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Introduction to integrable systems: open Toda lattice, KP-, and KdV-hierarchies

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ABSTRACT. The goal of this crash course is to make a brief introduction into the beautiful world of integrable hierarchies. We do not intend to give a general survey of integrable system theory but rather want to use few known examples to introduce notions and initial circle of ideas specific for integrable equations. We used [A] as a reference for the first section, [P] for the second section, and [MJD] and [K] for the third section.

1. Classical mechanics, Poisson structures, Hamiltonian formalism

The classical Newton's equations for a particle with coordinates $q(t) = (q_1(t), \dots, q_n(t))$ in a potential force field $\mathbf{F}$ have a form

$$\ddot{q} = \mathbf{F}, \ \mathbf{F} = -\operatorname{grad} P,$$

where $P = P(q)$ is a potential energy.

We rewrite these equations as

$$\begin{cases} \dot{q} &= p \\ \dot{p} &= -\operatorname{grad} P, \end{cases}$$

where $p = (p_1, \dots, p_n)$ is *the momentum.*

The space $\mathbb{R}^{2n} = \{(p, q)\}$ is *the phase space* of the system.

Definition 1.1. *A Poisson structure* on a manifold M is a bilinear bracket $\{\cdot, \cdot\}$ on the space of functions $C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying

(1) skew-symmetry $\{F, G\} = -\{G, F\}$
(2) Leibnitz rule $\{F, G \cdot H\} = \{F, G\} + \{F, H\}$
(3) Jacobi identity $\{\{F, G\}H\} + \{\{G, H\}F\} + \{\{H, F\}G\} = 0$

Remark 1.2. Leibnitz rule is equivalent to the following formula: in any local coordinate system $\{x_i\}$ on M one can express the Poisson bracket as $\{F, G\} = \sum_i \dfrac{\partial F}{\partial x_i}\{x_i, G\}$.

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Example 1.3. The standard Poisson structure on $\mathbb{R}^{2n} = \{(p,q)\}$ is given by

$$\{F,G\} = \sum_i \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$

This Poisson structure corresponds to a bi-vector field

$$\nu = \sum_i \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i},$$

such that $\{F,G\}(x) = \langle dF(x) \wedge dG(x), \nu(x)\rangle$, where $\langle *, * \rangle$ is the natural pairing between $\Lambda^2 T_x^*(X)$ and $\Lambda^2 T_x(X)$.

A Poisson bracket transforms a covector field into a vector field, more exactly it induces a linear map $\Psi_x : T_x^*M \to T_xM$. For any two covectors $\xi, \eta \in T_x^*M$ let $\xi = dF(x)$, $\eta = dG(x)$. Then Ψ_x is determined uniquely from the equality $\eta(\Psi_x\xi) = \{F,G\}(x)$. Hence, a Poisson bracket transforms any function H into a vector field $\operatorname{sgrad} H = \Psi_x(dH(x))$.

1.1. Hamiltonian equation

Given *a hamiltonian function (or hamiltonian)* $H(x)$ we define the Hamiltonian flow (or Hamiltonian equation) as

$$\dot{x} = \{x, H\} \tag{1.1}$$

For $x = (p,q)$ we have $H = H(p,q), \dot{p} = \{p,H\}, \dot{q} = \{q,H\}$.

Remark 1.4. For any function $f(x)$ we have $\dot{f} = \sum_i \frac{\partial f}{\partial x_i}\dot{x}_i = \sum_i \frac{\partial f}{\partial x_i}\{x_i, H\} = \{f, H\}$ by the Leibnitz rule.

Example 1.5. Consider $x = (p,q) \in \mathbb{R}^{2n}$, equipped with the standard Poisson structure $\{F,G\} = \sum_i \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$, and the hamiltonian $H(p,q) = \frac{p^2}{2} + P(q)$ (the total energy). Then the flow equations take the form

$$\begin{cases} \dot{q} = p \\ \dot{p} = -\operatorname{grad} P \end{cases}$$

We recognize Newton's equations of motion.

1.2. Integrals of motion

Let $I(x)$ be a function such that $\{I,H\} = 0$. Then I is preserved under the Hamiltonian flow with hamiltonian H. Indeed, $\dot{I} = \{I,H\} = 0$ and $I(t) = I(p(t),q(t)) = Constant$.

Definition 1.6. Such I is called *a first integral of motion*. A collection $I_1, \ldots, I_k$ of the first integrals such that $\{I_j, I_l\} = 0$ for all j and l form *integrals in involution*. Integrals are called *independent* if their gradients are linearly independent at a generic point of M.

1.3. Liouville's theorem

A $2n$ dimensional Poisson manifold M^{2n} is *symplectic* if the bivector field ν giving the Poisson structure is nondegenerate at every point of M.

Theorem 1.7. Let a symplectic M^{2n} have n functions $F_1, \dots, F_n$ in involution, i.e., $\{F_i, F_j\} \equiv 0$, for all i, j. Consider $M_f := \{x | F_i(x) = f_i,\ i = 1 \dots n\}$. Assume that all F_i are independent on M_f (i.e. $dF_i(x)$ are linearly independent for all $x \in M_f$). Then,

(1) M_f is a smooth manifold invariant with respect to hamiltonian flow $\operatorname{sgrad} F_1$.
(2) If M_f is compact and connected then

$$M_f \approx T^n = \{(\varphi_1, \dots, \varphi_n) mod 2\pi\}.$$

(3) The flow in the phase space with hamiltonian $H = F_1$ determines on M_f a quasi-periodic motion in angle coordinates $\varphi_1, \dots, \varphi_n$. Namely, $\dot{\varphi}_i = \omega_i$, where $\omega = (\omega_1, \dots, \omega_n)$ depends on f.
(4) Hamiltonian equations with hamiltonian $H = F_1$ are solved in quadratures (i.e. there are analytic formulas for the motion).

For the system with n degrees of freedom (dimension of the phase space is $2n$) there are at most n independent integrals in involution $H_1, \dots, H_n$. They are the *action variables.* If $H = H_1$ is a hamiltonian then there are n additional coordinates (called *angle variables*) $\varphi_1, \dots, \varphi_n$ such that $0 = \{\varphi_i, \varphi_j\} = \{H_i, H_j\},\ \{\varphi_i, H_j\} = 0$ for $i \neq j$, $\{\varphi_i, H_i\} = \delta_{ij}$.

The solution of the Cauchy problem with initial condition $(H_j(0), \varphi_j(0))$ is

$$\begin{aligned} H_j(t) &= H_j(0) \\ \varphi_j(t) &= \varphi_j(0) + H_j(0)t. \end{aligned}$$

2. Open Toda lattice

We consider the evolution of n labeled interacting particles on a line where only ith and $i+1$st particles interact. The corresponding hamiltonian is given by

$$H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + g^2 \sum_{j=1}^{n-1} exp(2(q_j - q_{j+1})),$$

where q_j is the coordinate of jth particle, $p_j = \dot{q_j}$ is its momentum. The Poisson structure in coordinates (p_i, q_i) is standard.

By the change of variables $q_j \to q_j + a_j$ we can always make $g = 1$. The corresponding differential equations take the form

$$\begin{cases} \dot{q_1} &= p_1 \\ \dots & \\ \dot{q_n} &= p_n \\ \dot{p_1} &= -2e^{2(q_1-q_2)} \\ \dot{p_2} &= -2e^{2(q_2-q_3)} + 2e^{2(q_1-q_2)} \\ \dots & \\ \dot{p_n} &= 2e^{2(q_{n-1}-q_n)} \end{cases} \tag{2.1}$$

Note that the system is invariant under the shift $q_i \to q_i + a$, and therefore we can reduce the dimension of the system by 1. Moreover, recall that the total momentum of the closed system is preserved. Without loss of generality we will assume that $\sum_i p_i = 0$ so that the dimension of the system becomes $2n - 2$.

To solve the system we want to find integrals in involution. To find the integrals we look for the Lax form of 2.1. Namely, we are looking for a way to construct the matrix L such that matrix Lax equation

$$\dot{L} = [L, L_+] \tag{2.2}$$

is equivalent to 2.1. Then the coefficients of the characteristic polynomial of L give integrals of motion. Such a matrix $L = L(p_i, q_i)$ can be found in a tridiagonal form,

$$L = \begin{pmatrix} p_1 & e^{q_1-q_2} & 0 & \dots & \\ e^{q_1-q_2} & p_2 & e^{q_2-q_3} & \dots & \\ & & \dots & & \\ & & & \dots & e^{q_{n-1}-q_n} \\ & & \dots & e^{q_{n-1}-q_n} & p_n \end{pmatrix} \tag{2.3}$$

and L_+ is a skew-symmetrization of L

$$L_+ = \begin{pmatrix} 0 & e^{q_1-q_2} & 0 & \dots & \\ -e^{q_1-q_2} & 0 & e^{q_2-q_3} & \dots & \\ & & \dots & & \\ & & & \dots & e^{q_{n-1}-q_n} \\ & & \dots & -e^{q_{n-1}-q_n} & 0 \end{pmatrix} \tag{2.4}$$

Home Assignment: Check that Lax equation 2.2 is equivalent to 2.1.

Denote $a_i = exp(q_i - q_{i+1})$ for $i = [1, n-1]$, $b_j = p_j$ for $j = [1, n]$. As above we have $\sum_i b_i = 0$.

The flow in coordinates of a_i, b_i is

$$\begin{cases} \dot{a}_k &= a_k(b_k - b_{k+1}) \\ \dot{b}_k &= 2(-a_k^2 + a_{k-1}^2) \end{cases} \tag{2.5}$$

The Poisson structure in coordinates a_i, b_i has the form $\{b_j, a_{j-1}\} = -a_{j-1}$, and $\{b_j, a_j\} = a_j$, while all other brackets are zeroes.

Remark 2.1. This Poisson bracket is a particular case of Kirillov-Konstant Poisson structure on an orbit of a coadjoint action by the Borel subgroup B of SL_n on the dual space b^* of the Borel subalgebra $b \subset sl_n$.

Here,

- B is the subgroup of the non-degenerate upper triangular matrices.
- $\mathfrak{b}$ is the subalgebra of the upper-triangular matrices.
- $\langle X, Y\rangle = Const \cdot Tr(XY)$ is the non-degenerate symmetric Killing form.
- We identify $\mathfrak{b}^*$ with the space Sym_n of symmetric $n \times n$ matrices with zero trace, it is dual to $\mathfrak{b}$ with respect to the Killing form $\langle *, *\rangle$.

Co-adjoint action of B on $\mathfrak{b}^*$ is given by the following formula. For $X \in B$ and $y \in \mathfrak{b}^*$, we have $ad^*_X(y) = \mathrm{sym}\,(XyX^{-1})$, where $\mathrm{sym} : sl_n \to Sym_n$ is the projection onto Sym_n along $\mathfrak{b}$.

The Kirillov-Konstant Poisson bracket is defined as follows. For any $f, g \in C^\infty(b^*; \mathbb{R})$ we define $\{f, g\}_{KK}(x) = \langle x, [df(x), dg(x)]\rangle$, where $[df(x), dg(x)]$ is the Lie bracket of $df(x)$ and $dg(x)$ in the Lie algebra $\mathfrak{b}$ (note that both $df(x), dg(x) \in \mathfrak{b}$). Orbits of ad^*_B are symplectic leaves of $\{,\}_{KK}$ in $\mathfrak{b}^*$.

Symplectic leaf of a Poisson manifold M is an equivalence class of points in M such that one can reach from one point to another in the same class along piecewise trajectories of hamiltonian flows. Tridiagonal matrices form an orbit, i.e. a symplectic leaf in $\mathfrak{b}^* \simeq Sym_n$. The Kirillov-Kostant Poisson bracket on the space of tridiagonal matrices coincides with the Toda lattice Poisson bracket written in coordinates a_i, b_i. Hamiltonian $H(x) = \frac{1}{2}Tr(x^2)$ induces the hamiltonian flow which coincides with the open Toda lattice flow.

Home Assignment: check that hamiltonian flow on tridiagonal matrices equipped with Kirillliov-Konstant Poisson bracket with hamiltonian $H(x) = \frac{1}{2}Tr(x^2)$ is given by 2.5.

Remark 2.2. The Cauchy problem $\dot{L} = [L, L_+],\ L(0) = L_0$ has a solution of the form $L(t) = u(t)L_0u(t)^{-1}$, where $u(t)$ is an orthogonal matrix.

Remark 2.3. The characteristic polynomial is preserved under Toda flow.

Remark 2.4. Functions $H_k(L) = \frac{1}{k}Tr(L^k),\ k = [1, n]$ are integrals of motion.

Let $det(\lambda I - L) = \prod_{k=1}^n(\lambda - \lambda_k) = \sum_{k=1}^n(-1)^k J_k(L)\lambda^{n-k}$.

Remark 2.5. Functions $J_k(L), k = [1, n]$ are integrals of motion.

Remark 2.6. Functions $\lambda_k(L), k = [1, n]$ are integrals of motion.

Now we show that restrictions of functions invariant under sl_n-conjugation Poisson commute with each other.

Let f be a functions on sl_n, and let $\tilde{f}$ denote its restriction $f|_{\mathfrak{b}^*} = f|_{Sym_n}$. Recall that $sl_n = \mathfrak{b} \oplus \mathfrak{n}$, where $\mathfrak{n}$ denotes the Lie subalgebra of skew-symmetric matrices, $\mathfrak{n} = Sym_n^{\perp} = (\mathfrak{b}^*)^{\perp}$ with respect to the Killing form. Note that $d\tilde{f} = \pi_{\mathfrak{b}} f$, where $\pi_{\mathfrak{b}}$ is the orthogonal projection onto $\mathfrak{b}$. If f and g are invariant under conjugation then $\langle df(x), [x,u]\rangle = \langle dg(x), [x,u]\rangle = 0$ for any $x \in \mathfrak{b}, u \in sl_n$. Then $\{\tilde{f}, \tilde{g}\}(x) = \langle x, [\pi_{\mathfrak{b}}(df(x)), \pi_{\mathfrak{b}}(dg(x))]\rangle$. Rewrite the latter using $\pi_{\mathfrak{b}}(df(x)) = df(x) - \pi_{\mathfrak{n}}(df(x))$, and similarly for g. More exactly,

$$\begin{aligned} \langle x, [\pi_{\mathfrak{b}}(df(x)), \pi_{\mathfrak{b}}(dg(x))]\rangle &= \langle x, [df(x) - \pi_{\mathfrak{n}}(df(x)), dg(x) - \pi_{\mathfrak{n}}(dg(x))]\rangle \\ &= \langle x, [\pi_{\mathfrak{n}}(df(x)), \pi_{\mathfrak{n}}(dg(x))]\rangle = 0 \end{aligned}$$

since $[\pi_{\mathfrak{n}}(df(x)), \pi_{\mathfrak{n}}(dg(x))] \in \mathfrak{n} = (\mathfrak{b}^*)^{\perp}$.

Remark 2.7. Three collections $\{H_k\}$, $\{J_k\}$, $\{\lambda_k\}$ are collections of integrals in involution.

2.1. Solution of Toda lattice

We found above three different complete sets of integrals in involution. We can choose any set as a collection of action variables and find the corresponding angle variables φ_k. This will lead to a complete solution of the corresponding differential equations. However, there is a simpler method to solve Toda equations that we are going to discuss now.

2.1.1. Factorization method

As we saw above we can look for a solution in the form

$$L(t) = u(t)L_0u(t)^{-1}, \ u(0) = I. \tag{2.6}$$

In order to find $L(t)$ it is enough to compute $u(t)$.

Factorize $exp(L_0 \cdot t)$ into the product

$$exp(L_0t) = b(t) \cdot u(t), \tag{2.7}$$

where $u(t) \in SO(n)$ is orthogonal, $b(t) \in B$ is upper-triangular. Then substituting $u(t)$ into 2.6 gives the solution.

Proof. Let $L(t) = u(t)L_0u(t)^{-1}$. Its derivative is $\dot{L} = \dot{u}L_0u(t)^{-1} - u(t)L_0u(t)^{-1}\dot{u}(t)u(t)^{-1} = \dot{u}(t)u(t)^{-1} \cdot u(t)L_0u(t)^{-1} - u(t)L_0u(t)^{-1} \cdot \dot{u}(t)u(t)^{-1} = \big[\dot{u}(t)u(t)^{-1}, L(t)\big]$.

It remains to show that

$$\dot{u}(t)u(t)^{-1} = \pi_+(u(t)L_0u(t)^{-1}),$$

where π_+ is the projection along the upper triangular matrices onto the subspace of skew-symmetric matrices. Let $exp(L_0t) = b(t)u(t)$. Then $\dot{b}u + b\dot{u} = exp(L_0t)L_0 = b(t)u(t)L_0$. Hence, $b^{-1}\dot{b} + \dot{u}u^{-1} = uL_0u^{-1} = L(t)$. Note that, $\dot{u}u^{-1}$ is skew-symmetric, while $b^{-1}\dot{b}$ is upper-triangular. This observation accomplishes the proof. □

Corollary 2.1. $q_k(t) = q_k(0) + \frac{1}{2}\ln\frac{\Delta_{n+1-k}(t)}{\Delta_{n-k}(t)}$, *where* Δ_j *is the determinant of the lower right* $j \times j$ *submatrix of* $exp(2L_0t)$.

2.1.2. Moment method

Let $f(\lambda) = R_{nn}(\lambda) = \left((\lambda I - L)^{-1}\right)_{nn}$.

$$f(\lambda) = \sum_{i=1}^{n} \frac{r_k^2}{\lambda - \lambda_k},$$

where $r_k = u_n^{(k)}$, $u^{(k)} = (u_1^{(k)}, \ldots, u_n^{(k)})$ is an eigenvector of L, $Lu^{(k)} = \lambda_k u^{(k)}$, and $(u^{(k)}, u^{(k)}) = 1$, $u_n^{(k)} > 0$.

Remark 2.8. $\sum_{k=1}^n r_k^2 = 1$

First we compute $r_k(t)$. Functions $r_k(t)$ satisfy differential equations.

Let $L(t) = u(t)\Lambda u(t)^{-1}$, where $\Lambda = diag(\lambda_1, \ldots, \lambda_n)$. We show that $\dot{u^{(k)}} = -Mu^{(k)}$, where $M = L_+$. Then $\dot{L} = \dot{u}\Lambda u^{-1} - u\Lambda u^{-1} \cdot \dot{u}u^{-1} = [\dot{u}u^{-1}, L]$. Hence, $\dot{u}u^{-1} = M$, and $\dot{u} = Mu$.

Then, $\dot{r_k} = \dot{u}_n^{(k)} = -\left(Mu^{(k)}\right)_n = a_{n-1}u_{n-1}^{(k)}$.

On the other hand, $Lu^{(k)} = \lambda_k u^{(k)}$.

$a_{n-1}u_{n-1}^{(k)} = (\lambda_k - b_n)r_k$,

$b_n = L_{nn} = \sum_{k=1}^n \lambda_k r_k^2$.

Then, $\dot{r_k} = (\lambda_k - \sum_j \lambda_j r_j^2)r_k$, $\lambda_1 < \cdots < \lambda_n$.

Solution to this equation is $r_k^2(t) = \dfrac{r_k^2(0)e^{2\lambda_k t}}{\sum_{j=1}^n r_j^2(0)e^{2\lambda_j t}}$. To restore the solution of Toda lattice it remains to recover a_j and b_j in terms of λ_j, r_k^2. Here we follow Moser's procedure of continuous fractions. Define,

$$f(\lambda) = f_n(\lambda) = \cfrac{1}{\lambda - b_n - \cfrac{a_{n-1}^2}{\lambda - b_{n-1} - \cfrac{a_{n-2}^2}{\lambda - b_{n-2} - \cfrac{\cdots}{\cdots - \cfrac{a_1^2}{\lambda - b_1}}}}}.$$

Let Δ_k be the upper left $k \times k$ minor of $(\lambda I - L)$. Then, $\Delta_k = (\lambda - b_k)\Delta_{k-1} - a_{k-1}^2\Delta_{k-2}$. Set $s_k = \frac{\Delta_k}{\Delta_{k-1}}$, $s_1 = \Delta_1$, $\Delta_0 = 1$. By induction, $s_k = (\lambda - b_k) - a_{k-1}^2/s_{k-1}$. On the other hand, Moser showed that $s_k = f_k^{-1}$, where $f_k = f|_{k\times k \text{ upper left submatrix of } (\lambda I - L)}$.

Then, $f_k = \dfrac{1}{\lambda - b_k - a_{k-1}^2 f_{k-1}}$.

This approach allows one to obtain formulas

$$b_l(t) = \frac{d}{dt}\ln\left(\frac{Q_l(t)}{Q_{l-1}(t)}\right),\ a_l(t) = \frac{\sqrt{Q_{l-1}(t)Q_{l+1}(t)}}{Q_l(t)},$$

where $Q_l(t)$ is the principal $l \times l$ minor of the Hankel matrix $H(t) = (h_{j+k}(t))_{j,k=1}^n$ satisfying the linear system

$$\dot{H}(t) = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -p_0 & -p_1 & \dots & -p_{n-1} \end{pmatrix} \cdot H(t)$$

with initial data $h_i(0) = (L^i(0))_{00} (i = 1, \dots, 2n)$ and coefficients $p_0, \dots, p_{n-1}$ defined by $det(L - \lambda I) = \lambda^n + \sum_{j=0}^{n-1} p_j \lambda^j$.

In particular, for $n = 2$ we get $b_1 = \dfrac{\lambda_2 r_1^2 + \lambda_1 r_2^2}{r_1^2 + r_2^2}$, $b_2 = \dfrac{\lambda_1 r_1^2 + \lambda_2 r_2^2}{r_1^2 + r_2^2}$, $a_1 = \dfrac{(\lambda_2 - \lambda_1) r_1 r_2}{r_1^2 + r_2^2}$.

3. KP-, and KdV-hierarchies

1. KdV-equation $u_t = -3uu' - au'''$, where $u = u(t, x)$ is considered as a time evolution for function $u(x)$. We want to describe Hamiltonian formalism and Lax form for KdV.

We are looking for integrals in involutions. Integrals in involution induce commuting flows. Such system of commuting flows is called *an integrable hierarchy.*

2. KP-hierarchy (KdV is a specialization of KP-hierarchy)

Let $F(p_1, p_2, \dots)$ be a function of infinite number of variables $p_1, p_2, \dots$. KP-hierarchy is the collection of differential equations:

$$\left\{ \begin{array}{l} \dfrac{3}{4} \dfrac{\partial^2 F}{\partial p_2^2} = \dfrac{\partial}{\partial p_1} \left(\dfrac{\partial F}{\partial p_3} - \dfrac{3}{2} F \dfrac{\partial F}{\partial p_1} - \dfrac{1}{4} \dfrac{\partial^3 F}{\partial p_1^3} \right) \\ \dots \end{array} \right. \tag{3.1}$$

Below we discuss how to obtain these partial differential equations in terms of hamiltonian flows on the space of pseudo-differential symbols.

Definition 3.1. We define *a pseudo-differential symbol* $A := \sum_{k=-N}^{\infty} a_k \partial^{-k}$, $\left(\partial = \frac{d}{dx}, a_i(x) \in C^\infty(S^1)\right)$, $deg(A) = N$ if $a_N(x) \not\equiv 0$, and $a_n(x) \equiv 0 \ \forall n > N$. Its *differential part* is $A_+ = \sum_{k=-N}^{0} a_k \partial^{-k}$.

Pseudo-differential symbols form a noncommutative associative algebra with multiplication defined as composition of operators using the following commutation relation $\partial^n \circ f = \sum_{j \geq 0} \binom{n}{j} (\partial^j f) \circ \partial^{n-j}$, where $\binom{n}{j} = \frac{n(n-1)\dots(n-j+1)}{j(j-1)\dots 1}$ defined for all $n \in \mathbb{Z}$ and $j \in \mathbb{Z}_{>0}$.

Example 3.2. $S = \partial^2 + u$ is a Schrödinger operator. Then $L = S^{\frac{1}{2}} = \partial + \sum_{n=1}^{\infty} f_n \partial^{-n}$. The Coefficients f_n can be determined recurrently to give

$$L = \partial + \frac{1}{2} u \partial^{-1} - \frac{1}{4} u_x \partial^{-2} + \left(\frac{u_{xx}}{8} - \frac{u^2}{8} \right) \partial^{-3} + \dots$$

Definition 3.3. Ψ is the Lie algebra of pseudo-differential operators. The commutator is defined by $[A, B] = A \circ B - B \circ A$.

Definition 3.4. Residue $res : \Psi \to C^\infty(S^1)$, is given by$(\sum a_i \partial^{-i}) \mapsto a_{-1}(x)$.

Proposition 3.1. $res([A, B]) = 0,\ \forall A, B \in \Psi$

Definition 3.5. Let $Tr(A) = \int_{S^1} res(A)$, for $A \in \Psi$. This gives rise to a symmetric bilinear form on Ψ: $(A, B) = Tr(AB) = \int_{S^1} res(A \circ B)$.

We introduce the special Lie subalgebra $\Psi_{int} = \{\partial + \sum_{k=1}^{\infty} a_k(x)\partial^{-k}\}$.

There is a natural *Gelfand-Dikii Poisson structure* on Ψ_{int}. Let $L \in \Psi_{int}$. Note that tangent vector to Ψ_{int} has a form $\delta L = \sum_{k=-1}^{\infty} \delta a_k(x)\partial^{-k}$, $T_L\Psi_{int} = \{\sum_{k=-1}^{\infty} \delta a_k(x)\partial^{-k}\}$.

A cotangent vector is a differential operator $A = \sum_{m=1}^{N} a_m(x)\partial^m$. It determines the linear functional $F_A(\delta L) = \langle A, \delta L\rangle = Tr(\delta L \circ A)$.

It is enough to define the Poisson bracket on linear functionals as follows, then we can extend the Poisson bracket to other functions by the Leibniz rule.

$$\{F_A, F_B\}(L) = F_B\left((LA)_+L - L(AL)_+\right).$$

Assume now that a pseudo-differential $L \in \Psi_{int}$ depends on $p_2, p_3, \ldots$ For a hamiltonian $H_k = \frac{1}{k}Tr(L^k)$, $k \geq 2$ the corresponding Hamiltonian equation has the form

$$\frac{\partial L}{\partial p_k} = [(L^k)_+, L], \tag{3.2}$$

where X_+ is pure differential part of X. These equations form KP-hierarchy for L. They are related by the so called τ-function and it will be commented below.

3.1. Reduction of KP to KdV

Let $S = \partial^2 + u$ satisfy equations

$$\frac{\partial S}{\partial p_k} = [(S^{\frac{k}{2}})_+, S]$$

Remark 3.6. If S is a differential operator (DO) then $S^{\frac{k}{2}}$ is also a DO for even k. Then $(S^{\frac{k}{2}})_+ = S^{\frac{k}{2}}$, and $[(S^{\frac{k}{2}})_+, S] = 0$ for even k. Hence, S does not depend on p_k for even k.

All coefficients of $L = S^{\frac{1}{2}}$ are determined just by one function u (see Example 3.2).

KdV equation can be written in Lax form as

$$\dot{S} = [\left(S^{3/2}\right)_+, S] \tag{3.3}$$

Any solution of KP that does not depend on even p_{2k} is a solution of KdV.

3.2. τ-function

KP-hierarchy 3.2 describing commuting flows of a pseudo-differential operator L is an infinite collection of equations written for an infinite number of unknown functions, namely the coefficients of the pseudo-differential operator L. However, one can rewrite all these equations in terms of a unique unknown function. This is called the *τ-function.* Let $L = \partial + \sum_{j=1}^{\infty} u_i\partial^{-i} \in \Psi_{int}$ be a pseudo-differential operator of order 1.

Let $\mathbf{p}$ denote the infinite collection of variables $\mathbf{p} = (p_1, p_2, \dots)$ where $p_1 = x$. Through the dressing transformation $L = W\partial W^{-1}$ with $W = 1 + \sum_{i=1}^{\infty} w_i \partial^{-i}$ we obtain the equivalent Sato form of the hierarchy

$$\frac{\partial W}{\partial p_n} = (L^n)_+ W - W\partial^n. \tag{3.4}$$

The system 3.2 is also equivalent to the following eigenvalue problem

$$Lw = kw, \tag{3.5}$$

where $w = e^{\xi(\mathbf{p},k)}\left(1 + \frac{w_1}{k} + \frac{w_2}{k^2} + \dots\right)$, $\xi(\mathbf{p}, k) = \sum_{j=1}^{\infty} p_j k^j$. Note that $\frac{\partial}{\partial p_j} e^{\xi(\mathbf{p},k)} = k^j e^{\xi(\mathbf{p},k)}$. Consider the system of linear equations

$$\frac{\partial w}{\partial p_j} = (L^j)_+ w. \tag{3.6}$$

System 3.2 is the compatibility condition for 3.5 and 3.6.

The function w can be expressed in terms of a function τ as

$$w = \frac{\tau\left(p_1 - \frac{1}{k}, p_2 - \frac{1}{2k^2}, \dots\right)}{\tau(p_1, p_2, \dots)} e^{\xi(\mathbf{p},k)} \tag{3.7}$$

The constants $w_1, w_2, \dots$ can be written in terms of τ as follows,

$$\begin{aligned} w_1 &= -\frac{\partial \tau}{\partial p_1} \cdot \tau^{-1} \\ w_2 &= \frac{1}{2}\left(\frac{\partial^2 \tau}{\partial p_1^2} - \frac{\partial^2 \tau}{\partial p_2^2}\right) \cdot \tau^{-1}. \end{aligned}$$

So, one can consider KP-hierarchy as an infinite system of equations for one function τ.

Remark 3.7. The function u in the KdV-equation is $u = 2\frac{\partial^2}{\partial p_1^2} \log \tau$.

Remark 3.8. If F is a solution of KP system (3) then e^F is a τ-function.

Remark 3.9. KP-system 3.2 is equivalent to one Hirota bilinear difference equation on τ-function

$$\begin{aligned} 0 = {} & (\lambda_2 - \lambda_1)(\lambda_4 - \lambda_3)\tau(\mathbf{p} + [\lambda_1] + [\lambda_2])\tau(\mathbf{p} + [\lambda_3] + [\lambda_4]) - \\ & (\lambda_4 - \lambda_2)(\lambda_3 - \lambda_1)\tau(\mathbf{p} + [\lambda_1] + [\lambda_3])\tau(\mathbf{p} + [\lambda_2] + [\lambda_4]) + \\ & (\lambda_4 - \lambda_1)(\lambda_3 - \lambda_2)\tau(\mathbf{p} + [\lambda_1] + [\lambda_4])\tau(\mathbf{p} + [\lambda_2] + [\lambda_3]) \end{aligned}$$

where the *Miwa shift* is defined as

$$\mathbf{p} + [\lambda] = \left(p_1 + \lambda, p_2 + \frac{\lambda^2}{2}, p_3 + \frac{\lambda^3}{3}, \dots\right).$$

We want to describe all solutions of KP, (or, equivalently, all τ-functions). Note that $\tau \equiv 1$ is a τ-function. We obtain all other τ-functions by a certain "Lie group" action. More exactly, the algebra $\widehat{gl}(\infty)$ (a certain extension of $gl(\infty)$ described below) acts on the space of solutions of KP, the "corresponding Lie group $\widehat{GL}(\infty)$" acts on τ-functions and this action is transitive. In particular, any τ-function lies in the orbit of $\tau \equiv 1$. To describe all τ-functions we discuss below *boson-fermion correspondence* with applications to Sato Grassmannian and the KP-hierarchy.

Schur polynomials. We define elementary Schur polynomials $S_k(\mathbf{p})$ whose generating function is

$$\exp\left(\sum_{k=1}^{\infty} p_k z^k\right) = \sum_{k=0}^{\infty} S_k(\mathbf{p}) z^k$$

For $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_l)$, $\lambda_i \in \mathbb{Z}_{\geq 0}$ we define the Schur polynomial to be $S_\lambda(\mathbf{p})) = det\,(S_{\lambda_i - i + j}(\mathbf{p}))$.

Let $\bar{y} = y_1, \dots, y_N$. The Schur function is defined by

$$s_\lambda(\bar{y}) = \frac{det\left(y_j^{\lambda_i + N - i}\right)_{i,j=1}^N}{det\left(y_j^{N-i}\right)_{i,j=1}^N} = det(h_{\lambda_i + j - i})_{i,j=1}^N,$$

where $h_k = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq N} y_{i_1} \dots y_{i_k}$ is the complete homogeneous symmetric polynomial in y_i. Define $s_\lambda(y_1, y_2, \dots) = \lim_{N \to \infty} s_\lambda(\bar{y})$.

Set $p_k = \frac{1}{k} \sum_{i=1}^N y_i^k$. Then $S_\lambda(\mathbf{p}) = s_\lambda(y_1, y_2, \dots)$.

Example 3.10. $S_0 = 1$, $S_1 = p_1$, $S_2 = \frac{1}{2}p_1^2 + p_2, S_{1,1} = \frac{1}{2}p_1^2 - p_2$, $S_{2,1} = \frac{1}{3}p_1^3 - p_3, \dots$

Note that Schur polynomials form a basis in $\mathbb{C}[[\mathbf{p}]]$, i.e. $\mathbb{C}[[\mathbf{p}]] \simeq \overline{\oplus_\lambda \mathbb{C} S_\lambda(\mathbf{p})}$.

Let $V = \mathbb{C}[z^{-1}][[z]]$ be the space of formal Laurent series.

Let $\Lambda^{\frac{\infty}{2}} V = span\{v_\lambda := z^{k_1} \wedge z^{k_2} \wedge \dots\}$, where $k_i = \lambda_i - i$. Note that $k_i = -i$ for sufficiently large i.

Remark 3.11. $\mathbb{C}[[p_1, p_2, \dots]]$ is the bosonic Fock space, $\Lambda^{\frac{\infty}{2}} V$ is the fermionic Fock space.

Boson-fermion correspondence: $\mathbb{C}[[p_1, p_2, \dots]] \simeq \Lambda^{\frac{\infty}{2}} V,\ S_\lambda \leftrightarrow v_\lambda$.

Remark 3.12. Vacuum vector $v_\emptyset = z^{-1} \wedge z^{-2} \wedge z^{-3} \wedge \dots$ corresponds to $S_\emptyset = 1$.

Theorem 3.13. The function $\tau \in \mathbb{C}[[p_1, p_2, \dots]]$ is the exponential function of a solution of a KP-hierarchy if and only if, under the boson-fermion correspondence, its image can be represented by a decomposable wedge product:

$$\tau \leftrightarrow \varphi_1(z) \wedge \varphi_2(z) \wedge \dots,\ \varphi_i \in \mathbb{C}[z^{-1}][[z]]$$

Geometric reformulation: τ-functions form a cone over the Grassmannian $Gr_{\frac{\infty}{2}}(V)$ of half-infinite subspaces Plücker embedded into the projective space $\mathbb{P}\Lambda^{\frac{\infty}{2}}(V)$.

Plücker embedding is defined by quadratic (Plücker) relations. These algebraic equations written in terms of coefficients of Taylor series of τ are called the *Hirota bilinear equations.*

KP-hierarchy (3) coincides with the collection of these equations written for $F = \log(\tau)$.

Example 3.14. $(z^{-1}+z^2)\wedge z^{-2}\wedge z^{-3}\wedge\cdots = v_\emptyset + v_3 \leftrightarrow S_\emptyset + S_3 = 1+\frac{1}{6}(p_1^3+3p_1p_2+2p_3)$.
The function $\log(1+\frac{1}{6}(p_1^3+3p_1p_2+2p_3))$ is a solution of KP (3).

3.3. $\widehat{gl}(\infty)$

Definition 3.15. $gl(\infty)$ is the Lie algebra of DO on S^1.

Operator $A \in gl(\infty)$ acts on V linearly. With respect to the basis z^i, $i \in \mathbb{Z}$ operator A is written as an infinite matrix (a_{ij}): $A : z^j \mapsto \sum_i a_{ij} z^i$.

Lie algebra $gl(\infty)$ is graded by $\deg(z^i(\frac{\partial}{\partial z})^j) = i - j$. Operator A induces the operator $\widehat{A} : \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V$, $\hat{A}(z^{k_1}\wedge z^{k_2}\wedge\dots) = A(z^{k_1})\wedge z^{k_2}\wedge + z^{k_1}\wedge A(z^{k_2})\wedge\cdots+\dots$.

Remark 3.16. When $deg(A) = 0$, say $A = diag(\dots, a_{-1}, a_0, a_1, \dots)$, we can obtain divergence. Hence, we need *the normal ordering*, as follows. $\hat{A}(z^{k_1}\wedge z^{k_2}\wedge\dots) = \sum_{i=1}^{\infty}(a_{k_i} - a_{-i}) z^{k_1}\wedge z^{k_2}\wedge\dots$.

Example 3.17. $E = z\frac{\partial}{\partial z} \in gl(\infty)$, $E : z^k \mapsto kz^k$. Then $\hat{E}$ is the energy operator $\hat{E} : v_\lambda \mapsto |\lambda| v_\lambda$, where $|\lambda| = \sum \lambda_i$.

Remark 3.18. $A \mapsto \hat{A}$ is not a Lie algebra homomorphism.

Example 3.19. Abusing notations we denote by z^m the operator of multiplication by z^m. Then $[z^m, z^n] = 0$, but $[\widehat{z^m}, \widehat{z^n}] = n\delta_{m,-n}$.

More exactly, $\widehat{[A,B]} = [\hat{A}, \hat{B}]$ + scalar operator. Therefore, we have a representation of $\widehat{gl}(\infty)$ - a one-dimensional central extension of $gl(\infty)$.

Proposition 3.2. *The Lie group action of* $e^{t\hat{A}}$ *preserves the set of decomposable vectors.*

Proof. $e^{t\hat{A}}(\varphi_1\wedge\varphi_2\wedge\dots) = e^{tA}\varphi_1\wedge e^{tA}\varphi_2\wedge\dots$ if diagonal entries of A are all zeroes and $e^{t\hat{A}}(\varphi_1\wedge\varphi_2\wedge\dots) = e^{t(A-a_{-1})}\varphi_1\wedge e^{t(A-a_{-2})}\varphi_2\wedge\dots$ if $A = diag(\dots, a_{-1}, a_0, a_1, \dots)$. □

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