Proceedings of 19^{th} Gökova
Geometry-Topology Conference
pp. 1 – 42

Lectures on the equivalence of Heegaard Floer and Seiberg–Witten Floer homologies

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Abstract. This article gives a detailed account of the lectures delivered by the author on the construction, in joint work with Yi-Jen Lee and Clifford H. Taubes, of an isomorphism between Heegaard Floer and Seiberg–Witten Floer homologies of closed, connected, and oriented 3-manifolds.

1. Introduction

Over the last three decades, many breakthrough ideas have emerged from gauge theory and symplectic geometry with applications to topology. In the early 1980s, Donaldson revolutionized our understanding of the smooth topology of 4-manifolds using solutions of the anti-self-dual Yang–Mills equations [6]. In the late 1980s, Floer introduced the idea of generalizing Morse homology to certain infinite dimensional spaces. In this vein, Floer developed what came to be known as *Lagrangian Floer homology* [9] which associates to a pair of transversally intersecting Lagrangian submanifolds inside a symplectic manifold a relatively graded module. He also introduced an invariant of integer homology 3-spheres from Yang–Mills gauge theory, called *instanton Floer homology* [8], which he conjectured to be isomorphic to the Lagrangian Floer homology of the character varieties arising from a Heegaard decomposition. A more general version of this conjecture is known as the *Atiyah–Floer Conjecture*. Meanwhile, Donaldson introduced polynomial invariants of smooth 4-manifolds [7]. Instanton Floer homology carries Donaldson's invariants into a topological quantum field theory framework. In general, a topological quantum field theory (TQFT) in dimension n associates a module to every closed, oriented, and smooth n-manifold, and to every compact, oriented, and smooth $(n+1)$-manifold with boundary it assigns an element in the module associated to its boundary. These assignments are subject to certain axioms laid out by Atiyah [1]. As a consequence of these axioms, a TQFT associates to every closed, oriented, and smooth $(n+1)$-manifold an element of the ground ring. Even though the latter perspective has become useful in understanding the structure of Donaldson's invariants, these are still fairly hard to compute. This is why Donaldson's invariants were largely abandoned after the introduction of Seiberg–Witten equations in the mid 1990s [29].

The author is supported by a National Science Foundation Postdoctoral Research Fellowship under Award No. DMS-1103795.

Much like in Donaldson's theory, solutions of the Seiberg–Witten equations are used to define invariants of closed, oriented, and smooth 4-manifolds, called the *Seiberg–Witten invariants*, and a TQFT-like structure exists within Seiberg–Witten gauge theory as well. Namely, Seiberg–Witten Floer homology, developed by Kronheimer and Mrowka [17], associates to a given closed, connected, oriented, and smooth 3-manifold M three "graded" Abelian groups denoted by $\overline{HM}_*(M)$, $\widehat{HM}_*(M)$, and $\widetilde{HM}_*(M)$. These are invariants of smooth 3-manifolds, and they can be used to recapture, through homomorphisms induced by cobordisms, the Seiberg–Witten invariants of closed, oriented, and smooth 4-manifolds.

Even though the Seiberg–Witten invariants are more user-friendly than Donaldson's invariants, these are also defined using geometric PDEs. Therefore, finding efficient ways to compute these invariants has been a challenging task. Twelve years ago, in order to give a differential topological description of the Seiberg–Witten invariants, and motivated by the Atiyah–Floer Conjecture, Ozsváth and Szabó developed Heegaard Floer homology [26, 27]. Heegaard Floer homology is an invariant of closed, connected, oriented, and smooth 3-manifolds which appears as a version of Lagrangian Floer homology arising from a Heegaard decomposition of the 3-manifold. Similar to Seiberg–Witten Floer homology, Heegaard Floer homology of a closed, connected, oriented, and smooth 3-manifold M has three flavors, denoted by $HF^\infty(M)$, $HF^-(M)$, and $HF^+(M)$, and these are used to define invariants of closed, oriented, and smooth 4-manifolds.

Despite their different origins, Heegaard Floer homology and Seiberg–Witten Floer homology have the same formal properties—such as the existence of a module structure over the ring $\mathbb{Z}[U] \otimes \wedge^*(H_1(M;\mathbb{Z})/torsion)$ and a long-exact sequence relating the three flavors—and they yield identical results where they can both be calculated; for example,

$$HF^\infty(S^3) \cong \mathbb{Z}[U, U^{-1}] \cong \overline{HM}_*(S^3), \quad HF^-(S^3) \cong \mathbb{Z}[U] \cong \widehat{HM}_*(S^3),$$

$$HF^+(S^3) \cong \mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U] \cong \widetilde{HM}_*(S^3).$$

Furthermore, the groups in each theory decompose into a direct sum over the set of isomorphism classes of Spin^c structures on M, which is a $H^2(M;\mathbb{Z})$-torsor. Ozsváth and Szabó conjectured that these two invariants are in fact the same. Our main result settles this conjecture:

Theorem 1.1 (Main Theorem in [18])**.** *Let M be a closed, connected, and oriented 3-manifold and $\mathfrak{s}$ be a Spin^c structure on M. Then, there exists a commutative diagram*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & HF^-(M,\mathfrak{s}) & \longrightarrow & HF^\infty(M,\mathfrak{s}) & \longrightarrow & HF^+(M,\mathfrak{s}) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & \widehat{HM}_*(M,\mathfrak{s},c_b) & \longrightarrow & \overline{HM}_*(M,\mathfrak{s},c_b) & \longrightarrow & \widetilde{HM}_*(M,\mathfrak{s},c_b) & \longrightarrow \cdots \end{array}$$

where the top and the bottom rows are the standard long-exact sequences for Heegaard Floer homology and Seiberg–Witten Floer homology with balanced perturbations, while the vertical arrows are isomorphisms that preserve the relative gradings and intertwine the $\mathbb{Z}[U] \otimes \wedge^*(H_1(M;\mathbb{Z})/torsion)$-module structures.

The precise form of the conjectured relationship between Heegaard Floer and Seiberg–Witten Floer homologies, as stated in the above theorem, was first presented by Lee in [23]. Furthermore, she outlined a proof of the above theorem which essentially follows the same line of thought described in this article. The main difference, and the technical difficulty Lee ran into at the time, lies where one needs to handle the problems arising from the indefinite critical points of a self-indexing Morse function that is used to obtain a Heegaard decomposition of the 3-manifold.

The purpose of this expository article is to explain the content of the above theorem and to give an overview of its proof, which is joint work with Lee and Taubes, based on lectures delivered by the author at the 19th Gökova Geometry and Topology Conference. It is intended to serve as a guide for those who would like to read [19, 20, 21, 22] for more details. Our approach in the proof is to exploit the relationship between Seiberg–Witten theory and the theory of pseudo-holomorphic curves as established by Taubes. Roughly speaking, this relationship is based on the observation that, in a neighborhood of a pseudo-holomorphic curve, smooth sections of a particular line bundle belonging to solutions of the Seiberg–Witten equations are very close to being holomorphic and therefore their zero loci can be used to approximate the pseudo-holomorphic curve through some limit as the size of canonically chosen perturbations get large. That said, the construction of the isomorphisms in Theorem 1.1 requires the use of an auxiliary manifold Y and a particular type of geometry on it. The latter is used to define a twisted version of Hutchings's embedded contact homology (ECH), which we denote by *ech*. The proof of Theorem 1.1 is then broken into three parts. This is explained schematically as follows:

$$\begin{array}{ccc} \textit{ech} \text{ of } Y & \leftrightarrow & \text{Seiberg} - \text{Witten Floer cohomology of } Y \\ \updownarrow & & \updownarrow \\ \text{Heegaard Floer homology of } M & \cong & \text{Seiberg} - \text{Witten Floer homology of } M \end{array}$$

The organization of this paper is as follows: Section 2 describes the auxiliary manifold Y and its geometry. Section 3 defines *ech* and outlines the proof of its equivalence with Heegaard Floer homology. Section 4 describes a twisted version of Seiberg–Witten Floer cohomology and explains its equivalence with *ech*. Section 5 discuses a connected sum formula for Seiberg–Witten Floer homology and how it is used to relate the Seiberg–Witten Floer cohomology of Y with the Seiberg–Witten Floer homology of M. Finally, in Section 6, we conclude with some remarks.

Remark 1.2. Colin, Ghiggini, and Honda recently proved that ECH is isomorphic to $HF^+(-M)$ using open book decompositions [2, 3, 4, 5]. Together with Taubes's recent result on the equivalence of Seiberg–Witten Floer cohomology and ECH [34, 35, 36, 37, 38], their result provides an alternative way to see that $\widetilde{HM}_*(M)$ and $HF^+(M)$ are isomorphic. In fact, it might be possible to see, more directly, how the two approaches compare if one implements the construction described in the next section using a Heegaard diagram arising from an open book decomposition adapted to a contact structure. This way, one could try to relate the ECH and *ech* chain complexes in a natural way.

2. The auxiliary manifold and its geometry

Let M be a closed, connected, and oriented 3-manifold. We start by choosing a self-indexing Morse function $f : M \to \mathbb{R}$ with a single pair of definite critical points. In other words, f is a Morse function whose value at a critical point is equal to the index of that critical point. Such a function yields a Heegaard decomposition $M = U_0 \cup_\Sigma U_1$ where $U_0 = f^{-1}[0, 3/2]$ and $U_1 = f^{-1}[3/2, 3]$ are genus-G handlebodies, and Σ is a closed, connected, and oriented surface of genus G oriented as ∂U_0. Next, fix a gradient-like vector field for f, that is, a smooth vector field $\mathfrak{v}$ defined in the complement of the critical point set of f satisfying $df(\mathfrak{v}) > 0$. We rescale $\mathfrak{v}$ so as to satisfy $df(\mathfrak{v}) = 1$. The flow lines of $\mathfrak{v}$ that emanate from index-1 critical points of f intersect the surface Σ in G disjoint simple closed curves $\mathbf{A} = \{A_1, \ldots, A_{\mathrm{G}}\}$. Similarly, the flow lines of $-\mathfrak{v}$ that emanate from index-2 critical points of f intersect the surface Σ in G disjoint simple closed curves $\mathbf{B} = \{B_1, \ldots, B_{\mathrm{G}}\}$. We further choose $\mathfrak{v}$ so that $A_i \pitchfork B_j$ for each $i, j \in \{1, \ldots, \mathrm{G}\}$. Then the ordered tuple $(\Sigma, \mathbf{A}, \mathbf{B})$, called a Heegaard diagram, determines the manifold M up to diffeomorphism.

A Heegaard diagram for M is part of the data that is required to define the Heegaard Floer homology groups. The remaining part of the data consists of a base point and a Spin^c structure. A base point is a fixed point $z \in \Sigma \setminus \bigcup(\mathbf{A} \cup \mathbf{B})$. A Heegaard diagram with a base point is called a *pointed Heegaard diagram.* The relevance of a pointed Heegaard diagram can be explained as follows: a Spin^c structure on M, in the sense of Turaev [40], is a nowhere vanishing vector field on M up to homotopy outside a 3-ball. Since a base point lies outside all the $\mathbf{A}$ and $\mathbf{B}$ curves, there exists a unique integral curve η_0 of $\mathfrak{v}$ that connects the definite critical points of f and intersects Σ transversally at this base point. On the other hand, any G-tuple of intersection points $\{x_1, \ldots, x_{\mathrm{G}}\}$, where $x_i \in A_i \cap B_{\sigma(i)}$ for σ a permutation of $\{1, \ldots, \mathrm{G}\}$, gives rise to a G-tuple of integral curves $\{\eta_1, \ldots, \eta_{\mathrm{G}}\}$ of $\mathfrak{v}$ connecting pairs of indefinite critical points of f. One can change $\mathfrak{v}$ in a neighborhood of the curves $\eta_0, \eta_1 \ldots, \eta_{\mathrm{G}}$ so that the resulting vector field is nowhere vanishing. In this way, one assigns a Spin^c structure to $\{x_1, \ldots, x_{\mathrm{G}}\}$. As a result, the set of G-tuples of intersection points is partitioned into Spin^c equivalence classes.

Now, let $\mathfrak{s}$ be a Spin^c structure on M. The $\mathbf{A}$ and $\mathbf{B}$ curves divide the surface Σ into regions $\{\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_N\}$ where $\mathcal{D}_0$ denotes the region containing the base point z. A 2-chain $\mathcal{P}$ that is an integer linear combination of $\mathcal{D}_1, \ldots, \mathcal{D}_N$ and whose boundary is an integer linear combination of $\mathbf{A}$ and $\mathbf{B}$ curves is called a *periodic domain.* Every periodic domain $\mathcal{P}$ determines an element $H(\mathcal{P}) \in H_2(M; \mathbb{Z})$, and every element of $H_2(M; \mathbb{Z})$ can be represented by a periodic domain. A pointed Heegaard diagram $(\Sigma, \mathbf{A}, \mathbf{B}, z)$ is called *strongly $\mathfrak{s}$-admissible* if for every periodic domain $\mathcal{P}$ with $\langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle = 2n \geq 0$, there exists a coefficient in $\mathcal{P}$ larger than n. This condition ensures that there are finitely many pseudo-holomorphic curves, up to translation, that enter into the definition of the Heegaard Floer differential. The following lemma claims that strong admissibility is equivalent to existence of a certain area form on Σ.

Lemma 2.1 (Lemma 1.1 in [19]). *A pointed Heegaard diagram* $(\Sigma, \mathbf{A}, \mathbf{B}, z)$ *is strongly* $\mathfrak{s}$*-admissible if and only if there exists an area form* w_Σ *on* Σ *such that*

- $\int_\Sigma w_\Sigma = 2$,
- $\int_\mathcal{P} w_\Sigma = \langle c_1(\mathfrak{s}), H(\mathcal{P})\rangle$ *for each periodic domain* $\mathcal{P}$.

Proof. The proof is similar to the proof of Lemma 4.12 in [26], except here we use $\mathfrak{s}$-renormalized periodic domains. See also the proof of the second bullet in Lemma 5.3 in [25]. ☐

Our aim in this section is to outline the construction, as in Sections 1 and 2 of [19], of an auxiliary manifold Y and a particular kind of geometry starting from the data provided by the strongly $\mathfrak{s}$-admissible Heegaard diagram $(\Sigma, \mathbf{A}, \mathbf{B}, z)$. We start by describing the manifold Y. First, fix a pairing Λ of index-1 and index-2 critical points of f, i.e., each $\mathfrak{p} \in \Lambda$ is a pair $\mathfrak{p} = (p_1, p_2)$ where p_i is an index-i critical point of f. Assume, without loss of generality, that the pairing respects the indexing of the $\mathbf{A}$ and $\mathbf{B}$ curves. Use p to denote the index-0 critical point of f, and use q to denote the index-3 critical point of f. The Morse Lemma tells us that, for some $\delta_* \in (0, c_0^{-1})$ where $c_0 >> 1$, there exist coordinate balls centered at index-1 and index-2 critical points of f on which f can be written respectively as

$$\begin{aligned} f &= 1 + x^2 + y^2 - 2z^2, \\ f &= 2 - x^2 - y^2 + 2z^2, \end{aligned}$$

where $(x^2 + y^2 + z^2)^{\frac{1}{2}} \leq 10\delta_*$. Denote by (r_+, θ_+, ϕ_+) and (r_-, θ_-, ϕ_-) the spherical coordinates in coordinate balls respectively centered at an index-1 and an index-2 critical point of f. Fix $R > -100 \ln \delta_*$, and for each $\mathfrak{p} = (p_1, p_2) \in \Lambda$, consider the spherical shell $\mathcal{H}_\mathfrak{p}$ centered at the origin with coordinates (r, θ, ϕ) defined by $e^{-R}(7\delta_*)^{-1} \leq r \leq e^R 7\delta_*$. After deleting the $r_+ < e^{-2R}(7\delta_*)^{-1}$ and $r_- < e^{-2R}(7\delta_*)^{-1}$ parts of the respective coordinate balls centered at p_1 and p_2, identify the respective $e^{-2R}(7\delta_*)^{-1} \leq r_+ \leq 7\delta_*$ and $e^{-2R}(7\delta_*)^{-1} \leq r_- \leq 7\delta_*$ parts with $\mathcal{H}_\mathfrak{p}$ via

$$\begin{aligned} (r_+, \theta_+, \phi_+) &= (e^{-R} r, \theta, \phi), \\ (r_-, \theta_-, \phi_-) &= (e^{-R} r^{-1}, \pi - \theta, \phi). \end{aligned} \tag{1}$$

Similarly, there exist coordinate balls centered at p and q on which f can be written respectively as

$$\begin{aligned} f &= x^2 + y^2 + z^2, \\ f &= 3 - x^2 - y^2 - z^2, \end{aligned}$$

where $(x^2 + y^2 + z^2)^{\frac{1}{2}} \leq 10\delta_*$. Denote by (r_+, θ_+, ϕ_+) and (r_-, θ_-, ϕ_-) the spherical coordinates in coordinate balls respectively centered at p and q. Consider the spherical shell $\mathcal{H}_0$ with coordinates (r, θ, ϕ) defined by $e^{-R}(7\delta_*)^{-1} \leq r \leq e^R 7\delta_*$ centered at the origin. After deleting the $r_+ < e^{-2R}(7\delta_*)^{-1}$ and $r_- < e^{-2R}(7\delta_*)^{-1}$ parts of the respective coordinate balls centered at p and q, identify the respective $e^{-2R}(7\delta_*)^{-1} \leq r_+ \leq 7\delta_*$ and

$e^{-2R}(7\delta_*)^{-1} \le r_- \le 7\delta_*$ parts with $\mathcal{H}_0$ via (1). The resulting manifold Y is diffeomorphic to $M\#_{\mathrm{G}+1}S^1 \times S^2$, the connected sum of M with $\mathrm{G}+1$ copies of $S^1 \times S^2$.

Next, we describe the particular type of geometry on Y that will enable us to establish the equivalence between Heegaard Floer and Seiberg–Witten Floer homologies.

2.1. A stable Hamiltonian structure on Y

Our aim is to construct a geometric structure on Y that will be consistent with the Heegaard Floer data on M. As Taubes has already established a relationship between Seiberg–Witten Floer cohomology and embedded contact homology, the appropriate sort of geometric structure here is a stable Hamiltonian structure. A stable Hamiltonian structure on Y is a pair (a, w) of a smooth 1-form a and a closed 2-form w on Y such that

- $da = \mathrm{H}w$ for a smooth function $\mathrm{H} : Y \to \mathbb{R}$,
- $a \wedge w$ is a volume form on Y.

On one end of a spectrum, contact structures are an example of stable Hamiltonian structures with a being a contact 1-form and $w = da$, while surface bundles over the circle are another example on the other end. The stable Hamiltonian structure we will describe on Y combines the two.

We start by defining the stable Hamiltonian structure on $\mathcal{H}_{\mathfrak{p}}$ for each $\mathfrak{p} \in \Lambda$. In order to do this, first introduce a new variable by $u = \ln r$ where $u \in [-2R+\ln 7\delta_*, 2R-\ln 7\delta_*]$. Then fix positive numbers $\delta < c_0^{-1}\delta_*$ and $\mathrm{x}_0 < \delta^3$. With δ and x_0 chosen, we may also need to increase the lower bound for the parameter R, namely, $R > -c_0 \ln \mathrm{x}_0$. Next, fix a smooth non-increasing function $\chi : \mathbb{R} \to [0, 1]$ that equals 1 on $(-\infty, 0]$ and equals 0 on $[1, \infty)$. Also define the following three functions:

$$\mathrm{x}(u) = \mathrm{x}_0\chi(|u| - R - \ln\delta - 12),$$
$$\chi_+(u) = \chi(-u - \frac{R}{4}), \qquad \chi_-(u) = \chi(u - \frac{R}{4})$$

whose graphs are illustrated in Figure 1.

Now, consider the following contact 1-form on $\mathcal{H}_{\mathfrak{p}}$:

$$\mathrm{a} = (1 - 3\cos^2\theta)du - \sqrt{6}\cos\theta\sin^2\theta d\phi^1,$$

and define a 1-form on $\mathcal{H}_{\mathfrak{p}}$ by

$$a = \mathrm{x}(u)\mathrm{a} - 2\sqrt{6}(\chi_+e^{2(u-R)} + \chi_-e^{-2(u+R)})\cos\theta\sin^2\theta d\phi + df_*, \tag{2}$$

where

$$f_* = (\chi_+e^{2(u-R)} + \chi_-e^{-2(u+R)})(1 - 3\cos^2\theta).$$

With the choice of the parameters δ, x_0, and R as above, $a \wedge da$ is non-vanishing where $|u| \le R + \ln \mathrm{x}_0 - c_0$. Hence, we set $w = da$.

[1] This is the contact 1-form that appears on the boundary of a tubular neighborhood of a zero circle for a near-symplectic structure (see [32], also [12].)

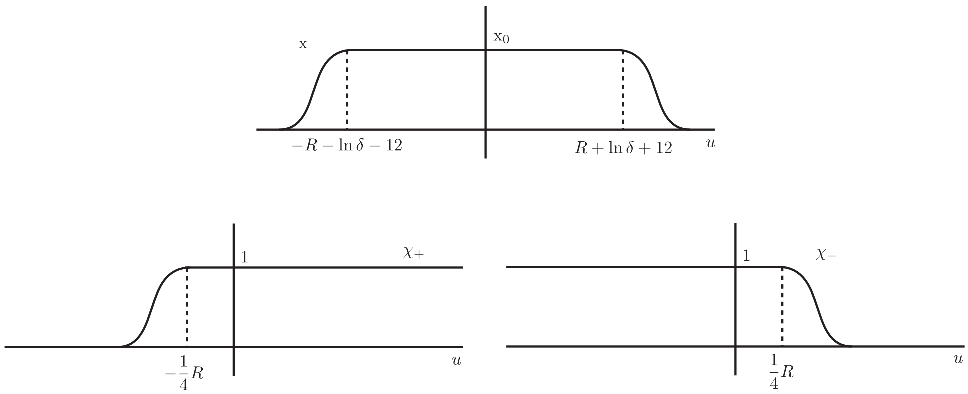

FIGURE 1. The functions x, χ_-, and χ_+.

In what follows, we describe the pair (a, w) on the complement in M of radius δ coordinate balls around all critical point of f. Denote the latter manifold by M_δ. For each index-1 or index-2 critical point of f, introduce new variables by $u_\pm = \ln r_\pm$. We will use an area form w_Σ on Σ, but with the opposite of the orientation of ∂U_0, supplied by Lemma 2.1 to construct the 2-form w on $f^{-1}([1,2]) \cap M_\delta$. To be more precise, use the integral curves of $\mathfrak{v}$ to identify $f^{-1}(1,2)$ with $(1,2) \times \Sigma$. Then pull back w_Σ via the projection from $[1,2] \times \Sigma$ onto Σ, and use the aforementioned identification to define a 2-form on $f^{-1}([1,2]) \cap M_\delta$. The latter agrees with w on the $\ln\delta < u_\pm < \ln 7\delta_*$ part of a coordinate ball around a given index-1 or index-2 critical point of f, therefore it extends w on $\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_\mathfrak{p}$ to the union of the latter with $f^{-1}([1,2]) \cap M_\delta$.

In order to extend the 1-form a to the union of $\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_\mathfrak{p}$ with $f^{-1}([1,2]) \cap M_\delta$, we need the following lemma.

Lemma 2.2 (Lemma 1.2 in [19]). *There exists a set of b_1, the first Betti number of M, points in $\Sigma \setminus \bigcup(\mathbf{A} \cup \mathbf{B})$ and a 1-form a_Σ defined on the complement of the base point z and these points such that*

- $da_\Sigma = w_\Sigma$,
- *The integral of a_Σ on each A_i or B_i is zero.*

Proof. We repeat the proof here for completeness. Consider the cohomology Mayer–Vietoris exact sequence for the Heegaard decomposition of M:

$$0 \to H^1(M;\mathbb{R}) \xrightarrow{\varphi} H^1(\Sigma;\mathbb{R}) \xrightarrow{\psi} Hom_\mathbb{Z}(F_{\mathbf{A}\cup\mathbf{B}}, \mathbb{R}) \xrightarrow{\delta} H^2(M;\mathbb{R}) \to 0,$$

where $F_{\mathbf{A}\cup\mathbf{B}}$ denotes the free $\mathbb{Z}$-module generated by $\mathbf{A}\cup\mathbf{B}$. The map φ is simply induced by the restriction of cochains, ψ is the map that sends the cohomology class of a closed 1-form on Σ to the hom-dual of $F_{\mathbf{A}\cup\mathbf{B}}$ defined by integrating this closed 1-form on curves in $\mathbf{A}\cup\mathbf{B}$, and $\delta(T)$ for $T \in Hom_\mathbb{Z}(F_{\mathbf{A}\cup\mathbf{B}}, \mathbb{R})$ is defined by $\delta(T)(H(\mathcal{P})) = T(\partial\mathcal{P})$. With the preceding understood, there exists a smooth 1-form a_z defined in the complement of the point z such that $w_\Sigma|_{\Sigma\setminus\{z\}} = da_z$, since $H^2(M\setminus\{z\};\mathbb{R})\cong 0$. Integrating a_z along curves in $\mathbf{A}\cup\mathbf{B}$ defines an element T_{a_z} of $Hom_{\mathbb{Z}}(F_{\mathbf{A}\cup\mathbf{B}},\mathbb{R})$. If $\delta(T_{a_z})=0$, then by exactness of the above sequence, there exists a closed 1-form on Σ such that the integral of the sum of the latter and a_z vanishes on curves in $\mathbf{A}\cup\mathbf{B}$. Any given $(\mathrm{R}_1,\dots,\mathrm{R}_N)\in\mathbb{R}^N$ gives rise to an element of $H^2(M;\mathbb{R})$ by sending a homology class represented by a periodic domain $\mathcal{P}=\sum_{i=1}^N c_i\mathcal{D}_i$ to $\sum_{i=1}^N c_i\mathrm{R}_i$. Therefore, we have a surjective homomorphism $\Phi:\mathbb{R}^N\to H^2(M;\mathbb{R})$. Now, choose a b_1-dimensional subspace of $\mathbb{R}^N$ whose restriction to Φ is an isomorphism. Fix points $\{z_i\}_{i=1,\dots,b_1}$ in the interiors of the corresponding regions. Denote by R_i the integral of a_z on the boundary of the region $\mathcal{D}_i$. As a result, $\delta(T_{a_z})=\Phi(\mathrm{R}_1,\dots,\mathrm{R}_N)$, and we can modify a_z in $\Sigma\setminus\{z,z_1,\dots,z_{b_1}\}$ so that the resulting 1-form a_z' has an associated $T_{a_z'}\in Hom_{\mathbb{Z}}(F_{\mathbf{A}\cup\mathbf{B}},\mathbb{R})$ with $\delta(T_{a_z'})=0$. □

With the 1-form a_Σ in hand, we can find a 1-form $\hat{a}_\Sigma$ on Σ that is zero near the base point z and the set of b_1 points as in Lemma 2.2, that agrees with a_Σ outside a collection of disjoint disks with small area centered at the same points, and $d\hat{a}_\Sigma=\mathrm{H}w_\Sigma$ for some smooth function $\mathrm{H}:\Sigma\to\mathbb{R}$ that equals 1 on the complement of these disks. Then Part 7 in Section 1c of [19] explains how to extend the 1-form a to the union of $\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}$ with $f^{-1}([1,2])\cap M_\delta$ so that its restriction to $f^{-1}([1,2])\cap M_\delta$ can be written as $\hat{a}_\Sigma+h(t)dt+d\varphi$ where t denotes the Euclidean coordinate on $[1,2]$, h is a function on $[1,2]$ that equals 1 near $t=1$ and $t=2$, and φ is a compactly supported t-dependent function on Σ. Furthermore, $da=\mathrm{H}w$ on $f^{-1}([1,2])\cap M_\delta$.

In order to extend (a,w) to the remainder of Y, first extend (a,w) to the union of $\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}$ with a neighborhood of $f^{-1}([1,2])\cap M_\delta$. This can be done by identifying neighborhoods of $\Sigma_1:=f^{-1}(1)\cap M_\delta$ and $\Sigma_2:=f^{-1}(2)\cap M_\delta$ with $[1-2\delta^2,1+\delta^2]\times\Sigma_1$ and $[2-\delta^2,2+2\delta^2]\times\Sigma_2$ respectively via the integral curves of $\mathfrak{v}$, since the latter is tangent to the boundary of the radius δ coordinate balls centered at index-1 and index-2 critical points of f. With the preceding understood, w can be extended so as to be equal to w_Σ on these neighborhoods. On the other hand, a can be extended by Lie transport via the gradient-like vector field $\mathfrak{v}$. Then a construction similar to the extension of (a,w) to the union of $\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}$ with $f^{-1}([1,2])\cap M_\delta$ extends the pair to $(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}})\cup M_\delta$. Finally, (a,w) extends to the union of $(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}})\cup M_\delta$ with $\mathcal{H}_0$ so as to restrict on $\mathcal{H}_0$ as follows:

- $a=2(\chi_+e^{2(|u|-R)}+\chi_-e^{-2(|u|+R)})du+\hat{a}_0$,
- $w=\sin\theta d\theta\wedge d\phi$,

where $\hat{a}_0$ is a certain smooth 1-form on S^2. To be more explicit, the latter is equal to $\frac{1}{2\pi}\sin\theta d\theta\wedge d\phi$ outside disks of small radius centered at points that are identified with the b_1+1 points in Lemma 2.2 via the flow of $\mathfrak{v}$.

We orient the manifold Y so that $a\wedge w>0$. Therefore, the inclusion of M_δ into Y is an orientation reversing embedding. By way of a summary, the stable Hamiltonian structure (a, w) and the associated Reeb vector field $\mathcal{R}$[2] have the following properties:

$$\left.\begin{array}{l}
\text{(a) On } \mathcal{H}_0,\ w = \sin\theta d\theta\wedge d\phi \text{ and } \mathcal{R} = \frac{1}{2(\chi_+ e^{2(u-R)}+\chi_- e^{-2(u+R)})}\frac{\partial}{\partial u}.\\
\text{(b) On } M_\delta,\ w \text{ restricts to each constant } f \text{ slice as an area form, and the vector}\\
\quad\ \text{field } \mathcal{R} \text{ on } M_\delta \text{ agrees with the gradient-like vector field } \mathfrak{v}.\\
\text{(c) Fix } \mathfrak{p}\in\Lambda. \text{ Then on } \mathcal{H}_\mathfrak{p}:\\
\quad - a = (\mathrm{x}+\mathfrak{g}')(1-3\cos^2\theta)du - \sqrt{6}\mathfrak{f}\cos\theta\sin^2\theta d\phi + 6\mathfrak{g}\cos\theta\sin\theta d\theta,\\
\quad - w = 6\mathrm{x}\cos\theta\sin\theta d\theta\wedge du - \sqrt{6}d(\mathfrak{f}\cos\theta\sin^2\theta d\phi),\\
\quad - \mathcal{R} = \frac{1}{(\mathfrak{f}(\mathrm{x}+\mathfrak{g})(1+9\cos^4\theta))^{\frac{1}{2}}}(\mathfrak{f}(1-3\cos^2\theta)\frac{\partial}{\partial u} - \sqrt{6}\mathrm{x}\cos\theta\frac{\partial}{\partial\phi} + \mathfrak{f}'\cos\theta\sin\theta\frac{\partial}{\partial\theta}),\\
\quad\ \text{where}\\
\qquad \mathfrak{f} = \mathrm{x} + 2(\chi_+ e^{2(u-R)} + \chi_- e^{-2(u+R)}), \qquad \mathfrak{g} = \chi_+ e^{2(u-R)} - \chi_- e^{-2(u+R)}.\\
\text{(d) Having written } H_2(Y;\mathbb{Z}) \cong H_2(M;\mathbb{Z})\oplus H_2(\mathcal{H}_0;\mathbb{Z})\oplus(\oplus_{\mathfrak{p}\in\Lambda}H_2(\mathcal{H}_\mathfrak{p};\mathbb{Z})), \text{ we}\\
\quad\ \text{have}\\
\quad - \langle [w],[S^2]\rangle = 2 \text{ for } [S^2] \text{ the positive generator of } H_2(\mathcal{H}_0;\mathbb{Z}),\\
\quad - \langle [w],[S^2]\rangle = 0 \text{ for } [S^2] \text{ the positive generator of } H_2(\mathcal{H}_\mathfrak{p};\mathbb{Z}) \text{ for some } \mathfrak{p}\in\Lambda,\\
\quad - \langle [w],F\rangle = \langle c_1(\mathfrak{s}),F\rangle \text{ for any } F\in H_2(M;\mathbb{Z}).
\end{array}\right\} \quad (3)$$

There is also a closed 1-form ν on Y which extends df on M_δ. To be more precise, the 1-form ν equals df on M_δ, it is given by $2(\chi_+ e^{2(|u|-R)} + \chi_- e^{-2(|u|+R)})$ on $\mathcal{H}_0$, and it is given by df_* on each $\mathcal{H}_\mathfrak{p}$. Furthermore, it satisfies $\nu\wedge w \geq 0$ with equality only where $u = 0$ and $1-3\cos^2\theta = 0$ on each $\mathcal{H}_\mathfrak{p}$. The 1-form ν is used to define an auxiliary 1-form $\hat{a}$ on Y as follows: let $\chi_\delta(u) = \chi(|u| - R - \ln\delta - 10)$. Then $\hat{a}$ is equal to ν on $M_\delta\cup\mathcal{H}_0$ and it is equal to $\chi_\delta a + (1-\chi_\delta)\nu$ on each $\mathcal{H}_\mathfrak{p}$. The 1-form $\hat{a}$ satisfies $\hat{a}(\mathfrak{v}) = 1$ on M_δ and $\hat{a}\wedge w > 0$.

2.2. Closed integral curves for the stable Hamiltonian structure

Here, we characterize closed integral curves of $\mathcal{R}$ that are relevant to our story. We describe these curves via their intersections with various parts of the manifold Y. In this regard, let K^{-1} denote the oriented 2-plane field that is the kernel of a and is oriented by w. It follows from properties (a) and (d) in (3) that the Euler class $e_{K^{-1}}$ of K^{-1} satisfies:

- $\langle e_{K^{-1}}, [S^2]\rangle = 2$ where $[S^2]$ is the positive generator of $H_2(\mathcal{H}_0;\mathbb{Z})$,
- $\langle e_{K^{-1}}, [S^2]\rangle = -2$ where $[S^2]$ is the positive generator of $H_2(\mathcal{H}_\mathfrak{p};\mathbb{Z})$ for some $\mathfrak{p}\in\Lambda$.

There exists a unique $\Gamma \in H_1(Y;\mathbb{Z})$ and a Spin^c structure $\underline{\mathfrak{s}}$ on Y which satisfy $c_1(\underline{\mathfrak{s}}) = e_{K^{-1}} + 2PD(\Gamma)$, and

- $\langle PD(\Gamma), [S^2]\rangle = 0$ where $[S^2]$ is the positive generator of $H_2(\mathcal{H}_0;\mathbb{Z})$,
- $\langle PD(\Gamma), [S^2]\rangle = 1$ where $[S^2]$ is the positive generator of $H_2(\mathcal{H}_\mathfrak{p};\mathbb{Z})$ for some $\mathfrak{p}\in\Lambda$,
- The Spin^c structure $\underline{\mathfrak{s}}$ on Y restricts to the Spin^c structure $\mathfrak{s}$ on M.

[2]This is the vector field that spans the kernel of w and is normalized by a, i.e., $\iota_\mathcal{R} w = 0$ and $a(\mathcal{R}) = 1$.

Since $c_1(\underline{\mathfrak{s}}) = e_{K^{-1}} + 2PD(\Gamma)$, the second bullet implies that $\underline{\mathfrak{s}}$ restricts to the trivial Spin^c structure on each $\mathcal{H}_{\mathfrak{p}}$. This is an important fact that will come up naturally as we proceed.

With $\Gamma \in H_1(Y;\mathbb{Z})$ as above, let $\mathcal{Z}_{ech}$ denote the set of all finite collections Θ of pairs (γ_i, m_i) where γ_i is a closed integral curve of $\mathcal{R}$, and m_i is a positive integer such that $\Gamma = \sum_i m_i[\gamma_i]$. Additional constraints are required to determine the set of generators for the embedded contact homology chain complex. In order to explain these constraints, we consider the linearized return map of the Reeb flow around a closed integral curve of $\mathcal{R}$. Having fixed a closed integral curve γ of $\mathcal{R}$ and a point $x_0 \in \gamma$, the latter is a linear map $P_{\gamma,x_0} : K^{-1}|_{x_0} \to K^{-1}|_{x_0}$ which can be regarded as an element of $SL(2,\mathbb{R})$ when a trivialization of K^{-1} over γ is chosen. The curve γ is called *non-degenerate* if no power of P_{γ,x_0} has an eigenvalue equal to 1. For a non-degenerate closed integral curve γ, either both eigenvalues of P_{γ,x_0} are real, in which case γ is called *hyperbolic*, or both eigenvalues are on the unit circle, in which case γ is called *elliptic*. The constraint on a closed integral curve γ_i to be a part of a generator for the embedded contact homology chain complex is that it is non-degenerate, and the constraint on m_i is that $m_i = 1$ if γ_i is hyperbolic. We shall denote by $\mathcal{Z}_{ech,M}$ the subset of $\mathcal{Z}_{ech}$ with these additional constraints. An element of the set $\mathcal{Z}_{ech,M}$ is called an *admissible orbit set*.

We now characterize closed integral curves of $\mathcal{R}$ that belong to a collection from $\mathcal{Z}_{ech}$. First of all, such a closed integral curve γ of $\mathcal{R}$ would have empty intersection with $\mathcal{H}_0$. Here is why: if γ were to intersect $\mathcal{H}_0$, then it follows from property (a) in (3) that it would do so as $[-R - \ln\delta_*, R + \ln\delta_*] \times \{\text{point} \in S^2\}$, and therefore has positive intersection with every constant u cross-sectional 2-sphere of $\mathcal{H}_0$ oriented by w. As a result, $\langle PD(\Gamma), [S^2]\rangle > 0$ where $[S^2]$ is the positive generator of $H_2(\mathcal{H}_0;\mathbb{Z})$, contrary to what is said in the first bullet above. In fact, γ lies in the union of $f^{-1}(1,2)$ and $\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}$ because otherwise γ would eventually have to intersect $\mathcal{H}_0$. Before we proceed, note that there exists a unique closed integral curve of $\mathcal{R}$ which intersects the Heegaard surface Σ at the base point z. We denote this closed integral curve by γ_z. It lies in $M_\delta \cup \mathcal{H}_0$ and intersects each level set of f as well as each constant u cross-sectional 2-sphere of $\mathcal{H}_0$ exactly once. The significance of the closed integral curve γ_z will be explained in the next section.

There are exactly two closed integral curves of $\mathcal{R}$ that sit entirely in $\mathcal{H}_{\mathfrak{p}}$ for each $\mathfrak{p} \in \Lambda$. These are denoted by $\gamma_{\mathfrak{p}}^+$ and $\gamma_{\mathfrak{p}}^-$, defined respectively by $u = 0$, $\cos\theta = \frac{1}{\sqrt{3}}$ and $u = 0$, $\cos\theta = -\frac{1}{\sqrt{3}}$ (see Figure 2). Both of these curves are hyperbolic and their associated linearized return maps have positive eigenvalues. On the other hand, if γ intersects $\mathcal{H}_{\mathfrak{p}}$ but does not lie entirely in it, then $\gamma\cap\mathcal{H}_{\mathfrak{p}}$ lies where $\cos^2\theta < \frac{1}{3}$, otherwise it would intersect $f^{-1}((0,1)\cup(2,3))\cap M_\delta$ and eventually $\mathcal{H}_0$. Moreover, it follows from property (c) in (3) that γ intersects each constant u cross-sectional sphere of $\mathcal{H}_{\mathfrak{p}}$ exactly once positively, and either $\cos\theta = 0$ or $0 < \cos^2\theta < \frac{1}{3}$ along $\gamma \cap \mathcal{H}_{\mathfrak{p}}$. In fact, if θ_+ and θ_- denote the polar angle coordinates at which γ intersects respectively the boundaries of the radius δ_* coordinate balls centered at the index-1 and the index-2 critical points of f belonging to $\mathfrak{p}$,

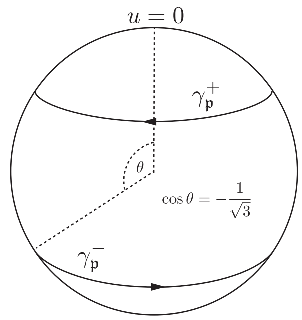

FIGURE 2. Integral curves that lie entirely in $\mathcal{H}_{\mathfrak{p}}$.

then $\theta_+ = \theta_-$; and the net change $\Delta\phi$ in value of the azimuthal angle coordinate as γ travels through the $|u| \le R + \ln\delta_*$ part of $\mathcal{H}_{\mathfrak{p}}$ can be computed via the integral

$$\Delta\phi = -\sqrt{6}\int_{[-R-\ln\delta_*, R+\ln\delta_*]} \frac{\mathrm{x}(u)}{\mathfrak{f}(u)}\frac{\cos\theta(u)}{1-3\cos^2\theta(u)}du,$$

where $\theta(u)$ is the unique solution of the equation

$$\cos\theta\sin^2\theta = 2{\delta_*}^2\cos\theta_\pm\sin^2\theta_\pm\frac{1}{\mathfrak{f}(u)}$$

with $1 - 3\cos^2\theta(u) > 0$.

Finally, we characterize the intersection of γ with M_δ. In this regard, let p_+ be an index-1 critical point and p_- be an index-2 critical point of f, and η denote a closed connected segment of γ that runs from the boundary of the radius δ coordinate ball centered at p_+ to the boundary of the radius δ coordinate ball centered at p_-. Suppose that $\eta \subset M_\delta$. Then η is in close proximity of an integral curve of $\mathfrak{v}$ that connects p_+ to p_-. The fundamental reason behind this is the fact that $\mathcal{R}$ agrees with $\mathfrak{v}$ in M_δ. With the preceding understood, a closed integral curve γ of $\mathcal{R}$ from a collection in $\mathcal{Z}_{ech}$ is broken into $2N$ closed connected segments $\{\eta^{\mathfrak{p}_1}, \eta_1, \dots, \eta^{\mathfrak{p}_N}, \eta_N\}$ where $1 \le N \le \mathrm{G}$ and

- $\eta^{\mathfrak{p}_i} = \gamma \cap \mathcal{H}_{\mathfrak{p}_i}$ is a connected segment that starts from the index-2 critical point end of $\mathcal{H}_{\mathfrak{p}_i}$ and stops at the index-1 critical point end of $\mathcal{H}_{\mathfrak{p}_i}$,
- η_i is a connected segment that lies in M_δ. It starts from the index-1 critical point end of $\mathcal{H}_{\mathfrak{p}_i}$ and stops at the index-2 critical point end of $\mathcal{H}_{\mathfrak{p}_{i+1}}$ for $1 \le i \le N-1$, while η_N connects the index-1 critical point end of $\mathcal{H}_{\mathfrak{p}_N}$ to the index-2 critical point end of $\mathcal{H}_{\mathfrak{p}_1}$.

(see Figures 3 and 4.) The above observations bring up the following definition to be used in the upcoming structure theorem for elements of the set $\mathcal{Z}_{ech,M}$. An *index 1-2 cycle* is a non-empty, finite, cyclically order set $\{\upsilon_1, \dots \upsilon_N\}$ consisting of closures of integral curves of $\mathfrak{v}$ that connect index-1 critical points of f to index-2 critical points of f in such a way that for $1 \le i \le N-1$ and the end point of υ_i and the starting point of

υ_{i+1} as well as the end point of υ_N the starting point of υ_1 define distinct pairs from Λ. The above observations point out that any closed integral curve of $\mathcal{R}$ that belongs to a collection from the set $\mathcal{Z}_{ech,M}$ specifies an index 1-2 cycle. Conversely, we have the following statement:

Proposition 2.3 (Proposition 2.7 in [19]). *Given an index 1-2 cycle $\{\upsilon_1, \ldots \upsilon_N\}$, the set of closed integral curves of $\mathcal{R}$ that correspond to it enjoy a 1-1 correspondence with the set $\mathbb{Z}^N$. To be more precise, a closed integral curve of $\mathcal{R}$ corresponding to $\{\upsilon_1, \ldots \upsilon_N\}$ undergoes a change of $\Delta\phi_i + 2\pi m_i$, where $0 \leq \Delta\phi_i < 2\pi$ and $m_i \in \mathbb{Z}$, in its azimuthal angle coordinate as it travels through $\mathcal{H}_{\mathfrak{p}_i}$. Moreover, all of these closed integral curves are non-degenerate, hyperbolic, and the sign of the eigenvalues for the corresponding linearized return maps are $(-1)^N$ times the product of the orientation signs of the integral curves $\{\upsilon_1, \ldots \upsilon_N\}$. These signs are determined by comparing the orientation of Σ with the orientation determined by the* **A** *and* **B** *curves. Two pairs of an index 1-2 cycle and an N-tuple of integers specify the same closed integral curve if and only if they differ by an action of a cyclic permutation of $\{1, \ldots, N\}$ on the indices.*

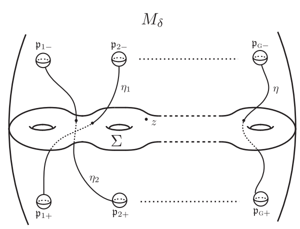

FIGURE 3. For each $i \in \{1, \ldots, \mathrm{G}\}$, the index-1 critical point $\mathfrak{p}_{i+}$ is paired with the index-2 critical point $\mathfrak{p}_{i-}$. Hence, the two segments on the left belong to a single closed integral curve, while the one on the right to another.

3. *ech* and its equivalence with Heegaard Floer homology

This section describes a twisted version of embedded contact homology for the stable Hamiltonian structure (a, w) on Y constructed from a strongly $\mathfrak{s}$-admissible Heegaard diagram $(\Sigma, \mathbf{A}, \mathbf{B}, z)$ arising from a self-indexing Morse function f on M as in the previous section, and justifies its equivalence with the Heegaard Floer homology of M.

3.1. The *ech* chain complex

The embedded contact homology chain complex is a free $\mathbb{Z}$-module generated by a set described as follows: Let Θ_0 be a 1-chain which represents Γ and does not intersect any closed integral curve from a collection in $\mathcal{Z}_{ech,M}$. Given $\Theta \in \mathcal{Z}_{ech,M}$, consider the relative homology classes of surfaces with boundary $\Theta - \Theta_0$. The latter is a

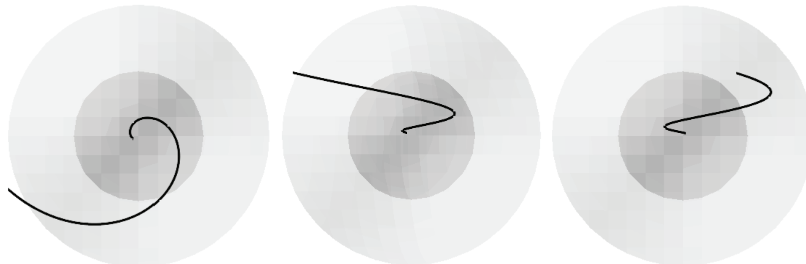

FIGURE 4. The intersection of a closed integral curve with a handle $\mathcal{H}_{\mathfrak{p}}$, as seen respectively (left to right) through the z-, y-, and x- axes.

$H_2(Y;\mathbb{Z})$-torsor, denoted by $H_2(Y,\Theta,\Theta_0)$. Next, consider pairs of the form (Θ, Z) consisting of $\Theta \in \mathcal{Z}_{ech,M}$ and $Z \in H_2(Y,\Theta,\Theta_0)$. Two such pairs (Θ,Z) and (Θ,Z') are deemed equivalent if $[Z-Z']\cdot[\gamma_z]=0$. We denote the set of equivalence classes under this relation by $\hat{\mathcal{Z}}_{ech,M}$. The latter admits an identification with $\mathcal{Z}_{ech,M}\times\mathbb{Z}$ via the surjection $H_2(Y,\Theta,\Theta_0)\to\mathbb{Z}$ sending Z to its algebraic intersection number with γ_z.

With the above understood, let ecc^{∞} denote the free $\mathbb{Z}$-module generated by $\hat{\mathcal{Z}}_{ech,M}$. We can define a submodule ecc^{-} as the one generated by the subset $\mathcal{Z}_{ech,M}\times\{\dots,-2,-1\}$ and denote the quotient of ecc^{∞} by ecc^{-} by ecc^{+}. Then, there exists a natural short-exact sequence of free $\mathbb{Z}$-modules:

$$0 \longrightarrow ecc^{-} \xrightarrow{i} ecc^{\infty} \xrightarrow{\pi} ecc^{+} \longrightarrow 0. \tag{4}$$

The relative grading and the differential on these free $\mathbb{Z}$-modules are defined with the help of a suitable almost complex structure J on $\mathbb{R}\times Y$. The latter is an automorphism of $T(\mathbb{R}\times Y)$ whose square is the negative of the identity, and it is subject to additional constraints. In this regard, let s denote the Euclidean coordinate on $\mathbb{R}$. Then,

- $J\frac{\partial}{\partial s}=\mathcal{R}$.
- J is invariant under translations along the $\mathbb{R}$-factor.
- J preserves the kernel of the 1-form $\hat{a}$, and is compatible with the restriction of w on this plane field.
- J is invariant under translations of the azimuthal angle coordinate ϕ on each $\mathcal{H}_{\mathfrak{p}}$.
- Let

$$e_1=-6\mathfrak{g}\cos\theta\sin\theta\frac{\partial}{\partial u}+(\mathrm{x}+\mathfrak{g}')(1-3\cos^2\theta)\frac{\partial}{\partial\theta},\ e_2=\frac{\partial}{\partial\phi}+\sqrt{6}\chi_\delta\mathfrak{f}\cos\theta\sin^2\theta\mathcal{R}.$$

Then $Je_1=\sigma^{-1}e_2$ with σ a positive function of u and θ on each $\mathcal{H}_{\mathfrak{p}}$. (5)

A few remarks are in order. First of all, for any $v\in\mathrm{Ker}(a)$ we have $v-\hat{a}(v)\mathcal{R}\in\mathrm{Ker}(\hat{a})$. Hence, $w(\cdot,J\cdot)$ is positive definite on the kernel of a. Furthermore, it follows from the latter that there exists $r\geq 1$ such that the 2-form $\omega=ds\wedge a+rw$ tames J. On the other hand, J is compatible with the 2-form $\hat{\omega} = ds \wedge \hat{a} + w$. In particular, $\hat{\omega}(\cdot, J\cdot)$ defines an s-independent Riemannian metric on $\mathbb{R} \times Y$.

Having fixed an almost complex structure J on $\mathbb{R}\times Y$ satisfying the above constraints, a J-holomorphic curve is a smooth map $u : (S, j) \to (\mathbb{R}\times Y, J)$ where (S, j) is a (punctured) compact Riemann surface and $du \circ j = J \circ du$. A J-holomorphic curve is called *somewhere injective* if there exists a point $x \in S$ such that $u^{-1}(u(x)) = \{x\}$ and the linear map $du|_x : TS|_x \to T(\mathbb{R} \times Y)|_{u(x)}$ is injective. A somewhere injective curve is an embedding in the complement of finitely many points, and is determined by its image. On the other hand, a J-holomorphic curve is called *multiply covered* if it can be written as the composition of a somewhere injective curve $u : (S', j') \to (\mathbb{R} \times Y, J)$ and a holomorphic (branched) covering map $\varphi : (S, j) \to (S', j')$. In either case, the image C is a closed subset of $\mathbb{R} \times Y$ with finite 2-dimensional Hausdorff measure having no totally disconnected components, and the complement of finitely many points in C is a submanifold with J-invariant tangent space. We impose the following additional constraints on C:

- The integral of w over C is finite.
- There is an s-independent bound for the integral of $ds \wedge a$ over $C \cap [s, s + 1] \times Y$ for any $s \in \mathbb{R}$.

The above two constraints together with the fact that ω tames J imply that constant s slices of C limit to a finite collection of closed integral curves of $\mathcal{R}$ as s tends to $\pm\infty$ (see [11] and [30]). To be more precise, there exists $s_0 \geq 1$ such that the $s > s_0$ part of C is a disjoint union of embedded cylinders on which the projection to the $\mathbb{R}$-factor has no critical points, similarly for the $s < -s_0$ part of C. Each such cylinder is called an *end* of C. It is called a *positive end* if it belongs to the $s > s_0$ part of C, and it is called a *negative end* if it belongs to the $s < -s_0$ part of C. Furthermore, constant s slices of each end of C converge pointwise to a closed integral curve of $\mathcal{R}$. With the preceding understood, we consider J-holomorphic curves whose images have ends at elements of $\mathcal{Z}_{ech,M}$. We will refer to such J-holomorphic curves as *admissible*. Hutchings defined the *ech* index for admissible J-holomorphic curves (see [13] and [14]). For a somewhere injective J-holomorphic curve, the *ech* index constitutes a lower bound for the Fredholm index of the associated deformation operator, and the two are equal if and only if the J-holomorphic curve is embedded. Given an admissible J-holomorphic curve whose image C has positive ends at $\Theta_+ \in \mathcal{Z}_{ech,M}$ and negative ends at $\Theta_- \in \mathcal{Z}_{ech,M}$, the *ech* index depends only on the relative homology class $[C] \in H_2(Y, \Theta_+, \Theta_-)$, and is denoted by $I([C])$. The *ech* index endows the *ech* chain complex with a relative $\mathbb{Z}$-grading that is well-defined modulo a non-negative integer p_M, the divisibility of $c_1(\mathfrak{s})$. For low *ech* index values Hutchings proved that admissible J-holomorphic curves consist of unions of embedded J-holomorphic curves together with possibly multiply covered $\mathbb{R}$-invariant cylinders, namely, $\mathbb{R} \times \Theta$ for $\Theta \in \mathcal{Z}_{ech,M}$. Therefore, from now on, we shall use the term J-holomorphic curve for finite collections of pairs $\{(C, n)\}$ where C is the image of a connected J-holomorphic curve in the above sense, and n is a positive integer indicating the covering multiplicity. The moduli space of J-holomorphic curves is endowed with the restriction of the weak*-topology on the space of currents. More precisely, given a J-holomorphic curve C, a neighborhood basis at C is indexed by pairs of the form (ϵ, ϖ) where ϵ is a positive real number and ϖ is a smooth 2-form on $\mathbb{R} \times Y$ such that a J-holomorphic curve C' is in one of these open neighborhoods if and only if

- $\sup_{x \in C} d(x, C') + \sup_{x \in C'} d(x, C) < \epsilon$,
- $|\int_C \varpi - \int_{C'} \varpi| < \epsilon$,

where $d(\cdot, \cdot)$ is the metric on $\mathbb{R} \times Y$ as defined by the Riemannian metric on $T(\mathbb{R} \times Y)$. There exists a version of Gromov Compactness for J-holomorphic curves in the current setting.

Theorem 3.1 (Proposition 5.5 in [19])**.** *Let $U \subset Y$ be an open set, $I \subset \mathbb{R}$ be a bounded, open interval, $\mathcal{L} \geq 1$, and $\{(C_k, n_k)\}$ be a sequence of J-holomorphic curves in $\mathbb{R} \times U$. Suppose that there exist $s \in \mathbb{R}$ and $\epsilon > 0$ such that*

$$\int_{C_k \cap [s-2, s+2] \times U_\epsilon} \hat{\omega} \leq \mathcal{L}$$

for each k, where U_ϵ denotes the set of points at a distance ϵ or less from the set U. Then, there exists a J-holomorphic curve $\{(C, n)\}$ and a subsequence of $\{(C_k, n_k)\}$ such that

- $\lim_{k \to \infty} (\sup_{x \in (\cup C) \cap I \times U} d(x, \cup C_k) + \sup_{x \in \cup C_k} d(x, (\cup C) \cap I \times U) = 0$,
- *Let ϖ be a smooth bounded 2-form defined in a neighborhood of $I \times U$. Then*

$$\lim_{k \to \infty} |\sum n \int_C \varpi - \sum n_k \int_{C_k} \varpi| = 0.$$

Note that translating a J-holomorphic curve along the $\mathbb{R}$-factor yields a J-holomorphic curve. This is because J is invariant under translations along the $\mathbb{R}$-factor. Therefore, the group $\mathbb{R}$ acts on the moduli space of J-holomorphic curves. This action is free away from the set of $\mathbb{R}$-invariant cylinders. Having said that, we now define the differential on the *ech* chain complex by describing the boundary of a generator $[\Theta_+, Z]$:

$$\partial_{ech}^{\infty}[\Theta_+, Z] = \sum_{\{\Theta_- \in \mathcal{Z}_{ech,M},\ W \in H_2(Y, \Theta_+, \Theta_-) \ |\ I(W)=1\}} \#\big(\mathcal{M}(W)/\mathbb{R}\big)[\Theta_-, Z - W]. \tag{6}$$

The quantity $\#\big(\mathcal{M}(W)/\mathbb{R}\big)$ refers to a signed count of admissible J-holomorphic curves (up to translation) representing the class W, which is finite thanks to the version of Gromov Compactness provided by Theorem 3.1. There is also a degree -2 chain map U on the *ech* chain complex that is defined by fixing a point $x \in \mathcal{H}_0$:

$$U[\Theta_+, Z] = \sum_{\{\Theta_- \in \mathcal{Z}_{ech,M},\ W \in H_2(Y, \Theta_+, \Theta_-) \ |\ I(W)=2\}} \#\mathcal{M}_x(W)[\Theta_-, Z - W]. \tag{7}$$

Here, $\mathcal{M}_x(W)$ denotes the moduli space of admissible J-holomorphic curves representing the class W that pass through the point $(0, x) \in \mathbb{R} \times Y$. Similarly, we can define the action of $\wedge^*(H_1(Y; \mathbb{Z})/torsion)$ by choosing a suitably generic basis of cycles that generate $H_1(Y; \mathbb{Z})/torsion$ and by a weighted sum of the algebraic intersection number of *ech* index-1 curves with these cycles in $\{0\} \times Y$.

Next we explain how admissible J-holomorphic curves foliate the complement in $\mathbb{R}\times Y$ of $\mathbb{R}\times\cup_{\mathfrak{p}\in\Lambda}(\gamma_{\mathfrak{p}}^{+}\cup\gamma_{\mathfrak{p}}^{-})$. This will be key to understanding the structure of the moduli space of J-holomorphic curves that enter, in particular, into the definition of the differential for the *ech* chain complex. We start by investigating admissible J-holomorphic curves that arise from level sets of the Morse function f and the handle $\mathcal{H}_0$. Since J preserves the kernel of $\hat{a}$ which restricts to df on M_δ, the intersection of level sets of f with M_δ are admissible J-holomorphic curves if $f\in(\delta^2,1-2\delta^2)\cup(1+\delta^2,2-\delta^2)\cup(2+2\delta^2,3-\delta^2)$, and are otherwise subsets of admissible J-holomorphic curves. These are described as follows (see Section 3(b) of [19]):

- A component $\mathcal{M}_0$ of the moduli space of admissible J-holomorphic curves consists of embedded 2-spheres and is $\mathbb{R}$-equivariantly diffeomorphic to $\mathbb{R}\times(-1,1)$. The part of $\mathcal{M}_0$ that is parametrized by $\mathbb{R}\times[-\frac{1}{2},\frac{1}{2}]$ consists of 2-spheres that arise from the constant u cross-sectional 2-spheres of $\mathcal{H}_0$ for $u\in[-R-\ln\delta,R+\ln\delta]$. The parts of $\mathcal{M}_0$ that are parametrized by $\mathbb{R}\times[\frac{1}{2},1)$ and $\mathbb{R}\times(-1,-\frac{1}{2}]$ respectively correspond to level sets of f for $f\in[\delta^2,1)$ and $f\in(2,3-\delta^2]$. Each admissible J-holomorphic curve from $\mathcal{M}_0$ has *ech* index 2.
- A component $\mathcal{M}_\Sigma$ of the moduli space of admissible J-holomorphic curves consists of embedded closed surfaces of genus G arising from the $f\in(1,2)$ level sets of f, and it is $\mathbb{R}$-equivariantly diffeomorphic to $\mathbb{R}\times(1,2)$. Each admissible J-holomorphic curve from $\mathcal{M}_\Sigma$ has *ech* index $2-2\mathrm{G}$.
- Two components, denoted by $\mathcal{M}_1$ and $\mathcal{M}_2$, of the moduli space of admissible J-holomorphic curves each consists of properly embedded 2-spheres with $2\mathrm{G}$ punctures, and each are $\mathbb{R}$-equivariantly diffeomorphic to $\mathbb{R}$. An admissible J-holomorphic curve from $\mathcal{M}_1$ projects onto M_δ as $f^{-1}(1)$, while an admissible J-holomorphic curve from $\mathcal{M}_2$ projects onto M_δ as $f^{-1}(2)$. These curves have $2\mathrm{G}$ negative ends each at a distinct $\gamma_{\mathfrak{p}}^{+}$ or $\gamma_{\mathfrak{p}}^{-}$, and no positive ends. Each admissible J-holomorphic curve from $\mathcal{M}_1$ or $\mathcal{M}_2$ has *ech* index $2-2\mathrm{G}$.

As for the admissible J-holomorphic curves contained in $\mathbb{R}\times\mathcal{H}_{\mathfrak{p}}$ for some $\mathfrak{p}\in\Lambda$, we have the following characterization:

- A component $\mathcal{M}_{\mathfrak{p}+}$ consists of embedded open disks that lie where $\cos\theta>\frac{1}{\sqrt{3}}$ and is $\mathbb{R}$-equivariantly diffeomorphic to $\mathbb{R}$. These curves have one positive end at $\gamma_{\mathfrak{p}}^{+}$, and no negative ends. They project onto $\mathcal{H}_{\mathfrak{p}}$ as the open disk defined by $u=0$ and $\cos\theta>\frac{1}{\sqrt{3}}$, and project onto $\mathbb{R}$ as the half infinite line $[s,\infty)$ for some $s\in\mathbb{R}$. Each admissible J-holomorphic curve from $\mathcal{M}_{\mathfrak{p}+}$ has *ech* index 1.
- A component $\mathcal{M}_{\mathfrak{p}-}$ consists of embedded open disks that lie where $\cos\theta<-\frac{1}{\sqrt{3}}$ and is $\mathbb{R}$-equivariantly diffeomorphic to $\mathbb{R}$. These curves have one positive end at $\gamma_{\mathfrak{p}}^{-}$, and no negative ends. They project onto $\mathcal{H}_{\mathfrak{p}}$ as the open disk defined by $u=0$ and $\cos\theta<-\frac{1}{\sqrt{3}}$, and project onto $\mathbb{R}$ as the half infinite line $[s,\infty)$ for some $s\in\mathbb{R}$. Each admissible J-holomorphic curve from $\mathcal{M}_{\mathfrak{p}+}$ has *ech* index 1.

- A component $\mathcal{M}_{\mathfrak{p}0}$ consists of embedded open annuli that lie where $\cos^2\theta < \frac{1}{3}$ and is $\mathbb{R}$-equivariantly diffeomorphic to $\mathbb{R}$. These curves have two positive ends at $\gamma_{\mathfrak{p}}^+$ and $\gamma_{\mathfrak{p}}^-$, and no negative ends. They project onto $\mathcal{H}_{\mathfrak{p}}$ as the open annulus defined by $u = 0$ and $\cos^2\theta < \frac{1}{3}$. Each admissible J-holomorphic curve from $\mathcal{M}_{\mathfrak{p}0}$ has *ech* index 0.

With the above understood, note that the only compact and connected admissible J-holomorphic curves are the ones contained in $\mathcal{M}_0$ and $\mathcal{M}_\Sigma$. This is because any such curve C has to intersect some curve from $\mathcal{M}_0$ or $\mathcal{M}_\Sigma$ nontrivially. Since J-holomorphic curves intersect with non-negative local intersection number and both C and the curve it intersects in $\mathcal{M}_0$ or $\mathcal{M}_\Sigma$ are compact, C has to intersect all curves from $\mathcal{M}_0$ or $\mathcal{M}_\Sigma$. But, this is possible only if the s-coordinate is unbounded on C, contradicting the assumption that C is compact.

Next, we make some preliminary observations about the admissible J-holomorphic curves that define the differential and various endomorphisms on the *ech* chain complex. First, we call an admissible J-holomorphic curve C an *ech-HF curve* if it does not contain any connected components from $\mathcal{M}_0$, $\mathcal{M}_\Sigma$, $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_{\mathfrak{p}+}$, $\mathcal{M}_{\mathfrak{p}-}$, or $\mathcal{M}_{\mathfrak{p}0}$. An *ech-HF* curve C satisfies the following:

- C has empty intersection with $\mathbb{R}\times\mathcal{H}_0$ and with the parts of $\mathbb{R}\times M_\delta$ where $f \le 1$ or $f \ge 2$.
- C has empty intersection with the part of $\mathbb{R}\times\mathcal{H}_{\mathfrak{p}}$ where $\cos^2\theta > \frac{1}{3}$ for each $\mathfrak{p}\in\Lambda$.
- A connected component of C intersecting $\mathbb{R}\times\mathcal{H}_{\mathfrak{p}}$ in the $\cos^2\theta = \frac{1}{3}$ locus is either the cylinder $\mathbb{R}\times\gamma_{\mathfrak{p}}^+$ or the cylinder $\mathbb{R}\times\gamma_{\mathfrak{p}}^-$.
- C has intersection number G with each curve from $\mathcal{M}_\Sigma$.

By way of an explanation, if C were to intersect $\mathbb{R}\times\mathcal{H}_0$, the part of $\mathbb{R}\times M_\delta$ where $f\le 1$ or $f \ge 2$, or the part of $\mathbb{R}\times\mathcal{H}_{\mathfrak{p}}$ where $\cos^2\theta > \frac{1}{3}$ for some $\mathfrak{p}\in\Lambda$, then it would have to intersect a curve from $\mathcal{M}_0$ since it does not contain any connected components from $\mathcal{M}_0$, $\mathcal{M}_1$, or $\mathcal{M}_2$. As a result, C must intersect each curve from $\mathcal{M}_0$, which is true even if C is non-compact due to the fact that the ends of C are disjoint from $\mathcal{H}_0$, the part of M_δ where $f \notin (1,2)$, and the part of each $\mathcal{H}_{\mathfrak{p}}$ where $\cos^2\theta > \frac{1}{3}$. The latter reasoning also explains why C has empty intersection with the curves from $\mathcal{M}_0$ located at $\{s\}\times Y$ for sufficiently large values of $|s|$, contradicting the previous conclusion. On the other hand, suppose C has a connected component that intersects $\mathbb{R}\times\mathcal{H}_{\mathfrak{p}}$ where $\cos^2\theta = \frac{1}{3}$ and $u = 0$, and that it is neither $\mathbb{R}\times\gamma_{\mathfrak{p}}^+$ nor $\mathbb{R}\times\gamma_{\mathfrak{p}}^-$. Then it also intersects the part of $\mathbb{R}\times\mathcal{H}_{\mathfrak{p}}$ where $\cos^2\theta > \frac{1}{3}$, hence a curve from $\mathcal{M}_0$, $\mathcal{M}_{\mathfrak{p}+}$, or $\mathcal{M}_{\mathfrak{p}-}$. Since C does not contain any connected components from $\mathcal{M}_0$, $\mathcal{M}_{\mathfrak{p}+}$, or $\mathcal{M}_{\mathfrak{p}-}$, this possibility is ruled out by an argument similar to the one above. As for the claim in the last bullet, let C have positive ends at $\Theta_+ \in \mathcal{Z}_{ech,M}$ and negative ends at $\Theta_- \in \mathcal{Z}_{ech,M}$. Since the intersection of Θ_+ and Θ_- with M_δ each determines a G-tuple of integral curves of the gradient-like vector field $\mathfrak{v}$ which intersect Σ in G distinct points, the part of C for sufficiently large values of $|s|$ intersects a unique curve from $\mathcal{M}_\Sigma$ in G distinct points. Therefore, C has intersection number G with every curve from $\mathcal{M}_\Sigma$ by the fact the intersection numbers are locally constant under small perturbations.

Finally, we describe the intersection of an *ech-HF* curve with $\mathbb{R} \times \mathcal{H}_{\mathfrak{p}}$ for any $\mathfrak{p} \in \Lambda$. Let C be an *ech-HF* curve and $\mathfrak{p} \in \Lambda$ be fixed.

- If no end of C is at $\gamma_{\mathfrak{p}}^{+}$ or $\gamma_{\mathfrak{p}}^{-}$, then C has intersection number 1 with each curve from $\mathcal{M}_{\mathfrak{p}0}$.
- If one end of C is at either $\gamma_{\mathfrak{p}}^{+}$ or $\gamma_{\mathfrak{p}}^{-}$, then C has intersection number 1 with each curve from $\mathcal{M}_{\mathfrak{p}0}$ except for one, with which it has intersection number 0.
- If there are two ends of C one of which is at $\gamma_{\mathfrak{p}}^{+}$ and the other is at $\gamma_{\mathfrak{p}}^{-}$, then C has intersection number 1 with each curve from $\mathcal{M}_{\mathfrak{p}0}$ except for two, with which it has intersection number 0.

(8)

To sum up, the intersection of an *ech-HF* curve with a given $\mathbb{R} \times \mathcal{H}_{\mathfrak{p}}$ is a J-holomorphic strip with 0, 1, or 2 punctures.

Before we end this subsection, a few remarks are in order. First, since closed integral curves of $\mathcal{R}$ that comprise elements of $\mathcal{Z}_{ech,M}$ are all hyperbolic, there are no multiply covered admissible J-holomorphic curves in $\mathbb{R} \times Y$. Second, having fixed a point x in $\mathcal{H}_0$, Hutchings's characterization of the *ech* index-2 curves as unions of embedded J-holomorphic curves with possibly multiply covered $\mathbb{R}$-invariant cylinders lets us argue that the only admissible *ech* index-2 curve that passes through $(0, x) \in \mathbb{R} \times Y$ is the union of an embedded $\mathbb{R}$-invariant cylinder over an element of $\mathcal{Z}_{ech,M}$ and the unique embedded 2-sphere ${S^2}_x$ from $\mathcal{M}_0$ that passes through this point. As a result, the U-map in (7) can be written as

$$U[\Theta, Z] = [\Theta, Z - [{S^2}_x]].$$

3.2. The *ech* and Heegaard Floer homology equivalence

In order to establish the correspondence between Heegaard Floer homology and *ech*, we use Lipshitz's cylindrical reformulation of Heegaard Floer homology [25]. The latter exploits the same data provided by a strongly $\mathfrak{s}$-admissible Heegaard diagram $(\Sigma, \mathbf{A}, \mathbf{B}, z)$. The generators of the Heegaard Floer chain complex appear as pairs of the form (υ, i) where $\upsilon = \{\upsilon_k = [1,2] \times \{x_k\}\}_{k=1}^{\textsc{g}}$ with $x_k \in A_k \cap B_{\sigma(k)}$ for σ a permutation of $\{1, \ldots, \textsc{g}\}$, and $i \in \mathbb{Z}$. Having identified $f^{-1}(1,2)$ with $(1,2) \times \Sigma$ via the gradient-like vector field $\mathfrak{v}$, each υ_k corresponds to an integral curve of $\mathfrak{v}$ whose closure connects an index-1 critical point of f to an index-2 critical point of f. Therefore, if we decompose σ into cycles, each cycle corresponds to what we formerly called an index 1-2 cycle. Let us denote the set of generators for the Heegaard Floer chain complex by $\mathcal{Z}_{HF}$, and introduce a set with four elements $\textsc{o} = \{0, 1, -1, \{1, -1\}\}$. Then, by Proposition 2.3, there exists a $1-1$ correspondence

$$\mathcal{Z}_{ech,M} \leftrightarrow \mathcal{Z}_{HF} \times \prod_{\mathfrak{p} \in \Lambda} (\mathbb{Z} \times \textsc{o}). \tag{9}$$

The above correspondence is canonical when the $\mathbb{Z}$-factors on the right-hand side are considered as affine spaces over $\mathbb{Z}$. For each $\mathfrak{p} \in \Lambda$, a $\mathbb{Z}$-factor on the right-hand side is related to the net change in the azimuthal angle coordinate as a closed integral curve from a given element of $\mathcal{Z}_{ech,M}$ traverses $\mathcal{H}_{\mathfrak{p}}$, and an identification with $\mathbb{Z}$ is fixed once a lift of the azimuthal angle coordinate to $\mathbb{R}$ is chosen[3]; whereas the O-factor signifies whether a given element of $\mathcal{Z}_{ech,M}$ contains none of, exactly one of, or both of $\gamma_{\mathfrak{p}}^{+}$ and $\gamma_{\mathfrak{p}}^{-}$. The above correspondence lifts to a $1-1$ correspondence

$$\hat{\mathcal{Z}}_{ech,M} \leftrightarrow \mathcal{Z}_{HF} \times \mathbb{Z} \times \prod_{\mathfrak{p}\in\Lambda} (\mathbb{Z} \times \mathrm{O}), \tag{10}$$

where the identification of $\hat{\mathcal{Z}}_{ech,M}$ with $\mathcal{Z}_{ech,M} \times \mathbb{Z}$, as is explained in the previous subsection, is used.

The differential and various endomorphisms of the Heegaard Floer chain complex defined by Lipshitz require the choice of an almost complex structure on $\mathbb{R} \times [1,2] \times \Sigma$ subject to a number of constraints. Having identified $f^{-1}(1,2)$ with $(1,2) \times \Sigma$ via the gradient-like vector field $\mathfrak{v}$, these constraints translate as follows: J_{HF} be an almost complex structure on $\mathbb{R} \times f^{-1}(1,2)$, and s denote the Euclidean coordinate on the $\mathbb{R}$-factor. Then, we require

$$\left.\begin{array}{l}\bullet\ J_{HF}\frac{\partial}{\partial s} = \mathfrak{v}.\\ \bullet\ J_{HF} \text{ is invariant under translations along the } \mathbb{R}\text{-factor.}\\ \bullet\ J_{HF} \text{ preserves the kernel of the 1-form } df, \text{ and it is compatible with the}\\ \quad \text{restriction of } w \text{ on this plane field.}\\ \bullet\ J_{HF} \text{ is invariant under translations along integral curves of the gradient-}\\ \quad \text{like vector field } \mathfrak{v} \text{ near } \mathbb{R}\times(\mathfrak{A}\cup\mathfrak{D}) \text{ where } \mathfrak{A} \text{ denotes the union of ascending}\\ \quad \text{manifolds of the index-1 critical points and } \mathfrak{D} \text{ denotes the union of descending}\\ \quad \text{manifolds of the index-2 critical points of } f.\end{array}\right\} \tag{11}$$

Note that the first three constraints in (11) agree on $\mathbb{R} \times (M_\delta \cap f^{-1}(1,2))$ with the first three constraints in (5). Moreover, these constraints indicate that J_{HF} is compatible with the symplectic form $ds \wedge df + w_\Sigma$, which restricts to $\hat{\omega}$ on $\mathbb{R} \times (M_\delta \cap f^{-1}(1,2))$. In addition to the above constraints, we require that

$$J_{HF}\frac{\partial}{\partial \phi_+} = \frac{\partial}{\partial \mathfrak{h}_+} \quad \text{and} \quad J_{HF}\frac{\partial}{\partial \phi_-} = \frac{\partial}{\partial \mathfrak{h}_-},$$

where

$$\mathfrak{h}_+ = 2e^{2u_+}\cos\theta_+ \sin^2\theta_+ \quad \text{and} \quad \mathfrak{h}_- = 2e^{2u_-}\cos\theta_- \sin^2\theta_-$$

on the intersection of $f^{-1}(1,2)$ and the radius $8\delta_*$ coordinate ball centered at respectively an index-1 and an index-2 critical point of f. Note also that this last constraint is consistent with the last two constraints listed above. With the preceding understood, Lipshitz considers J_{HF}-holomorphic curves S_0 in $\mathbb{R} \times (1,2) \times \Sigma$ satisfying the following properties:

- S_0 is the interior of a properly embedded surface S in $\mathbb{R} \times [1,2] \times \Sigma$ with $2\mathrm{G}$ boundary components half of which lie on $\mathbb{R} \times \{1\} \times \mathbf{A}$, and the other half lie on $\mathbb{R} \times \{2\} \times \mathbf{B}$.

[3]However, there is no canonical lift. This is partly why the isomorphisms in Theorem 1.1 fail to be canonical.

- S is the complement of $2\textsc{g}$ distinct points in the boundary of a compact surface $\bar{S}$ with boundary. The s coordinate tends to $+\infty$ on sequences of points that limit to any one of $\textsc{g}$ of these points, while it tends to $-\infty$ on sequences of points that limit to any one of the remaining $\textsc{g}$ points.
- $\int_{S_0} w_\Sigma$ is finite, and there exists a constant $\kappa_S \geq 1$ such that for any $s \in \mathbb{R}$ we have $\int_{S_0 \cap ([s,s+1]\times(1,2)\times\Sigma)} ds \wedge dt < \kappa_S$.
- The set of constant s slices of S converge pointwise to elements of $\mathcal{Z}_{HF}$ as s tends to $\pm\infty$.
- The associated deformation operator $D_S : L^2_1(N_S) \to L^2(N_S \otimes T^{0,1}S)$, where N_S denotes the normal bundle to S, has trivial cokernel.

The boundary of a given generator (υ_+, i) of the Heegaard Floer chain complex is given by

$$\partial^\infty_{HF}(\upsilon_+, i) = \sum_{\{\upsilon_- \in \mathcal{Z}_{HF}\ [S] \in \pi_2(\upsilon_+,\upsilon_-)\ |\ ind(D_S)=1\}} \#\big(\mathcal{M}([S])/\mathbb{R}\big)(\upsilon_-, i - n_z([S])), \tag{12}$$

where $ind(D_S)$ denotes the Fredholm index of S, $\#\big(\mathcal{M}([S])/\mathbb{R}\big)$ is a signed count, modulo translations along the $\mathbb{R}$-factor, of J_{HF}-holomorphic curves in the relative homology class $[S]$, and $n_z([S])$ is the algebraic intersection number of S with $\mathbb{R} \times [1,2] \times \{z\}$, which is non-negative. A degree -2 chain map U on the Heegaard Floer chain complex is defined by

$$U(\upsilon, i) = (\upsilon, i-1). \tag{13}$$

Having established a correspondence between generators of the Heegaard Floer and *ech* chain complexes, we briefly explain how to relate the respective differentials. Given two generators $\Theta_+, \Theta_- \in \mathcal{Z}_{ech,M}$ which can be written respectively as $(\upsilon_+, (m_+, \text{o}_+))$ and $(\upsilon_-, (m_-, \text{o}_-))$ using the identification in (9), and a J_{HF}-holomorphic curve S of Fredholm index 1 that limits to υ_+ at $+\infty$ and to υ_- at $-\infty$, we first construct a canonical approximation $C_0 = \{C_{S0}, \{C_{\mathfrak{p}0}\}_{\mathfrak{p}\in\Lambda}\}$ to the desired admissible J-holomorphic curve C with positive ends at Θ_+ at $+\infty$ and negative ends at Θ_- at $-\infty$. Here, C_{S0} denotes a J-holomorphic curve with $2\textsc{g}$ boundary components in $\mathbb{R} \times f^{-1}(1,2) \cap M_\delta$ and each $C_{\mathfrak{p}0}$ is a J-holomorphic curve with 2 boundary components in $\mathbb{R} \times \mathcal{H}_\mathfrak{p}$. The boundaries do not match, but there is a canonical $1-1$ correspondence between the boundary components of C_{S0} and the boundary components of $\bigcup_{\mathfrak{p}\in\Lambda} C_{\mathfrak{p}0}$. Moreover, the local boundary conditions on the deformation operators corresponding to C_{S0} and $C_{\mathfrak{p}0}s$ are coupled with respect to this correspondence. The curve C_{S0} looks very much like a J_{HF}-holomorphic curve of Fredholm index at most 1, while each $C_{\mathfrak{p}0}$ is one of the curves described in (8). As a matter of fact, C_{S0} arises as the image under the exponential map of a section of the bundle N_S. With the preceding understood, we construct a cobordism from the moduli space of such approximations to the component of the moduli space of admissible *ech* index 1 curves in $\mathbb{R} \times Y$. This cobordism maps to the interval $[0,1]$ in a smooth and proper manner where the pre-image of 0 is the moduli space of approximations, while the pre-image of 1 is the component of the moduli space of admissible *ech* index 1 curves; and it induces an $\mathbb{R}$-equivariant diffeomorphism between the two spaces. To be more explicit, we have the following possibilities:

- C_{S0} has Fredholm index 1 and $(m_+, \mathrm{o}_+) = (m_-, \mathrm{o}_-)$,
- C_{S0} has Fredholm index 0, $(m_+, \mathrm{o}_+) \neq (m_-, \mathrm{o}_-)$, and either
  - $m_+ = m_-$ and $|\mathrm{o}_+| = |\mathrm{o}_-| + 1$, or
  - $m_+ = m_- \pm 1$ and $\mathrm{o}_+ = \mathrm{o}_-$.

Meanwhile, given a coherent system of orientations for the low index components of the moduli space of J_{HF}-holomorphic curves, there exists a canonical choice for a coherent system of orientations for the corresponding components of the moduli space admissible J-holomorphic curves. The latter is induced in part by canonical orientations on the moduli space of holomorphic curves in $\mathbb{R}\times\mathcal{H}_{\mathfrak{p}}$. Consequently, we get $\partial^{\infty}_{ech} = \partial^{\infty}_{HF} + \sum_{\mathfrak{p}\in\Lambda} \partial_*$ where ∂_* denotes the differential on the free $\mathbb{Z}$-module generated by $\mathbb{Z} \times \mathrm{O}$ defined as follows:

- $\partial_*(m, 0) = 0$ for each $m \in \mathbb{Z}$,
- $\partial_*(m, 1) = (m, 0) + (m + 1, 0)$ for each $m \in \mathbb{Z}$,
- $\partial_*(m, -1) = (m, 0) + (m - 1, 0)$ for each $m \in \mathbb{Z}$,
- $\partial_*(m, \{1, -1\}) = (m, -1) - (m, 1) + (m + 1, -1) - (m - 1, 1)$ for each $m \in \mathbb{Z}$.

As is easily checked, the homology of the chain complex $(\mathbb{Z}(\mathbb{Z} \times \mathrm{O}), \partial_*)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Denote the latter graded group by $\hat{\mathrm{V}}$. The chains $(0, 0)$ and $(0, 1) - (1, -1)$ are closed and they represent two independent generators of the homology, the former in degree 0 and the latter in degree 1. Therefore,

Theorem 3.2 (Theorem 2.4 in [18])**.** *There exists a commutative diagram*

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & ech^- & \longrightarrow & ech^\infty & \longrightarrow & ech^+ & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & HF^-(M,\mathfrak{s}) \otimes_{\mathbb{Z}} \hat{\mathrm{V}}^{\otimes \mathrm{G}} & \longrightarrow & HF^\infty(M,\mathfrak{s}) \otimes_{\mathbb{Z}} \hat{\mathrm{V}}^{\otimes \mathrm{G}} & \longrightarrow & HF^+(M,\mathfrak{s}) \otimes_{\mathbb{Z}} \hat{\mathrm{V}}^{\otimes \mathrm{G}} & \longrightarrow \cdots
\end{array}$$

where the top row is the long-exact sequence for ech associated to (4) and the bottom row is the standard long-exact sequence for the Heegaard Floer homology of M tensored with $\hat{\mathrm{V}}^{\otimes \mathrm{G}}$, while the vertical arrows are isomorphisms, induced by the correspondence (9), that preserve the relative gradings and intertwine the $\mathbb{Z}[U] \otimes \wedge^*(H_1(M;\mathbb{Z})/torsion)$-module structures.

For precise statements of the results summarized in this section that lead to a proof of the above theorem, see Theorem 1.1 in [20].

4. Seiberg–Witten Floer cohomology and its equivalence with *ech*

We start this section by reviewing some of the background in Seiberg–Witten Floer homology. We encourage the uninformed reader to refer to the book by Kronheimer and Mrowka [17] for an extensive treatment of the subject. After we cover some of the background material, we move on to describing a twisted version of Seiberg–Witten

Floer cohomology for certain Spin^c structures on Y as in [21]. Finally, we explain the proof of the equivalence of the latter with *ech*. This last part is along the same lines as Taubes's construction of a canonical isomorphism between ECH and Seiberg–Witten Floer cohomology [34, 35, 36, 37, 38], and the reader should refer to [21] for further details.

4.1. Preliminaries in Seiberg–Witten Floer homology

Here, we describe the Seiberg–Witten Floer homology groups of M. Start by fixing a Riemannian metric on M. Then a Spin^c structure $\mathfrak{s}$ on M consists of the following data:

- A rank-2 Hermitian bundle $\mathbb{S}$ over M, called *spinor bundle*,
- A Clifford multiplication, i.e., $\mathfrak{cl} : T^*M \to End_{\mathbb{C}}(\mathbb{S})$ an isometry onto the group of traceless, skew-Hermitian endomorphisms such that $\mathfrak{cl}(e^1)\mathfrak{cl}(e^2)\mathfrak{cl}(e^3) = 1_{\mathbb{S}}$ for any local oriented orthonormal basis $\{e^1, e^2, e^3\}$ for T^*M.

The first Chern class of the bundle $\mathbb{S}$ is denoted by $c_1(\mathfrak{s})$. With regard to the second part of the data, given $x \in M$ and any oriented orthonormal basis $\{e^1, e^2, e^3\}$ for T^*_xM, one can choose a frame for the fiber $\mathbb{S}_x$ so that

$$\mathfrak{cl}(e^1) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \qquad \mathfrak{cl}(e^2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \mathfrak{cl}(e^3) = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Comparing the above definition to the definition of a Spin^c structure from Section 2, the orthogonal complement of a nowhere vanishing unit length vector field results in an oriented 2-plane field L on M which can be seen as the kernel of a unique nowhere vanishing smooth 1-form λ dual to that vector field. Then, the corresponding spinor bundle splits as $\mathbb{S}_L = \underline{\mathbb{C}} \oplus L$, and the Clifford multiplication is defined by

$$\mathfrak{cl}(\lambda) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

while for any smooth unit length 1-form μ orthogonal to λ, $\mathfrak{cl}(\mu)(1,0) = (0, \mu^\dagger)$ where $\mu^\dagger$ denotes the vector field dual to μ. Conversely, the latter data defines a unique smooth unit length 1-form λ on M with $\mathfrak{cl}(\lambda) = \left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right]$, whose dual vector field is the nowhere vanishing vector field we started with. The set of isomorphism classes of Spin^c structures on M is a principal homogeneous space for $H^2(M;\mathbb{Z})$. The action of a class $e \in H^2(M;\mathbb{Z})$ on a Spin^c structure $\mathfrak{s}$ with spinor bundle $\mathbb{S}$ and Clifford multiplication $\mathfrak{cl}$ results in a Spin^c structure $\mathfrak{s} \otimes e$ with spinor bundle $\mathbb{S} \otimes E$ where E is a Hermitian line bundle over M with $c_1(E) = e$, and with Clifford multiplication $\mathfrak{cl} \otimes 1_E$.

Having fixed a Spin^c structure $\mathfrak{s}$ on M, consider the space $\mathcal{C}(M, \mathfrak{s})$ consisting of pairs of the form $(\mathbb{A}, \Psi)$ where $\mathbb{A}$ is a Spin^c connection, i.e., a Hermitian connection on $\mathbb{S}$ that satisfies the Leibniz rule

$$\nabla_{\mathbb{A}}(\mathfrak{cl}(\lambda)\Psi) = \mathfrak{cl}(\nabla\lambda)\Psi + \mathfrak{cl}(\lambda)(\nabla_{\mathbb{A}}\Psi),$$

with $\nabla\lambda$ denoting the covariant derivative with respect to the Levi-Civita connection, and Ψ is a smooth section of the spinor bundle $\mathbb{S}$. A Spin^c connection $\mathbb{A}$ on $\mathbb{S}$ induces a Hermitian connection $\mathbb{A}^t$ on the determinant bundle $det(\mathbb{S})$. Conversely, a Hermitian connection on $det(\mathbb{S})$ together with the Levi-Civita connection on TM induce a Spin$^{\mathrm{c}}$ connection on $\mathbb{S}$. The Lie group $\mathcal{G} = C^\infty(M, S^1)$ acts on $\mathcal{C}(M, \mathfrak{s})$ by the rule

$$\begin{array}{rcl} \mathcal{G} \times \mathcal{C}(M, \mathfrak{s}) & \to & \mathcal{C}(M, \mathfrak{s}) \\ (u, (\mathbb{A}, \Psi)) & \mapsto & (\mathbb{A} - u^{-1} du \otimes 1_{\mathbb{S}}, u\Psi). \end{array}$$

A configuration $(\mathbb{A}, \Psi)$ is called *reducible* if $\Psi = 0$, otherwise it is called *irreducible*. Note that the action of $\mathcal{G}$ on $(\mathbb{A}, \Psi)$ has non-trivial stabilizer if and only if $(\mathbb{A}, \Psi)$ is reducible.

With the preceding understood, the *Seiberg–Witten equations* read:

$$\begin{array}{rcl} *F_{\mathbb{A}^t} & = & \Psi^\dagger \tau \Psi, \\ \mathcal{D}_{\mathbb{A}} \Psi & = & 0. \end{array} \tag{14}$$

Here, $*F_{\mathbb{A}^t}$ is the Hodge dual of the curvature form of $\mathbb{A}^t$, $\Psi^\dagger \tau \Psi : T^*M \to i\mathbb{R}$ is defined by $\Psi^\dagger \tau \Psi(\lambda) = \Psi^\dagger \mathfrak{cl}(\lambda) \Psi$, and $\mathcal{D}_{\mathbb{A}}$ is the Dirac operator defined to be the composition

$$C^\infty(M, \mathbb{S}) \xrightarrow{\nabla_{\mathbb{A}}} C^\infty(M, T^*M \otimes \mathbb{S}) \xrightarrow{\mathfrak{cl}} C^\infty(M, \mathbb{S}).$$

The Seiberg–Witten equations arise as the variational equations of the *Chern–Simons–Dirac functional*. Having fixed a base Spin$^{\mathrm{c}}$ connection $\mathbb{A}_0$ on $\mathbb{S}$, the latter can be written as

$$\mathfrak{csd}(\mathbb{A}, \Psi) = -\frac{1}{8} \int_M (\mathbb{A}^t - \mathbb{A}_0{}^t) \wedge (F_{\mathbb{A}^t} + F_{\mathbb{A}_0{}^t}) + \frac{1}{2} \int_M \langle \mathcal{D}_{\mathbb{A}} \Psi, \Psi \rangle. \tag{15}$$

Note that the moduli space of solutions to the Seiberg–Witten equations is invariant under the action of $\mathcal{G}$, while the value of the Chern–Simons–Dirac functional changes as follows:

$$\mathfrak{csd}(u \cdot (\mathbb{A}, \Psi)) - \mathfrak{csd}(\mathbb{A}, \Psi) = 2\pi^2 (c_1(\mathfrak{s}) \cup [u]) \cap [M],$$

for every $u \in \mathcal{G}$, where $[u] \in H^1(M; \mathbb{Z})$ is represented by $-\frac{i}{2\pi} u^{-1} du$. In particular, the Chern–Simons–Dirac functional is gauge invariant if and only if $c_1(\mathfrak{s})$ is torsion.

One can perturb the Seiberg–Witten equations using a closed 2-form ϖ:

$$\begin{array}{rcl} *F_{\mathbb{A}^t} & = & \Psi^\dagger \tau \Psi + i * \varpi, \\ \mathcal{D}_{\mathbb{A}} \Psi & = & 0. \end{array} \tag{16}$$

The above perturbed Seiberg–Witten equations are obtained from a perturbed version of the Chern–Simons–Dirac functional defined by

$$\mathfrak{csd}_\varpi(\mathbb{A}, \Psi) = \mathfrak{csd}_\varpi(\mathbb{A}, \Psi) + \frac{1}{4} \int_M (\mathbb{A}^t - \mathbb{A}_0{}^t) \wedge i\varpi. \tag{17}$$

The perturbed Chern–Simons–Dirac functional changes under the action of $\mathcal{G}$ as follows:

$$\mathfrak{csd}_\varpi(u \cdot (\mathbb{A}, \Psi)) - \mathfrak{csd}_\varpi(\mathbb{A}, \Psi) = 2\pi^2 ((c_1(\mathfrak{s}) + \frac{1}{2\pi}[\varpi]) \cup [u]) \cap [M].$$

The cohomology class $c = 2\pi^2 (c_1(\mathfrak{s}) + \frac{1}{2\pi}[\varpi]) \in H^2(M; \mathbb{R})$ is called the *period class*. Note that if the period class is non-zero, then the perturbed Seiberg–Witten equations in (16) have no reducible solutions. If ϖ is an exact 2-form, then the resulting perturbation is called an *exact* perturbation, otherwise it is called a *non-exact* perturbation. A non-exact perturbation is called

- *Balanced* if c is zero.
- *Positively/Negatively monotone* if $c = \kappa 2\pi^2 c_1(\mathfrak{s})$ for some $\kappa > 0$ or $\kappa < 0$ respectively.

In order to define the Seiberg–Witten Floer chain complexes, Kronheimer and Mrowka *blow up* (see Section of [17]) the quotient $\mathcal{B}(M, \mathfrak{s}) = \mathcal{C}(M, \mathfrak{s})/\mathcal{G}$ along the singular locus consisting of gauge equivalence classes of reducible solutions. Note that the latter set is non-empty if and only if the period class is zero. In this case, the resulting Banach manifold $\mathcal{B}^\sigma(M, \mathfrak{s})$ with boundary admits a vector field induced by the L^2-gradient of the perturbed Chern–Simons–Dirac functional in (17). This vector field is tangent to the boundary of $\mathcal{B}^\sigma(M, \mathfrak{s})$ and its critical points are pre-images of gauge equivalence classes of solutions to the equations in (16) under the blow-down map. The critical points that map to gauge equivalence classes of reducible solutions to the equations in (16) fall into two categories depending on whether or not the outward normal direction at a critical point on the boundary is contained in the corresponding stable manifold. These are called *boundary stable* and *boundary unstable*, respectively. The Seiberg–Witten Floer chain complexes $\overline{CM}_*(M, \mathfrak{s}, c_{[\varpi]})$, $\widehat{CM}_*(M, \mathfrak{s}, c_{[\varpi]})$, and $\widecheck{CM}_*(M, \mathfrak{s}, c_{[\varpi]})$ are the free $\mathbb{Z}$-modules generated respectively by boundary stable and boundary unstable critical points, irreducible and boundary unstable critical points, and irreducible and boundary stable critical points; the respective differentials $\bar{\partial}$, $\hat{\partial}$, and $\check{\partial}$ are defined via a suitable count of trajectories of the aforementioned vector field. Here, $c_{[\varpi]}$ indicates the period class of the perturbation defined by the closed 2-form ϖ. In order to ensure transversality, one may need to use additional abstract perturbations. We will elaborate on this matter in the next subsection. The homologies of the chain complexes $(\overline{CM}_*(M, \mathfrak{s}, c_{[\varpi]}), \bar{\partial})$, $(\widehat{CM}_*(M, \mathfrak{s}, c_{[\varpi]}), \hat{\partial})$, and $(\widecheck{CM}_*(M, \mathfrak{s}, c_{[\varpi]}), \check{\partial})$ are respectively denoted by $\overline{HM}_*(M, \mathfrak{s}, c_{[\varpi]})$, $\widehat{HM}_*(M, \mathfrak{s}, c_{[\varpi]})$, and $\widecheck{HM}_*(M, \mathfrak{s}, c_{[\varpi]})$. Each of these groups is relatively graded by $\mathbb{Z}/p_M$ where p_M denotes the divisibility of $c_1(\mathfrak{s})$, each has the structure of a $\mathbb{Z}(U) \otimes \wedge^*(H_1(M; \mathbb{Z})/torsion)$-module, and they fit into a long exact sequence

$$\cdots \to \widehat{HM}_* \xrightarrow{p_*} \overline{HM}_* \xrightarrow{i_*} \widecheck{HM}_* \xrightarrow{j_*} \widehat{HM}_* \to \cdots$$

for each $(M, \mathfrak{s}, c_{[\varpi]})$, analogous to the homology long exact sequence for a pair of a manifold and its boundary.

4.2. A twisted Seiberg–Witten Floer cohomology on Y

Having fixed a Spin^c structure $\mathfrak{s}$ on M, and a strongly $\mathfrak{s}$-admissible Heegaard diagram $(\Sigma, \mathbf{A}, \mathbf{B}, z)$ arising from a self-indexing Morse funcrion f and a gradient-like vector field $\mathfrak{v}$, we described in Section 2 a 3-manifold $Y \simeq M \#_{\scriptscriptstyle G} S^1 \times S^2$, a stable Hamiltonian structure (a, w) with associated Reeb vector field $\mathcal{R}$, and an auxiliary 1-form $\hat{a}$ on Y. Next, we choose a Riemannian metric on Y so that

- $*\hat{a} = w$,
- $|\mathcal{R}| = 1$.

Choosing such a Riemannian metric on Y is equivalent to choosing a compatible complex structure on the 2-plane field $K^{-1} = ker(\hat{a})$ oriented by w. The latter determines a canonical Spin^c structure $\mathfrak{s}_0$ on Y satisfying

- $\langle c_1(\mathfrak{s}_0), [S^2] \rangle = 2$ where $[S^2]$ is the positive generator of $H_2(\mathcal{H}_0; \mathbb{Z})$,
- $\langle c_1(\mathfrak{s}_0), [S^2] \rangle = -2$ where $[S^2]$ is the positive generator of $H_2(\mathcal{H}_{\mathfrak{p}}; \mathbb{Z})$ for some $\mathfrak{p} \in \Lambda$,
- $\mathfrak{s}_0 \otimes PD(\Gamma) = \underline{\mathfrak{s}}$.

The corresponding spinor bundle $\mathbb{S}_0$ splits into $\pm i$-eigenbundles of $\mathfrak{cl}(\hat{a})$ as $\underline{\mathbb{C}} \oplus K^{-1}$, where the left most summand is the $+i$-eigenbundle, and there exists a canonical (modulo action of $C^\infty(Y, S^1)$) Spin^c connection $\mathbb{A}_0$ on $\mathbb{S}_0$ such that $\mathcal{D}_{\mathbb{A}_0}(1_{\underline{\mathbb{C}}}, 0) = 0$. With the preceding understood, the spinor bundle $\underline{\mathbb{S}}$ associated to the Spin^c structure $\underline{\mathfrak{s}}$ splits into $\pm i$-eigenbundles of $\mathfrak{cl}(\hat{a})$ as $E \oplus (E \otimes K^{-1})$ where E is a Hermitian line bundle over Y with $c_1(E) = PD(\Gamma)$, and for any Spin^c connection $\mathbb{A}$ on $\underline{\mathbb{S}}$ we can write $\mathbb{A}^t = \mathbb{A}_0^t + 2A$ where A is a Hermitian connection on E. Now, we introduce a perturbed version of the Seiberg–Witten equations:

$$
\begin{aligned}
*F_A &= r(\psi^\dagger \tau \psi - i * w) - *\frac{1}{2} F_{\mathbb{A}_0}, \\
\mathcal{D}_A \psi &= 0,
\end{aligned}
\tag{18}
$$

where $r > 0$ is a real number and ψ is a smooth section of $\underline{\mathbb{S}}$. If we write $\Psi = (2r)^{1/2}\psi$, the equations in (18) would become the same as the equations in (16) with $\varpi = -2rw$. Note that since $c_1(\underline{\mathfrak{s}}) = [w]$ in $H^2(Y; \mathbb{R})$, the above equations admit no reducible solutions for $r > \pi$; in fact, the equations are negatively monotone perturbed. In this case, a solution (A, ψ) of the equations in (18) is non-degenerate if the kernel of the following first order, elliptic, self-adjoint operator on $C^\infty(Y; iT^*Y \oplus \underline{\mathbb{S}} \oplus i\mathbb{R})$ has trivial kernel:

$$
\mathcal{L}_{(A,\psi)}(b, \eta, \phi) = \begin{pmatrix} *db - d\phi - \left(\frac{r}{2}\right)^{1/2}(\psi^\dagger \tau \eta + \eta^\dagger \tau \psi) \\ \mathcal{D}_A \eta + (2r)^{1/2}(\mathfrak{cl}(b)\psi + \phi\psi) \\ *d * b - \left(\frac{r}{2}\right)^{1/2}(\eta^\dagger \psi - \psi^\dagger \eta) \end{pmatrix}. \tag{19}
$$

One can achieve non-degeneracy of all irreducible solutions to the perturbed Seiberg–Witten equations in (18) via an additional exact perturbation with small norm from a certain Banach space Ω of smooth 1-forms. Having chosen $\mu \in \Omega$, the resulting equations would read

$$
\begin{aligned}
*F_A &= r(\psi^\dagger \tau \psi - i * w) - i * d\mu - *\frac{1}{2} F_{\mathbb{A}_0}, \\
\mathcal{D}_A \psi &= 0.
\end{aligned}
\tag{20}
$$

Given a smooth section ψ of $\underline{\mathbb{S}}$, write $\psi = (\alpha, \beta)$. Recall that there exists a distinguished closed integral curve of $\mathcal{R}$ that intersects the Heegaard surface Σ at the base point z, denoted γ_z. Fix a Hermitian connection A_E on E that is flat on $\mathcal{H}_0$ and has holonomy 1 around γ_z, and a smooth non-decreasing function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\varphi(x) = 0$ for $0 \leq x < \frac{7}{16}$ and $\varphi(x) = 1$ for $x \geq \frac{9}{16}$. Then, given a pair (A, ψ) of a Hermitian connection A on E and a smooth section ψ of $\underline{\mathbb{S}}$, define

$$\begin{aligned} \hat{A} &:= A - \frac{1}{2}\varphi(|\alpha|^2)|\alpha|^{-2}(\bar{\alpha}\nabla_A \alpha - \alpha \nabla_A \bar{\alpha}), \\ \mathrm{x}_z(A,\psi) &:= \int_{\gamma_z} \frac{i}{2\pi}(\hat{A} - A_E). \end{aligned}$$

The function x_z is not invariant under the action of $C^\infty(Y,S^1)$, but it is invariant under the action of a subgroup $\mathcal{G}_{M_\Lambda}$ that is the kernel of the homomorphism $C^\infty(Y,S^1) \to \mathbb{Z}$ sending $u \in C^\infty(Y,S^1)$ to $\int_{\gamma_z} \frac{i}{2\pi} u^{-1} du$. A pair (A,ψ) of a Hermitian connection A on E and a smooth section of $\underline{\mathbb{S}}$ is called *holonomy non-degenerate* if $\mathrm{x}_z(A,\psi) - \frac{1}{2} \notin \mathbb{Z}$. There exists a residual set in Ω such that for every μ in this set and $r \in (\pi,\infty)$ in the complement of a discrete countable set dependent upon μ, all solutions to the equations in (20) are irreducible, non-degenerate, and holonomy non-degenerate. Moreover, the set $\mathcal{Z}_{SW,r}$ consisting of gauge equivalence classes of solutions to (20) is finite. With the preceding understood, the set $\hat{\mathcal{Z}}_{SW,r}$ consisting of $\mathcal{G}_{M_\Lambda}$-orbits of solutions to (20) admits a $1-1$ correspondence

$$\hat{\mathcal{Z}}_{SW,r} \leftrightarrow \mathcal{Z}_{SW,r} \times \mathbb{Z}, \tag{21}$$

where projection onto the $\mathbb{Z}$-factor is defined by the function x_z. Having said that, we define twisted versions of the Seiberg–Witten Floer cohomology on Y as follows: let $C^\infty_{SW,r}$ and $C^-_{SW,r}$ denote the free $\mathbb{Z}$-modules generated respectively by $\hat{\mathcal{Z}}_{SW,r}$ and the subset of $\hat{\mathcal{Z}}_{SW,r}$ identified with $\mathcal{Z}_{SW,r} \times \{\cdots,-2,-1\}$ via (21); and let $C^+_{SW,r}$ denote the quotient $C^\infty_{SW,r}/C^-_{SW,r}$. The relative grading on $C^\infty_{SW,r}$ is defined to be the negative of the spectral flow for a generic path of self-adjoint Fredholm operators starting and ending at the version of the operator in (19) for a solution of the equations in (20). Roughly speaking, the spectral flow of a generic path of self-adjoint Fredholm operators starting and ending at the associated versions of the operator in (19) is the net number of negative eigenvalues becoming positive.

The upcoming definition of the differential on the above defined Seiberg–Witten Floer chain complexes use solutions to the following flow equations:

$$\begin{aligned} \frac{\partial}{\partial s}A(s) &= -*F_A(s) + r(\psi(s)^\dagger \tau \psi(s) - i*w) - *\frac{1}{2}F_{\mathbb{A}_0} - i*d\mu, \\ \frac{\partial}{\partial s}\psi(s) &= -\mathcal{D}_{A(s)}\psi(s), \end{aligned} \tag{22}$$

A solution $(A(s),\psi(s))$ of the above equations with $\lim_{s\to\pm\infty}(A(s),\psi(s)) = (A_\pm,\psi_\pm)$ being solutions to the equations in (20) is called an *instanton.* An instanton $(A(s),\psi(s))$ connecting two irreducible, non-degenerate solutions to the equations in (20) is called non-degenerate if the operator $\frac{\partial}{\partial s} + \mathcal{L}_{(A(s),\psi(s))}$ on $C^\infty(\mathbb{R}\times Y; iT^*Y \oplus \underline{\mathbb{S}} \oplus i\mathbb{R})$ has trivial cokernel. In order to make sure that all instantons are non-degenerate, we need to use certain generic perturbations from a Banach space $\mathcal{P}$ of *tame perturbations* as defined by Kronheimer and Mrowka in Section 11 of [17]. These are smooth gauge invariant functions on the configuration space with values in $iT^*M \oplus \mathbb{S}$. In particular, Ω is contained in $\mathcal{P}$. We suppress additional tame perturbations from the notation for sake of exposition. Given $r \in (\pi, \infty)$ and $\mu \in \Omega$ so that all solutions to the equations in (20) are irreducible and non-degenerate, we may choose tame perturbations to vanish to second order on any solution of the equations in (20) and at all points of a non-degenerate instanton with Fredholm index at most 2. We denote the subset of such perturbations in $\mathcal{P}$ by $\mathcal{P}_\mu$. There exists a residual subset of $\mathcal{P}_\mu$ such that all instantons are non-degenerate. For the remainder of this section, we assume that $\mu \in \Omega$, $r \in (\pi, \infty)$, and $\mathfrak{q} \in \mathcal{P}_\mu$ are chosen so that all solutions to the equations in (20) are irreducible, non-degenerate, holonomy non-degenerate, and all instanton solutions to the equations in (22) are non-degenerate. Having said that, a Fredholm index-0 instanton arises as a constant map from $\mathbb{R}$ to the moduli space of solutions to the equations in (20). The differential on $C^\infty_{SW,r}$ is defined by

$$\partial^{SW,r}_\infty[A_+, \psi_+] = \sum_{[A_-,\psi_-] \in \hat{\mathcal{Z}}_{SW,r}} \sigma_1([A_+, \psi_+], [A_-, \psi_-])[A_-, \psi_-],$$

where $\sigma_1([A_+, \psi_+], [A_-, \psi_-])$ is a signed count, modulo the action of the Lie group $\mathcal{G}_{M_\Lambda}$ and translations along the $\mathbb{R}$-factor, of Fredholm index-1 instantons $(A(s), \psi(s))$ with $\lim_{s\to+\infty}(A(s), \psi(s)) = (A_+, \psi_+)$ and $\lim_{s\to-\infty}(A(s), \psi(s)) = u \cdot (A_-, \psi_-)$ for some u in $\mathcal{G}_{M_\Lambda}$. There exists a certain closed 2-form on Y representing the Poincaré dual of $[\gamma_z]$ such that

$$\mathrm{x}_z(A_+, \psi_+) - \mathrm{x}_z(A_-, \psi_-) = \int_{\mathbb{R}\times Y} \frac{i}{2\pi} F_{\hat{A}(s)} \wedge \varpi_z \geq 0 \tag{23}$$

(see Proposition 7.1 in [21]). Therefore, the differential $\partial^{SW,r}_\infty$ preserves the filtration induced by the function x_z, and hence induces differentials on $C^-_{SW,r}$ and $C^+_{SW,r}$. The resulting homologies are denoted respectively by $H^\infty_{SW,r}$, $H^-_{SW,r}$, and $H^+_{SW,r}$. The latter fit into a long-exact sequence induced by the natural short-exact sequence

$$0 \longrightarrow C^-_{SW,r} \xrightarrow{i} C^\infty_{SW,r} \xrightarrow{\pi} C^+_{SW,r} \longrightarrow 0. \tag{24}$$

Meanwhile, a degree 2 chain map on $C^\infty_{SW,r}$ is defined by

$$U[A_+, \psi_+] = \sum_{[A_-,\psi_-] \in \hat{\mathcal{Z}}_{SW,r}} \sigma_{2,x}([A_+, \psi_+], [A_-, \psi_-])[A_-, \psi_-],$$

where $\sigma_{2,x}([A_+, \psi_+], [A_-, \psi_-])$ is a signed count, modulo the action of the Lie group $\mathcal{G}_{M_\Lambda}$ and translations along the $\mathbb{R}$-factor, of Fredholm index-2 instantons $(A(s), \psi(s))$ such that $\lim_{s\to+\infty}(A(s), \psi(s)) = (A_+, \psi_+)$ and $\lim_{s\to-\infty}(A(s), \psi(s)) = u \cdot (A_-, \psi_-)$ for some u in $\mathcal{G}_{M_\Lambda}$, and $\psi(0)(x) = (0, \beta(0)(x))$. Similarly, we can define degree 1 chain maps on $C^\infty_{SW,r}$ for any given element of $H_1(Y; \mathbb{Z})/torsion$ by a weighted sum of the algebraic intersection numbers of $\alpha(s)^{-1}(0)$ from Fredholm index-1 instantons $(A(s), \psi(s) = (\alpha(s), \beta(s)))$ with the cylinder over a cycle representing that element. As a result, $H^\infty_{SW,r}$, $H^-_{SW,r}$, and $H^+_{SW,r}$ each has the structure of a $\mathbb{Z}[U] \otimes \wedge^*(H_1(Y; \mathbb{Z})/torsion)$-module.

For any one of the flavors $H^\infty_{SW,r}$, $H^-_{SW,r}$, and $H^+_{SW,r}$, there are canonical isomorphisms between the homology groups corresponding to two different choices of data $(r_0, \mu_0, \mathfrak{q}_0)$

and $(r_1, \mu_1, \mathfrak{q}_1)$. In order to prove this, consider a generic path $\{(r_s, \mu_s, \mathfrak{q}_s)\}_{s\in[0,1]}$ of such data to be used with the equations in (20). Then, Kronheimer and Mrowka explain in Section 25 of [17] how to construct a chain map between the corresponding versions of the chain complexes by counting Fredholm index-0 instantons. These chain maps induce isomorphisms on homology that are independent of the choice of the generic path $\{(r_s, \mu_s, \mathfrak{q}_s)\}_{s\in[0,1]}$. Therefore, the homology groups corresponding to the perturbation data $(r_0, \mu_0, \mathfrak{q}_0)$ and $(r_1, \mu_1, \mathfrak{q}_1)$ are canonically isomorphic; in particular, the subscript r can be dropped from the notation. For the remainder of this article we denote the three flavors of the twisted version of Seiberg–Witten Floer cohomology of Y by H^{∞}_{SW}, H^{-}_{SW}, and H^{+}_{SW}, respectively.

With the preceding understood, the aim of this section is to explain what goes into the proof of the following theorem:

Theorem 4.1 (Theorem 3.4 in [18]). *There exists a commutative diagram*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & ech^- & \longrightarrow & ech^\infty & \longrightarrow & ech^+ & \longrightarrow & \cdots \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
\cdots & \longrightarrow & H^-_{SW} & \longrightarrow & H^\infty_{SW} & \longrightarrow & H^+_{SW} & \longrightarrow & \cdots
\end{array}$$

where the top row is the long-exact sequence for ech associated to (4) and the bottom row is the long-exact sequence for the twisted version of the Seiberg–Witten Floer cohomology of Y associated to (24) while the vertical arrows are canonical isomorphisms that reverse the sign of the relative gradings and intertwine the $\mathbb{Z}[U]\otimes\wedge^*(H_1(M;\mathbb{Z})/torsion)$-module structures.

The proof of the above theorem is along the same lines as Taubes's proof of the equivalence between ECH and Seiberg–Witten Floer cohomology [34, 35, 36, 37, 38], but in many ways it also borrows from the ideas involved in the proof of the equivalence between periodic Floer homology and Seiberg–Witten Floer cohomology by Lee and Taubes [24]. In particular, a priori estimates on the energy of instanton solutions to (22) follow from arguments that are exact analogs of those in the latter. The overarching idea is to filter the embedded contact homology chain complex via the symplectic action and define monomorphisms from each filtration level into the Seiberg–Witten Floer cochain complex. The desired isomorphism then follows from taking the direct limit. Next we explain how the aforementioned homomorphisms are constructed in the present setting. For precise statements of the results summarized in this section that lead to a proof of Theorem 4.1, see Theorem 1.5 in [21].

A filtration on the chain complex ecc^∞ is defined by a $\mathbb{Z}$-valued function on $\hat{\mathcal{Z}}_{ech,M}$. For a given $[\Theta, Z]\in\hat{\mathcal{Z}}_{ech,M}$ where Θ is identified with $(\upsilon, \{(m_\mathfrak{p}, o_\mathfrak{p})\}_{\mathfrak{p}\in\Lambda})$ via (9), this function assigns to it the integer $\sum_{\mathfrak{p}\in\Lambda}|m_\mathfrak{p}| + 2|o_\mathfrak{p}|_\mathrm{o}$ where $|0|_\mathrm{o} = 0$, $|\pm 1|_\mathrm{o} = 1$, and $|\{1,-1\}|_\mathrm{o} = 2$. Given a finitely generated subgroup $\mathbb{H}$ of H^∞_{SW}, there exists $L_\mathbb{H} > 0$ such that for any $L \geq L_\mathbb{H}$ there exists $L' \geq L$ with the following significance: suppose $r > \pi$ is sufficiently large and $(\mu, \mathfrak{q}) \in \Omega\times\mathcal{P}$ is suitably generic subject to the constraints discussed in the previous subsection. Denote by $\hat{\mathcal{Z}}^{L'}_{ech,M}$ the subset of $\hat{\mathcal{Z}}_{ech,M}$ consisting of elements with filtration level less or equal L' and by $ecc^{\infty,L'}$ the free $\mathbb{Z}$-module that it generates. Then, there exists an injective $\mathbb{Z}$-equivariant map $\hat{\Phi}_r : \hat{\mathcal{Z}}^{L'}_{ech,M} \to \hat{\mathcal{Z}}_{SW,r}$ that induces a monomorphism

$$\mathbb{L}_r : ecc^{\infty,L'} \to C^{\infty}_{SW,r},$$

which intertwines the respective differentials and the $\mathbb{Z}[U]\otimes\wedge^*(H_1(Y;\mathbb{Z})/torsion)$-actions. Furthermore, the image of the induced map on homology contains $\mathbb{H}$. The assertion about the respective differentials and the $\mathbb{Z}[U]\otimes\wedge^*(H_1(Y;\mathbb{Z})/torsion)$-actions requires also the construction of an $\mathbb{R}$-equivariant map Ψ_r between the moduli spaces of J-holomorphic curves and instantons. The latter map together with the interpretation of the right-hand side of (23) as the algebraic intersection number of a J-holomorphic curve in $\mathbb{R}\times Y$ with the cylinder $\mathbb{R}\times\gamma_z$ verifies the claim about the remaining two equivalences and the long-exact sequences. The construction of both $\hat{\Phi}_r$ and Ψ_r make use of solutions to the vortex equations on $\mathbb{C}$. Next we talk about these equations and properties of their solutions.

4.3. Vortices on $\mathbb{C}$

Given a pair of a smooth imaginary-valued 1-form A_0 and a smooth complex-valued function α_0 on $\mathbb{C}$, the vortex equations on $\mathbb{C}$ read:

$$\begin{aligned} *dA_0 &= -i(1-|\alpha_0|^2), \\ \bar{\partial}_{A_0}\alpha_0 &= 0. \end{aligned} \tag{25}$$

Here, we assume that the complex plane is equipped with the standard Euclidean metric, and $\bar{\partial}_{A_0}$ denotes the ∂-bar operator on smooth sections of the trivial complex line bundle over $\mathbb{C}$. Recall that a smooth imaginary valued 1-form on $\mathbb{C}$ determines a holomorphic structure on the trivial complex line bundle over $\mathbb{C}$. These equations are subject to a pair of constraints:

- $|\alpha_0|^2 \leq 1$,
- $\int_{\mathbb{C}}(1-|\alpha_0|^2)dvol = 2\pi m$ for some positive integer m.

A few quick observations are in order. First, if (A_0,α_0) is a solution of the equations in (25) subject to the constraint in the first bullet, then it follows from the strong maximum principle that either $|\alpha_0| < 1$ or $|\alpha_0| \equiv 1$. Second, $|\alpha_0|$ has no non-zero local minimum unless $|\alpha_0| \equiv 1$. Assuming the contrary, in a neighborhood of a local minimum of $|\alpha_0|$ we can write $|\alpha_0| = e^u$ for a smooth function $u < 0$. Then, the top equation in (25) implies that $-\Delta u = 1-e^{2u}$. At a non-zero local minimum of $|\alpha_0|$, the right-hand side of the latter equation is strictly positive, while the left-hand side is strictly negative. Therefore we get a contradiction. On the other hand, using the constraint in the second bullet above, we deduce that $1-|\alpha_0|^2 \leq c_0 e^{-\sqrt{2}dist(\cdot,\alpha_0^{-1}(0))}$ where $c_0 > 1$ is a constant depending only on the integer m.

The group $C^{\infty}(\mathbb{C};S^1)$ acts on the space $C^{\infty}(\mathbb{C};iT^*\mathbb{C}\oplus\underline{\mathbb{C}})$ according to the following rule:

$$u\cdot(A_0,\alpha_0) = (A_0 - u^{-1}du, u\alpha_0).$$

Note that the set of solutions to the equations in (25) remain unchanged under this action. Meanwhile, the zeroes of α_0 are isolated and they appear as roots of the polynomial

$$z^m + \sigma_1 z^{m-1} + \cdots + \sigma_{m-1} z + \sigma_m,$$

where each σ_k is a function on $C^\infty(\mathbb{C}; iT^*\mathbb{C} \oplus \underline{\mathbb{C}})$ defined by

$$\sigma_k(A, \alpha_0) = \frac{1}{2\pi} \int_{\mathbb{C}} z^k (1 - |\alpha_0|^2) dvol$$

Denote by $\mathfrak{C}_m$ the space of $C^\infty(\mathbb{C}; S^1)$-orbits of solutions to the equations in (25) subject to the above constraints. Then $\mathfrak{C}_m$ is diffeomorphic to $\mathbb{C}^m$ via a map sending $[A_0, \alpha_0]$ to $(\sigma_1(A_0, \alpha_0), \ldots, \sigma_m(A_0, \alpha_0))$. The tangent space to $\mathfrak{C}_m$ at a point $[A_0, \alpha_0]$ is given by pairs of smooth complex-valued L^2 functions in the kernel of the following bounded complex-linear Fredholm operator on $C^\infty(\mathbb{C}; \mathbb{C} \oplus \mathbb{C})$,

$$\vartheta(q, \zeta) = (\partial q + 2^{-1/2} \bar{\alpha}_0 \zeta, \bar{\partial}_{A_0} \zeta + 2^{-1/2} \alpha_0 q).$$

The tangent space admits a Hermitian inner product defined by

$$\langle (q, \zeta), (q', \zeta') \rangle = \frac{1}{\pi} \int_{\mathbb{C}} (\bar{q} q' + \bar{\zeta} \zeta') dvol.$$

which induces a Kähler metric on $\mathfrak{C}_m$. This metric coincides with the standard Kähler metric on $\mathbb{C}^m$ if $m = 1$. The formal L^2-adjoint of the operator ϑ is defined by

$$\vartheta^\dagger(p, \xi) = (-\bar{\partial} p + 2^{-1/2} \bar{\alpha}_0 \xi, -\partial_{A_0} \xi + 2^{-1/2} \alpha_0 p).$$

Hence,

$$\vartheta \vartheta^\dagger(p, \xi) = (-\partial \bar{\partial} p + 2^{-1/2} |\alpha_0|^2 \xi, -\bar{\partial}_{A_0} \partial_{A_0} \xi + 2^{-1/2} |\alpha_0|^2 p),$$

when (A_0, α_0) defines a solution of the equations in (25). For more details on properties of solutions to the vortex equations, the reader is encouraged to refer to [39] and [15].

Given $\nu \in C^\infty(S^1; \mathbb{R})$ and $\mu \in C^\infty(S^1; \mathbb{C})$, we can define a Hamiltonian function $\hbar$ on $\mathfrak{C}_m$ as follows:

$$\hbar([A_0, \alpha_0]) = \frac{1}{4\pi} \int_{\mathbb{C}} (2\nu |z|^2 + \mu \bar{z}^2 + \bar{\mu} z^2)(1 - |\alpha_0|^2) dvol.$$

The imaginary part of the Kähler metric on $\mathfrak{C}_m$ defines a symplectic form, therefore the function $\hbar$ defines a Hamiltonian vector field on $\mathfrak{C}_m$ whose integral curves $\mathfrak{c}(t)$ obey

$$\frac{i}{2} \mathfrak{c}_* (\frac{d}{dt})^{(1,0)} + \nabla^{(1,0)} \hbar|_{\mathfrak{c}(t)} = 0, \tag{26}$$

where $\nabla^{(1,0)}$ denotes the holomorphic part of the gradient. A solution $\mathfrak{c}(t)$ of the equation in (26) satisfying $\mathfrak{c}(t + T) = \mathfrak{c}(t)$ for some $T > 0$ is called *periodic*. Given a smooth map $\mathfrak{c} : S^1 \to \mathfrak{C}_m$, the bundle $\mathfrak{c}^* T_{1,0} \mathfrak{C}_m$ admits a Hermitian connection arising from the Levi-Civita connection on $T_{1,0} \mathfrak{C}_m$. A periodic solution $\mathfrak{c}$ of the equation in (26) is called *non-degenerate* if the linearization of the Hamiltonian flow, namely the operator on $C^\infty(S^1; \mathfrak{c}^* T_{1,0} \mathfrak{C}_m)$ sending ζ to $\frac{i}{2} \nabla_{S^1} \zeta + (\nabla_\zeta \nabla^{(1,0)} \hbar)|_{\mathfrak{c}}$, has trivial kernel. Here ∇_{S^1} denotes the covariant derivative for the Hermitian connection on $\mathfrak{c}^* T_{1,0} \mathfrak{C}_m$.

Now let γ be a closed integral curve of the Reeb vector field $\mathcal{R}$, and fix a unitary trivialization of $K^{-1}|_\gamma$. Using the latter and the exponential map, identify a tubular neighborhood of γ in Y with a tubular neighborhood of $\gamma \times \{0\}$ in $\gamma \times \mathbb{C}$. Let $\varrho > 0$ be the radius of the latter, and (t, z) denote the coordinates on $\gamma \times \mathbb{C}$ where t is the affine coordinate on γ with values in $\mathbb{R}/L_\gamma\mathbb{Z}$, L_γ being the length of γ, and z is the complex coordinate on $\mathbb{C}$. We can arrange that the linearization of the aforementioned identification along $\gamma \times \{0\}$ sends $\frac{\partial}{\partial t}$ to $\frac{L_\gamma}{2\pi}\mathcal{R}$. In these coordinates, w can be written as

$$w = \frac{i}{2} dz \wedge d\bar{z} - 2(\nu z + \mu \bar{z})(d\bar{z} \wedge dt) - 2(\nu \bar{z} + \mu z)(dz \wedge dt) + \cdots ,$$

where ν is a real-valued function on S^1, since w is a closed 2-form, μ is a complex-valued functions on S^1, and the norms of the unwritten terms are bounded by $c_0|z|^2$. With the preceding understood, the next lemma is crucial to the construction of the map $\hat{\Phi}_r$.

Lemma 4.2 (Lemma 3.2 in [21]). *Suppose that $\Theta \in \mathcal{Z}_{ech,M}$ and γ is a closed integral curve of $\mathcal{R}$ from Θ. Then for $m = 1$ there exists a unique periodic solution of the equation in (26) associated to the pair (ν, μ) arising from a tubular neighborhood of γ as above. This solution arises as the constant map from S^1 to $\mathfrak{C}_m$ represented by a pair (A_0, α_0) such that $\alpha_0^{-1}(0) = 0$, and it is non-degenerate. Moreover, for any $\mathfrak{p} \in \Lambda$ and $\gamma \in \{\gamma_\mathfrak{p}^+, \gamma_\mathfrak{p}^-\}$, there are no periodic solutions to the equations in (26) if $m > 1$.*

4.4. Construction of the map $\hat{\Phi}_r$

The map $\hat{\Phi}_r$ arises as the $\mathbb{Z}$-equivariant covering of a map Φ_r from $\mathcal{Z}^{L'}_{ech,M}$ to $\mathcal{Z}_{SW,r}$. The latter map is constructed in two steps. First, use periodic solutions to the equation in (26) for $m = 1$ in order to associate to each $\Theta \in \mathcal{Z}^{L'}_{ech,M}$ the gauge equivalence class of a pair (A_Θ, ψ_Θ) of a Hermitian connection on E and a smooth section of $\mathbb{S}$ that nearly solves the equations in (20). Second, use perturbation theory for linear operators to show existence and uniqueness, up to gauge, of an honest solution of the equations in (20) associated to Θ. Various properties of the map Φ_r are summarized in the following proposition.

Proposition 4.3 (Proposition 3.1 in [21]). *There exists $\kappa > \pi$ such that for any $\mathrm{E} > 1$ and $L > \kappa\mathrm{E}$, there exists $\kappa_L > \kappa$ satisfying the following: fix $r \geq \kappa_L$ and $\mu \in \Omega$ with small norm. Then there exists an injective map $\Phi_r : \mathcal{Z}^L_{ech,M} \to \mathcal{Z}_{SW,r}$ whose image contains gauge equivalence classes of $(A, \psi = (\alpha, \beta))$ with*

$$\mathrm{M}(A, \psi = (\alpha, \beta)) = r \int_Y (1 - |\alpha|^2) dvol < \mathrm{E}.$$

Moreover, if $(A, \psi = (\alpha, \beta))$ is a solution of the equations in (20) whose gauge equivalence class is $\Phi_r(\Theta)$ for some $\Theta \in \mathcal{Z}^L_{ech,M}$, then

- *$(A, \psi = (\alpha, \beta))$ is both non-degenerate and holonomy non-degenerate.*
- *$\mathrm{M}(A, \psi = (\alpha, \beta)) < 2\pi \sum_{\gamma \in \Theta} L_\gamma + \kappa^{-1}$.*

- *$\alpha^{-1}(0)$ is a disjoint union of embedded closed curves which are in $1-1$ correspondence with components of Θ, and each lies in a $\kappa r^{-1/2}$ radius tubular neighborhood of its partner from Θ.*
- *The map Φ_r extends to an injective map from $\mathcal{Z}^L_{ech,M} \times \mathbb{Z} \to \mathcal{Z}_{SW,r} \times \mathbb{Z}$ which is identity when restricted to the $\mathbb{Z}$-factors. The latter map defines a $\mathbb{Z}$-equivariant map $\hat{\Phi}_r : \hat{\mathcal{Z}}^L_{ech,M} \to \hat{\mathcal{Z}}_{SW,r}$ via the identification in (9) and the one defined by x_z.*

A priori estimates on solutions to the equations in (20)—such as those in the next lemma—are crucial to the proof of the above proposition.

Lemma 4.4 (Lemma 2.1 in [21])**.** *There exists $\kappa > 1$ such that if $r \geq \kappa$ and $\mu \in \Omega$ has small norm, then a solution $(A, \psi = (\alpha, \beta))$ of the equations in (20) satisfies the following:*

- $|\alpha| \leq 1 + \frac{\kappa}{r}$,
- $|\beta|^2 \leq \frac{\kappa}{r}(1 - |\alpha|^2) + \frac{\kappa^3}{r^2}$,
- $|\nabla^A \alpha|^2 \leq \kappa r(1 - |\alpha|^2) + \kappa^3$,
- $|\nabla^{\mathbb{A}_0^t + A} \beta|^2 \leq \kappa(1 - |\alpha|^2) + \frac{\kappa^3}{r}$.

Furthermore, for any $i \geq 1$, there exists $\kappa_i > 1$ such that if $r \geq \kappa_i$ and $\mu \in \Omega$ has small norm, then

- $|(\nabla^A)^i \alpha| + r^{1/2} |(\nabla^{\mathbb{A}_0^t + A})^i \beta| \leq \kappa_i r^{i/2}$.

The proof of the above lemma is an application of the Bochner–Weitzenböck formula for the Dirac operator and standard elliptic regularity arguments. The above estimates are exact analogs of those in Lemmas 2.2-2.4 of [33] and in Section 2e of [31]. An improvement over the above lemma's estimates is stated as follows:

Lemma 4.5 (Lemma 2.3 in [21])**.** *There exists $\kappa > 1$ with the following property: suppose $r \geq \kappa$ and $\mu \in \Omega$ has small norm. Let $(A, \psi = (\alpha, \beta))$ be a solution of the equations in (20), and $Y_* \subset Y$ denote the subset of points where $1 - |\alpha|^2 \geq \kappa^{-1}$. Then*

$$|1 - |\alpha|^2| \leq e^{-\frac{\sqrt{r} dist(\cdot, Y_*)}{\kappa}} + \kappa r^{-1}.$$

In addition to the above a priori estimates on a solution $(A, \psi = (\alpha, \beta))$ of the equations in (20), we deduce several facts about the set $\alpha^{-1}(0)$. What follows is part of the content of Proposition 2.4 in [21]. There exists $\kappa > \pi$ with the following property: suppose $r \geq \kappa$ and $\mu \in \Omega$ has small norm. Let $(A, \psi = (\alpha, \beta))$ be a solution of the equations in (20), and $Y_r \subset Y$ denote the subset of points with distance $\kappa^2 r^{-1/2}$ from the curves in $\cup_{\mathfrak{p} \in \Lambda} \{\gamma^+_{\mathfrak{p}}, \gamma^-_{\mathfrak{p}}\}$. Then, α is transversal to the zero section of the Hermitian bundle E over the closure of Y_r. Moreover, $\alpha^{-1}(0)$ consists of at most G components each of which is either a properly embedded arc or a circle, and it has the following peoperties:

- The tangent line to each component has distance at most $\kappa r^{-1/2}$ from the line along the Reeb vector field $\mathcal{R}$.
- The intersection of each component with any $\mathcal{H}_{\mathfrak{p}}$ lies where $1 - 3\cos^2\theta > 0$.

- The intersection of $\alpha^{-1}(0)$ with $M_\delta \subset Y_r$ consists of G properly embedded segments that connect, in a $1-1$ manner, the boundaries of the radius δ coordinate balls around index-1 and index-2 critical points of the Morse function f.

Next we briefly explain how the map Φ_r as in Proposition 4.3 is constructed, following the general philosophy due to Taubes. The first of the two steps in the construction is to use non-degenerate periodic solutions to the equation in (26) to construct a pair of a Hermitian connection on E and a smooth section of $\underline{\mathbb{S}}$ on a neighborhood of each component of Θ, and then extend it to the rest of Y in a canonical manner. This is made possible by Lemma 4.2. The resulting pair nearly solves the equations in (20). In order to explain how such a pair is constructed, we start with a preliminary digression. The point in $\mathfrak{C}_1$ identified with the origin in $\mathbb{C}$ via the identification described in the previous subsection is called the *symmetric vortex*, because it is invariant under the action of S^1 by rotations on $\mathbb{C}$. There exists a unique pair (A_0, α_0) which defines the symmetric vortex and can be written as

- $A_0 = A_{\underline{\mathbb{C}}} - \frac{1}{2}\mathrm{a}_0(z^{-1}dz - \bar{z}^{-1}d\bar{z})$,
- $\alpha_0 = |\alpha_0|\frac{z}{|z|}$,

where $A_{\underline{\mathbb{C}}}$ is the trivial flat connection on the trivial complex line bundle over $\mathbb{C}$, and a_0 is a real-valued function on $\mathbb{C}$. Meanwhile, the following equation has a unique smooth L^2 solution

$$\vartheta\vartheta^\dagger(p,\xi) = -\Big(\frac{1}{\sqrt{2}}(1-|\alpha_0|^2), \partial_{A_0}\alpha_0\Big),$$

which can be written as $(p,\xi) = (2^{1/2}z\alpha_0^{-1}\partial_{A_0}\alpha_0, -\bar{z}\bar{\alpha}_0^{-1}(1-|\alpha_0|^2)$. Now, given Θ in $\mathcal{Z}^L_{ech,M}$ and $\gamma \in \Theta$, define a function $r_\gamma : \mathbb{C} \to \mathbb{C}$ by $r_\gamma(z) = (\frac{L_\gamma}{2\pi}r)^{1/2}z$. Fix $\rho \in (0, \frac{1}{100}\varrho)$ and let U_γ and U'_γ denote respectively the radius 4ρ and radius ρ tubular neighborhoods of γ. We choose ρ in such a way that $U_\gamma \cap U_{\gamma'} = \emptyset$ for any distinct pair $\gamma, \gamma' \in \Theta$. Next, denote by $U_0 = Y \setminus \cup_{\gamma\in\Theta}U'_\gamma$, and let $\chi_\gamma = \chi(|z|/\rho - 1)$. Note that the restriction of the Hermitian bundle E to any $U \in \{U_0\} \cup \{U_\gamma\}_{\gamma\in\Theta}$ admits a Hermitian trivialization so that the trivializing unit length section of the bundle $E|_{U_0}$ is identified with the section $\frac{z}{|z|}$ of the bundle $E|_{U_\gamma}$ under the corresponding trivialization. With the preceding understood, write a Hermitian connection A on $E|_U$ as $A = A_{\underline{\mathbb{C}}} + \mathrm{a}_U$ where a_U is an imaginary-valued 1-form on U, and a smooth section of $\underline{\mathbb{S}}|_U$ as (α_U, β_U). We will define a pair (A_Θ, ψ_Θ) of a Hermitian connection on E and a smooth section of $\underline{\mathbb{S}}$ via their restrictions to each $U \in \{U_0\} \cup \{U_\gamma\}_{\gamma\in\Theta}$.

- If $U = U_0$, then $\mathrm{a}_{U_0} = 0$ and $\psi_0 = (\alpha_{U_0}, \beta_{U_0}) = (1, 0)$.
- If $U = U_\gamma$ for some $\gamma \in \Theta$, then

$$\mathrm{a}_U = \chi_\gamma\big[i2^{1/2}\nu r^*_\gamma p\,dt - \frac{1}{2}r^*_\gamma\big(\mathrm{a}_0(z^{-1}dz - \bar{z}^{-1}d\bar{z})\big)\big] - (1-\chi_\gamma)u_r^{-1}du_r$$

  and $(\alpha_U, \beta_U) = (\chi_\gamma r^*_\gamma\alpha_0 + (1-\chi_\gamma)u_r, \chi_\gamma i\mu r^{-1/2}r^*_\gamma\xi)$ where $u_r = r^*_\gamma(z/|z|)$. Denote the corresponding pair of Hermitian connection on $E|_{U_\gamma}$ and smooth section of $\underline{\mathbb{S}}|_{U_\gamma}$ by (A_γ, ψ_γ).

Since the transition function between trivializations of $E|_U$ and $E|_{U_\gamma}$ for $\gamma \in \Theta$ is given by $z/|z|$ over $U \cap U_\gamma$, the collection $\{(A_0, \psi_0), \{(A_\gamma, \psi_\gamma)\}_{\gamma\in\Theta}\}$ defines the desired pair (A_Θ, ψ_Θ).

With (A_Θ, ψ_Θ) in hand, write a solution (A, ψ) of the equations in (20) as

$$(A, \psi) = (A_\Theta, \psi_\Theta) + ((2r)^{1/2} b, \eta),$$

where b is an imaginary-valued 1-form on Y and η is a smooth section of $\underline{\mathbb{S}}$. The pair (A, ψ) solves the equations on (20) if and only if $\mathfrak{b} = (b, \eta, \phi) \in C^\infty(Y; iT^*Y \oplus \underline{\mathbb{S}} \oplus i\mathbb{R})$ solves a certain linear first order elliptic differential equation. This equation can be written as

$$\mathcal{L}_{(A_\Theta,\psi_\Theta)}\mathfrak{b} + r^{1/2}\mathfrak{b} * \mathfrak{b} = \mathfrak{u},$$

where $\mathfrak{b} \mapsto \mathfrak{b} * \mathfrak{b}$ is a quadratic bundle map of $iT^*Y \oplus \underline{\mathbb{S}} \oplus i\mathbb{R}$ and $\mathfrak{u}$ has bounded norm. A careful investigation of the eigenvalue equation $\mathcal{L}_{(A_\Theta,\psi_\Theta)}\mathfrak{b} = \lambda\mathfrak{b}$ yield a solution of the above equations (See Appendix to [21] for details). The latter is used to define $\Phi_r(\Theta)$ for sufficiently large $r > \pi$.

We end this section with a few brief remarks as to the properties of the map Φ_r listed in Proposition 4.3. The second and the third bullets of Proposition 4.3 follow from the construction of the map Φ_r and a priori estimates from Lemmas 4.4 and 4.5. The assertion in the first bullet of Proposition 4.3 that the image of the map Φ_r is represented by pairs that are holonomy non-degenerate follows from Lemma 4.2 and the observation that $\hat{A}$ is flat with covariant constant section $\alpha/|\alpha|$ near γ_z. Therefore, $\mathrm{x}_z(A, \psi)$ is an integer since the connection A_E on E was chosen to have trivial holonomy around γ_z. As for the fourth bullet, fix a cycle $Z \in H_2(Y; \Theta, \Theta_0)$ so that it has zero algebraic intersection number with the curve γ_z. Then, the construction of the map Φ_r associates to the equivalence class $[\Theta, Z]$ the $\mathcal{G}_{M_\Lambda}$-orbit of a solution (A, ψ) of the equations in (20) with $\mathrm{x}_z(A, \psi) = 0$.

5. Back to the Seiberg–Witten Floer homology of M

The goal of this section is to explain the relation between the Seiberg–Witten Floer homology of M and the twisted Seiberg–Witten Floer cohomology of Y as described in Section 4.2. More precisely, we discuss the following filtered connected sum theorem for Seiberg–Witten Floer homology:

Theorem 5.1 (Theorem 4.1 in [18]). *There exists a commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & H^-_{SW} \arrow[r] \arrow[d] & H^\infty_{SW} \arrow[r] \arrow[d] & H^+_{SW} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \widehat{HM}_*(M, \mathfrak{s}, c_b) \otimes_{\mathbb{Z}} \hat{\mathrm{V}}^{\otimes \mathrm{G}} \arrow[r] & \overline{HM}_*(M, \mathfrak{s}, c_b) \otimes_{\mathbb{Z}} \hat{\mathrm{V}}^{\otimes \mathrm{G}} \arrow[r] & \widetilde{HM}_*(M, \mathfrak{s}, c_b) \otimes_{\mathbb{Z}} \hat{\mathrm{V}}^{\otimes \mathrm{G}} \arrow[r] & \cdots
\end{tikzcd}$$

where the top row is the long-exact sequence for the twisted version of the Seiberg–Witten Floer cohomology of Y associated to (24) and the bottom row is the standard long-exact sequence for the Seiberg–Witten Floer homology of M with balanced perturbations tensored with $\hat{\mathrm{V}}^{\otimes \mathrm{G}}$, while the vertical arrows are isomorphisms that reverse the sign of the relative gradings and intertwine the $\mathbb{Z}[U] \otimes \wedge^*(H_1(M; \mathbb{Z})/torsion)$-module structures.

The above theorem follows from an S^1-equivariant formulation of Seiberg–Witten Floer homology and a connected sum formula. Before we state these results, we need some notation and terminology. In this regard, let (C_*, ∂_*) be a chain complex of $\mathbb{Z}$-modules that admits a degree -2 chain map U. Then one may consider the mapping cone of the U-map defined by

$$(S_U(C_*), S_U(\partial_*)) := (C_* \otimes_{\mathbb{Z}} \mathbb{Z}[W]/(W^2), \partial_* \otimes \sigma + U \otimes W),$$

where $\sigma(p + qW) := p - qW$ and $\mathbb{Z}[W]/(W^2)$ is graded so that W is of degree 1. Hence, multiplication by $1 \otimes W$ defines a degree 1 chain map. Conversely, introduce the $\mathbb{Z}$-modules

$$\begin{aligned} \mathrm{V}^\infty &:= \mathbb{Z}[U, U^{-1}], \\ \mathrm{V}^- &:= U\mathbb{Z}[U], \\ \mathrm{V}^+ &:= \mathbb{Z}[U, U^{-1}]/U\mathbb{Z}[U]. \end{aligned}$$

Suppose (C_*, ∂_*) is a chain complex of $\mathbb{Z}$-modules that admits a degree 1 chain map W with $W^2 = 0$. Then we can define the following chain complexes:

$$\begin{aligned} E^\infty(C_*) &:= C_* \otimes_{\mathbb{Z}} \mathrm{V}^\infty, \\ E^-(C_*) &:= C_* \otimes_{\mathbb{Z}} \mathrm{V}^-, \\ E^+(C_*) &:= C_* \otimes_{\mathbb{Z}} \mathrm{V}^+, \end{aligned}$$

with the differential $E(\partial_*) := (\partial_* \otimes 1 + W \otimes U)$ and the U-map defined as multiplication by $1 \otimes U$ which is of degree -2. Note that there exists a short-exact sequence

$$0 \to \mathrm{V}^- \xrightarrow{i} \mathrm{V}^\infty \xrightarrow{\pi} \mathrm{V}^+ \to 0, \tag{27}$$

which induces a short-exact sequence

$$0 \to E^-(C_*) \xrightarrow{id \otimes i} E^\infty(C_*) \xrightarrow{id \otimes \pi} E^+(C_*) \to 0,$$

with associated long-exact sequence of homologies

$$\cdots \to H_*(E^-(C_*)) \to H_*(E^\infty(C_*)) \to H_*(E^+(C_*)) \to H_*(E^-(C)) \to \cdots . \tag{28}$$

Now, the next result is a kind of Kozsul duality in the sense of [10].

Proposition 5.2 (Proposition 4.8 in [22]). *Let (C_*, ∂_*) be a chain complex of $\mathbb{Z}$-modules that admits a degree -2 chain map U. Then there are isomorphisms*

$$H_*(E^\circ S_U(C_*)) \cong H_*(C_* \otimes_{\mathbb{Z}[U]} \mathrm{V}^\circ),$$

where $\circ \in \{\infty, -, +\}$. Furthermore, these isomorphisms are natural with respect to the associated long-exact sequences. On the other hand, let (C_*, ∂_*) be a chain complex of $\mathbb{Z}$-modules that admits a degree 1 chain map Y. Then there is an isomorphism

$$H_*(S_U E^-(C_*)) \cong H_*(C_*).$$

With the preceding understood, the first main ingredient in the proof of Theorem 5.1 is the following S^1-equivariant formulation of Seiberg–Witten Floer homologies due to Lee (cf. [23]):

Proposition 5.3 (Proposition 5.9 in [22]). *Let (C_*, ∂_*) denote the Seiberg–Witten Floer chain complex $(\widehat{CM}_*(M, \mathfrak{s}, c_b), \hat{\partial})$ where c_b indicates the use of a balanced perturbation. Then there exists a commutative diagram*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_*(E^- S_U(C_*)) & \longrightarrow & H_*(E^\infty S_U(C_*)) & \longrightarrow & H_*(E^+ S_U(C_*)) & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \widehat{HM}_*(M, \mathfrak{s}, c_b) & \longrightarrow & \overline{HM}_*(M, \mathfrak{s}, c_b) & \longrightarrow & \widetilde{HM}_*(M, \mathfrak{s}, c_b) & \longrightarrow & \cdots
\end{array}$$

where the top row is the long-exact sequence for equivariant homologies as in (28) and the bottom row is the long-exact sequence for the Seiberg–Witten Floer homology of M with balanced perturbations, while the vertical arrows are isomorphisms that preserve the relative gradings and intertwine the $\mathbb{Z}[U] \otimes \wedge^*(H_1(M; \mathbb{Z})/torsion)$-module structures.

Remark 5.4. Kronheimer and Mrowka prove that the chain complexes $(\widehat{CM}_*(M, \mathfrak{s}, c_b), \hat{\partial})$ and $(\widehat{CM}_*(M, \mathfrak{s}, c_-), \hat{\partial})$, where c_- denotes the period class for some negatively monotone perturbation, are chain homotopy equivalent (see Theorem 31.5.1 in [17]). Therefore, in the above proposition, one can also take C_* to be $(\widehat{CM}_*(M, \mathfrak{s}, c_-), \hat{\partial})$.

Next we discuss the second main ingredient in the proof of Theorem 5.1, which is a connected sum formula for Seiberg–Witten Floer homology. Before we state the connected sum formula, we need to introduce further notation. Let M_1 and M_2 be two closed, connected, and oriented 3-manifolds. Fix Spin^c structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ on M_1 and M_2 respectively. Denote by $\mathfrak{s}_\#$ and $\mathfrak{s}_\sqcup$ respectively the unique Spin^c structures on $M_1 \# M_2$ and $M_1 \sqcup M_2$ induced by $\mathfrak{s}_1$ and $\mathfrak{s}_2$. Given $\Omega_1 \in H^2(M_1; \mathbb{R})$ and $\Omega_2 \in H^2(M_2; \mathbb{R})$ the cohomology classes for perturbations of the Seiberg–Witten equations on M_1 and M_2, denote by $\Omega_\#$ and $\Omega_\sqcup$ the corresponding cohomology classes in $H^2(M_1 \# M_2; \mathbb{R})$ and $H^2(M_1 \sqcup M_2; \mathbb{R})$ respectively.

Proposition 5.5 (Proposition 6.2 in [22]). *Suppose that $\Omega_\#$ yields a negatively monotone perturbation. Then the chain complexes*

$$\widehat{CM}_*(M_1 \# M_2, \mathfrak{s}_\#, c_{\Omega_\#}) \qquad \textit{and} \qquad S_{U_\sqcup}(\widehat{CM}_*(M_1 \sqcup M_2, \mathfrak{s}_\sqcup, c_{\Omega_\sqcup}))$$

are chain homotopy equivalent. Furthermore, the induced isomorphism on homology intertwines the $\mathbb{Z}[U] \otimes \wedge^*(H_1(M_1; \mathbb{Z})/torsion)) \otimes \wedge^*(H_1(M_2; \mathbb{Z})/torsion)$-module structures and preserves the relative gradings.

Kronheimer and Mrowka defined Seiberg–Witten Floer chain complexes for connected 3-manifolds. In the above proposition and in what follows, we abuse the notation to denote by $(\widehat{CM}_*(M_1 \sqcup M_2, \mathfrak{s}_\sqcup, c_{\Omega_\sqcup}), \hat{\partial}_\sqcup)$ the tensor product of $(\widehat{CM}_*(M_1, \mathfrak{s}_1, c_{\Omega_1}), \hat{\partial}_{M_1})$ and $(\widehat{CM}_*(M_2, \mathfrak{s}_2, c_{\Omega_2}), \hat{\partial}_{M_2})$, with $U_\sqcup = U_1 \otimes 1 - 1 \otimes U_2$. This is acceptable since at least one of Ω_1 and Ω_2 yields a negatively monotone perturbation.

Now, Theorem 5.1 follows from an application of Proposition 5.5 to the following two cases:

- Take $M_1 = S^1 \times S^2$, $\mathfrak{s}_1$ so that $\langle c_1(\mathfrak{s}_1), [S^2]\rangle = 2$, and w_1 a nowhere zero closed 2-form representing $c_1(\mathfrak{s}_1)$ that is harmonic with respect to the product Riemannian metric on $S^1 \times S^2$. Meanwhile, take $M_2 = M$, $\mathfrak{s}_2$ any Spin^c structure, and w_2 a closed 2-form with non-degenerate zeros and $c_1(\mathfrak{s}_2) = [w_2]$ that is harmonic with respect to a suitable Riemannian metric on M, and has non-degenerate zeros in the case $c_1(\mathfrak{s}_2)$ is non-torsion. Denote by Y_0 the connected sum $M\#S^1 \times S^2$.
- Take $M_1 = Y_i = M\#_{i+1}S^1 \times S^2$ for $i \in \{0, 1, \ldots, \mathrm{G} - 1\}$, $\mathfrak{s}_1$ a Spin^c structure with $c_1(\mathfrak{s}_1)$ non-torsion, and w_1 a closed 2-form representing $c_1(\mathfrak{s}_1)$ that is harmonic with respect to a suitable Riemannian metric on Y_i. Meanwhile, take $M_2 = S^1 \times S^2$, $\mathfrak{s}_2$ is the trivial Spin^c structure, and $w_2 \equiv 0$.

In either of the above cases, take $\Omega_1 = rc_1(\mathfrak{s}_1)$, $\Omega_2 = rc_1(\mathfrak{s}_2)$, and hence $\Omega_\# = rc_1(\mathfrak{s}_\#)$ for $r > \pi$ sufficiently large. Note that in each Y_i for $i \in \{0, 1, \ldots, \mathrm{G} - 1\}$ and in $Y = Y_{\mathrm{G}}$, there exists a special simple closed curve γ that represents the generator of $H_1(S^1 \times S^2; \mathbb{Z})$ for the $S^1 \times S^2$ summand with non-trivial Spin^c structure. This cycle introduces a local system of coefficients Π_γ (see Section 22.6 in [22]). With the preceding understood, there is also a filtered variant of Proposition 5.5:

Proposition 5.6 (Proposition 6.2 in [22]). *Let M_1 and M_2 be as in either of the above two cases. Then for each $\circ \in \{\infty, -, +\}$ there exists an isomorphism between*

$$H_*(\widehat{CM}_*(M_1\#M_2, \mathfrak{s}_\#, c_{\Omega_\#}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^\circ) \cong H_*(S_{U_\sqcup}(\widehat{CM}_*(M_1 \sqcup M_2, \mathfrak{s}_\sqcup, c_{\Omega_\sqcup}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^\circ)).$$

Furthermore, these isomorphisms preserve the relative gradings, they intertwine the $\mathbb{Z}[U] \otimes \wedge^*(H_1(M_1; \mathbb{Z})/torsion)) \otimes \wedge^*(H_1(M_2; \mathbb{Z})/torsion)$-module structures, and are natural with respect to the long-exact sequences induced by (27).

In the case of the first bullet above, the equations in (20) on M_1 have a unique solution up to gauge. This solution can be written as $(A, (\alpha, \beta)) = (0, ((2r^{-1/2}, 0))$. Thus, for each $\circ \in \{\infty, -, +\}$ we have $\widehat{CM}_*(M_1, \mathfrak{s}_1, c_{\Omega_1}; \Pi_\gamma \otimes \mathrm{V}^\circ)) \cong \mathrm{V}^\circ$. Then the chain complex

$$S_{U_\sqcup}(\widehat{CM}_*(S^1 \times S^2 \sqcup M, \mathfrak{s}_\sqcup, c_{\Omega_\sqcup}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^\circ)) = \mathrm{V}^\circ \otimes_{\mathbb{Z}} \widehat{CM}_*(M, \mathfrak{s}_2, c_{\Omega_2}) \otimes_{\mathbb{Z}} \mathbb{Z}[W]/(W^2),$$

with differential $S_{U_\sqcup}(\hat{\partial}_\sqcup) = 1 \otimes \hat{\partial}_M \otimes \sigma + (U_1 \otimes 1 - 1 \otimes U_2) \otimes W$, can be rewritten as

$$E^\circ(\widehat{CM}_*(M, \mathfrak{s}_2, c_{\Omega_2}) \otimes_{\mathbb{Z}} \mathbb{Z}[W]/(W^2), \hat{\partial}_M \otimes \sigma - U_2 \otimes W),$$

in other words, $E^\circ S_{U_2}(\widehat{CM}_*(M, \mathfrak{s}_2, c_{\Omega_2}), \hat{\partial}_M)$. Using Propositions 5.3 and 5.6, we obtain isomorphisms

$$\begin{aligned} H_*(\widehat{CM}_*(Y_0, \mathfrak{s}_\#, c_{\Omega_\#}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^\infty) &\cong \overline{HM}_*(M, \mathfrak{s}, c_b), \\ H_*(\widehat{CM}_*(Y_0, \mathfrak{s}_\#, c_{\Omega_\#}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^-) &\cong \widehat{HM}_*(M, \mathfrak{s}, c_b), \\ H_*(\widehat{CM}_*(Y_0, \mathfrak{s}_\#, c_{\Omega_\#}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^+) &\cong \widetilde{HM}_*(M, \mathfrak{s}, c_b), \end{aligned}$$

which respect the associated long-exact sequences. To be more explicit, the long-exact sequence for the groups on the left arises from the short-exact sequence in (27).

With regard to the second bullet above, the Seiberg–Witten equations for the trivial Spin^c structure on $S^1 \times S^2$ equipped with a constant curvature product Riemannian metric have only reducible solutions, and the set of solutions modulo gauge is identified with the circle $H^1(S^1 \times S^2; i\mathbb{R})/2\pi i H^1(S^1 \times S^2; \mathbb{Z})$. Then, as is explained in Section 36 of [17], $\widehat{CM}_*(S^1 \times S^2, \mathfrak{s}_2) \cong \mathbb{Z}[U_2] \oplus \mathbb{Z}[U_2][1]$ as graded $\mathbb{Z}$-modules. Therefore, the chain complex $(S_{U_\sqcup}(\widehat{CM}_*(Y_i \sqcup S^1 \times S^2, \mathfrak{s}_\sqcup, c_{\Omega_\sqcup}; \Pi_\gamma \otimes \mathrm{V}^\circ)), S_{U_\sqcup}(\hat{\partial}_\sqcup))$ can be written as

$$\widehat{CM}_*(Y_i, \mathfrak{s}_1, c_{\Omega_1}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^\circ)) \otimes_{\mathbb{Z}} (\mathbb{Z}[U_2] \oplus \mathbb{Z}[U_2][1]) \otimes_{\mathbb{Z}} \mathbb{Z}[W]/(W^2),$$

with differential

$$\hat{\partial}_{Y_i} \otimes 1 \otimes \sigma - (U_1 \otimes U_2) \otimes W.$$

A straightforward computation shows that

$$H_*(S_{U_\sqcup}(\widehat{CM}_*(Y_i \sqcup S^1 \times S^2, \mathfrak{s}_\sqcup, c_{\Omega_\sqcup}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^\circ),)) \cong H_*(\widehat{CM}_*(Y_i, \mathfrak{s}_1, c_{\Omega_1}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^\circ) \otimes_{\mathbb{Z}} \hat{\mathrm{V}}.$$

In conclusion,

$$\begin{aligned}
H_*(\widehat{CM}_*(Y_{\mathrm{G}}, \mathfrak{s}_\#, c_{\Omega_\#}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^\infty) &\cong \overline{HM}_*(M, \mathfrak{s}, c_b) \otimes_{\mathbb{Z}} \hat{\mathrm{V}}^{\otimes \mathrm{G}},\\
H_*(\widehat{CM}_*(Y_{\mathrm{G}}, \mathfrak{s}_\#, c_{\Omega_\#}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^-) &\cong \widehat{HM}_*(M, \mathfrak{s}, c_b) \otimes_{\mathbb{Z}} \hat{\mathrm{V}}^{\otimes \mathrm{G}},\\
H_*(\widehat{CM}_*(Y_{\mathrm{G}}, \mathfrak{s}_\#, c_{\Omega_\#}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^+) &\cong \widetilde{HM}_*(M, \mathfrak{s}, c_b) \otimes_{\mathbb{Z}} \hat{\mathrm{V}}^{\otimes \mathrm{G}}.
\end{aligned}$$

Here, Y_{G} is diffeomorphic to the manifold Y, but with opposite orientation. It is equipped with a closed 2-form and a Riemannian metric as induced by the initial data in the second bullet above. Keeping in mind that there are canonical isomorphisms between the Seiberg–Witten Floer cohomology groups described here on Y and the corresponding filtered Seiberg–Witten Floer homology groups on $\overline{Y}$, the orientation reversed copy of Y, we have $H_*(\widehat{CM}_*(Y_{\mathrm{G}}, \mathfrak{s}_\#, c_{\Omega_\#}; \Pi_\gamma \otimes_{\mathbb{Z}} \mathrm{V}^\circ) \cong H^\circ_{SW}$, which follows by analyzing the Seiberg–Witten equations on the product cobordism $\mathbb{R} \times \overline{Y}$ equipped with a suitable self-dual 2-form and a Riemannian metric interpolating between the data on Y_{G} and the stable Hamiltonian structure data on $\overline{Y}$. This last equivalence finishes our outline of the proof of Theorem 5.1.

Remark 5.7. The orientation of the manifold Y fixed in this article and in [19, 20, 21] is the opposite of the orientation used in [18] and in [22]. To be more precise, the convention used in the latter is that the inclusion of M_δ into Y is an orientation preserving map.

6. Final Remarks

As is mentioned in the Introduction, both Heegaard Floer and Seiberg–Witten Floer homologies fit in the general framework of topological quantum field theories, but for a few important caveats. Having fixed a commutative ring with unity $\mathfrak{R}$, a TQFT $\mathcal{F}$ in dimension 3 associates to every closed, oriented, and smooth 3-manifold M a module $\mathcal{F}(M)$ over $\mathfrak{R}$, and to every compact, oriented, and smooth 4-manifold with boundary M it associates an element $\mathcal{F}(Z) \in \mathcal{F}(M)$. In particular, $\mathcal{F}$ associates the ground ring

$\mathfrak{R}$ to the empty 3-manifold. According to Atiyah [1], $\mathcal{F}$ should also satisfy the following axioms:

(1) It is functorial with respect to orientation preserving diffeomorphisms of 3- and 4-manifolds.
(2) It is involutary, in other words, the module associated to $-M$ is the dual of the one associated to M.
(3) It is multiplicative.

Let us explain the multiplicative axiom. Suppose that Z is a cobordism from M_1 to M_2, i.e., $\partial Z = -M_1 \sqcup M_2$. In this case, the multiplicative axiom requires that

$$\mathcal{F}(Z) \in \mathcal{F}(-M_1) \otimes_{\mathfrak{R}} \mathcal{F}(M_2).$$

Using the involutary axiom, the latter can be rewritten as $\mathcal{F}(Z) \in Hom_{\mathfrak{R}}(\mathcal{F}(M_1), \mathcal{F}(M_2))$. The multiplicative axiom also requires that cobordism maps are transitive under the composition of cobordisms. Note that if Z is a cobordism from $\emptyset$ to M, then $\mathcal{F}(Z)$ is in $Hom_{\mathfrak{R}}(\mathfrak{R}, \mathcal{F}(M)) \cong \mathcal{F}(M)$; and if Z is closed, then $\mathcal{F}(Z) \in \mathfrak{R}$. Alternatively, given a closed, oriented, and smooth 4-manifold X, one cuts X open along a closed, oriented, and smooth 3-manifold M so that it becomes the composition of two cobordisms $X = Z_1 \cup_M Z_2$ where Z_1 is a cobordism from $\emptyset$ to M and Z_2 is a cobordism from M to $\emptyset$. Then, $\mathcal{F}(X) \in \mathfrak{R}$ is obtained via the pairing $\langle \mathcal{F}(Z_1), \mathcal{F}(-Z_2) \rangle$.

Here are a few points where Heegaard Floer and Seiberg–Witten Floer homologies differ from the classical picture of a TQFT explained above. First of all, the homology groups and the homomorphisms between them are defined only for connected 3-manifolds and connected cobordisms between them. Second, neither the Heegaard Floer homology groups $HF^\infty(M)$, $HF^-(M)$, $HF^+(M)$ nor the Seiberg–Witten Floer homology groups $\overline{HM}_*(M)$, $\widehat{HM}_*(M)$, $\widetilde{HM}_*(M)$ as defined by the unperturbed Seiberg–Witten equations in (14) are functorial since an infinite number of Spin^c structures on a given cobordism may contribute non-trivially to its induced homomorphism. Hence, one should instead consider the completions $\mathbf{HF}^\infty(M)$, $\mathbf{HF}^-(M)$, $\mathbf{HF}^+(M)$ and $\overline{HM}_\bullet(M)$, $\widehat{HM}_\bullet(M)$, $\widetilde{HM}_\bullet(M)$[4] of these groups with respect to the filtrations induced by their $\mathbb{Z}[U]$-module structures. Even so, Heegaard Floer homology is not natural in any obvious way, i.e., it assigns to a closed, connected, and oriented 3-manifold a trio of groups up to isomorphism. This issue has recently been addressed by Juhász and Thurston in [16]. Granted naturality, the invariants for closed, connected, oriented, and smooth 4-manifolds with $b_2^+ > 0$ obtained via the above recipe using any of these groups are trivial. Therefore, a slightly different recipe was employed by Ozsváth and Szabó to define the closed 4-manifold invariants.

[4]Having fixed a Spin^c structure $\mathfrak{s}$ on M, Kronheimer and Mrowka proved that there are isomorphisms

$$\begin{aligned}\overline{HM}_\bullet(M,\mathfrak{s}) &\cong H_*(\overline{CM}_\bullet(M,\mathfrak{s},c_b),\bar{\partial})\\ \widehat{HM}_\bullet(M,\mathfrak{s}) &\cong H_*(\widehat{CM}_\bullet(M,\mathfrak{s},c_b),\hat{\partial})\\ \widetilde{HM}_\bullet(M,\mathfrak{s}) &\cong H_*(\widetilde{CM}_\bullet(M,\mathfrak{s},c_b),\check{\partial}),\end{aligned}$$

(see Theorem 31.1.1 in [17]).

Given a closed, connected, and oriented 3-manifold M and a Spin^c structure $\mathfrak{s}$ on M, recall the long-exact sequence for the Heegaard Floer homology groups:

$$\cdots \to HF^-(M,\mathfrak{s}) \xrightarrow{i_*} HF^\infty(M,\mathfrak{s}) \xrightarrow{\pi_*} HF^+(M,\mathfrak{s}) \xrightarrow{\delta} HF^-(M,\mathfrak{s}) \to \cdots .$$

The above long-exact sequence is induced by a short-exact sequence of chain complexes, and δ is the connecting homomorphism. The image of δ, equivalently, the kernel of i_*, is a finite rank subgroup of $HF^-(M,\mathfrak{s})$ denoted by $HF^-_{red}(M,\mathfrak{s})$. The latter is isomorphic, by exactness, to the cokernel of π_*, which is denoted by $HF^+_{red}(M,\mathfrak{s})$. If Z is a cobordism from M_1 to M_2 with $b_2^+ > 0$, then the map $HF^-(Z,\mathfrak{t}) : HF^-(M_1,\mathfrak{s}_1) \to HF^-(M_2,\mathfrak{s}_2)$ factors through $HF^-_{red}(M_2,\mathfrak{s}_2)$, while the map $HF^+(Z,\mathfrak{t}) : HF^+(M_1,\mathfrak{s}_1) \to HF^+(M_2,\mathfrak{s}_2)$ factors through $HF^+_{red}(M_1,\mathfrak{s}_1)$. With the preceding understood, let X be a closed, oriented, and smooth 4-manifold with $b_2^+ > 1$ and $\mathfrak{s}$ be a Spin^c structure on X. Then, Ozsváth and Szabó define the 4-dimensional invariant $\Phi_{X,\mathfrak{s}}$ as follows: fix a closed, connected, oriented, and smooth 3-manifold M embedded in X that separates X into two pieces each with $b_2^+ > 0$, and such that $H_2(M;\mathbb{Z})$ has trivial image in $H_2(X;\mathbb{Z})$. Upon excising an open ball in each of these pieces, we obtain the cobordisms Z_1 from S^3 to M and Z_2 from M to S^3. Finally, define $\Phi_{X,\mathfrak{s}}$ as the composition of the maps

$$\begin{aligned} HF^-(Z_1,\mathfrak{s}|Z_1) &: HF^-(S^3) \to HF^-_{red}(M,\mathfrak{s}|M), \\ HF^+(Z_2,\mathfrak{s}|Z_2) &: HF^+_{red}(M,\mathfrak{s}|M) \to HF^+(S^3), \end{aligned}$$

while identifying $HF^-_{red}(M,\mathfrak{s}|M)$ with $HF^+_{red}(M,\mathfrak{s}|M)$ via the connecting homomorphism δ. An analog of the preceding construction, perhaps using a local system of coefficients, can be repeated using the Seiberg–Witten Floer homology groups to capture the Seiberg–Witten invariants of X, which are defined by a signed count of solutions of the Seiberg–Witten equations on X (cf. Section 3.8 in [17]). As a matter of fact, Ozsváth and Szabó conjectured the following:

Conjecture 6.1 ([28])**.** *Let X be a closed, connected, oriented, and smooth 4-manifold with $b_2^+ > 1$ and $\mathfrak{s}$ be a Spin^c structure on X. Then, the invariant $\Phi_{X,\mathfrak{s}}$ agrees with the Seiberg–Witten invariant for X in the Spin^c structure $\mathfrak{s}$.*

Granted the above claim about the Seiberg–Witten invariants of closed, connected, oriented, and smooth 4-manifolds, and the fact that Heegaard Floer and Seiberg–Witten Floer homologies are isomorphic, what remains to be shown to prove Conjecture 6.1 is the *naturality* of these isomorphisms, i.e., that the isomorphisms between Heegaard Floer and Seiberg–Witten Floer homology groups intertwine respective cobordism maps. In order to prove naturality, one would like to define cobordism maps for *ech* and compare these to cobordism maps for Heegaard Floer and Seiberg–Witten Floer homologies. In order to do so, one needs to understand the component of the moduli space of pseudo-holomorphic curves in cobordisms with *ech* index 0. In general, as explained by Hutchings, the latter may have a very complicated structure as there may be *ech* index-0 broken curves with negative *ech* index levels. Luckily, such complications do not arise in the context described in this article thanks to the fact that all Reeb orbits in our setting are hyperbolic and the only homologically trivial Reeb orbits are positive hyperbolic. Therefore, contrary to the case of embedded contact homology, it should be possible to give a definition of cobordism maps for *ech* solely via pseudo-holomorphic curves. This is part of the content of work in progress by the author.

Acknowledgements: The author would like to thank the anonymous referee for numerous comments that helped improve the exposition.

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