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On the construction of the sutured Floer complex

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Abstract. We review the recent construction and basic properties of a refinement of sutured Floer homology given by the authors in [1], defined for weakly balanced sutured manifolds (X, τ), and provide calculations that illustrate the differences from the original construction by Juhász [9]. For any weakly balanced sutured manifold (X, τ) we construct an algebra $\mathbb{A}_\tau$ associated to its boundary and introduce a chain complex $\mathrm{CF}(X, \tau)$ with coefficients in $\mathbb{A}_\tau$. The *filtered* chain homotopy type of $\mathrm{CF}(X, \tau)$ is an invariant of the sutured manifold (X, τ). In this paper we review some of the results from [1], on which the second author spoke at 2013 Gökova Geometry/Topology Conference.

1. Introduction

Heegaard Floer theory grew out of gauge theory, as a collection of powerful invariants for smooth 3-manifolds, 4-manifolds, knots, links and contact structures in the past decade through the works of Ozsváth and Szabó [14, 15, 18, 13, 16, 17]. It has been a source of important developments in low dimensional topology and knot theory. Heegaard Floer theory is defined as a version of Lagrangian Floer homology in a suitable symmetric product of a Heegaard surface. Using counts of holomorphic curves, the theory associates to a closed 3-manifold X four flavors of Heegaard Floer chain complexes $\widehat{CF}$, CF^-, CF^+ and CF^∞, whose chain homotopy types are diffeomorphism invariants. The *minus chain complex*, CF^-, together with the U action determines the other chain complexes. Moreover, 4-manifold invariants were constructed as some TQFT type homomorphism between the homology groups of the chain complexes associated with the negative and positive boundary components. The knot and link invariants are strong enough to detect the genus, fibered-ness and Thurston norm. Associated with contact structures over closed oriented 3-manifolds X, Ozsv́ath and Szabó introduced an invariant of the contact structure and X as an element of $\widehat{\mathrm{HF}}(-X)$ which detects overtwistedness and Giroux torsion [5].

Juhász (in [9]) introduced an extension of the notion of Heegaard Floer homology for non-closed 3-manifolds with certain boundary decorations, *sutured manifolds*, which extends the *hat* version of the previous construction. His construction is called *sutured Floer homology*. Sutured Floer homology has found several important applications in

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knot theory, contact geometry and the theory of foliations. The applications include the detection of fibred-ness and tautness and bounds on the depth of the foliation by Juhász [8, 10]. Honda, Kazez and Matić defined an invariant of the contact 3-manifold with convex boundary as an element of Juhász's sutured Floer homology [7] which vanishes if the contact structure is overtwisted or has Giroux torsion. An alternative approach for extending Heegaard Floer theory to closed three-manifolds with (parametrized) boundary was taken by Lipshitz, Ozsváth and Thurston [11, 12], which resulted in the introduction of bordered Heegaard Floer homology.

In [1] we introduced an extension of Juhász's construction and defined a *minus theory* associated with a sutured manifolds (X,τ) which is balanced in a sense weaker than Juhász's definition. Associated with the boundary of X (which is decorated with the sutures τ) we construct an algebra $\mathbb{A}=\mathbb{A}_\tau$. Let $\overline{X}$ denote the three-manifold with boundary which is obtained from X by filling the sutures. Fix a Spin^c structure $\mathfrak{s}$ in $\mathrm{Spin}^c(\overline{X})$. The *Sutured Floer chain complex* $\mathrm{CF}(X,\tau,\mathfrak{s})$ is then constructed as a module over $\mathbb{A}$, together with a differential $\partial:\mathrm{CF}(X,\tau,\mathfrak{s})\to\mathrm{CF}(X,\tau,\mathfrak{s})$. The relations in the algebra correspond to the disks with boundary entirely in $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, and which obstruct $\partial^2=0$. The sutured Floer chain complex is filtered by $\mathbb{H}=\mathrm{H}^2(X,\partial X;\mathbb{Z})$. We prove

Theorem 1.1. *With the above notation fixed, the filtered $(\mathbb{A},\mathbb{H})$ chain homotopy type of the filtered chain complex $\mathrm{CF}(X,\tau,\mathfrak{s})$ is an invariant of the weakly balanced sutured manifold (X,τ) and the Spin^c class $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})$. In particular, for any $\underline{\mathfrak{s}}\in\mathfrak{s}\subset\mathrm{Spin}^c(X,\tau)$ the chain homotopy type of the summand $\mathrm{CF}(X,\tau,\underline{\mathfrak{s}})$ is an invariant of $(X,\tau,\underline{\mathfrak{s}})$.*

The sutured Floer chain complex generalizes the existing Heegaard Floer homology groups, as well as the knot and link Floer homologies.

Example 1.1. Let

$$X=D^3=\{(x_1,x_2,x_3)\in\mathbb{R}^3\mid x_1^2+x_2^2+x_3^2\le 1\}$$

and let $\tau=\{\tau_1,\tau_2,\tau_3\}$ be the sutures on $\partial X=S^2$, where τ_1,τ_2 and τ_3 are the intersections of S^2 with the planes $x_3=\frac{1}{2},0$ and $-\frac{1}{2}$, respectively. There is a distinguished Spin^c structure $\mathfrak{s}_0$ over $\overline{X}$ in this case. The algebra associated with the sutured manifold (X,τ) is

$$\mathbb{A}=\frac{\mathbb{Z}[\mathfrak{u}_1,\mathfrak{u}_2,\mathfrak{u}_3]}{\langle(\mathfrak{u}_1-\mathfrak{u}_3)(\mathfrak{u}_2-1)\rangle}.$$

Moreover, for an appropriate Heegaard diagram we find that the chain complex is generated by two generators x and y:

$$\mathrm{CF}(X,\tau,\mathfrak{s}_0)=\mathbb{A}\langle x,y\rangle$$
$$\partial x=(\mathfrak{u}_1-\mathfrak{u}_3)y\ ,\ \partial y=(\mathfrak{u}_2-1)x.$$

Thus the chain homotopy type of $\mathrm{CF}(X,\tau,\mathfrak{s}_0)$ is non-trivial, while

$$\mathrm{SFH}(X,\tau,\mathfrak{s}_0)=0.$$

This is one of the simplest examples where Juhász's sutured Floer homology is trivial, while the sutured Floer chain complex is nontrivial.

The connected sum theorem for Ozsváth-Szabó complexes associated with closed 3-manifolds, as well as the stabilization theorem for link Floer homology may be generalized in this setting. This note is a report on these results.

2. Basic definitions

In this section we recall some basic definitions which are used in the construction of the sutured Floer chain complex.

2.1. Sutured manifolds and Heegaard diagrams

Since we only deal with sutured manifolds without toroidal sutures, we restrict the (standard) definition to this special class.

Definition 2.1. A *sutured manifold* without toroidal sutures is a pair (X, τ) where X is a compact, oriented 3-manifold with boundary and $\tau = \{\tau_1, ..., \tau_\kappa\}$ is a set of disjoint oriented simple closed curves (the *sutures*) on ∂X, which divides ∂X into positive and negative parts. More precisely, $\mathfrak{R}(\tau) = \mathfrak{R}^+(\tau) \cup \mathfrak{R}^-(\tau)$ where $\mathfrak{R}(\tau) = \partial X - \tau$ and $\mathfrak{R}^+(\tau)$ ($\mathfrak{R}^-(\tau)$) is the union of components such that the orientation induced on τ as the boundary of $\mathfrak{R}^+(\tau)$ agrees with (is opposite to) the orientation of τ.

Definition 2.2. A sutured manifold (X, τ) is called *weakly balanced* if any component of X has non-empty intersection with $\mathfrak{R}^+(\tau)$ and $\mathfrak{R}^-(\tau)$ and $\chi(\mathfrak{R}^+(\tau)) = \chi(\mathfrak{R}^-(\tau))$.

Note that by filling the sutures of any sutured manifold (X, τ) with attaching 2-handles, we may construct a sutured manifold with an empty set of sutures on its boundary, denoted by $\overline{X}$.

Definition 2.3. A *Heegaard diagram* for a weakly balanced sutured manifold (X, τ) is a 4-tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ consisting of a closed surface Σ, ℓ-tuples of disjoint simple closed curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and a set $\mathbf{z}$ of κ marked points on Σ such that X is obtained from $\Sigma^\circ \times [-1, 1]$ by attaching 2-handles to $\boldsymbol{\alpha} \times \{-1\}$ and $\boldsymbol{\beta} \times \{1\}$ while τ is obtained as $\partial\Sigma^\circ \times \{0\}$.

Note that if (X, τ) is balanced, then $(\Sigma^\circ, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a sutured Heegaard diagram for (X, τ) as Juhász defined in [9].

2.2. Spin^c structures and Whitney disks

Let (X, τ) be a weakly balanced sutured manifold. Consider a nowhere vanishing vector field v_τ on ∂X which points outward on $\mathfrak{R}^+(\tau) - \mathrm{nhd}(\tau)$ and points inward on $\mathfrak{R}^-(\tau) - \mathrm{nhd}(\tau)$, where $\mathrm{nhd}(\tau) \subset \partial X$ is a small tubular neighborhood of τ. Furthermore, assume that under the identification $\mathrm{nhd}(\tau) = \tau \times [-1, 1]$, $v_\tau|_{\mathrm{nhd}}(\tau)$ is the unit tangent vector of the interval $[-1, 1]$.

Definition 2.4. The space of *relative* Spin^c *structures* on the sutured manifold (X, τ) is the space of homology classes of nowhere vanishing vector fields on X which agree with v_τ on ∂X. Note that two such vector fields are called homologous if they are homotopic relative to boundary outside a ball contained in X°.

Assume $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ is a Heegaard diagram for (X, τ). Let $\mathrm{Sym}^\ell = \frac{\Sigma^{\times \ell}}{S_\ell}$ be the ℓ-th symmetric product of Σ and consider the real tori $\mathbb{T}_\alpha = \alpha_1 \times ... \times \alpha_\ell$ and $\mathbb{T}_\beta = \beta_1 \times ... \times \beta_\ell$ inside it. We define a map

$$\underline{\mathfrak{s}} = \underline{\mathfrak{s}}_{\mathbf{z}} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathrm{Spin}^c(X, \tau)$$

by considering a Morse function compatible with the Heegaard diagram on (X, τ), looking at any intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ as a set of flow lines joining index-1 critical points to index-3 critical points of the corresponding Morse function and perturbing the gradient vector field of the Morse function in a neighborhood of the flow lines associated with $\mathbf{x}$ in order to get a nowhere vanishing vector field on X with the desired properties. Similarly, we define a map

$$\mathfrak{s} : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \longrightarrow \mathrm{Spin}^c(\overline{X}).$$

Definition 2.5. Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $D \subset \mathbb{C}$ be the unit disk. By a *Whitney disk* connecting $\mathbf{x}$ to $\mathbf{y}$ we mean a continuous map $\phi : D \to \mathrm{Sym}^\ell(\Sigma)$ which satisfies the following properties

- $\phi(-i) = \mathbf{x}$ and $\phi(i) = \mathbf{y}$.
- $\phi\{z \in \partial D | \mathrm{Re}(z) \geq 0\} \subset \mathbb{T}_\alpha$ and $\phi\{z \in \partial D | \mathrm{Re}(z) \leq 0\} \subset \mathbb{T}_\beta$.

For any two intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ we will denote the set of homotopy classes of Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$ by $\pi_2(\mathbf{x}, \mathbf{y})$.

2.3. Algebra and admissibility

Associated with the sutures on the boundary of any weakly balanced sutured manifold $(X, \tau = \{\tau_1, ..., \tau_\kappa\})$, we construct the algebra $\mathbb{A}_\tau$ as a quotient of the $\mathbb{Z}$-algebra $\mathbb{Z}[\kappa] := \mathbb{Z}[\mathfrak{u}_1, ..., \mathfrak{u}_\kappa]$ as follows. Assume that

$$\mathfrak{R}^-(\tau) = \bigcup_{i=1}^{k} R_i^- \quad \text{and} \quad \mathfrak{R}^+(\tau) = \bigcup_{j=1}^{l} R_j^+,$$

where $\{R_i^-\}_i$ and $\{R_j^+\}_j$ are the connected components of $\mathfrak{R}^-(\tau)$ and $\mathfrak{R}^+(\tau)$, respectively. Associated with these connected components consider the elements

$$\mathfrak{u}_i^- := \prod_{\tau_j \subset \partial R_i^-} \mathfrak{u}_j, \quad i = 1, ..., k \quad \text{and} \quad \mathfrak{u}_i^+ := \prod_{\tau_j \subset \partial R_i^+} \mathfrak{u}_j, \quad i = 1, ..., l,$$

in $\mathbb{Z}[\kappa]$. Let $\mathcal{I}_\tau$ denote the *relations ideal*

$$\mathcal{I}_\tau = \left\langle \mathfrak{u}^+(\tau) - \mathfrak{u}^-(\tau) \right\rangle + \left\langle \mathfrak{u}_i^+ \mid g_i^+ > 0 \right\rangle + \left\langle \mathfrak{u}_j^- \mid g_j^- > 0 \right\rangle$$

where $\mathfrak{u}^-(\tau) = \sum_{i=1}^k \mathfrak{u}_i^-$, $\mathfrak{u}^+(\tau) = \sum_{j=1}^l \mathfrak{u}_j^+$ and g_i^- and g_j^+ denote the genus of R_i^- and R_j^+ respectively for $i = 1, ..., k$ and $j = 1, ..., l$. Now we define the algebra $\mathbb{A} = \mathbb{A}_\tau$ as the quotient

$$\mathbb{A}_\tau := \frac{\mathbb{Z}[\kappa]}{\mathcal{I}_\tau}.$$

This algebra will be the ring of coefficients in the definition of the sutured Floer chain complex.

Note that the second and the third summands of the relations ideal, $\mathcal{I}_\tau$, are needed to deal with the α and β boundary degenerations which have positive coefficient in any non-zero genus component of $\Sigma - \alpha$ or $\Sigma - \beta$ and we need all the relations to make the differential map ∂, which will be defined in the next section, satisfy $\partial^2 = 0$.

The algebra $\mathbb{A}_\tau$ admits a filtration by the $\mathbb{Z}$-module $\mathbb{H} := H^2(X, \partial X, \mathbb{Z})$ as follows. Let $G(\mathbb{A}_\tau) \subset \mathbb{A}_\tau$ be the set of monomials $\prod_{i=1}^\kappa \mathfrak{u}_i^{a_i}$ inside $\mathbb{A}_\tau$. We define a map

$$\chi : G(\mathbb{A}_\tau) \longrightarrow \mathbb{H}$$

by setting

$$\chi\Big(\prod_{i=1}^{\kappa} \mathfrak{u}_i^{a_i}\Big) := a_1\mathrm{PD}[\tau_1] + ... + a_\kappa \mathrm{PD}[\tau_\kappa], \qquad \text{for} \quad a_1, ..., a_\kappa \in \mathbb{Z}^{\geq 0}.$$

It is easy to see that $\chi(1) = 0$ and $\chi(UV) = \chi(U) + \chi(V)$ for any two monomials $U, V \in G(\mathbb{A})$.

In order to define the sutured Floer chain complex we use a special family of Heegaard diagrams which have the following *admissibility* property.

Definition 2.6. Let (X, τ) be a weakly balanced sutured manifold and $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$. A Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ for (X, τ) is called $\mathfrak{s}$-*admissible* if for any nontrivial periodic domain $\mathcal{P}$ which satisfies $\langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle = 0$ one of the following happens

- There is a point $w \in \Sigma$ such that $n_w(\mathcal{P}) < 0$.
- $\mathcal{P} \geq 0$ and $\mathfrak{u}(\mathcal{P})$ is trivial in $\mathbb{A}_\tau$.

For any Spin^c class $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$ there is a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ for (X, τ) which is $\mathfrak{s}$-admissible. For the proof please see Lemma 4.5 in [1].

3. Sutured Floer chain complex

Let (X, τ) be a weakly balanced sutured manifold. Fix a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$. Associated with this Spin^c structure consider an $\mathfrak{s}$-admissible Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ for (X, τ). Let $\mathrm{CF}(X, \tau, \mathfrak{s})$ be the free $\mathbb{A}_\tau$-module generated by those intersection points of the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ which correspond to the Spin^c class $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X})$.

For any two intersection points $\mathbf{x},\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ we may define a map from the set $\pi_2^+(\mathbf{x},\mathbf{y})$ of positive Whitney disks ϕ connecting $\mathbf{x}$ to $\mathbf{y}$ to $G(\mathbb{A}_\tau)$ as follows.

$$\mathfrak{u}_{\mathbf{z}}:\coprod_{\mathbf{x},\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\pi_2^+(\mathbf{x},\mathbf{y})\longrightarrow G(\mathbb{A}_\tau),$$
$$\mathfrak{u}_{\mathbf{z}}(\phi):=\prod_{i=1}^{\kappa}\mathfrak{u}_i^{n_{z_i}(\phi)}.$$

Note that $n_{z_i}(\phi)$ is the coefficient of the domain $\mathcal{D}(\phi)$ associated with ϕ in the domain containing z_i. We define the differential ∂ of the complex $\mathrm{CF}(X,\tau,\mathfrak{s})$ by

$$\partial:\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s})\longrightarrow\mathrm{CF}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{z};\mathfrak{s})$$
$$\partial(\mathbf{x}):=\sum_{y\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\{\phi\in\pi_2^+(\mathbf{x},\mathbf{y})\mid\mu(\phi)=1\}}\big(\mathfrak{m}(\phi)\mathfrak{u}(\phi)\big)\cdot\mathbf{y},$$

where $\mathfrak{m}(\phi)=\#\widehat{\mathcal{M}}(\phi)$ is the signed number of the holomorphic representatives of the class ϕ up to translation.

The map $\underline{\mathfrak{s}}_{\mathbf{z}}$ assigns to any intersection point $\mathbf{x}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ an element of $\mathrm{Spin}^c(X,\tau)$ and the map χ assigns to any monomial in $G(\mathbb{A}_\tau)$ an element of $\mathbb{H}$. Therefore, these maps give $\mathrm{CF}(X,\tau,\mathfrak{s})$ the structure of a filtered $(\mathbb{A}_\tau,\mathbb{H})$ chain complex. In particular,

$$\mathrm{CF}(X,\tau,\mathfrak{s})=\bigoplus_{\underline{\mathfrak{s}}\in\mathfrak{s}}\mathrm{CF}(X,\tau,\underline{\mathfrak{s}}).$$

Theorem 3.1. *With the above notation fixed, the filtered* $(\mathbb{A}_\tau,\mathbb{H})$ *chain homotopy type of the filtered chain complex* $\mathrm{CF}(X,\tau,\mathfrak{s})$ *is an invariant of the weakly balanced sutured manifold* (X,τ) *and the* Spin^c *class* $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})$*. In particular, for any* $\underline{\mathfrak{s}}\in\mathfrak{s}\subset\mathrm{Spin}^c(X,\tau)$ *the chain homotopy type of the summand* $\mathrm{CF}(X,\tau,\underline{\mathfrak{s}})$ *is an invariant of* $(X,\tau,\underline{\mathfrak{s}})$.

Proof. The proof of the invariance under the choice of the path of almost complex structures, as well as the proof of the invariance under isotopies and handle slides supported away from the marked points is almost similar to the proof Ozsváth-Szabó for the Heegaard Floer chain complex in [14]. We need to keep track of the marked points and that the constructed chain homotopy equivalence respects the decomposition under the relative Spin^c classes. Moreover, we need to introduce a generalization of the holomorphic m-gon maps for the proof of the invariance under handle slides. Note that the the choice of the algebra is very crucial in the proof. See the proof of Theorem 5.11 of [1] for details. □

Therefore, for any $\mathbb{A}$-module $\mathbb{B}$, the chain homotopy type of the complex

$$\mathrm{CF}(X,\tau,\mathfrak{s};\mathbb{B})=\mathrm{CF}(X,\tau,\mathfrak{s})\otimes_{\mathbb{A}}\mathbb{B}$$

is an invariant of (X,τ) as well. We denote the homology of $\mathrm{CF}(X,\tau,\mathfrak{s};\mathbb{B})$ by $\mathrm{HF}(X,\tau,\mathfrak{s};\mathbb{B})$. Moreover, if $\mathbb{B}$ is also filtered by $\mathbb{H}$ and the action of $\mathbb{A}$ on $\mathbb{B}$ respects the filteration of the monomials of $\mathbb{A}$ by elements of $\mathbb{H}$, the complex is equipped with a filtration by $\mathrm{Spin}^c(X,\tau)$. If this is the case,

$$\mathrm{HF}(X,\tau,\mathfrak{s};\mathbb{B}) = \bigoplus_{\underline{\mathfrak{s}}\in\mathfrak{s}\subset\mathrm{Spin}^c(X,\tau)} \mathrm{HF}(X,\tau,\underline{\mathfrak{s}};\mathbb{B}).$$

For example, we may consider the the coefficient ring $\mathbb{B}_\tau$ which is equal to the quotient of $\mathbb{A}_\tau$ by *homologically trivial monomials*, i.e.,

$$\mathbb{B}_\tau = \frac{\mathbb{Z}[\kappa]}{\left\langle \prod_{i=1}^{\kappa} \mathfrak{u}_i^{n_i} \neq 1 \mid n_i \in \mathbb{Z}^{\geq 0} \ \& \ \sum_{i=1}^{\kappa} n_i[\tau_i] = 0 \text{ in } \mathrm{H}_1(X;\mathbb{Z})/\mathrm{Tors}\right\rangle}.$$

Thus the homotopy type of the complex $\mathrm{CF}(X,\tau,\mathfrak{s};\mathbb{B})$ is also an invariant of (X,τ). Moreover, multiplication by homologically trivial monomials respects the filtration by relative Spin^c structures. Therefore it has the structure of a $(\mathbb{B}_\tau,\mathbb{H})$ filtered chain complex. Using this coefficient ring we obtain a refinement of Theorem 1.4 from [9] as follows.

Proposition 3.2. *An irreducible balanced sutured manifold (X,τ) is taut if and only if the filtered $(\mathbb{B}_\tau,\mathbb{H})$ chain homotopy type of the complex $\mathrm{CF}(X,\tau;\mathfrak{s};\mathbb{B}_\tau)$ is non-trivial for some $\mathfrak{s}\in\mathrm{Spin}^c(\overline{X})$.*

4. Examples

In this section we explain the above construction in some special cases. Furthermore, we show that the sutured Floer chain complex generalizes Juhász's sutured Floer homology and Ozsváth-Szabó chain complexes for closed 3-manifold, knots and links.

Example 4.1. Let (X,τ) be a balanced sutured manifold in the sense of [9]. In this case, $\mathbb{Z}$ has an $\mathbb{A}$-module structure by letting nontrivial monomials of $\mathbb{A}$ act trivially on $\mathbb{Z}$ and may thus be regarded as a coefficient ring. For $\underline{\mathfrak{s}}\in\mathrm{Spin}^c(X,\tau)$, the sutured Floer homology of Juhász is recovered as

$$\mathrm{SFH}(X,\tau,\underline{\mathfrak{s}}) = H_*(\mathrm{CF}(X,\tau,\underline{\mathfrak{s}};\mathbb{Z}),\partial) = \mathrm{HF}(X,\tau,\underline{\mathfrak{s}};\mathbb{Z}).$$

Example 4.2. Let Y be a closed, oriented three-manifold. Associated with Y we may construct the sutured manifold $(X,\tau=\{\tau_1,...,\tau_n\}) = Y(n)$ by removing n disjoint balls from Y and placing a suture on each one of the resulting sphere boundary components. In this case we have

$$k=l=n,\quad g_i^+ = g_i^- = 0 \ \text{ and } \ \mathfrak{u}_i^+ = \mathfrak{u}_i^- = \mathfrak{u}_i,\quad \text{for } i=1,...,n.$$

Therefore, the algebra $\mathbb{A}$ is equal to $\mathbb{Z}[\mathfrak{u}_1,...,\mathfrak{u}_n]$. Furthermore, any Heegaard diagram corresponding to (X,τ) is a multi-pointed Heegaard diagram for Y. Thus we recover the multi-pointed Ozsváth-Szabó complex associated with the closed three-manifold Y. Each $\mathfrak{u}_i$ is a homologically trivial monomial, and the action of $\{\mathfrak{u}_i\}_i$ on the complex corresponds to the so-called U-action. Note that the above construction gives a complex with $\mathbb{Z}$ rather that $\mathbb{Z}/2\mathbb{Z}$ coefficients and refines the Ozsváth and Szabó chain complex in [16].

Example 4.3. Let L be a n-component link inside a closed three-manifold Y. Associated with the link L we may construct the sutured manifold $(X, \tau = \{\tau_1, ..., \tau_{2n}\}) = Y(L)$ by removing a neighbourhood of L from Y and placing a pair of parallel sutures $\{\tau_{2i-1}, \tau_{2i}\}$ on the i-th torus component of the boundary for any $1 \leq i \leq n$. Furthermore, we may assume τ_{2i-1} and τ_{2i} are oppositely oriented meridians for the i-th component of L. With the above notation,

$$k = l = n, \quad g_i^+ = g_i^- = 0 \quad \text{and} \quad \mathfrak{u}_i^+ = \mathfrak{u}_i^- = \mathfrak{u}_{2i-1}\mathfrak{u}_{2i}, \quad \text{for } i = 1, ..., n.$$

Thus, the algebra $\mathbb{A}$ is equal to $\mathbb{Z}[\mathfrak{u}_1, \mathfrak{u}_2, ..., \mathfrak{u}_{2n}]$. Furthermore, any Heegaard diagram corresponding to (X, τ) is a link Heegaard diagram as in [16] for the link L inside Y. Once again, this generalizes the construction of [16] in several ways (including the fact that the coefficient ring is improved to $\mathbb{Z}$). In particular, when L is a knot (i.e., $n = 1$), the algebra is equal to $\mathbb{Z}[\mathfrak{u}_1, \mathfrak{u}_2]$ and the sutured Floer chain complex is equal to the minus version of the knot Floer chain complex. Note that multiplication by the homologically trivial monomial $\mathfrak{u}_1\mathfrak{u}_2$ gives the U-action on the knot Floer complex.

Example 4.4. Let K be a homologically trivial knot inside a closed three-manifold Y, and let S be a connected Seifert surface for K of genus g. We may construct a sutured manifold $(X, \tau = \{\tau_1\})$ associated with the Seifert surface S by removing a product neighbourhood of S from Y and adding a copy of K as the single suture on the boundary of $Y - \mathrm{nd}(S)$. In this case we have

$$k = l = 1, \quad g_1^+ = g_1^- = g \quad \text{and} \quad \mathfrak{u}_1^+ = \mathfrak{u}_1^- = \mathfrak{u}_1.$$

If K is the unknot and $g = 0$, we will have $\mathbb{A} = \mathbb{Z}[\mathfrak{u}_1]$ and we will recover the Ozsváth-Szabó chain complex associated with the closed three-manifold Y. But if K is not the unknot and $g > 0$, the corresponding algebra would be $\mathbb{Z}$ and we recover the complex of Juhász. However, a slightly stronger version of the construction allows us to use the coefficient ring $\mathbb{Z}[\mathfrak{u}_1]/\langle \mathfrak{u}_1^2 \rangle$ when the genus g is bigger than 1. Considering the improved algebra the following question seems interesting.

Question. Would the improved invariant help us distinguish different Seifert surfaces of the same knot?

5. Properties

5.1. Simple stabilization

By a *simple stabilization* on a sutured manifold (X, τ) we mean adding two parallel oriented simple closed curves $-\tau_{\kappa+1}$ and $\tau_{\kappa+2}$ to the sutures in a genus zero connected component of $\mathfrak{R}(\tau)$ in a specific way. We assume that $-\tau_{\kappa+1}$ and $\tau_{\kappa+2}$ are inside $R_l^+ \subset \mathfrak{R}^+(\tau)$ and parallel to the suture $\tau_\kappa \in \tau$ in the common boundary of R_l^+ with $R_k^-(\tau) \subset \mathfrak{R}^-(\tau)$, which is also a genus zero component. Moreover, τ_κ and $\tau_{\kappa+1}$ bound an annulus R^+ in $\mathfrak{R}^+(\widehat{\tau})$ while $\tau_{\kappa+1}$ and $\tau_{\kappa+2}$ bound an annulus $R^- \subset \mathfrak{R}^-(\widehat{\tau})$, where $\widehat{\tau} = \tau \cup \{\tau_{\kappa+1}, \tau_{\kappa+2}\}$, as illustrated in Figure 1.

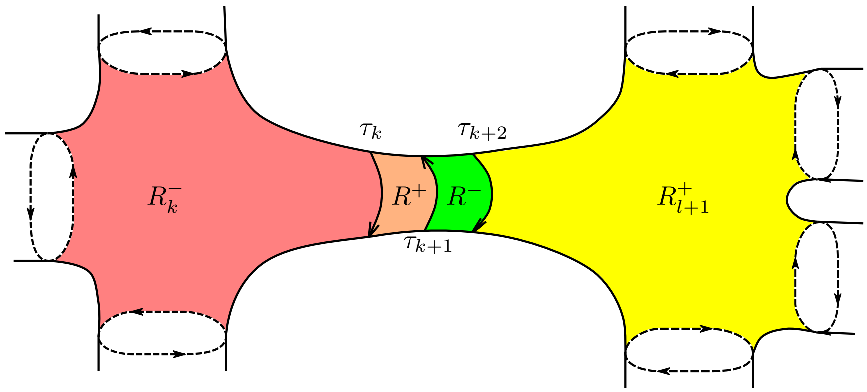

Figure 1. The simple stabilization of a sutured manifold. Here we have $R_l^+ = R_{l+1}^+ \coprod R^- \coprod R^+$.

The stabilization theorem of [16] is generalized as follows, while the ring of coefficients is improved from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$.

Theorem 5.1. *Let (X, τ) be a weakly balanced sutured manifold. With the above notation fixed, for any given* Spin^c *class*

$$\mathfrak{s} \in \mathrm{Spin}^c(\overline{X}^{\widehat{\tau}}) = \mathrm{Spin}^c(\overline{X}),$$

the filtered chain homotopy type of the complex $\mathrm{CF}(X, \widehat{\tau}, \mathfrak{s})$ *is the same as the filtered chain homotopy type of the chain complex obtained by equipping the module*

$$\mathrm{CF}(X, \tau, \mathfrak{s}; \mathbb{A}_{\widehat{\tau}}) \oplus \mathrm{CF}(X, \tau, \mathfrak{s}; \mathbb{A}_{\widehat{\tau}})$$

with the differential

$$\widehat{\partial} = \begin{pmatrix} \partial_\tau & \mathfrak{u}_{\kappa+1} - \mathfrak{u} \\ \mathfrak{u}_\kappa - \mathfrak{u}_{\kappa+2} & -\partial_\tau \end{pmatrix}, \qquad \text{where} \qquad \mathfrak{u} := \prod_{\tau_\kappa \neq \tau_i \in \partial R_l^+} \mathfrak{u}_i \in \mathbb{A}_{\widehat{\tau}}.$$

Proof. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$ be a Heegaard diagram associated with (X, τ). We may introduce a Heegaard diagram $(\Sigma, \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \hat{\mathbf{z}})$ for the stabilized sutured manifold $(X, \hat{\tau})$ as the connected sum

$$\left(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z} \cup \{w\} - \{z_\kappa\}\right) \# \left(S, \alpha_{\ell+1}, \beta_{\ell+1}, \{v, z_\kappa, z_{\kappa+1}, z_{\kappa+2}\}\right),$$

where S is a sphere, v and w are the connected sum points and the Heegaard diagram $\left(S, \alpha_{\ell+1}, \beta_{\ell+1}, \{v, z_\kappa, z_{\kappa+1}, z_{\kappa+2}\}\right)$ is illustrated in Figure 2. Moreover, the marked point w belongs to the same domain as z_k in $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$. Note that $\mathbb{T}_{\hat{\alpha}} \cap \mathbb{T}_{\hat{\beta}} = (\mathbb{T}_\alpha \cap \mathbb{T}_\beta) \times \{x, y\}$. Therefore, we have a module splitting

$$\mathrm{CF}\left(\Sigma, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\mathbf{z}}, \mathfrak{s}\right) = C_x \oplus C_y,$$

where C_x and C_y are the submodules of $\mathrm{CF}\left(\Sigma, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\mathbf{z}}, \mathfrak{s}\right)$ generated by the intersection points containing x and y respectively. By choosing the point w sufficiently close to one of the β curves and stretching the connected sum neck we compute the differential of the chain complex $\mathrm{CF}\left(\Sigma, \widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, \widehat{\mathbf{z}}, \mathfrak{s}\right)$. To see the details of the proof one can refer to the proof of Proposition 7.5 of [1].

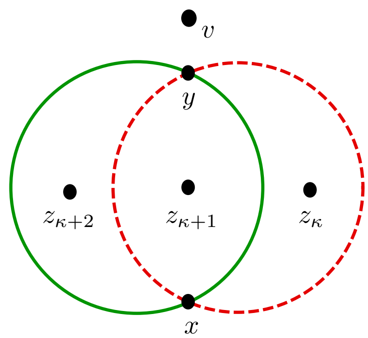

FIGURE 2. The marked point v is used as the connected sum point of the current diagram (on a Riemann sphere) with the old Heegaard diagram.

□

Remark 5.1. *Note that $\mathbb{A}_{\widehat{\tau}}$ does not admit the structure of a natural $\mathbb{A}_\tau$-module. Although the module $\mathrm{CF}(X, \tau, \mathfrak{s}; \mathbb{A}_{\widehat{\tau}})$, defined from an admissible Heegaard diagram for (X, τ), does not have the structure of a chain complex, the homomorphism*

$$\partial_\tau : \mathrm{CF}(X, \tau, \mathfrak{s}; \mathbb{A}_{\widehat{\tau}}) \longrightarrow \mathrm{CF}(X, \tau, \mathfrak{s}; \mathbb{A}_{\widehat{\tau}})$$

satisfies

$$\partial_\tau \circ \partial_\tau = (\mathfrak{u}^+(\tau) - \mathfrak{u}^-(\tau)) \cdot \mathrm{Id}.$$

Nevertheless, this suffices for $\widehat{\partial}$ to be a differential.

5.2. Connected sum formula

Let (X_1, τ^1) and (X_2, τ^2) be weakly balanced sutured manifolds. Fix the sutures $\sigma_i \in \tau^i$ for $i = 1, 2$. The connected sum of (X_1, τ^1) and (X_2, τ^2) along σ_1 and σ_2 is a sutured manifold (X, τ) where X is the boundary connected sum of X_1 and X_2 constructed by identifying disks $D_i \subset \partial X_i$ such that $D_1 \cap \mathfrak{R}^\bullet(\tau^1)$ is identified with $D_2 \cap \mathfrak{R}^\bullet(\tau^2)$ for $\bullet \in \{+, -\}$ and $\tau = (\tau^1 - \sigma_1) \cup \tau_0 \cup (\tau^2 - \sigma_2)$. Assume that D_i intersects $\sigma_i \in \tau^i$ in a single arc and remains disjoint from the other sutures in τ^i for $i = 1, 2$ and τ_0 is constructed by concatenating $\sigma_1 - (\sigma_1 \cap D_1)$ and $\sigma_2 - (\sigma_2 \cap D_2)$.

We generalize the Ozsváth-Szabó connected sum formula for closed 3-manifolds in [15] to a connected sum formula for sutured manifolds.

Theorem 5.2. *With the above notation fixed, for any pair of* Spin^c *structures* $\mathfrak{s}^i$ *in* $\mathrm{Spin}^c(\overline{X_i})$ *for* $i = 1, 2$ *the filtered chain homotopy type of the filtered* $(\mathbb{A}, \mathbb{H})$ *chain complexes*

$$\mathrm{CF}(X, \tau, \mathfrak{s}^1 \# \mathfrak{s}^2; \mathbb{A}) \quad \text{and} \quad \mathrm{CF}(X_1, \tau^1, \mathfrak{s}^1; \mathbb{A}) \otimes_{\mathbb{A}} \mathrm{CF}(X_2, \tau^2, \mathfrak{s}^2; \mathbb{A})$$

are the same, where $\mathbb{A} = \mathbb{A}(\tau^1, \tau^2; \sigma_1, \sigma_2)$.

The algebra $\mathbb{A}(\tau^1, \tau^2; \sigma_1, \sigma_2)$ is discussed in Theorem 9.5 of [1].

Proof. The proof is similar to the proof of Ozsváth and Szabó for the connected sum of closed three-manifolds.
We first study the special case when (X_2, τ^2) is a sutured manifold which corresponds to a special Heegaard diagram where each β curve is obtained by a small Hamiltonian isotopy on the corresponding α curve and cuts it in a pair of canceling intersection points. Using the top generator of $\mathrm{CF}(X_2, \tau^2; \mathbb{A})$ we define a filtered chain map from $\mathrm{CF}(X_1, \tau^1, \mathfrak{s}^1; \mathbb{A})$ to $\mathrm{CF}(X, \tau, \mathfrak{s}^1 \# \mathfrak{s}^0; \mathbb{A})$. This map and an appropriate holomorphic triangle map define a filtered chain map

$$\Gamma : \mathrm{CF}(X_1, \tau^1, \mathfrak{s}^1; \mathbb{A}) \otimes \mathrm{CF}(X_2, \tau^2, \mathfrak{s}^2; \mathbb{A}) \to \mathrm{CF}(X, \tau, \mathfrak{s}^1 \# \mathfrak{s}^2; \mathbb{A})$$

when (X_2, τ^2) is an arbitrary weakly balanced sutured manifold. Finally, with an appropriate energy filtration we show that Γ is a homotopy equivalence of filtered chain complexes. To see the details of the proof one can see the proof of Proposition 9.3 in [1]. $\square$

In the remainder of this paper we use the above connected sum formula to study a family of examples.

Example 5.2. Let G be a tree inside a closed, oriented 3-manifold X. Fix a coloring of the vertices $V(G)$ of G by $+$ and $-$ such that no two adjacent vertices share the same sign. Therefore, $V(G) = V^-(G) \coprod V^+(G)$. By adding isolated vertices to G we may assume that $\#|V^-(G)| = \#|V^+(G)|$. Note that we may turn G into a directed graph by putting a direction on each edge $e \in G$ such that it starts from an element of $V^-(G)$ and ends at an element of $V^+(G)$. Associated with the graph G inside X and its coloring we may construct the sutured manifold $(X(G), \tau(G))$ as follows. Let $X(G) = X - \mathrm{nhd}(G)$, while $\tau(G)$ is obtained by placing an oriented simple closed curve on the boundary of $X(G)$ corresponding to the meridian of each edge of G.

Fix a degree one vertex v of G, which is connected to $G - v$ by an edge e. Let 1 denote the connected path with two vertices. The 3-manifold $X(G)$ is obtained by removing a ball from X and its boundary is a 2-sphere. Therefore, it is easy to see that the weakly balanced sutured manifold $(X(G), \tau(G))$ is obtained as the connected sum of the sutured manifolds $(S^3(G), \tau(G))$ and $(X(1), \tau(1))$ along the sutures which correspond to the edge e and the single edge of the graph 1.

By Theorem 5.2, for $\mathfrak{s} \in \mathrm{Spin}^c(\overline{X(1)})$, the filtered chain homotopy type of the chain complex $\mathrm{CF}(X(G), \tau(G), \mathfrak{s})$ is the same as

$$\mathrm{CF}(X(1), \tau(1), \mathfrak{s}; \mathbb{A}_G) \otimes_{\mathbb{Z}[\mathfrak{u}]} \mathrm{CF}(S^3(G), \tau(G), \mathfrak{s}_0),$$

where $\mathfrak{s}_0$ is the unique Spin^c class in $\mathrm{Spin}^c(\overline{S^3(G)})$ and $\mathbb{A}_G$ is a $\mathbb{Z}[\mathfrak{u}]$-module via the algebra homomorphism sending $\mathfrak{u}$ to $\mathfrak{u}_e$. Consequently, $\mathrm{CF}(X(G), \tau(G), \mathfrak{s})$ only depends on $\mathrm{CF}(X(1), \tau(1), \mathfrak{s}) = \mathrm{CF}^-(X, \mathfrak{s})$ and the tree G.

Example 5.3. Let G be the connected path with four vertices in S^3. We may color its vertices with $+$ and $-$, and the number of $+$ vertices is equal to the number of $-$ vertices. The algebra associated with the sutured manifold $(S^3(G), \tau(G))$ is

$$\mathbb{A} = \frac{\mathbb{Z}[\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3]}{\langle (\mathfrak{u}_1 - \mathfrak{u}_3)(\mathfrak{u}_2 - 1) \rangle}.$$

We may construct a Heegaard diagram corresponding to $(S^3(G), \tau(G))$ as follows. Consider the three marked points z_1, z_2 and z_3 on the two-sphere S^2. Furthermore, let α and β be simple closed curves on S^2 which bound disks containing the marked points z_1, z_2 and z_2, z_3, respectively and intersect each other in two points $\{x, y\}$. It is easy to that $(S^2, \alpha, \beta, \mathbf{z} = \{z_1, z_2, z_3\})$ is a Heegaard diagram for $(S^3(G), \tau(G))$. We then have

$$\begin{aligned}
&\mathrm{CF}(S^2, \alpha, \beta, \mathbf{z}) = \mathbb{A}\langle x, y \rangle, \\
&\partial x = (\mathfrak{u}_1 - \mathfrak{u}_3) y, \\
&\partial y = (\mathfrak{u}_2 - 1) x.
\end{aligned}$$

Thus the chain homotopy type of $\mathrm{CF}(S^3(G), \tau(G), \mathfrak{s}_0)$ is non-trivial, while

$$\mathrm{SFH}(S^3(G), \tau(G), \mathfrak{s}_0) = 0.$$

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