$ is admissible then the inverse map is given by the composition

$$\begin{array}{ccccc} \mathbb{V}(\mathbf{\Sigma}) & \to & \mathbb{V}(\mathbf{\Sigma}_1)_{\mathbf{\Sigma}_1} \otimes_{\mathbb{C}} {}_{\mathbf{\Sigma}_1}\mathbb{V}(\mathbf{\Sigma}_2) & \to & \mathbb{V}(\mathbf{\Sigma}_1)_{\_} \otimes_{\Delta(\gamma,\omega_\gamma)} {}_{\_}\mathbb{V}(\mathbf{\Sigma}_2) \\ [\boldsymbol{M}] & \mapsto & \mathrm{Id}'_{\mathbf{\Sigma}_1} \otimes [\boldsymbol{M}] & \mapsto & \mathrm{Id}'_{\mathbf{\Sigma}_1} \otimes [\boldsymbol{M}] \end{array}$$

where $\mathrm{Id}'_{\mathbf{\Sigma}_1}$ denotes the class of the cylinder of $\mathbf{\Sigma}_1$ pinched at $\gamma \times [0,1]$. The first map consists in viewing M as the glueing of a collar over $\mathbf{\Sigma}_1$ in M and its complement which is diffeomorphic to the whole M; the second map is the quotient defining the tensor product over $\Delta(\gamma, \omega_\gamma)$.

This map is well defined as by definition, ${}_{\mathbf{\Sigma}_1}\mathbb{V}(\mathbf{\Sigma}_2) = \mathbb{V}(\mathbf{\Sigma}_1 \cup_\gamma \mathbf{\Sigma}_2) = \mathbb{V}(\mathbf{\Sigma})$ holds. □

As a corollary we get the TQFT vector space of a decorated surface $\mathbf{\Sigma}$ which is obtained by closing an admissible decorated cobordism $\mathbf{\Sigma}_\gamma$ from (γ, ω_γ) to itself by identification of the two copies of γ. Here, we have a base point on each component of the boundary curve. We then define the TQFT bimodule $\mathbb{V}({}_{\_}(\mathbf{\Sigma}_\gamma)_{\_})$ over $\Delta(\gamma, \omega_\gamma)$.

Corollary 5.3 (Trace corollary)**.** *Let $\mathbf{\Sigma}$ be a connected closed surface obtained by closing an admissible cobordism (identification of the two connected copies of γ), then we have an isomorphism:*

$$HH_0(\Delta(\gamma, \omega_\gamma), \mathbb{V}({}_{\_}(\mathbf{\Sigma}_\gamma)_{\_}) \cong \mathbb{V}(\mathbf{\Sigma}) \ .$$

The above theorem and corollary cannot be used directly for the computation. Following again the method in [3] we will have a colored version of the splitting theorem and trace corollary thanks to a Morita reduction of the algebroid of a curve. Using the theorem in [3, Appendix A] we establish the following:

Theorem 5.4. *The algebroid of a curve $\gamma = \sqcup(\gamma_i, \omega_\gamma)$ is Morita equivalent to a tensor product of basic algebras $\otimes_i \mathbb{A}_{\alpha_i}$, where $\alpha_i = \omega_\gamma(\gamma_i)$. Here, we view this algebra as an algebroid with $\otimes_i \mathbf{P}_{\alpha_i}$ as its unique object.*

The above theorem is basically a special case of Morita's theorem showing that the category Mod − A of representations of a finite dimensional algebra A is Morita equivalent to that of representations of $End(P)$ if P is a projective module over A generating Mod − A.

We deduce the colored splitting and colored trace corollaries. We state them for connected curve which is enough for proving Theorem 5.1; they also hold for multicurves.

Corollary 5.5 (Colored splitting)**.** *If $\mathbf{\Sigma} = \mathbf{\Sigma}_1 \cup_\gamma \mathbf{\Sigma}_2$ is a closed decorated surface which splits along the connected curve γ as a cobordism $\mathbf{\Sigma}_1$ from $\emptyset$ to (γ, ω_γ) followed by a cobordism $\mathbf{\Sigma}_2$ from (γ, ω_γ) to $\emptyset$, and suppose that $\mathbf{\Sigma}_1$ is admissible then we have an isomorphism*

$$\mathbb{V}(\mathbf{\Sigma}_1 \cup (\mathrm{disc}, (\mathbf{P}_\alpha, +))) \ \otimes_{\mathbb{A}_\alpha} \ \mathbb{V}(((\mathrm{disc}, (\mathbf{P}_\alpha, -)) \cup \mathbf{\Sigma}_2) \cong \mathbb{V}(\mathbf{\Sigma}) \ .$$

Corollary 5.6 (Colored trace)**.** *Let $\mathbf{\Sigma}$ be a connected closed surface obtained by closing along a connected curve (γ, ω_γ) an admissible cobordism $\mathbf{\Sigma}_\gamma$ from (γ, ω_γ) to itself, then we have an isomorphism:*

$$HH_0(\mathbb{A}_\alpha, \mathbb{V}((\mathrm{disc}, (\mathbf{P}_\alpha, -)) \cup \mathbf{\Sigma}_\gamma \cup (\mathrm{disc}, (\mathbf{P}_\alpha, +))) \cong \mathbb{V}(\mathbf{\Sigma}) \ .$$

In both statements $\alpha = \omega_\gamma(\gamma)$. Note that in the last statement we need a base point on each boundary component, and have to be careful with the relative homology class when computing left and right actions.

Theorem 5.1 is now proved by using the sphere with 2 or 3 points and the above results.

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Univ Paris Diderot, Sorbonne Paris Cité, IMJ-PRG, UMR 7586 CNRS, F-75013, Paris, France
E-mail address: christian.blanchet@imj-prg.fr

Univ Toulouse III Paul Sabatier, Institut de Mathmatique de Toulouse F-31062, Toulouse, France
E-mail address: francesco.costantino@math.univ-toulouse.fr

Mathematics & Statistics, Utah State University, Logan, Utah 84322, USA
E-mail address: nathan.geer@gmail.com

Univ. Bretagne - Sud, UMR 6205, LMBA, F-56000 Vannes, France
E-mail address: bertrand.patureau@univ-ubs.fr