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Action of the cork twist on Floer homology

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Abstract. We utilize the Ozsváth-Szabó contact invariant to detect the action of involutions on certain homology spheres that are surgeries on symmetric links, generalizing a previous result of Akbulut and Durusoy. Potentially this may be useful to detect different smooth structures on 4-manifolds obtained by the cork twisting operation.

1. Introduction

Any two different smooth structures of a closed simply connected 4-manifold are related to each other by a cork twisting operation [16], and the cork can be assumed to be a Stein manifold [4] (see [9] and [10] for applications). A quick way to generate corks, which was used in [11], is from symmetric links as follows: Let L be a link in S^3 with two components $K_1 \cup K_2$. Suppose that L satisfies the following:

(**1**) Both components K_1 and K_2 are unknotted.
(**2**) There is an involution of S^3 exchanging K_1 and K_2.
(**3**) The linking number of K_1 and K_2 is ± 1 (for some choice of orientations).

From this we can construct a 4–manifold $W(L)$ by carving out a disk bounded by K_1 from a 4–ball, and attaching a 2–handle along K_2 with framing 0. Therefore a handlebody diagram of $W(L)$ is given by a planar projection of L decorated by a dot on K_1 (cf. [1]), and a 0 on top of K_2. We also require that the 4–manifold $W(L)$ admits an additional structure:

(**4**) The handlebody of $W(L)$ described above is induced by a Stein structure.

The last condition can be reformulated as follows:

(**4'**) Regard $K_2 \subset S^1 \times S^2 = \partial S^1 \times B^3$ equipped with the unique Stein fillable contact structure. Then the maximal Thurston-Bennequin number of K_2 is at least $+1$.

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We will call links satisfying conditions (1) – (4) *admissible*. Some examples of admissible links are given in Figure 1. These examples were first studied in [9]. Note that the Hopf link is not admissible as it does not satisfy condition **(4)** even though the corresponding 4–manifold is B^4 which admits a Stein structure. Condition **(3)** ensures that $W(L)$ a contractible 4–manifold. Hence its boundary is a homology sphere. By condition **(2)**, we have an involution τ on $\partial W(L)$ that is obtained by exchanging the components of L. The significance this involution is indicated by the following theorem:

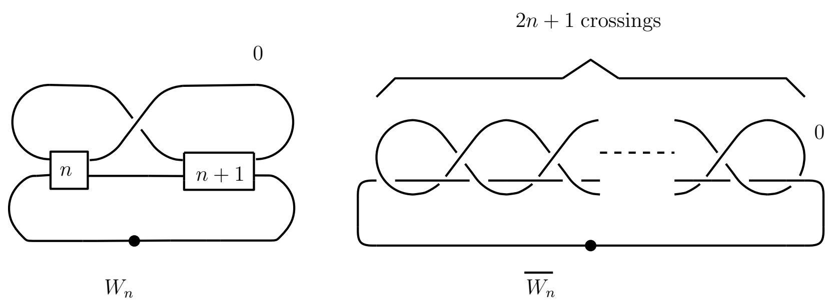

Figure 1. Examples of corks associated to certain symmetric links

Theorem 1.1. *Let L be an admissible link. The involution $\tau : \partial W(L) \to \partial W(L)$ acts non trivially on the Heegaard Floer homology group $HF^+(-\partial W(L))$*

This result generalizes [6], and it implies a result from [11]; namely τ does not extend to $W(L)$ as a diffeomorphism, even though it extends as a homeomorphism. Therefore $W(L)$ is a cork in the sense of [9]. The involution τ is called the *cork twist.* Theorem 1.1 was proved for the Mazur manifold W_1 in [2], and [26] (for instanton Floer homology), and [6] (for Heegaard Floer homology). Unlike the arguments in these papers, we do not explicitly find the Floer homology of $\partial W(L)$, and calculate the homomorphism the cork twist induces. Instead, we show that the homomorphism permutes two different distinguished Floer homology classes. These classes $c^+(\xi)$ are naturally associated to a cork via the induced contact structure ξ on the boundary. This suggests that a cork should be considered with its contact homology class $(W, c^+(\xi))$, to be used as a tool for checking nontriviality of its involution. In the course of our proof, we will incorporate several techniques developed in [9], [8] and [25].

Organization of this paper is as follows. We will review some standard facts about Stein manifolds, and Heegaard Floer homology respectively in the next two sections. We shall deduce Theorem 1.1 from a slightly stronger result (Theorem 4.1). This result along with some easy consequences are discussed in section 4.

2. Stein manifolds and their symplectic compactifications

Our aim in this section is to review the proof of the embedding theorem of Stein manifolds into closed symplectic manifolds in dimension four as given in [8]. See [18] for an alternative proof. We assume that the reader is familiar with the basics of contact geometry, open book decompositions and Lefschetz fibrations (cf. [19]). We start by recalling the topological characterization of Stein manifolds.

Theorem 2.1 ([12]). *Let $W = B^4 \cup$ (1-handles) $\cup$ (2-handles) be a 4-dimensional handlebody with one 0-handle and no 3 or 4-handles. Then:*

(1) *The standard Stein structure on B^4 can be extended over the 1-handles so that the manifold $W_1 := B^4 \cup$ (1-handles) becomes Stein.*
(2) *If each 2-handle is attached to ∂W_1 along a Legendrian knot with framing one less than the Thurston-Bennequin framing, then the complex structure on W_1 can be extended over the 2-handles making W a Stein manifold.*
(3) *The handle decomposition of W is induced by a strictly plurisubharmonic Morse function.*

By Theorem 2.1 (see also [15]), we represent Stein manifolds by special kind of handlebody diagrams which contain handles of index up to two and the attaching circles of the two handles are all Legendrian (i.e., they have horizontal cusps instead of vertical tangencies and the smaller slope strand is over the bigger slope strand at each crossing). For a Legendrian knot, the Thurston-Bennequin number (*tb* for short) is defined to be the writhe minus half of the number of cusps. In these special diagrams we understand that the framing on each 2–handle is one less than *tb* as in item (2) in Theorem 2.1. By abuse of language, a handle decomposition as in Theorem 2.1 is called a Stein structure.

Similarly for a given contact manifold (Y, ξ), one can attach 1–handles and $tb - 1$ framed 2–handles to $Y \times \{1\} \subset Y \times [0, 1]$, in order to form a *Stein cobordism* built on (Y, ξ).

The embedding theorem relies on the fact that Stein manifolds are equivalent to *positive allowable Lefschetz fibrations* ($PALF$ for short). Recall that a $PALF$ on a 4–manifold W is a Lefschetz fibration over a disk whose fibers have non-empty boundary and vanishing cycles are all non-separating curves. The restriction of a $PALF$ on the boundary $Y = \partial W$ is an open book decomposition whose monodromy can be written as a product of right handed Dehn twists. By [7], every Stein 4–manifold admits a $PALF$ (and every $PALF$ has a Stein structure). The construction is algorithmic where input is a handlebody decomposition of a Stein 4–manifold W as described in Theorem 2.1 and the output is a $PALF$ of W which is unique up to positive stabilization. Moreover, it is proved in [25] that the open book induced by this $PALF$ is compatible (in the sense of [14]) with the contact structure ξ induced by the Stein structure.

Given a Stein manifold W, fix a compatible $PALF$ as in the previous paragraph. Then one can extend this $PALF$ to a Lefschetz fibration over a closed manifold. Here is a sketch of what to do: First, recall the chain relation in the mapping class group of a surface. Let Σ_g be a surface of genus g. Let $\beta_1, \beta_2, \cdots, \beta_{2g}$ be a set of non-separating simple closed curves such that the following hold:

- $|\beta_i \cap \beta_j| = 1$ if $|i-j| = 1$,
- $|\beta_i \cap \beta_j| = 0$ if $|i-j| > 1$.

Proposition 2.2. *Let t_α denote the mapping class of the right handed Dehn twist about a curve α. Then following relation holds in the mapping class group of Σ_g:*

$$(t_{\beta_1} t_{\beta_2} \cdots t_{\beta_{2g}})^{4g+2} = 1. \tag{2.1}$$

Now on the Stein manifold W, we first choose a $PALF$ which induces an open book on $Y = \partial W$ with connected binding. This open book is compatible with the induced contact structure ξ. Then we attach a 2–handle along the binding with 0–framing relative to the page framing. Denote the corresponding cobordism by $V_0 : Y \to Y_0$. By [13], V_0 can be equipped with a symplectic structure extending the one defined on a collar neighborhood of Y. On the other hand, Y_0 is a surface bundle over the circle whose monodromy can be written as a product of right handed Dehn twists along non-separating curves. Let F denote a generic fiber in Y_0. Next, we use the chain relation of Proposition 2.2 to trivialize the monodromy by attaching -1 framed 2–handles: Write the monodromy of Y_0 as a product of right handed Dehn twists $t_{\gamma_1} \cdots t_{\gamma_n}$, where each γ_i is a non-separating curve on F. There is a diffeomorphism of F identifying γ_i with β_1 for each $i = 1, \cdots, n$. Using this diffeomorphism and the chain relation, we can write $t_{\gamma_i}^{-1}$ as a product of right handed Dehn twists. Attaching 2–handles as necessary, we can trivialize the monodromy. Finally we attach a copy $F \times D^2$ to get a cobordism $V_1 : Y_0 \to \emptyset$. Note that V_1 itself admits a Lefschetz fibration (with closed fibers) over disk. The closed 4–manifold $X := W \cup V_0 \cup V_1$ naturaly admits a Lefshcetz fibration over S^2. It is also possible to show that the Lefschetz fibration has a section, and hence symplectic, and that the construction can be made to guarantee that $b_2^+(X) \geq 2$. In other words, $V = V_0 \cup V_1$ is a *concave symplectic filling* for (Y, ξ). We summarize this construction in the following statement.

Theorem 2.3 ([7])**.** *Every Stein fillable contact manifold (Y, ξ) admits a concave symplectic filling $V = V_0 \cup V_1$, where V_0 is the cobordism $Y \to Y_0$ corresponding to a 2–handle attachment along the binding of an open book compatible with ξ, and V_1 admits a Lefschetz fibration over disk with closed fibers, which extends the fibration on Y_0. Moreover, V can be chosen in such a way that $b_2^+(V) \geq 2$.*

3. Heegaard Floer homology

Heegaard Floer homology ([21], [22]) is a type of Lagrangian Floer homology for the symmetric product of a Heegaard surface of a 3–manifold. There are several versions denoted by $\widehat{HF}(Y,\mathfrak{t})$, $HF^+(Y,\mathfrak{t}), HF^-(Y,\mathfrak{t}), HF^\infty(Y,\mathfrak{t})$. All of these groups are invariants of a 3-manifold Y with a Spin^c structure $\mathfrak{t}$. When $c_1(\mathfrak{t})$ is torsion these groups are $\mathbb{Q}$-graded. Each one admits an endomorphism U of degree -2 which makes all of them $\mathbb{Z}[U]$ modules (the action of U on $\widehat{HF}$ is trivial). They also satisfy the property that any Spin^c cobordism $(M,\mathfrak{s}) : (Y_1,\mathfrak{t}_1) \to (Y_2,\mathfrak{t}_2)$, from $(Y_1,\mathfrak{t}_1)$ to $(Y_2,\mathfrak{t}_2)$, induces a homomorphism

$$F^\circ_{(M,\mathfrak{s})} : HF^\circ(Y_1,\mathfrak{t}_1) \to HF^\circ(Y_2,\mathfrak{t}_2) \tag{3.1}$$

where HF° represents any of $\widehat{HF}, HF^+, HF^-$, or HF^∞. When both $c_1(\mathfrak{t}_1)$ and $c_1(\mathfrak{t}_2)$ are torsion, these homomorphisms shift degree by

$$d(M,\mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 - 3\sigma(M) - 2\chi(M)}{4}. \tag{3.2}$$

These homomorphisms also satisfy the following *composition law*: given two Spin^c cobordisms $(M_1,\mathfrak{s}_1) : (Y_1,\mathfrak{t}_1) \to (Y_2,\mathfrak{t}_2)$ and $(M_2,\mathfrak{s}_2) : (Y_2,\mathfrak{t}_2) \to (Y_3,\mathfrak{t}_3)$, and $F^\circ_{M_1,\mathfrak{s}_1}$ and $F^\circ_{M_2,\mathfrak{s}_2}$ are the induced homomorphisms, their composition is given by

$$F^\circ_{M_2,\mathfrak{s}_2} \circ F^\circ_{M_1,\mathfrak{s}_1} = \sum_{\mathfrak{s}\in \mathrm{Spin}^c(M_1\cup W_2):\mathfrak{s}|_{M_i}=\mathfrak{s}_i} F^\circ_{M_1\cup M_2,\mathfrak{s}}. \tag{3.3}$$

The following long exact sequence exists for every Spin^c 3–manifold $(Y,\mathfrak{t})$ and it is natural under cobordism–induced homomorphisms.

$$\cdots \xrightarrow{\ \delta\ } HF^-(Y,\mathfrak{t}) \xrightarrow{\ \iota\ } HF^\infty(Y,\mathfrak{t}) \xrightarrow{\ \pi\ } HF^+(Y,\mathfrak{t}) \xrightarrow{\ \delta\ } \cdots \tag{3.4}$$

The connecting homomorphism δ is an isomorphim between coker(π) and $\ker \iota$. We denote these groups respectively by $HF^+_{\mathrm{red}}(Y,\mathfrak{t})$ and $HF^-_{\mathrm{red}}(Y,\mathfrak{t})$, and call both of them by the same name: the *reduced* Heegaard Floer homology.

Heegaard Floer homology also provides a 4-manifold invariant (c.f. [23]). To review its definition first recall the mixed homomorphisms. Let $(M,\mathfrak{s}) : (Y_1,\mathfrak{t}_1) \to (Y_2,\mathfrak{t}_2)$ be a Spin^c cobordism with $b_2^+(M) \geq 2$. Every such cobordism admits an admissible cut, i.e., M can be decomposed as a union of two codimension zero sub-manifolds M_1 and M_2 with $b_2^+(M_i) \geq 1$, $i = 1,2$, and $\delta H^1(N) \subseteq H^2(M)$ is trivial where N is the common boundary of these two sub-manifolds. Let $\mathfrak{s}_i = \mathfrak{s}|_{M_i}$ and $\mathfrak{t} = \mathfrak{s}|_N$. Then we have the following commutative diagram.

$$\begin{tikzcd}
HF^+(Y_1,\mathfrak{t}_1) \arrow[r, "\delta"] \arrow[d, "F^+_{M_1,\mathfrak{s}_1}"] & HF^-(Y_1,\mathfrak{t}_1) \arrow[r, "\iota"] \arrow[d, "F^-_{M_1,\mathfrak{s}_1}"] & HF^\infty(Y_1,\mathfrak{t}_1) \arrow[d, "F^\infty_{M_1,\mathfrak{s}_1}=0"] \\
HF^+(N,\mathfrak{t}) \arrow[r, "\delta"] \arrow[d, "F^+_{M_2,\mathfrak{s}_2}"] & HF^-(N,\mathfrak{t}) \arrow[r, "\iota"] \arrow[d, "F^-_{M_2,\mathfrak{s}_2}"] & HF^\infty(N,\mathfrak{t}) \arrow[d, "F^\infty_{M_2,\mathfrak{s}_2}=0"] \\
HF^+(Y_2,\mathfrak{t}_1) \arrow[r, "\delta"] & HF^-(Y_2,\mathfrak{t}_1) \arrow[r, "\iota"] & HF^\infty(Y_2,\mathfrak{t}_1)
\end{tikzcd}$$

The mixed homomorphism

$$F^{\mathrm{mix}}_{(M,\mathfrak{s})} : HF^{-}(Y_1, \mathfrak{t}_1) \to HF^{+}(Y_2, \mathfrak{t}_2)$$

is defined by the composition $F^{+}_{(M,\mathfrak{s}_2)} \circ \delta^{-1} \circ F^{-}_{(M,\mathfrak{s}_1)}$. In [23], it is proved that the mixed homomorphism is independent of the admissible cut.

To define the 4–manifold invariant, we need to recall the Heegaard-Floer homology groups of the 3–sphere S^3 with its unique Spin^c structure:

$$HF^{+}_{n}(S^3) = \begin{cases} \mathbb{Z} & \text{If n is even and } n \geq 0 \\ 0 & \text{If n is odd,} \end{cases}$$

$$HF^{-}_{n}(S^3) = \begin{cases} \mathbb{Z} & \text{If n is even and } n \leq -2 \\ 0 & \text{If n is odd,} \end{cases}$$

Now, let X be a closed 4–manifold with $b_2^+(X) \geq 2$ and $\mathfrak{s}$ be a Spin^c structure on X. For simplicity, assume that $b_1(X) = 0$. Puncture X at two points and regard it as a cobordism from the 3–sphere to itself. Let $\Theta^{\pm}_{(n)}$ denote the generator of $HF^{\pm}_{n}(S^3)$. The *Ozsváth-Szabó* 4–manifold invariant is a linear map $\Phi_{X,\mathfrak{s}} : \mathbb{Z}[U] \to \mathbb{Z}$ which is described as follows: $\Phi_{X,\mathfrak{s}}(U^n)$ is characterized uniquely by the formula

$$F^{\mathrm{mix}}_{X,\mathfrak{s}}(U^n \Theta^{-}_{(-2)}) = (\Phi_{X,\mathfrak{s}}(U^n))\Theta^{+}_{(0)}.$$

The 4-manifold invariant is zero on elements of degree not equal to $d(X, \mathfrak{s})$. A Spin^c structure $\mathfrak{s}$ on X is called a *basic class* if $\Phi_{X,\mathfrak{s}} \not\equiv 0$. Finding the set of all basic classes of a given 4-manifold is an important problem in low-dimensional topology. The *adjunction inequality* gives a very strong restriction on the set of basic classes.

Theorem 3.1 ([23])**.** *Let X be a closed 4–manifold. Let $\Sigma \subset X$ be a homologically non-trivial embedded surface with genus $g \geq 1$ and with non-negative self-intersection number. Then for each* Spin^c *structure $\mathfrak{s} \in \mathrm{Spin}^c(X)$ for which $\Phi_{X,\mathfrak{s}} \neq 0$, we have that*

$$|\langle c_1(\mathfrak{s}), [\Sigma]\rangle| + [\Sigma] \cdot [\Sigma] \leq 2g - 2. \tag{3.5}$$

The following is another version of the adjunction inequality along with a non-vanishing result of the 4–manifold invariant for Lefschetz fibrations.

Theorem 3.2 ([24])**.** *Let $\pi : X \to S^2$ be a relatively minimal Lefschetz fibration over sphere with generic fiber F of genus $g > 1$, and $b_2^+ > 1$. Then for the canonical* Spin^c *structure $\mathfrak{s}$ the map $F^{\mathrm{mix}}_{X,\mathfrak{s}}$ sends the generator of $HF^{-}_{-2}(S^3)$ to the generator of $HF^{+}_{0}(S^3)$ (and vanishes on the rest of $HF^{-}(S^3)$). In particular $\mathfrak{s}$ is a basic class. For any other* Spin^c *structure $\mathfrak{t} \neq \mathfrak{s}$ with $\langle c_1(\mathfrak{t}), [F]\rangle \leq 2 - 2g = \langle c_1(\mathfrak{s}), [F]\rangle$, the map $F^{\mathrm{mix}}_{X,\mathfrak{t}}$ vanishes.*

Given a contact structure ξ on Y, let $\mathfrak{t}_\xi$ be the induced Spin^c structure. A Heegaard Floer (co-)homology class $c^+ \in HF^+(-Y, \mathfrak{t}_\xi)$, which is an invariant of the isotopy class of ξ, is constructed in [20] as follows: Take an adapted open book decomposition for ξ which has connected binding. One can always find such an open book by doing positive stabilizations to any adapted open book as necessary. Let Y_0 denote the result of the

0–surgery on the binding, and $V_0 : Y \to Y_0$ be the associated cobordism. Naturally, Y_0 admits a fibration over circle. Let $\mathfrak{t}_0$ the Spin^c structure corresponding to the tangent plane distribution of fibers.

Proposition 3.3 ([24]). *$HF^+(-Y_0, \mathfrak{t}_0) = \mathbb{Z}$.*

Let c be a generator of $HF^+(-Y_0, \mathfrak{t}_0)$. It can be shown that there is a unique extention $\mathfrak{s}$ of the Spin^c structure $\mathfrak{t}_0$ over the cobordism W. The contact invariant is defined to be the image of c under the homomorphism which is induced by the Spin^c cobordism $(V_0, \mathfrak{s})$.

Definition 3.4. $c^+(\xi) := F^+_{V_0,\mathfrak{s}}(c) \in HF^+(-Y, \mathfrak{t}_\xi)/(\pm 1)$.

Note that we turned V_0 upside down in this construction. This invariant is independent of the choice of the adapted open book decomposition used in its definition.

By using this definition along with the adjunction inequality and the symplectic compactification theorem, it can be proven that the contact invariant of a Stein fillable contact structure is in the image of the mixed homomorphism of some concave filling (c.f. [25]). From Theorem 2.3, we know that every Stein fillable contact manifold (Y, ξ) admits a concave symplectic filling $V = V_0 \cup V_1$ where V_0 is a cap off cobordism and V_1 is a Lefschetz fibration over a disk. Let $\mathfrak{s}$ be the canonical Spin^c structure.

Lemma 3.5 ([25]). *$F^{\mathrm{mix}}_{V,\mathfrak{s}}(\Theta^-_{(-2)}) = \pm c^+(\xi)$ if $c_1(\xi)$ is torsion.*

This lemma will play a key role in our argument. Its proof relies on the special topology of the concave filling constructed in Theorem 2.3. One would hope to prove it for arbitrary concave fillings (with $b_2^+ \geq 2$), but the authors do not know how to do it in full generality.

We are going to need a variant of Lemma 3.5 where one is allowed to add any Stein cobordism to the concave filling.

Lemma 3.6. *Let (Y, ξ) be a Stein fillable contact manifold with torsion $c_1(\xi)$. Let M be any Stein cobordism built on (Y, ξ) which does not contain any 1–handles. Then M can be extended to a concave filling V of (Y, ξ) such that $F^{\mathrm{mix}}_{V,\mathfrak{s}}(\Theta^-_{(-2)}) = \pm c^+(\xi)$.*

Proof. Let (Y_1, ξ_1) be the convex end of M. Take a Stein filling of (Y, ξ) and glue it to M in order to obtain a Stein filling of (Y_1, ξ_1). Pick a $PALF$ of this Stein manifold and apply the algorithm in the proof of the Theorem 2.3 to find a concave filling $V = V_0 \cup V_1$ of (Y_1, ξ_1). If we glue this to M, we get a concave filling V' of (Y, ξ). By changing the order of some 2–handle attachments we can write this as $V' = V_0 \cup V_1'$, where $V_1' = M \cup V_1$ which is a Lefschetz fibration on disk. Now apply Lemma 3.5. □

It is possible to define a relative version of the Ozsváth–Szabó 4–manifold invariant in the presence of a contact structure on the boundary. For, let W be a 4–manifold with connected boundary and ξ be a contact structure on ∂W. Given a Spin^c structure $\mathfrak{s}$ on W, the relative invariant $\Phi_{W,\mathfrak{s}}(\xi) \in \mathbb{Z}/\pm 1$ is uniquely characterized by the formula

$$F^+_{W,\mathfrak{s}}(c^+(\xi)) = \Phi_{W,\mathfrak{s}}(\xi)\Theta^+_{(0)}.$$

Again we puncture W and turn it upside down to regard it as a cobordism from $-\partial W$ to S^3. A twisted version of this invariant is conjecturally equivalent to the relative Seiberg–Witten invariant defined in [17].

4. Main theorem

With all the necessary tools in hand, we are ready to prove our main result. Henceforth suppose that W is a cork corresponding to an admissible link L. Let ξ be the induced contact structure for some choice of a Stein structure on W. Let τ be the involution on ∂W obtained by exchanging the components of L. Let ξ' be the pull back contact structure $\tau^*\xi$.

Theorem 4.1. *The contact invariants $c^+(\xi)$ and $c^+(\xi')$ in $HF^+(-\partial W)$ are distinct. Moreover, both elements descend non-trivially to $HF^+_{\mathrm{red}}(-\partial W)$.*

Proof. The trick is to inflate the cork using a Stein handle so that the cork twist changes the framing on the handle. This trick was first used in [10] to generate exotic Stein manifold pairs. We start by attaching a 2-handle to ∂W along a trefoil with framing 1 as in the left hand side of Figure 2. Thanks to the non-trivial linking with the 1-handle, this handle attachment induces a Stein cobordism M built on $(\partial W, \xi)$. On the right hand side of the same figure, however, the handle attachment can not be realized as a Stein cobordism, because the maximum Thurston-Bennequin number of trefoil is 1.

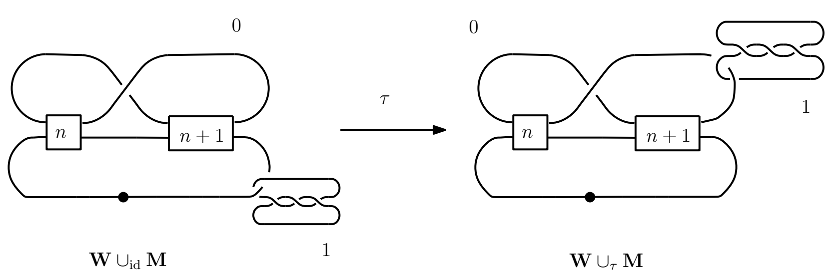

FIGURE 2

Next we apply Lemma 3.6 in order to extend M to a concave filling V of $(\partial W, \xi)$ whose mixed homomorphism hits the contact invariant $c^+(\xi)$.

The symplectic manifold $X := M \cup V$ admits a relatively minimal Lefschetz fibration. Let $\mathfrak{s}$ denote its canonical Spin^c structure. By Theorem 3.2, Lemma 3.6 and the compostion law, we have

$$\begin{aligned}\Theta^+_{(0)} &= F^{\mathrm{mix}}_{W\cup V,\mathfrak{s}}(\Theta^-_{(-2)})\\ &= F^+_{W,\mathfrak{s}}\circ F^{\mathrm{mix}}_{V,\mathfrak{s}}(\Theta^-_{(-2)})\\ &= \pm F^+_{W,\mathfrak{s}}(c^+(\xi)).\end{aligned}$$

In particular $F^+_{W,\mathfrak{s}}(c^+(\xi))\neq 0$. On the other hand if we remove the cork W from X and reglue it using the cork twist τ, we obtain the manifold $X' := W\cup_\tau V$ which is homeomorphic to X. The right hand side of Figure 2 is $W\cup_\tau M$ and it embeds into X'. In that figure the trefoil represents an embedded torus of self intersection 1, which violates the adjunction inequality (Theorem 3.1). Therefore X' has no basic classes. This implies

$$\begin{aligned}0 &= F^{\mathrm{mix}}_{W\cup_\tau V,\mathfrak{s}}(\Theta^-_{(-2)})\\ &= F^+_{W,\mathfrak{s}}\circ\tau^*\circ F^{\mathrm{mix}}_{V,\mathfrak{s}}(\Theta^-_{(-2)})\\ &= \pm F^+_{W,\mathfrak{s}}(\tau^*(c^+(\xi))).\end{aligned}$$

This proves that $c^+(\xi)$ and $c^+(\xi')=\tau^*(c^+(\xi))$ are distinct. To prove the last statement, note that up to sign the U–equivariant involution $\tau^* : HF^+(-\partial W)\to HF^+(-\partial W)$ fixes the image of $HF^\infty(-\partial W)$ under the homomorphism π in Equation 3.4. Since $c^+(\xi)$ and $c^+(\xi')$ are not fixed by τ they descend non-trivially to $\mathrm{coker}(\pi)$. □

The following corollary was previously proved in [4] by using different techniques.

Corollary 4.2. *The contact structures ξ and ξ' are homotopic, contactomorphic but not isotopic.*

Proof. We use the homotopy classification of 2–plane fields on a 3–manifold (see [15]). Since ∂W is an integral homology sphere, ξ and ξ' have the same two dimensional invariant. These two have also the same 3–dimensional invariant because they can be connected by a Stein cobordism which is topologically trivial: Simply take the symplectizations of $(\partial W,\xi)$ and $(\partial W,\xi')$, and glue two ends using τ. This proves that ξ and ξ' are homotopic. By definition τ defines a contactomorphism between these two contact structures. Theorem 4.1 shows that they are not isotopic. □

Next corollary was first proved in [3] for the Mazur manifold, and in [11] in the general form.

Corollary 4.3. *The cork twist $\tau : \partial W\to\partial W$ does not extend inside of W as a diffeomorphism.*

Proof. Let $\mathfrak{s}$ be the unique Spin^c structure on W. Let ξ be the contact structure induced by a Stein structure. The proof of Theorem 4.1 shows that $F_{W,\mathfrak{s}}(c^+(\xi))=\Theta^+_{(0)}$ and $F_{W,\mathfrak{s}}(\tau^*c^+(\xi))=0$. This shows that the relative invariants $\Phi_{W,\mathfrak{s}}(\xi)$ and $\Phi_{W,\mathfrak{s}}(\tau^*\xi)$ are different. □

The following corollary was first proved in [5] by using adjunction inequality. Let us first introduce a terminology. Two Spin^c manifolds $(X, \mathfrak{s})$ and $(X', \mathfrak{s}')$ are said to be fake copies of each other if they are homeomorphic but not diffeomorphic.

Corollary 4.4. *The cork W can be symplectically embedded in some closed symplectic 4–manifold X so that removing W and regluing it via cork twist produces a fake copy X, with its canonical* Spin^c *structure.*

Proof. Pick a concave symplectic filling V of $(\partial W, \xi)$ as in Theorem 2.3. The manifold $X = W \cup V$ is simply connected and symplectic. Let $\mathfrak{s}$ be the canonical Spin^c structure of X. Now, remove the cork and reglue it using cork twist. The manifold $X' = W \cup_\tau V$ is simply connected and has the same intersection form as X. By Freedman's theorem there is a homeomorphism $f : X \to X'$. Let $\mathfrak{s}' = f_*(\mathfrak{s})$. We will prove that $\mathfrak{s}'$ is not a basic class. By Lemma 3.5

$$\begin{aligned} F^{\mathrm{mix}}_{X',\mathfrak{s}'}(\Theta^-_{(-2)}) &= F^+_{W,\mathfrak{s}} \circ \tau^* \circ F^{\mathrm{mix}}_{V,\mathfrak{s}}(\Theta^-_{(-2)}) \\ &= \pm F^+_{W,\mathfrak{s}}(\tau^*(c^+(\xi))) \\ &= 0. \end{aligned}$$

□

Remark 4.5. *Note that if the inflated cork $W \cup_{id} M$ (of Theorem 1.1) embeds symplectically into a symplectic manifold X, then the cork twist produces a fake copy of X since in the cork twisted manifold the adjunction inequality fails. This is what is used in* [5] *and* [11].

Remark 4.6. *Let us formulate Corollary 4.4 in other terms: There is a concave filling V of $(\partial W, \xi)$ such that attaching V to W in two different ways produce two closed* Spin^c *4–manifolds $(X, \mathfrak{s})$ and $(X', \mathfrak{s}')$ that are fake copies of each other. A large family of concave fillings satisfy this condition and presumably this also holds for all concave fillings (with $b_2^+(V) \geq 2$). Proving the latter, however, requires a generalization of Lemma 3.5 for arbitrary concave fillings. Once this is done, one would be able to show the following:*

Conjecture: If a cork embeds symplectically into any symplectic manifold X, such that $b_2^+(X) \geq 2$, then the cork twist produces a fake copy of X.

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