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Nonexistence of rational homology disk weak fillings of certain singularity links

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Abstract. We show that the Milnor fillable contact structures on the links of singularities having resolution graphs from some specific families that have members with arbitrarily large numbers of nodes do not admit weak symplectic fillings having the rational homology of the 4-disk. This result provides further evidence toward the conjecture that no such weak symplectic filling exists once the mininal resolution tree has at least two nodes.

1. Introduction

Isolated complex surface singularities and their smoothings (if they exist) play an important role in constructing interesting smooth 4-manifolds. In particular, suppose that $(S, 0)$ is an isolated surface singularity with resolution graph Γ and M is the Milnor fibre of a smoothing of S. Suppose, furthermore, that (X, ω) is a symplectic 4-manifold and $C_1, \ldots, C_n \subset (X, \omega)$ are symplectic submanifolds intersecting each other according to Γ (and ω-orthogonally). It was shown in [13] that

$$Z = (X - \nu(C_1 \cup \ldots \cup C_n)) \cup M$$

is also symplectic. In many cases the underlying smooth structure is *exotic*, that is, homeomorphic but not diffeomorphic to some standard smooth 4-manifold. This construction was discovered and exploited by Fintushel and Stern [6] (and extended by J. Park [15]) for cyclic quotient singularities and Milnor fibres admitting the rational homology of a disk — this is Fintushel and Stern's *rational blow-down* procedure.

A smoothing with Milnor fibre M satisfying $H_*(M; \mathbb{Q}) \cong H_*(D^4; \mathbb{Q})$ is called a *rational homology disk smoothing* (or $\mathbb{Q}$HD smoothing for short). The list of singularities with $\mathbb{Q}$HD smoothings was conjectured by Wahl (cf. [8]); this conjecture was verified in [2] for weighted homogeneous singularities, that is, singularities which admit resolution graphs with one node. (Recall that a *node* in a graph in this context means a vertex with valency at least three.) Using different methods, this result was extended by Wahl [18] to singularities with resolution graphs having two nodes, each with valency 3. In [14] a complete solution of the problem is claimed: it is shown that a singularity with minimal resolution graph admitting at least two nodes does not have a $\mathbb{Q}$HD smoothing, hence the result of [2] implies the resolution of Wahl's conjecture.

In fact, the result verified in [2] is slightly stronger: the argument provides a classification of resolution graphs with one node which are rational and for which the corresponding Milnor fillable contact structure on the link admits a weak symplectic $\mathbb{Q}$HD filling. (The methods of [18] and [14], being algebro-geometric in nature, do not provide any information about weak symplectic fillings.) For completeness, let us recall the definition of weak and strong symplectic fillings (cf. also [7, Definition 5.1.1] or [12, Definition 12.1.1]).

Definition 1.1. For a given contact 3-manifold (Y,ξ) (where ξ is assumed to be cooriented), the compact symplectic 4-manifold (X,ω) is a **weak symplectic filling** if $\partial X = Y$ as oriented manifolds (where X is oriented by the volume form $\omega\wedge\omega$), and $\omega|_\xi > 0$. The compact symplectic 4-manifold (X,ω) is a **strong symplectic filling** if $\partial X = Y$ and there is a Liouville vector field V on X near ∂X satisfying $\xi = \ker(\iota_V\omega|_{TY})$. Here $\iota_V\omega$ denotes the contraction of the 2-form ω with the vector field V and the vector field being Liouville means that the Lie derivative $\mathfrak{L}_V\omega$ of the 2-form ω along V is ω.

Remark 1.2. Obviously a strong filling is a weak filling; moreover, for a strong filling the symplectic structure ω of the filling near the boundary ∂X is exact. According to [4, Proposition 3.1] (see also [5, Lemma 2.1] or [10]) this is essentially the only obstruction for a weak filling to be strong: if ω is exact near ∂X then (according to [4]) we can attach a topologically trivial symplectic cobordism $(\widetilde{X},\tilde{\omega})$ to $(\partial X,\omega)$ which is now a strong symplectic filling of (Y,ξ). Notice that if $b_1(Y)=0$ (as will be always the case for the 3-manifolds considered in the following), then the exactness hypothesis near ∂X is automatically satisfied.

The argument presented in [17] shows that if Γ is the resolution graph of a singularity with Milnor fillable contact structure admitting a weak symplectic $\mathbb{Q}$HD filling and Γ admits more than one node, then Γ is a member of one of the inductively defined families $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{C}$ of graphs described in [17]. (Here we will not repeat the simple definition of the graph families, see [17] or [14].)

In the present work we describe three families of resolution graphs Γ_n^C, Γ_n^B and Γ_n^A (one from each of the families $\mathcal{C}$, $\mathcal{B}$ and $\mathcal{A}$ of [17])), such that Γ_n^* (with $*=C,B,A$) has $n+2$ nodes and, although infinitely many of them pass the combinatorial constraints of [16, 17], the corresponding Milnor fillable contact structures on the plumbing 3-manifolds admit no weak symplectic $\mathbb{Q}$HD fillings (with the additional assumption $n\neq 0,3,5$ or 6 for the family Γ_n^A). The proof slightly extends the idea of the proof of the main result of [2], and relies on the same fundamental theorem of McDuff concerning symplectic 4-manifolds containing symplectic spheres of self-intersection $+1$.

The novelty of the present work is that in [2] we managed to find a concave filling of the Milnor fillable contact 3-manifold at hand that was diffeomorphic to the tubular neighbourhood of a configuration of J-holomorphic curves (for some almost complex

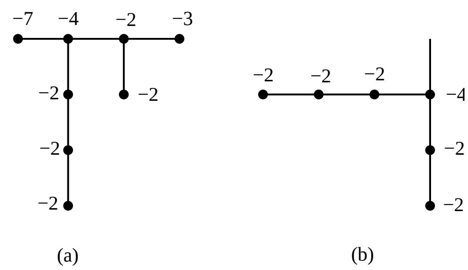


FIGURE 1. The graph Γ_0^C is given by (a). Repeatedly substituting the chain of three (-2)'s with the graph of (b), we get the sequence of graphs Γ_n^C with $n+2$ nodes.

structure J), while now we merely identify some curves in the concave filling. As it turns out, for the chosen families of graphs this slightly weaker information is sufficient to exclude the existence of weak $\mathbb{Q}$HD fillings.

This result provides further evidence for the conjectural strong relationship between smoothings of a rational surface singularity and weak symplectic fillings of the Milnor fillable contact structure on its link, cf. [3].

Acknwledgements: A. Stipsicz was partially supported by the *Lendület program ADT*, ERC Advanced Grant LDTBud and OTKA K112735. We would like to thank the organizers of the Gökova Geometry Conference for providing a very stimulating atmosphere in the conference.

2. Main Results

2.1. A family of graphs in $\mathcal{C}$

Consider the sequence of graphs Γ_n^C in the family $\mathcal{C}$ of [17] obtained inductively as follows: Γ_0^C is the graph given in Figure 1(a). For $n \geq 1$, Γ_n^C is the graph obtained from Γ_{n-1}^C by replacing the chain of three (-2)-vertices with the subgraph given in Figure 1(b). Let Y_n^C denote the associated plumbed 3-manifold and let ξ_n^C denote the unique Milnor fillable contact structure on Y_n.

Proposition 2.1. *For any $n \geq 0$, the contact 3-manifold (Y_n^C, ξ_n^C) does not admit a weak $\mathbb{Q}$HD filling, and hence no singularity with resolution graph Γ_n^C, for $n \geq 0$, admits a rational homology disk smoothing.*

Proof. It is easy to see that in the 16-point blow-up of the Hirzebruch surface $\mathcal{F}_{-4}$ with zero-section of self-intersection -4 (and hence infinity-section of self-intersection 4) we can find the configuration of rational curves indicated in Figure 2. Notice that

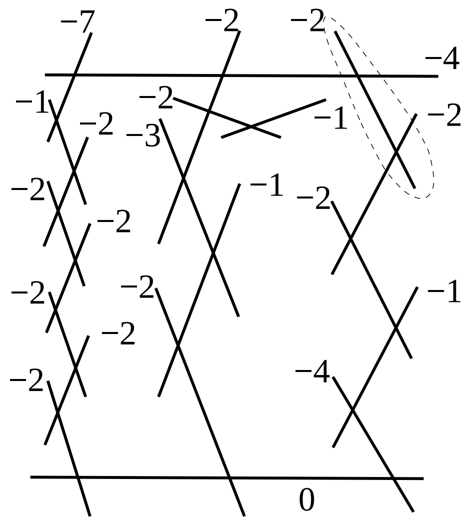


Figure 2. Rational curves in the 16-fold blow-up of the Hirzebruch surface $\mathcal{F}_{-4}$.

this collection of rational curves contains a configuration of curves intersecting each other according to Γ_0^C.

Now consider the (-2)-curve enclosed by the dashed circle. By sequentially blowing up five times one can transform the (-2)-curve into a (-4)-curve with a chain of five rational curves having self-intersections $-2,-2,-2,-1,-5$ attached to it (the chain is attached to the (-4)-curve at the first (-2)-curve). It follows that, in $\mathcal{F}_{-4}\#21\overline{\mathbb{CP}^2}$, we can find a configuration of rational curves intersecting each other according to Γ_1^C. Repeating this process for the first (-2)-curve in each new chain of five rational curves having self-intersections $-2,-2,-2,-1,-5$, we see that, for any $n\geq 0$, we can find a configuration of rational curves intersecting each other according to Γ_n^C in the rational surface $\mathcal{F}_{-4}\#(5n+16)\overline{\mathbb{CP}^2}$.

Since the curves intersecting according to the graph Γ_n^C in the rational surface given by $\mathcal{F}_{-4}\#(5n+16)\overline{\mathbb{CP}^2}$ admit an ω-convex neighbourhood (with symplectic form ω being the Kähler form on the blown up Hirzebruch surface), with Milnor fillable contact structure as induced structure on the boundary, the complement, which we will denote by W_n, provides a strong concave filling of (Y_n^C,ξ_n^C). Notice that W_n contains the configuration of rational curves indicated in Figure 3, although it is not diffeomorphic to the tubular neighbourhood of these curves. Now blow up W_n at the intersection point between the 0-curve and the (-2)-curve that belongs to the chain of six (-2)-curves. Then blow down the image of the 0-curve and the image of the (-2)-curve intersecting it to obtain a new symplectic 4-manifold W_n', which contains the configuration of curves depicted in

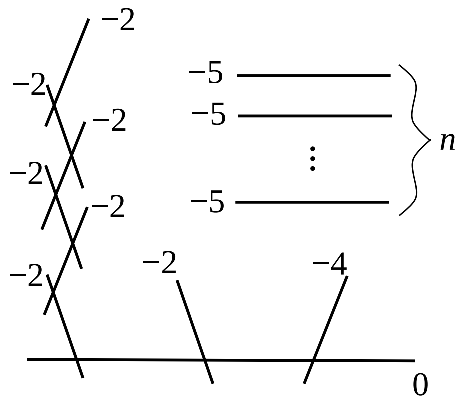


FIGURE 3. A configuration of rational curves contained in the strong concave filling W_n of (Y_n^C, ξ_n^C).

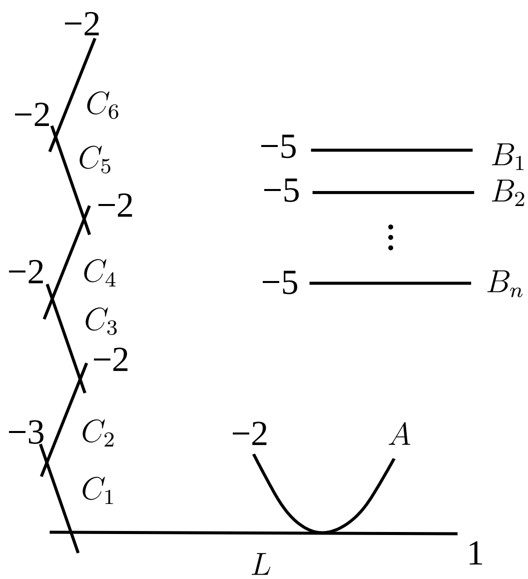


FIGURE 4. A configuration of rational curves contained in the strong concave filling W'_n of (Y_n^C, ξ_n^C).

Figure 4 and is again a strong concave filling of (Y_n^C, ξ_n^C). Here the tangency between the $(+1)$-curve L and the (-2)-curve A is an ordinary tangency, that is, the intersection multiplicity is 2.

Suppose now that X is a weak symplectic $\mathbb{Q}$HD filling of (Y_n^C, ξ_n^C). Since Y_n^C is a rational homology 3-sphere, we can perturb the symplectic structure on X in a neighbourhood of the boundary so that it becomes a strong symplectic filling of (Y_n^C, ξ_n^C). Glue X and W_n' along Y_n^C to obtain a closed symplectic 4-manifold Z. Since $b_2(W_n') = 2n+9$ and X is a rational homology disk, it follows that $b_2(Z)$ is also $2n+9$. Since Z contains a rational curve with self-intersection number 1, it follows from McDuff's Theorem [9] that Z is a rational symplectic 4-manifold and hence is diffeomorphic to $\mathbb{CP}^2\#(2n+8)\overline{\mathbb{CP}^2}$. In fact, McDuff's Theorem implies that, for a generic tame almost complex structure J, in the complement of the $(+1)$-curve L we can find $2n+8$ disjoint embedded symplectic 2-spheres with self-intersection number -1 (also called *symplectic* (-1)*-curves*). Arguing as in [11], [1] and [2], we can blow down the pair (Z, L) to the pair $(\mathbb{CP}^2, \text{line})$ while preserving the pseudoholomorphicity of A and C_1. Since the curves $C_2, \ldots, C_6$ and $B_1, \ldots, B_n$ are homologically essential and disjoint from the $(+1)$-curve L, we must blow them down, while the curves A and C_1 will descend to a conic and a line, respectively, in $\mathbb{CP}^2$. It follows that if we choose a compatible almost complex structure for which the above curves are J-holomorphic, and the almost complex structure is generic among such structures, then there must be $n+3$ additional (-1)-curves in Z. Since J-holomorphic curves intersect positively, the geometric intersections can be computed via homological arguments.

We now show that it is impossible to position $n+3$ additional (-1)-curves so that the pair (Z, L) can be blown down to $(\mathbb{CP}^2, \text{line})$ as above. Suppose for a contradiction that it is possible to find $n+3$ suitably placed additional (-1)-curves. Then there must be a (-1)-curve E_1 that intersects one of the curve C_i with $i \geq 2$. It is not hard to see by repeated blow-downs that i must be equal to 6 and the intersection is a single transverse intersection. Moreover, E_1 cannot intersect any other curve of Figure 4. Since, if E_1 also intersects B_i, for some i, then, after blowing down all (-1)-curves away from L, B_i will become a curve of positive self-intersection in the complement of L, a contradiction. Similarly, if E_1 intersects A, then, as A will become a conic in $\mathbb{CP}^2$, the images of C_1 and A in $\mathbb{CP}^2$ will intersect only once transversely, again a contradiction. Thus, after blowing down E_1 and sequentially blowing down five more times, we arrive at the configuration depicted in Figure 5. Let Z' denote the resulting symplectic 4-manifold. We must also have $n+2$ additional suitably placed (-1)-curves in Z'. The argument is now as follows.

Note that the total weight, that is, the sum of the self-intersection numbers, of all the given curves in Z', including the additional (-1)-curves, is $-6n-5$. After blowing down $2n+2$ (-1)-curves the total weight must become 6: the self-intersections of L, the image of C_1 and the image of A. Note also that the total weight can increase by at most 3 for each blow down, in case a (-1)-curve intersects at most two other curves, with at most one exception where the total weight increases by 4, when a (-1)-curve intersects three other curves, since at the end there is at most one double intersection. Thus the total weight can increase by at most $6n+7$. Thus the total weight can increase to at most $-6n-5+6n+7 = 2$, which contradicts the fact that it must become 6 after all blow

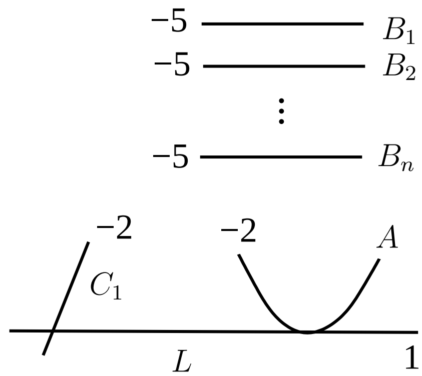


FIGURE 5. A configuration of rational curves contained in the symplectic 4-manifold Z'.

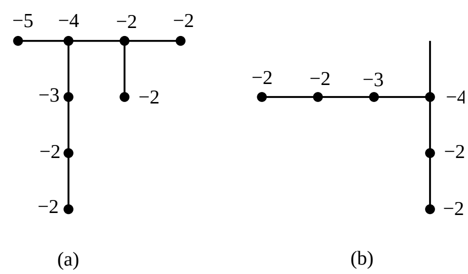


FIGURE 6. The graph Γ_0^B is given by (a). Repeatedly substituting the chain of three vertices with self-intersections $-3, -2, -2$ with the graph of (b), we get the sequence of graphs Γ_n^B with $n+2$ nodes.

downs. This contradiction shows that the hypothetical weak symplectic $\mathbb{Q}$HD filling X of (Y_n^C, ξ_n^C) cannot exist. $\square$

2.2. A family of graphs in $\mathcal{B}$

Consider the sequence of graphs Γ_n^B in the family $\mathcal{B}$ of [17] obtained inductively as follows: Γ_0^B is the graph given in Figure 6(a). For $n \geq 1$, Γ_n^B is the graph obtained from Γ_{n-1}^B by replacing the chain of three vertices with self-intersections $-3, -2, -2$ with the subgraph given in Figure 6(b). Let Y_n^B denote the associated plumbed 3-manifold and let ξ_n^B denote the unique Milnor fillable contact structure on Y_n^B.

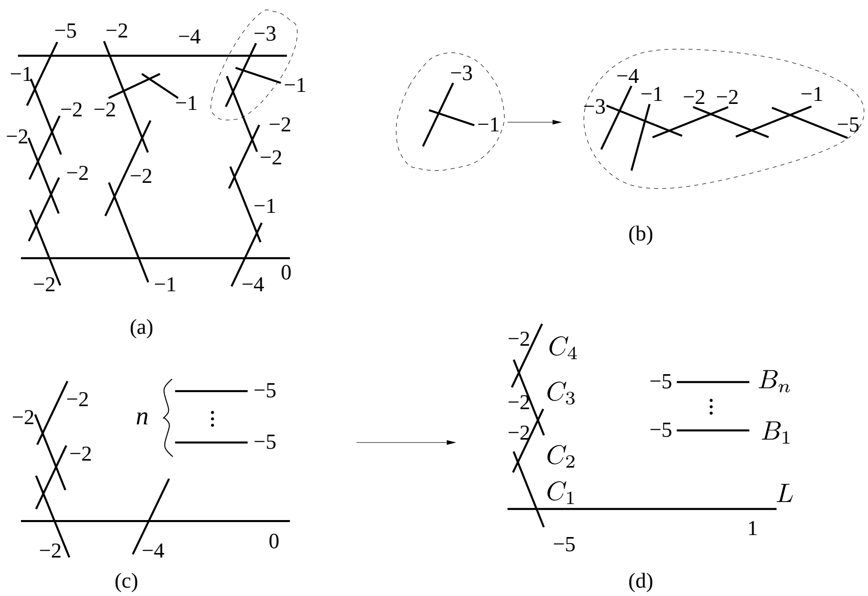


FIGURE 7. Rational curves in $\mathcal{F}_{-4}\#(5n+14)\overline{\mathbb{CP}^2}$ (in (a) and (b)), and in the strong concave fillings W_n (shown in (c)) and in W_n' (in (d)).

Proposition 2.2. *For any $n \geq 0$, the contact 3-manifold (Y_n^B, ξ_n^B) does not admit a weak $\mathbb{Q}$HD filling, and hence no singularity with resolution graph Γ_n^B, for $n \geq 0$, admits a rational homology disk smoothing.*

Proof. As in the case of Γ_n^C, we can find a configuration of rational curves intersecting each other according to Γ_n^B in the rational surface $\mathcal{F}_{-4}\#(5n+14)\overline{\mathbb{CP}^2}$ (see Figures 7(a) and (b)). It follows that an ω-convex neighbourhood of these curves provides a strong filling of (Y_n^B, ξ_n^B). The complement of this neighbourhood, denoted W_n, is therefore a strong concave filling of (Y_n^B, ξ_n^B). One can check that W_n contains the configuration of rational curves indicated in Figure 7(c), although it is not diffeomorphic to the tubular neighbourhood of these curves. Now blow up W_n at the intersection point of the 0-curve and the (-2)-curve intersecting it, and blow down the image of the 0-curve. Repeating this process a further two times, the (-4)-curve is transformed into a (-1)-curve. Blow down this (-1)-curve to obtain a new symplectic 4-manifold W_n', which contains the configuration of curves depicted in Figure 7(d).

Suppose now that X is a weak symplectic $\mathbb{Q}$HD filling of (Y_n^B, ξ_n^B). Since Y_n^B is a rational homology 3-sphere, we can perturb the symplectic structure on X in a neighbourhood of the boundary so that it becomes a strong symplectic filling of (Y_n^B, ξ_n^B). Glue X and W_n' along Y_n^B to obtain a closed symplectic 4-manifold Z. Since $b_2(W_n') = 2n+7$ and

X is a rational homology disk, it follows that $b_2(Z)$ is also $2n+7$. Since Z contains a rational curve with self-intersection number 1, it follows from McDuff's Theorem that Z is a rational symplectic 4-manifold and hence is diffeomorphic to $\mathbb{CP}^2\#(2n+6)\overline{\mathbb{CP}^2}$. In fact, McDuff's Theorem implies that, for a generic tame almost complex structure J, in the complement of the $(+1)$-curve L we can find $2n+6$ disjoint symplectic (-1)-curves. Arguing as before, we can blow down the pair (Z, L) to the pair $(\mathbb{CP}^2, \text{line})$ while preserving the pseudoholomorphicity of C_1. Since the curves C_2, C_3, C_4 and $B_1, \ldots, B_n$ of Figure 7(d) are homologically essential and disjoint from the $(+1)$-curve L, we must blow them down, while the curve C_1 will descend to a line in $\mathbb{CP}^2$. It follows that if we choose a compatible almost complex structure for which the above curves are J-holomorphic, and the almost complex structure is generic among such structures, then there must be $n+3$ additional (-1)-curves in Z. Since J-holomorphic curves intersect positively, the geometric intersections can be computed via homological arguments.

We now show that it is impossible to position $n+3$ additional (-1)-curves so that the pair (Z, L) can be blown down to $(\mathbb{CP}^2, \text{line})$ as above. Suppose for a contradiction that it is possible to find $n+3$ suitably placed additional (-1)-curves. Then there must be an an exceptional curve E_1 intersecting C_2 or C_4. Suppose, first, that it intersects C_2. Then, after repeatedly blowing down, C_1 becomes a (-2)-curve. Now denote the remaining exceptional curves $E_2, \ldots, E_{n+3}$. After blowing these down, denote the images of the curves $C_1, B_1, \ldots, B_n$ by $\widetilde{C}_1, \widetilde{B}_1, \ldots, \widetilde{B}_n$. Assume that $\widetilde{B}_1$ is the first of these to get blown down. Then at most two of the $\widetilde{B}_i$ can intersect it (or one $\widetilde{B}_i$ and $\widetilde{C}_1$). It follows that at least two exceptional curves E_i intersect B_1 and no other curves. So the total weight increases by only 2 when each of these is blown down. One can now use an elementary calculation to show that this case cannot occur.

Suppose, now, that E_1 intersect C_4. It is easy to rule out the case that E_1 is disjoint from the B_i. So suppose that it intersect B_1. Repeatedly blowing down, we can collapse B_1. There are $n-1$ (-5)-curves $B_2, \ldots, B_n$ left, and the curve C_1 becomes a (-3)-curve. Now arguing as above, and using the fact that when the last curve is blown down the total weight increases by only 2, one can exclude this case also. It follow that the hypothetical weak symplectic $\mathbb{Q}$HD filling X of (Y_n^B, ξ_n^B) cannot exist. $\square$

2.3. A family of graphs in $\mathcal{A}$

Consider the sequence of graphs Γ_n^A in the family $\mathcal{A}$ of [17] obtained inductively as follows: Γ_0^A is the graph given in Figure 8(a). For $n \geq 1$, Γ_n^A is the graph obtained from Γ_{n-1}^A by replacing the chain of three vertices with self-intersection $-4, -2, -2$ with the subgraph given in Figure 8(b). Let Y_n^A denote the associated plumbed 3-manifold and let ξ_n^A denote the unique Milnor fillable contact structure on Y_n^A.

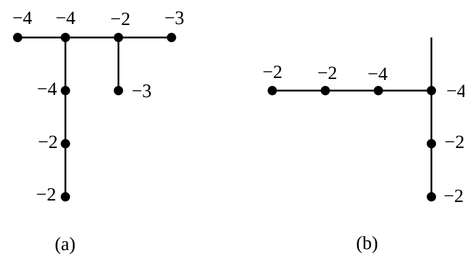


FIGURE 8. The graph Γ_0^A is given by (a). Repeatedly substituting the chain of three vertices with self-intersections $-4,-2,-2$ with the graph of (b), we get the sequence of graphs Γ_n^A with $n+2$ nodes.

Proposition 2.3. *For any nonnegative integer $n \neq 0,3,5$ or 6, the contact 3-manifold (Y_n^A, ξ_n^A) does not admit a weak $\mathbb{Q}$HD filling, and hence no singularity with resolution graph Γ_n^A, for $n \neq 0,3,5$ or 6, admits a rational homology disk smoothing.*

Proof. As before, we can find a configuration of rational curves intersecting each other according to Γ_n^A in the rational surface $\mathcal{F}_{-4}\#(5n+16)\overline{\mathbb{CP}^2}$ (see Figures 9(a) and (b)). It follows that an ω-convex neighbourhood of these curves provides a strong filling of (Y_n^A, ξ_n^A). The complement of this neighbourhood, denoted W_n, is therefore a strong concave filling of (Y_n^A, ξ_n^A). One can check that W_n contains the configuration of rational curves indicated in Figure 9(c), although it is not diffeomorphic to the tubular neighbourhood of these curves. Now blow up W_n at the intersection point between the 0-curve and the (-2)-curve that belongs to the chain of three (-2)-curves. Then blow down the image of the 0-curve and the image of the (-2)-curve intersecting it to obtain a new symplectic 4-manifold W_n', which contains the configuration of curves depicted in Figure 9(d).

Suppose now that X is a weak symplectic $\mathbb{Q}$HD filling of (Y_n^A, ξ_n^A). Since Y_n^A is a rational homology 3-sphere, we can perturb the symplectic structure on X in a neighbourhood of the boundary so that it becomes a strong symplectic filling of (Y_n^A, ξ_n^A). Glue X and W_n' along Y_n^A to obtain a closed symplectic 4-manifold Z. Since $b_2(W_n') = 2n+9$ and X is a rational homology disk, it follows that $b_2(Z)$ is also $2n+9$. Since Z contains a rational curve with self-intersection number 1, it follows from McDuff's Theorem that Z is a rational symplectic 4-manifold and hence is diffeomorphic to $\mathbb{CP}^2\#(2n+8)\overline{\mathbb{CP}^2}$. In fact, McDuff's Theorem implies that, for a generic tame almost complex structure J, in the complement of the $(+1)$-curve L we can find $2n+8$ disjoint symplectic (-1)-curves. Arguing as before, we can blow down the pair (Z, L) to the pair $(\mathbb{CP}^2, \text{line})$ while preserving the pseudoholomorphicity of A_1 and C_1. Since the curves A_2, A_3, C_2, C_3 and $B_1, \ldots, B_n$ are homologically essential and disjoint from the $(+1)$-curve L, we must

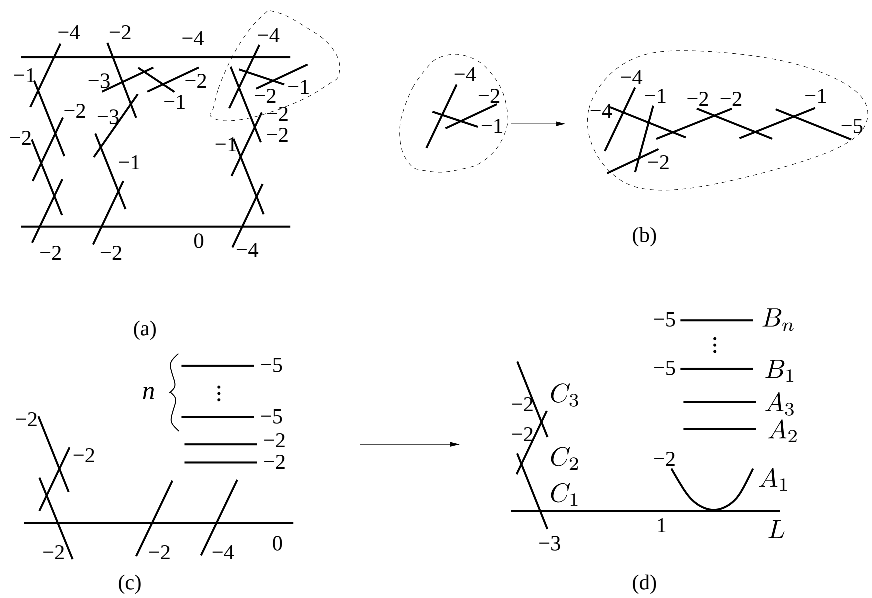


FIGURE 9. Rational curves in $\mathcal{F}_{-4}\#(5n+14)\overline{\mathbb{CP}^2}$ (in (a) and (b)), and in the strong concave fillings W_n (shown in (c)) and in W_n' (in (d)).

blow them down, while the curves A_1 and C_1 will descend to a conic and a line, respectively, in $\mathbb{CP}^2$. It follows that if we choose a compatible almost complex structure for which the above curves are J-holomorphic, and the almost complex structure is generic among such structures, then there must be $n+4$ additional (-1)-curves in Z. Since J-holomorphic curves intersect positively, the geometric intersections can be computed via homological arguments.

We claim that if $n \neq 0, 3, 5$ or 6, then it is impossible to position $n+4$ additional (-1)-curves so that the pair (Z, L) can be blown down to $(\mathbb{CP}^2, \text{line})$ as above. To see this, suppose that it is possible to find $n+4$ suitably placed additional (-1)-curves. Then there must be an exceptional curve E_1 intersecting only the labelled curves C_3 and A_1; C_2; C_3 and one of the (-5)-curves, say B_n; or C_2 and A_1. We now consider each of these cases in turn.

Case 1: There exists an exceptional curve E_1 intersecting only C_3 and A_1.

Blow down E_1 and then sequentially blow down the images of C_3 and C_2 so that we are in the situation depicted in the right hand picture of Figure 10(1). Denote the remaining exceptional curves $E_2, \dots, E_{n+4}$. Since the image of A_2 must eventually get blown down, there must be an exceptional curve, say E_2, that intersects A_2 and at most one other labelled curve. Thus when E_2 is blown down and then the image of A_2 is blown down,

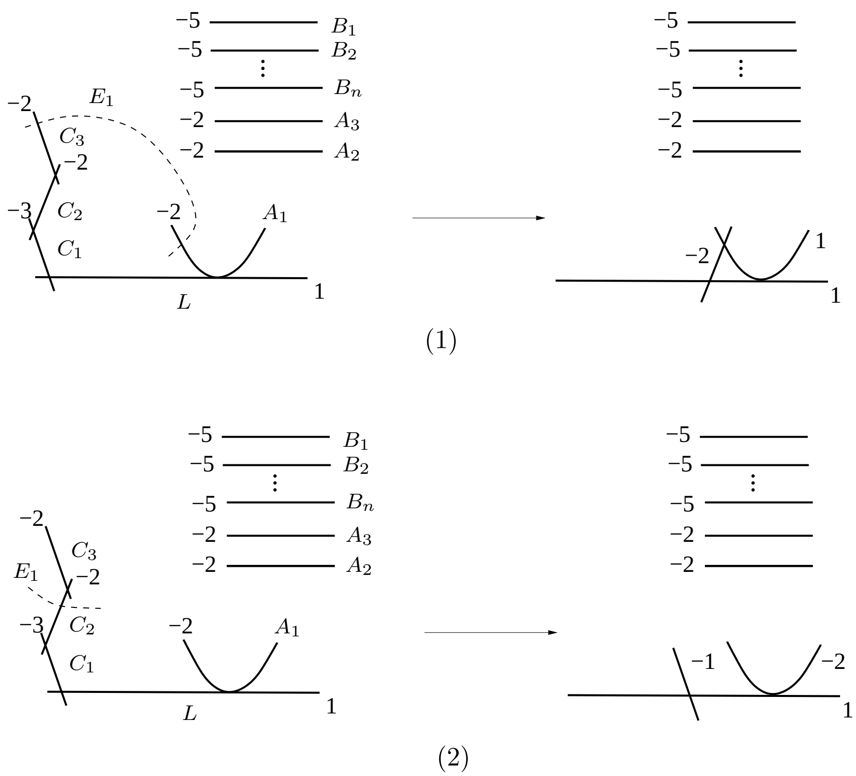
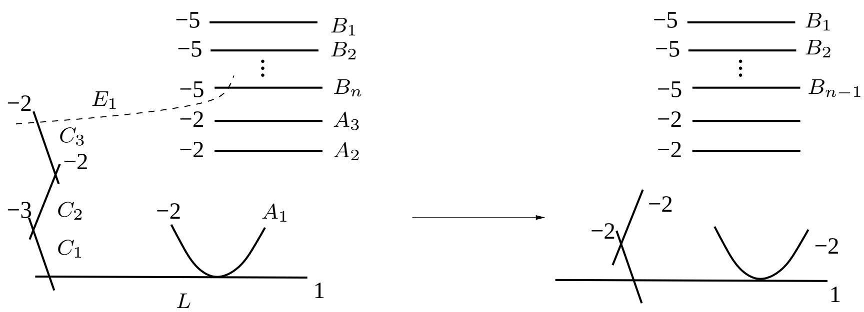


FIGURE 10. Possible positions for the (-1)-curve E_1 in the proof of Proposition 2.3.

the total weight can increase by at most 5. The situation is similar for A_3. It follows that each exceptional curve E_i must intersect exactly two labelled curves (so that the total weight increases by 3 when it is blown down) and that, furthermore, when all the exceptional curves E_i are blown down, the dual graph obtained from the images of the curves B_i is linear (otherwise the total weight will increase by less than 3 for at least one of the leaves when the image of the corresponding curve is blown down). Moreover, the ends of the chain formed from the images of the curves B_i must intersect the images of the curves A_2 and A_3. It is now easy to see that, when $n \neq 0$ or 3, it is impossible to sequentially blow down in the complement of L such a configuration of curves to a pair of lines and a conic in $\mathbb{CP}^2$.

Case 2: There exists an exceptional curve E_1 intersecting only C_2.

Blow down E_1 and then sequentially blow down the images of C_2 and C_3 so that we are in the situation depicted in the right hand picture of Figure 10(2). Denote the remaining



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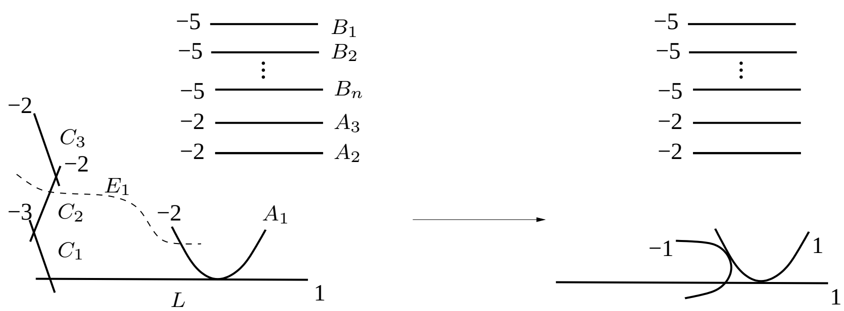

(4)

FIGURE 11. Possible positions for the (-1)-curve E_1 in the proof of Proposition 2.3.

exceptional curves $E_2, \ldots, E_{n+4}$. Let E_2 and E_3 denote the necessarily distinct exceptional curves that intersect A_2 and A_3, respectively. If E_2 and E_3 intersect precisely one further curve each, then the total weight will increase by at most 10 when E_2, E_3 and the images of A_2 and A_3 are blown down. It is easy to see by an elementary calculation that such a case cannot occur. So suppose that, without loss of generality, E_2 intersect two further curves. Then they must be the images of C_1 and A_1. Thus, when E_2 is blown down and then the image of A_2 is blown down, the images of C_1 and A_1 will intersect with multiplicity 2. In this case, E_3 can intersect at most one further curve, so that when E_3 is blown down and then the image of A_3 is blown down the total weight increases by at most 5. Now arguing as in Case 1, one can show, when all exceptional curves E_i are blown down, the dual graph obtained from the images of the curves B_i is linear. It is now easy to see that this will lead to a contradiction when the blowing down process is carried out.

Case 3: There exists an exceptional curve E_1 intersecting only C_3 and B_n.

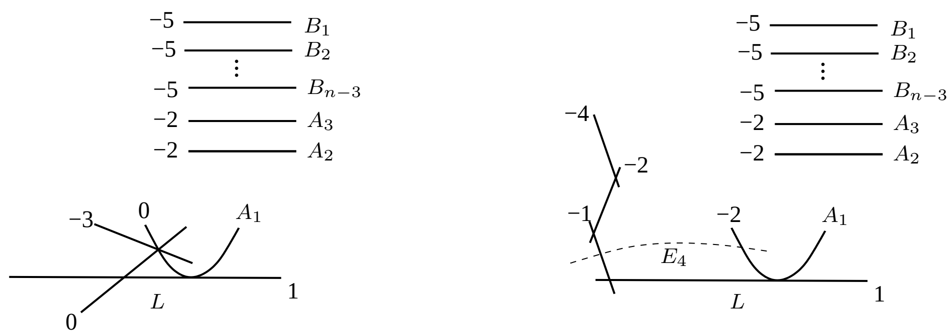

FIGURE 12. Possible positions for the (-1)-curves in the proof of Proposition 2.3.

Blow down E_1 and then sequentially blow down the images of C_3 and C_2 so that we are in the situation depicted in the right hand picture of Figure 11(3). There are now three possibilities that may arise:

(3a) There exists an exceptional curve E_2 that intersects only B_n and A_1. Arguing as in Case 1, one can show that, if $n \neq 5$, this case cannot occur.

(3b) There exists an exceptional curve E_2 that intersects only B_n. In this case one can argue as in Case 2 to show that this case does not occur.

(3c) There exists an exceptional curve E_2 that intersects only B_n and one further (-5)-curve, say B_{n-1}. In this case, after blowing down E_2 and then the image of B_n, C_1 will become a (-1)-curve and the image of B_{n-1} will become a (-3)-curve intersecting it. There are now two possibilities: there is an exceptional curve E_3 that intersects B_{n-1} and A_1 only, or there is no such curve.

(3ci) There is an exceptional curve E_3 that intersects only B_{n-1} and A_1. Blowing down E_3, the image of B_{n-1} will be transformed into a (-2)-curve intersecting the images of C_1 and A_1. It is easy to see that the only possibility now that does not lead to a contradiction is that there is an exceptional curve E_4 that intersects B_{n-1} and a further (-5)-curve, say B_{n-2}. Assume that such an exceptional curve E_4 exists. Blow down E_4 and the image of B_{n-1}, and denote the remaining exceptional curves $E_5, \ldots, E_{n+4}$. After the image of B_{n-2} is blown down, the images of C_1 and A_1 will have a double intersection, as shown by the left diagram of Figure 12 (the image of B_{n-2} is the (-3)-curve). It follows that each connected component of the collection trees formed from the (-5)-curves B_i, $i \leq n-3$, after all remaining exceptional curves E_i are blown down, must intersect the the image of $B_{n-2} \cup A_1$ at most once transversely. If this collection of trees is nonempty, the total weight will increase by at most 2 when each leaf which is disjoint from the image of $B_{n-2} \cup A_1$ is blown down. Further, there must be exactly $n-3$ exceptional curves that connect pairs of (-5)-curves B_i, $i \leq n-3$, or such a curve B_i and the image of $B_{n-2} \cup A_1$. As there must be two further distinct exceptional curves which intersect A_2 and A_3, there must be one further exceptional curve, say E_5, which intersects only one additional curve. When the images of A_2 and A_3 and the exceptional curve E_5 are blown down, the total weight increase by at most 2 in each case. Now arguing as before we get a contradiction if $n \neq 3$.

(3cii) There is no exceptional curve which intersects B_{n-1} and A_1. In this case there must be an exceptional curve E_3 that intersects B_{n-1} and one further (-5)-curve, say B_{n-2}. Blow E_3 down. Then no further exceptional curve can intersect B_{n-1} since such a curve would necessarily have to intersect A_1, a possibility which we excluded. Since the images of A_1 and C_1 will intersect twice at the end of the blowing down process, there must be an exceptional curve, say E_4, that intersect C_1 and A_1. The situation is now as depicted in the right diagram of Figure 12. Arguing as before, after all exceptional curves have been blown down, the images of the remaining (-5)-curves will form a chain with one end connected to the image of B_{n-2} and the other end to the image of A_1. Taking the (-2)-curves into account, one can check that if $n \neq 3$ or 6, no such possibility can arise.

Case 4: There exists an exceptional curve E_1 intersecting only C_2 and A_1.

The situation here is as depicted in Figure 11(4). Arguing as in other cases, one can show here that, if $n \neq 0$, this case also cannot occur. This completes the proof. □

Remark 2.4. We expect that the cases $n = 0, 3, 5$ and 6 also provide Milnor fillable contact 3-manifolds (Y_n^A, ξ_n^A) with no weak symplectic $\mathbb{Q}HD$ fillings, although the above arguments fall short for proving this claim for them.

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