

Moduli spaces of surfaces and monodromy invariants

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ABSTRACT. Bidouble covers $\pi : S \rightarrow Q := \mathbb{P}^1 \times \mathbb{P}^1$ of the quadric and their natural deformations are parametrized by connected families $\mathfrak{N}_{a,b,c,d}$ depending on four positive integers a, b, c, d . We shall call these surfaces abcd-surfaces. In the special case where $b = d$ we call them abc-surfaces.

Such a Galois covering π admits a small perturbation yielding a general 4-tuple covering of Q with branch curve Δ , and two natural Lefschetz fibrations obtained from a small perturbation of the composition $p_i \circ \pi$, $i = 1, 2$, p_i being the i -th projection of Q onto $\mathbb{P}^1$.

The first and third author showed that the respective mapping class group factorizations corresponding to the first Lefschetz fibration are equivalent for two abc-surfaces with the same values of $a + c, b$, a result which implies the diffeomorphism of two such surfaces.

We report on a more general result of the three authors implying that the first braid monodromy factorization corresponding to Δ determines the three integers a, b, c in the case of abc-surfaces. We provide in this article a new proof of the non equivalence of two such factorizations for different values of a, b, c .

We finally show that, under certain conditions, although the first Lefschetz fibrations are equivalent for two abc-surfaces with the same values of $a + c, b$, the second Lefschetz fibrations need not be equivalent.

These results rally around the question whether abc-surfaces with fixed values of $a + c, b$, although diffeomorphic but not deformation equivalent, might be not canonically symplectomorphic.

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1. Introduction: Moduli spaces of surfaces of general type: deformation, differentiable and symplectic types.

Let X be a smooth complex projective variety, i.e., X is a smooth compact complex submanifold of $\mathbb{P}^n := \mathbb{P}_{\mathbb{C}}^n$ of complex dimension d , and X is connected.

Observe that X is a compact oriented real manifold of real dimension $2d$, so, for instance, a complex surface gives rise to a real 4-manifold.

For $d = 1$, X is a complex algebraic curve (a real surface, called also Riemann surface) and its basic invariant is the genus $g = g(X)$, defined as the dimension of the vector space $H^0(\Omega_X^1)$ of holomorphic 1-forms.

The situation for curves is ‘easy’, since the genus determines the topological and the differentiable manifold underlying X : its intuitive meaning is the ‘number of handles’ that one must add to a sphere in order to obtain X as a topological space.

The rough classification of curves is the following :

- $g = 0$: $X \cong \mathbb{P}_{\mathbb{C}}^1$, topologically X is a sphere S^2 of real dimension 2.
- $g = 1$: $X \cong \mathbb{C}/\Gamma$, with Γ a discrete subgroup $\cong \mathbb{Z}^2$: X is called an elliptic curve, and topologically we have a real 2-torus $S^1 \times S^1$.
- $g \geq 2$: then we have a ‘curve of general type’, and topologically we have a connected sum of g 2-tori.

Every curve is a deformation of a special hyperelliptic curve, the non singular projective model of the affine curve

$$\{(x, z) \in \mathbb{C}^2 \mid z^2 = (x-1)(x-2)\dots(x-(2g+2))\}.$$

The main theorem for curves says more precisely that we have a **Moduli space** $\mathfrak{M}_g$ which parametrizes the isomorphism classes $[C]$ of compact complex curves C of genus g .

$\mathfrak{M}_g$ is a Zariski open set of a complex projective variety, is connected, and has complex dimension $(3g-3) + a(g)$, where $a(g)$ is the complex dimension of the group of complex automorphisms ($a(0) = 3, a(1) = 1, a(g) = 0$, for $g \geq 2$).

When we pass to complex dimension $d = 2$, some features do generalize, others do not.

The first generalizations of the genus were given by Clebsch, Noether, Enriques and Castelnuovo through the dimensions of certain vector spaces of holomorphic tensor differential forms

$$q(X) := \dim_{\mathbb{C}} H^0(\Omega_X^1), p_g(X) := \dim_{\mathbb{C}} H^0(\Omega_X^d), P_m(X) := \dim_{\mathbb{C}} H^0(\Omega_X^d \otimes^m).$$

$q(X)$ is called the irregularity of X , $p_g(X) = P_1(X)$ is called the geometric genus, while $P_m(X)$ is called the m -th plurigenus.

These invariants suffice to give the Castelnuovo-Enriques classification of algebraic surfaces.

The rough classification of projective surfaces S is the following :

- $P_{12} = 0$: S is birational to a product $C \times \mathbb{P}_{\mathbb{C}}^1$, where C is a complex curve of genus $q(S)$.

- $P_{12} = 1$: the surface is birational to a surface S which is an analogue of an elliptic curve, in the sense that there is a holomorphic section of $H^0(\Omega_S^{2 \otimes 12})$ which is nowhere vanishing.
- $P_{12} \geq 2$ and S is properly elliptic, i.e., the rational map ϕ_{12} associated to the sections of $H^0(\Omega_S^{2 \otimes 12})$ maps to a curve and the general fibres are elliptic curves.
- $P_{12} \geq 2$ and S is of general type, i.e., ϕ_{12} maps birationally to a surface X , called the canonical model of S .

Surfaces of special type are well understood, but surfaces of general type still offer us a lot of intriguing and fascinating open problems. In each birational class there is for them a unique smooth projective surface (up to isomorphism) S , called minimal model, with the property that every birational holomorphic map S onto another smooth surface S' is a biholomorphism. This minimal model offers therefore also a unique topological and differentiable model. As does the genus of an algebraic curve, the above numerical invariants are determined by the topological structure of S :

- $2q(S)$ is the first Betti number of S , $2q(S) = b_1(S)$, and
- if one defines $\chi(S) := 1 - q(S) + p_g(S)$, we have Noether's formula $12\chi(S) = K_S^2 + e(S)$, where $e(S) = c_2(S)$ is the topological Euler Poincaré characteristic, while $K_S^2 = c_1^2(S)$; moreover
- the signature $\sigma(S) := b^+(S) - b^-(S)$ of the intersection form $q_S : H^2(S, \mathbb{Z}) \rightarrow \mathbb{Z}$ equals, by virtue of Hodge's index theorem,

$$\sigma(S) = 1/3(K_S^2 - 2e(S))$$

and finally we have Kodaira's Riemann-Roch formula (holding for surfaces of general type) which says that

- the plurigenus $P_m(S)$, for $m \geq 2$, equals $\frac{m(m-1)}{2}K_S^2 + \chi(S)$.

Things seem 'on the surface' to generalize nicely: because also for algebraic surfaces of general type there exist similar moduli spaces $\mathfrak{M}_{x,y}$ (by the results of [Bom73], [Gie77]).

Here, $\mathfrak{M}_{x,y}$ is quasi projective and parametrizes isomorphism classes of minimal (smooth projective) surfaces of general type S such that $\chi(S) = x$, $K_S^2 = y$.

The fact that $\mathfrak{M}_{x,y}$ is quasi-projective implies that $\mathfrak{M}_{x,y}$ has a finite number of connected components, which parametrize deformation classes of surfaces of general type; and, by a classical theorem of Ehresmann ([Ehr43]), deformation equivalent varieties are diffeomorphic.

Hence, fixed the two numerical invariants $\chi(S) = x$, $K_S^2 = y$, which are determined by the topology of S , we have a finite number of differentiable types, and a fortiori a finite number of topological types.

These two numbers $\chi(S)$, K_S^2 are determined by the topology. But they do not determine the topology, since the fundamental group is not encoded in such invariants (and is not even invariant under the action of the absolute Galois group, cf. [Ser64], and [BCG07] for new examples).

For this reason one likes to restrict to the case of simply connected surfaces. In this case, these two numbers almost determine the topology. This follows by Michael Freedman's big Theorem of 1982 ([Free82]), showing that there are indeed at most two topological structures if the surface S is assumed to be simply connected.

Topologically, our 4-manifold is then obtained from very simple building blocks, one of them being the K3 surface, where:

Definition 1.1. *A **K3** surface is a smooth surface of degree 4 in $\mathbb{P}_{\mathbb{C}}^3$.*

Now, a complex manifold carries a natural orientation corresponding to the complex structure, and, in general, given an oriented differentiable manifold M , M^{opp} denotes the same manifold, but endowed with the opposite orientation. This said, one can explain the corollary of Freedman's theorem for the topological manifolds underlying simply connected (compact) complex surfaces as follows:

- if S is EVEN, i.e., the intersection form on $H^2(S, \mathbf{Z})$ is even: then S is homeomorphic to a connected sum of copies of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ and copies of a K3 surface if the signature $\sigma(S)$ is negative, and of copies of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ and of copies of a K3 surface with reversed orientation if the signature is positive.
- if S is ODD : then S is a connected sum of copies of $\mathbb{P}_{\mathbb{C}}^2$ and of $\mathbb{P}_{\mathbb{C}}^{2, opp}$.

Kodaira and Spencer defined quite generally ([K-S58]) for compact complex manifolds X, X' the equivalence relation called **deformation equivalence**: this, for surfaces of general type, means that the corresponding isomorphism classes yield points in the same connected component of the moduli space $\mathfrak{M}$.

The cited theorem of Ehresmann guarantees that $\text{DEF} \Rightarrow \text{DIFF}$: indeed a bit more holds, namely, deformation equivalence implies the existence of a diffeomorphism carrying the canonical class K_X to the canonical class $K_{X'}$.

In the 80's the work of Simon Donaldson ([Don83], [Don86], [Don90], [Don92], see also [D-K90]) showed that homeomorphism and diffeomorphism differ drastically for projective surfaces.

The introduction of Seiberg-Witten invariants (see [Wit94], [Don96], [Mor96]) showed then more easily that a diffeomorphism $\phi : S \rightarrow S'$ between minimal surfaces of general type satisfies $\phi^*(K_{S'}) = \pm K_S$.

Contrary to the conjecture made by the first author in [Kat83], Friedman and Morgan made the

FRIEDMAN-MORGAN'S SPECULATION (1987) (see [F-M88]):

differentiable and deformation equivalence coincide for surfaces.

We abbreviate this speculation with the acronym $\text{DEF} = \text{DIFF}$.

The question was answered negatively in every possible way ([Man01], [KK02], [Cat03], [CW04], [BCG07], see also [Cat08] for a rather comprehensive survey):

Theorem 1.2. *(Manetti 1998, Kharlamov-Kulikov 2001, Catanese 2001, Catanese-Wajnryb 2004, Bauer-Catanese-Grunewald 2005)*

The Friedman- Morgan speculation does not hold true.

The counterexamples obtained by Catanese and Wajnryb are the only ones which are simply connected, and one also has (as also in Manetti's examples) non deformation equivalent surfaces S, S' such that there exists an orientation preserving diffeomorphism $\phi : S \rightarrow S'$ with $\phi^*(K_{S'}) = K_S$.

The simply connected examples used, called **abc-surfaces**, are a special case of a class of surfaces which the first author introduced in 1982 ([Cat84]), namely, bidouble covers of the quadric and their natural deformations.

We shall briefly describe these surfaces in the next section.

But, before, let us explain why we believe that these surfaces deserve further attention. Recall the

Definition 1.3. *A pair (X, ω) of a real manifold X , and of a real differential 2- form ω is called a **Symplectic pair** if*

i) ω is a symplectic form, i.e., ω is closed ($d\omega = 0$) and ω is nondegenerate at each point (thus X has even real dimension).

*A symplectic pair (X, ω) is said to be **integral** iff the De Rham cohomology class of ω comes from $H^2(X, \mathbb{Z})$, or, equivalently, there is a complex line bundle L on X such that ω is a first Chern form of L .*

ii) An almost complex structure J on X (a differentiable endomorphism of the real tangent bundle of X satisfying $J^2 = -1$) is said to be compatible with ω if $\omega(Jv, Jw) = \omega(v, w)$, and the quadratic form $\omega(v, Jv)$ is strictly positive definite.

For long time (before the examples of Kodaira and Thurston) the basic examples of symplectic manifolds were given by symplectic submanifolds of Kähler manifolds, in particular of the flat space $\mathbb{C}^n$ and of projective space $\mathbb{P}_{\mathbb{C}}^N$, endowed with the Fubini-Study form

$$\omega_{FS} := \frac{i}{2\pi} \partial \bar{\partial} \log |z|^2.$$

It was observed recently by the first author ([Cat02], [Cat06]):

Theorem 1.4. *A minimal surface of general type S has a symplectic structure (S, ω) , unique up to symplectomorphism, and invariant for smooth deformation, with $\text{class}(\omega) = K_S = -c_1(S)$.*

*This symplectic structure is called the **canonical symplectic structure**.*

The above result is, in the case where K_S is ample, a rather direct consequence of the well known Moser's lemma ([Mos65]), since then it suffices to pull-back $1/m$ of the Fubini-Study metric by an embedding of S by the sections of $H^0(S, \Omega_S^2 \otimes^m)$.

It is the case where K_S is not ample which needs a non trivial proof.

An important consequence of the techniques of symplectic approximation of singularities employed in these papers is the following theorem ([Cat02], [Cat09], see also [Cat08] for a survey including the basics concerning the construction of the Manetti surfaces)

Theorem 1.5. *Manetti's surfaces yield examples of surfaces of general type which are not deformation equivalent but are canonically symplectomorphic.*

The following questions are still wide open.

Questions: 1) Are there (minimal) surfaces of general type which are orientedly diffeomorphic through a diffeomorphism carrying the canonical class to the canonical class, but, endowed with their canonical symplectic structure, are not canonically symplectomorphic?

2) Are there such simply connected examples?

3) Are there diffeomorphic abc-surfaces which are canonically symplectomorphic, thus yielding a counterexample to Can. $\text{Sympl} = \text{Def}$ also in the simply connected case? Are there diffeomorphic abc-surfaces which are not canonically symplectomorphic, thus providing an answer to Question 1)?

The difficult problem of understanding the canonical symplectic structures of abc-surfaces ([CLW]) is relying on the work of Auroux and Katzarkov ([A-K00]), who described some invariants of integral symplectic pairs of real dimension 4. We postpone to a later section the description of their method.

2. abcd-Surfaces and their moduli spaces

All the curves are deformations of hyperelliptic curves C , and the nature of the double cover $\pi : C \rightarrow \mathbb{P}^1$ allows an easy understanding of the topology of C .

Double covers of the projective plane $\mathbb{P}^2$ or of $\mathbb{P}^1 \times \mathbb{P}^1$ are however too simple surfaces, with very special behaviour; while it turned out that bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$, introduced in [Cat84], exhibited a quite non special behaviour concerning properties of the moduli spaces of surfaces of general type.

Bidouble covers of the quadric are smooth projective complex surfaces S endowed with a (finite) Galois covering $\pi : S \rightarrow Q := \mathbb{P}^1 \times \mathbb{P}^1$ with Galois group $(\mathbb{Z}/2\mathbb{Z})^2$.

More concretely, they are defined by a single pair of equations

$$\begin{aligned} z^2 &= f_{(2a,2b)}(x_0, x_1; y_0, y_1) \\ w^2 &= g_{(2c,2d)}(x_0, x_1; y_0, y_1) \end{aligned}$$

where $a, b, c, d \in \mathbb{N}^{\geq 3}$ and the notation $f_{(2a,2b)}$ denotes that f is a bihomogeneous polynomial, homogeneous of degree $2a$ in the variables x , respectively of degree $2b$ in the variables y .

These surfaces are simply connected and minimal of general type, and they were introduced in [Cat84] in order to show that the moduli spaces $\mathfrak{M}_{\chi, K^2}$ of smooth minimal surfaces of general type S with $K_S^2 = K^2$, $\chi(S) := \chi(\mathcal{O}_S) = \chi$ need not be equidimensional or irreducible (and indeed the same holds for the open and closed subsets $\mathfrak{M}_{\chi, K^2}^{00}$ corresponding to simply connected surfaces).

In fact, for the dimension of the moduli space $\mathfrak{M}_{\chi, K^2}$ one can only give lower and upper bounds, as shown by the following theorem (cf. [Cat84])

Theorem 2.1. *The dimension of an irreducible component $\mathfrak{M}'_{\chi, K^2}$ of $\mathfrak{M}_{\chi, K^2}$ is subject to the following inequalities:*

$$10\chi - 2K^2 \leq \dim(\mathfrak{M}'_{\chi, K^2}) \leq 10\chi + 3K^2 + 108.$$

And the deformations of bidouble covers of the quadric were used to show that the moduli spaces $\mathfrak{M}_{\chi, K^2}^{00}$ could have as many irreducible components of different dimensions as possible.

These irreducible components were determined as follows: given our four integers $a, b, c, d \in \mathbb{N}^{\geq 3}$, consider the so called natural deformations of these bidouble covers of *simple type* $(2a, 2b), (2c, 2d)$, defined by equations ¹

$$\begin{aligned} (***) \quad z_{a,b}^2 &= f_{2a,2b}(x, y) + w_{c,d} \phi_{2a-c, 2b-d}(x, y) \\ w_{c,d}^2 &= g_{2c,2d}(x, y) + z_{a,b} \psi_{2c-a, 2d-b}(x, y) \end{aligned}$$

where f, g, ϕ, ψ , are bihomogeneous polynomials, belonging to respective vector spaces of sections of line bundles (recall that we defined $Q := \mathbb{P}^1 \times \mathbb{P}^1$):

$$\begin{aligned} f &\in H^0(Q, \mathcal{O}_Q(2a, 2b)), \\ \phi &\in H^0(Q, \mathcal{O}_Q(2a-c, 2b-d)) \text{ and} \\ g &\in H^0(Q, \mathcal{O}_Q(2c, 2d)), \\ \psi &\in H^0(Q, \mathcal{O}_Q(2c-a, 2d-b)). \end{aligned}$$

In this way one defines an open² subset $\mathfrak{N}'_{a,b,c,d}$ of the moduli space, whose closure $\overline{\mathfrak{N}}'_{a,b,c,d}$ is an irreducible component of $\mathfrak{M}_{\chi, K^2}$, where $\chi = 1 + (a-1)(b-1) + (c-1)(d-1) + (a+c-1)(b+d-1)$, and $K^2 = 8(a+c-2)(b+d-2)$.

We record here for further use: $\chi - \frac{1}{8}K^2 = ab + cd$.

These calculations can be derived from the formula $\mathcal{O}_S(K_S) = \pi^* \mathcal{O}_Q(a+c-2, b+d-2)$, which was later used in [Cat86] to show that the index of divisibility $r(S)$ of K_S in $H^2(S, \mathbb{Z})$ equals the $G.C.D.(a+c-2, b+d-2)$. This result showed that many of these irreducible components $\overline{\mathfrak{N}}'_{a,b,c,d}$ would belong to distinct connected components of the moduli space (since the divisibility index $r(S)$ is constant on a connected component).

The **abc-surfaces** are obtained as the special case where $b = d$, and the upshot of [CW04] is that, once the values of the integers b and $a+c$ are fixed, one obtains diffeomorphic surfaces.

It is relevant to observe that the family $\mathfrak{N}'_{a,b,c,d}$ is contained as a dense open subset in a larger irreducible family $\mathfrak{N}_{a,b,c,d}$, where coverings of degree 4 of other rational surfaces $\mathbb{F}_{2h} := Proj(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2h))$ have also to be considered.

In [Cat84] it was shown that two irreducible components of the moduli space, such as $\overline{\mathfrak{N}}'_{a,b,c,d}$, even when they have the same dimension, are distinct as soon as the 4-tuples are not trivially equivalent by an obvious $(\mathbb{Z}/2)^2$ -symmetry (exchanging the two factors of $\mathbb{P}^1 \times \mathbb{P}^1$, or the two branch divisors). In fact an irreducible component defines a surface over the generic point, and if two families coincide, then they should define isomorphic surfaces over the generic point. In particular, these two surfaces should be isomorphic as surfaces over the algebraic closure of the function field of the generic point.

¹in the following formula, a polynomial of negative degree is identically zero.

²deformation theory shows that this set is open

In this particular case, the corresponding surfaces were shown to be non isomorphic since this isomorphism would yield an isomorphism of the canonical images, and the canonical images were shown to have non equivalent singularities.

In general, it is much more difficult to show that two irreducible components yield distinct connected components of the moduli space.

In practice, in order to show that a certain open set $\mathfrak{N}$ of the moduli space is a connected component, one has to prove that the set $\mathfrak{N}$ is also closed.

In order to do so one has to look at all families of surfaces parametrized by a smooth curve T , and such that all the surfaces S_t , $t \in T \setminus \{t_0\}$ belong to $\mathfrak{N}$: if one can show that also the limit surface S_{t_0} stays in $\mathfrak{N}$ then $\mathfrak{N}$ is then closed, hence a connected component.

Of course, if two surfaces S, S' are not diffeomorphic (e.g., if their divisibility indices of the canonical class satisfy $r(S) \neq r(S')$), then they lie in different connected components.

Boris Moishezon proposed instead to use, as an invariant of a connected component $\mathfrak{N}$ of the moduli space, the geometry of pluricanonical projections, which we shall describe in the final section.

3. Diffeomorphisms of abc-surfaces and Lefschetz pencils

The main new result of [CW04] was the following

Theorem 3.1. (Catanese-Wajnryb) *Let S be an (a, b, c) - surface (i.e., an (a, b, c, b) surface) and S' be an $(a + 1, b, c - 1)$ -surface. Moreover, assume that $a, b, c - 1 \geq 2$. Then S and S' are diffeomorphic.*

This result was then put together with the following more technical result, in order to exhibit simply connected surfaces which are diffeomorphic but not deformation equivalent.

Theorem 3.2. (Catanese-Wajnryb) *Let S, S' be simple bidouble covers of $\mathbb{P}^1 \times \mathbb{P}^1$ of respective types $((2a, 2b), (2c, 2b))$, and $(2a + 2k, 2b), (2c - 2k, 2b)$, and assume*

- (I) a, b, c, k are strictly positive even integers with $a, b, c - k \geq 4$
- (II) $a \geq 2c + 1$,
- (III) $b \geq c + 2$ and either
- (IV1) $b \geq 2a + 2k - 1$ or
- (IV2) $a \geq b + 2$

Then S and S' are not deformation equivalent.

The second theorem uses techniques which have been developed in a series of papers by the first author and by Marco Manetti during a long period of time, and it belongs more to algebraic geometry than to geometry and topology.

The essential thrust is to show, under the above technical conditions, that $\mathfrak{N}_{a,b,c,b}$ is indeed a connected component of the moduli space.

The substantial inequality is the inequality (II), which guarantees that for all natural deformations one has $\psi \equiv 0$ (since a polynomial of negative degree is identically zero), i.e., the natural deformations have the following form

$$(\ast \ast \ast) \quad z_{a,b}^2 = f_{2a,2b}(x, y) + w_{c,b} \phi_{2a-c,b}(x, y)$$

$$w_{c,b}^2 = g_{2c,2b}(x, y).$$

The above equations show that the covering $\pi : S \rightarrow Q$ is an iterated double covering, i.e., we first take an intermediate double covering Y determined by taking the square root w of g , and then we obtain (through the quadratic equation for z) S as a double covering $S \rightarrow Y$.

The main idea is that, while taking limits, we can recover a smooth rational surface $\mathbb{F}_{2h}$ as the limit of $Q = Y/(\mathbb{Z}/2) = (S/(\mathbb{Z}/2))/(\mathbb{Z}/2)$.

We mention here however that the following question is rather open in general:

Question: For which values of a, b, c, d is $\mathfrak{N}_{a,b,c,d}$ a connected component of the moduli space?

A more detailed exposition for both theorems can be found in the Lecture Notes of the C.I.M.E. courses ‘Algebraic surfaces and symplectic 4-manifolds’ (see especially [Cat08]).

The above result in differential topology Theorem 3.1 is based instead on a refinement of Lefschetz theory obtained by Kas ([Kas80]).

This refinement allows us to encode the differential topology of a 4-manifold X Lefschetz fibred over $\mathbb{P}_{\mathbb{C}}^1$ (i.e., $f : X \rightarrow \mathbb{P}_{\mathbb{C}}^1$ has the property that all the fibres are smooth and connected, except for a finite number which have exactly one nodal singularity) into an equivalence class of a factorization of the identity in the Mapping class group $\mathcal{M}ap_g$ of a compact curve C_0 of genus g .

Recall that the mapping class group, introduced by Max Dehn ([Dehn38]) in the 30’s, is defined for each orientable manifold M as

$$\mathcal{M}ap(M) := Diff^+(M)/Diff^0(M),$$

where $Diff^+(M)$ is the group of orientation preserving diffeomorphisms, while $Diff^0(M)$ is the connected component of the identity, the so called subgroup of the diffeomorphisms which are **isotopic to the identity**.

In the case of a complex curve of genus g , Dehn found that the group was generated by the so called Dehn twists around simple closed loops Γ , which are diffeomorphisms equal to the identity outside of a neighbourhood of Γ , and rotate Γ by the antipodal map.

In our situation, let $F : X \rightarrow \mathbb{P}_{\mathbb{C}}^1$ be a Lefschetz fibration, and let $b_1, \dots, b_m$ be the critical values of f . For each b_i the fibre $C_i := f^{-1}(b_i)$ has a nodal singularity P_i , and the local monodromy around b_i is given, by a theorem of Picard and Lefschetz, by the Dehn twist around the vanishing cycle in a nearby smooth fibre $C_{b'_i}$.

If one fixes a base point b_0 in the base $\mathbb{P}_{\mathbb{C}}^1$, and simple paths γ'_i not crossing each other from b_0 to b'_i , one obtains diffeomorphisms, well defined up to isotopy, of the standard fibre $C_0 := C_{b_0}$ with the smooth fibres $C_{b'_i}$, and then a sequence of Dehn twists $\tau_1, \dots, \tau_m$.

Assuming that the paths γ'_i are ordered in counterclockwise order, by adding a small circle around the point b_i run counterclockwise starting from b'_i , one obtains a geometric

basis γ_i for the fundamental group

$$\pi(\mathbb{P}_\mathbb{C}^1 \setminus \{b_1, \dots, b_m\}, b_0) = \langle \gamma_1, \dots, \gamma_m \mid \gamma_1 \cdots \gamma_m = 1 \rangle.$$

By virtue of the relation $\gamma_1 \cdots \gamma_m = 1$ we obtain a sequence of Dehn twists $\tau_1, \dots, \tau_m$ on $C_0 = C_{b_0}$ whose product $\tau_1 \cdots \tau_m$ equals the identity. We obtain a factorization of the identity in $\mathcal{Map}_g$ once we fix a diffeomorphism of the fibre $C_0 = C_{b_0}$ with a fixed compact complex curve of genus g .

This means that this factorization is only defined up to simultaneous conjugation in the group $\mathcal{Map}_g$, i.e., one can replace, for $\phi \in \mathcal{Map}_g$, each τ_i by

$$\tau_i^\phi := \phi^{-1} \tau_i \phi.$$

Moreover, there is another ambiguity for the factorization, due namely to the choice of the generators $\gamma_1, \dots, \gamma_m$ for the fundamental group: this ambiguity amounts to an action of the braid group in m strings on the factorization, leading to the so called Hurwitz equivalence for factorizations, and which will be described in the next section.

Having said this, one can state the theorem of Kas:

Theorem 3.3. *Two Lefschetz fibrations F_1, F_2 of the same genus $g \geq 2$ are diffeomorphic (there is a diffeomorphism φ with $F_1 = F_2 \circ \varphi$) iff the corresponding factorizations in the mapping class group $\mathcal{Map}_g$ are equivalent via Hurwitz equivalence and simultaneous conjugation.*

Moreover, if, for $j = 1, 2$, S_j is a complex surface and $F_j : S_j \rightarrow \mathbb{P}^1$ is homotopic to a holomorphic map, then the diffeomorphism φ sends the canonical class $-c_1(S_1)$ in $H^2(S_1, \mathbb{Z})$ to the canonical class $-c_1(S_2)$.

Proof. We have only added a second part to the statement, based on the Seiberg Witten theorem mentioned above, and on the fact that φ sends fibres to fibres in an orientation preserving fashion. Hence φ is orientation preserving, sends the class of the fibre to the class of a fibre, $c_1(S_1)$ to $\pm c_1(S_2)$: we conclude that the sign is $+1$, since the intersection number of a fibre with $-c_1(S_j)$ equals $2g - 2 > 0$.

Q.E.D.

The concrete application of Kas' theorem present two big difficulties:

- 1) how to write such a monodromy factorization?
- 2) how to verify that two factorizations are equivalent?
- 3) how to verify that two factorizations are not equivalent?

For the first question, we shall show how we can calculate such factorizations, using the reality principle: trying to write fibrations which are defined over $\mathbb{R}$, and possibly with all the critical values being real (if not, they come into conjugate pairs). In this case the paths we choose can be explicitly described. In particular, it helps to write the branch curves as smoothings of rather singular but explicitly given curves, union of graphs of rational functions.

The second question is quite hard, but, at least in the case of abc-surfaces, a big help came from a simple geometric idea.

In fact, consider our surfaces as bidouble covers Z

$$\begin{aligned} (***) \quad z_{a,b}^2 &= f'_{2a,2b}(x, y) \\ w_{c,b}^2 &= g'_{2c,2b}(x, y) \end{aligned}$$

branched on a union of horizontal and of vertical lines: in this case the smooth surface S is the blow up of Z at the nodes lying over the nodes of $C' = \{f'(x, y) = 0\}$ and of $D' = \{g'(x, y) = 0\}$.

Changing the bidegree of f' from $(2a, 2b)$ to $(2a + 2, 2b)$, and the bidegree of g' from $(2c, 2b)$ to $(2c - 2, 2b)$ amounts geometrically to removing two vertical lines from the curve D' and putting them back into the curve C' . Here we exploit the fortunate circumstance that $b = d$: we take the horizontal lines of C' as the lines $y = j$, $j = 1, \dots, 2b$, and the horizontal lines of D' as the lines $y = -j$, $j = 1, \dots, 2b$.

Consider then the manifold with boundary M , resolution of the branched cover of $\Delta \times \mathbb{P}^1$, where Δ is the unit disk in $\mathbb{C}$, described by

$$\begin{aligned} z^2 &= \Pi_1^{2b}(y - i) \\ w^2 &= (x^2 - \epsilon^2)\Pi_1^{2b}(y + i). \end{aligned}$$

Then S' is obtained from S removing M and attaching it back on the boundary of Δ after acting with the automorphism ψ of order two such that $\psi(x, y, z, w) = (x, -y, w, z)$.

The result is that the factorizations for S and S' differ just in this beginning part, where the factors have been conjugated by ψ . Geometrically, one says that the fibrations of S and S' are obtained as two different fibre sums, i.e., one adds to the same factorization (the final part) a beginning factorization, once as it is, and another time conjugated by ψ . There is a very nice lemma, due to Auroux (mentioned also parenthetically by Kas), which gives a criterion for equivalence of these two beginning pieces of the respective factorizations:

Lemma 3.4. (Auroux) *Let τ be a Dehn twist and let F be a factorization of a central element $\phi \in \mathfrak{Map}_g$, $\tau_1 \circ \tau_2 \circ \dots \circ \tau_m = \phi$.*

If there is a factorization F' such that F is Hurwitz equivalent to $\tau \circ F'$, then $(F)^\tau := \tau^{-1}F\tau$ is Hurwitz equivalent to F .

In particular, if F is a factorization of the identity, $\psi = \Pi_h \tau'_h$, and $\forall h \exists F'_h$ such that F is Hurwitz equivalent to $\tau'_h \circ F'_h$, then the fibre sum with the Lefschetz pencil associated with F yields the same Lefschetz pencil as the fibre sum twisted by ψ .

Proof. If $\cong$ denotes Hurwitz equivalence, then

$$(F)^\tau \cong \tau \circ (F')^\tau \cong F' \circ \tau \cong (\tau)^{(F')^{-1}} \circ F' = \tau \circ F' \cong F.$$

Q.E.D.

Corollary 3.5. *Notation as above, assume that F , $\tau_1 \circ \tau_2 \circ \dots \circ \tau_m = \phi$, is a factorization of the Identity and that ψ is a product of the Dehn twists τ_i appearing in F . Then a fibre sum with the Lefschetz pencil associated with F yields the same result as the same fibre sum twisted by ψ .*

Proof. We need only to verify that for each h , there is F'_h such that $F \cong \tau_h \circ F'_h$.

But this is immediately obtained by applying $h-1$ Hurwitz moves, the first one between τ_{h-1} and τ_h , and proceeding further to the left till we obtain τ_h as first factor.

Q.E.D.

Hence in [CW04] the problem was reduced to proving that the isotopy class of ψ was a product of certain Dehn twists appearing in the first part of the factorization.

Verifying isotopy of these diffeomorphisms was accomplished by constructing chains of loops in the complex fibre curve C_0 , which lead to a dissection of C_0 into open cells. The problem was then solved choosing several associate Coxeter elements to express the given diffeomorphism ψ as a product of those Dehn twists.

In the next section we shall instead concentrate on Question 3), which is related to our recent work.

4. How to distinguish factorizations up to Hurwitz equivalence, and stable equivalence

In this section the set up will be more algebraically oriented: the reason for doing so stems from the fact that in our applications we shall deal with two different type of groups, not only mapping class groups, but also braid groups, and moreover we shall have to deal, in the second case, with two different types of equivalence relations, the first one being the equivalence relation generated by simultaneous conjugation and by Hurwitz equivalence, the second one, introduced by Auroux and Katzarkov, is called stable equivalence and is more general.

We shall look carefully into the problem of distinguishing classes of factorizations, and for this reason we shall introduce a technical novelty allowing a new effective method for disproving stable-equivalence.

Assume that we have a group G . Then a factorization in G of length m is nothing else than an element in the Cartesian product G^m .

The product function associates to such an element $(\alpha_1, \dots, \alpha_m) \in G^m$ its product $\alpha_1 \cdots \alpha_m = a$.

The inverse image of the element a under the product function is called the set of **factorizations of a in G of length m** and we write such a factorization of a in G with the notation

$$(*) \alpha_1 \circ \cdots \circ \alpha_m = a.$$

There is a natural action of the group $Aut(G)$ on the set of length m factorizations, in particular G acts by conjugation, hence the Centralizer $Z(a)$ acts on the factorizations of a by simultaneous conjugation,

$$(\alpha_1 \circ \cdots \circ \alpha_m = a) \mapsto (\alpha_1^g \circ \cdots \circ \alpha_m^g = a), \quad \forall g \in Z(a),$$

where as before we set $a^g := g^{-1}ag$.

Hurwitz equivalence of factorizations is the equivalence generated by Hurwitz moves, where the i -th Hurwitz move leaves all the α_j 's with $j \neq i, i+1$ unchanged, while it

replaces α_{i+1} with α_i , and α_i with $\alpha_i \alpha_{i+1} \alpha_i^{-1}$, so that the product of the factorization remain unchanged.

Hurwitz equivalence classes are nothing else than orbits for the action of the braid group $\mathcal{B}_m$ on the set of factorizations of a given element a .

Recall (cf. e.g. [Bir74]):

Theorem 4.1. *The braid group $\mathcal{B}_n$ is the subgroup of $\mathfrak{Map}(\mathbb{C} - \{1, \dots, n\})$ given by the diffeomorphisms which are the identity outside of a circle with centre the origin and radius $2n$. It has the Artin presentation*

$$\langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \forall i \leq n - 2 \rangle.$$

Through this isomorphism the standard Artin generators σ_j of $\mathcal{B}_n$ ($j = 1, \dots, n - 1$) correspond to standard half-twists: σ_j is the diffeomorphism of $(\mathbb{C}, \{1, \dots, n\})$ isotopic to the homeomorphism of $(\mathbb{C}, \{1, \dots, n\})$ which is a rotation of 180 degrees on the disc with centre $j + 1/2$ and radius $1/2$, on a circle with the same centre and radius $1/2 + t/4$ is the identity for $t \geq 1$, and a rotation of angle $180(1 - t)$ degrees if $0 \leq t \leq 1$.

Since $\mathcal{B}_n$ is a subgroup of $\mathfrak{Map}(\mathbb{C} - \{1, \dots, n\})$ we have an obvious action of $\mathcal{B}_n$ on the free group $\pi_1(\mathbb{C} - \{1, \dots, n\})$. Taking as base point the complex number $p := -2ni$, one has a geometric basis $\gamma_1, \dots, \gamma_n$ on which $\mathcal{B}_n$ acts through the

Hurwitz action of the braid group

- $\sigma_i(\gamma_i) = \gamma_{i+1}$
- $\sigma_i(\gamma_i \gamma_{i+1}) = \gamma_i \gamma_{i+1}$ hence $\sigma_i(\gamma_{i+1}) = \gamma_{i+1}^{-1} \gamma_i \gamma_{i+1}$
- $\sigma_i(\gamma_j) = \gamma_j$ for $j \neq i, i + 1$.

The Hurwitz action leaves the product $\gamma_1 \gamma_2 \dots \gamma_n$ invariant.

We can summarize the above discussion in the following

Definition 4.2. *Let G be a group. Consider the partition of its Cartesian product G^n given by the product map, which to an ordered n -tuple $(g_1, g_2, \dots, g_n)$ assigns the product $g := g_1 \cdot g_2 \cdot \dots \cdot g_n \in G$.*

*The subsets of the partition of G^n are called the **factorizations of an element $g \in G$** . The orbits of the action of $\mathcal{B}_n$, which preserves the partition of G^n , are called **Hurwitz equivalence classes of factorizations**.*

*Two factorizations are said to be **M-equivalent** (this means: equivalent by Hurwitz equivalence and simultaneous conjugation, and the M - stands for monodromy) if they are in the same $\mathcal{B}_n \times G$ -orbit, where G acts by (we use the notation previously introduced $a^b := b^{-1}ab$) the transformation*

$$Int_b: (g_1 \circ g_2 \circ \dots \circ g_n \mapsto (g_1)^b \circ (g_2)^b \circ \dots \circ (g_n)^b.$$

Remark 4.3. *Given a group homomorphism $\phi : G \rightarrow G'$, and a factorization in G , $g_1 \circ g_2 \circ \dots \circ g_n = a$, we obtain a corresponding factorization*

$$\phi(g_1) \circ \phi(g_2) \circ \dots \circ \phi(g_n) = \phi(a),$$

through the application of ϕ^n .

ϕ^n preserves Hurwitz equivalence, and simultaneous conjugation. Therefore, a basic method to disprove equivalence of factorizations is to disprove equivalence of homomorphic factorizations in a simpler group G' .

This is essentially the main underlying idea.

There are some obvious invariants for a factorization class:

Proposition 4.4. *Consider the set of Hurwitz equivalence classes of factorizations of a fixed element $a \in G$:*

$$(*) \alpha_1 \circ \cdots \circ \alpha_m = a.$$

Then the basic invariants of a Hurwitz factorization class are:

- i) the subgroup H generated by the elements $\alpha_1, \dots, \alpha_m$*
- ii) the function, defined on the set $\mathcal{C}_H$ of H -conjugacy classes in H , assigning to each conjugacy class $A \in \mathcal{C}_H$ the integer $\sigma(A) \in \mathbb{N}$ counting how many of the elements $\alpha_1, \dots, \alpha_m$ belong to A .*

Consider instead the set of M -equivalence classes of factorizations of a fixed element $a \in G$:

$$(*) \alpha_1 \circ \cdots \circ \alpha_m = a.$$

Then the basic invariants of an M -factorization class are, $Z(a)$ being the centralizer of the element $a \in G$:

- 1) the $Z(a)$ -conjugacy class of the subgroup H generated by the elements $\alpha_1, \dots, \alpha_m$, in particular the isomorphism class of H as an abstract group.*
- 2) the $Z(a)$ class of the function $\sigma: \mathcal{C}_H \rightarrow \mathbb{N}$, in particular, the cardinalities $\sigma^{-1}(n)$, for $n \in \mathbb{N}$.*

In the holomorphic case the above invariants are well defined; however, in the symplectic world Auroux and Katzarkov had to introduce a broader equivalence of factorizations, where the length of the factorization is not fixed a priori. They called it m -equivalence, but we prefer to call it stable-equivalence, or S-equivalence.

Definition 4.5. *Consider a group G , an element $a \in G$, and a set $\mathcal{B}$ of elements $\beta_j \in G$.*

*We define **stable-equivalence** with respect to $\mathcal{B}$ simply by considering the equivalence relation generated by Hurwitz equivalence, simultaneous conjugation and creation/cancellation of consecutive factors $\beta_j \circ \beta_j^{-1}$.*

Then **refined invariants** of the stable equivalence class of a factorization are obtained as follows:

- I) let H be the subgroup of G generated by the α_i 's and let $\hat{H}$ be the subgroup of G generated by the α_i 's and by the β_j 's (in our case, H will be called the monodromy group, and $\hat{H}$ the stabilized monodromy group).

Then the $Z(a)$ -conjugacy class of $\hat{H}$ is a first invariant, in particular the isomorphism class of $\hat{H}$.

- II) Let $\mathcal{C}_H$ be the set of H -conjugacy classes A in the group H , and let $\hat{\mathcal{C}}$ be the set of $\hat{H}$ -conjugacy classes in the group $\hat{H}$, so that we have a natural map $\mathcal{C} \rightarrow \hat{\mathcal{C}}$, with $A \mapsto \hat{A}$.

Write $\hat{\mathcal{C}}$ as a disjoint union $\hat{\mathcal{C}}^+ \cup \hat{\mathcal{C}}^- \cup \hat{\mathcal{C}}^0$, where $\hat{\mathcal{C}}^0$ is the set of conjugacy classes of elements a which are conjugate to their inverse a^{-1} , and where $\hat{\mathcal{C}}^-$ is the set of the inverse conjugacy classes of the classes in $\hat{\mathcal{C}}^+$.

Associate now to the factorization $\alpha_1 \circ \dots \circ \alpha_m = 1$ the function $s : \hat{\mathcal{C}}^+ \rightarrow \mathbb{Z}$ such that $s(c)$, for $c \in \hat{\mathcal{C}}^+$, is the algebraic number of occurrences of c in the sequence of conjugacy classes of the α_j 's (i.e., an occurrence of c^{-1} counts as -1 for $s(c)$).

The $Z(a)$ class of the function $s : \hat{\mathcal{C}}^+ \rightarrow \mathbb{Z}$ is our second and most important invariant. From it one can derive an easier invariant, i.e., the $Z(a)$ class of the function induced by the absolute value $|s|$ on the set $\hat{\mathcal{C}}^{+-}$ of pairs of distinct conjugacy classes, one inverse of the other. In particular, the values $|s|^{-1}(n)$ for $n \in \mathbb{N}$.

Remark 4.6. *The calculation of the function $s : \hat{\mathcal{C}}^+ \rightarrow \mathbb{Z}$ presupposes however a detailed knowledge of the group $\hat{H}$. For this reason in our paper [CLW] we resorted to using some coarser derived invariant.*

Substrategy I.

Assume that we can write $\{\alpha_1, \dots, \alpha_m\}$ as a disjoint union $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{D} \cup \mathcal{A}'_1 \cup \mathcal{A}'_2$, such that the set $\mathcal{A}_j$, $j = 1, 2$, is contained in a conjugacy class $A_j \subset H$, the set $\mathcal{A}'_j$, $j = 1, 2$, is contained in the conjugacy class $A_j^{-1} \subset H$. Assume that the elements in $\mathcal{A}_1 \cup \mathcal{A}_2$ are contained in a conjugacy class C in G such that $C \cap C^{-1} = \emptyset$ but that the set $\mathcal{D}$ is disjoint from the union of the two conjugacy classes of G , $C \cup C^{-1}$.

If we then prove that $\hat{A}_1 \neq \hat{A}_2$ (this of course implies $A_1 \neq A_2$) we may assume without loss of generality that $\hat{A}_1, \hat{A}_2 \in \hat{\mathcal{C}}^+$, and then the unordered pair of positive numbers $(|s(\hat{A}_1)|, |s(\hat{A}_2)|)$ is our derived numerical invariant of the factorization (and can easily be calculated from the cardinalities of the four sets as $(||\mathcal{A}_1| - |\mathcal{A}'_1||, ||\mathcal{A}_2| - |\mathcal{A}'_2||)$).

Substrategy II.

This is the strategy to show that $\hat{A}_1 \neq \hat{A}_2$ and goes as follows.

Assume further that we have another subgroup of G , $\tilde{H} \supset \hat{H}$, and a group homomorphism $\rho : \tilde{H} \rightarrow \Sigma$ such that $\rho(\mathcal{B}) = 1$. Assume also that the following key property holds.

Key property.

For each element $\alpha_j \in A_j \subset H$ there exists an element $\tilde{\alpha}_j \in \tilde{H}$ such that

$$\alpha_j = \tilde{\alpha}_j^2,$$

and moreover that this element is unique in $\tilde{H}$ (a fortiori, it will suffice that it is unique in G).

Proving that $\rho(\tilde{\alpha}_1)$ is not conjugate to $\rho(\tilde{\alpha}_2)$ under the action of $\rho(H) = \rho(\hat{H}) \subset \Sigma$ shows finally that α_1 is not conjugate to α_2 in $\hat{H}$.

Since, if there is $h \in \hat{H}$ such that $\alpha_1 = h^{-1}\alpha_2h$, then $\tilde{\alpha}_1^2 = \alpha_1 = (h^{-1}\tilde{\alpha}_2h)^2$, whence $\tilde{\alpha}_1 = h^{-1}\tilde{\alpha}_2h$, a contradiction.

In our concrete case, we are able to determine the braid monodromy group H , and we observe that, since we have a so called cuspidal factorization, all the factors α_i

belong to only four conjugacy classes in the group G (the classes of $\sigma_1, \sigma_1^2, \sigma_1^{-2}, \sigma_1^3$ in the braid group), corresponding geometrically to vertical tangencies of the branch curve, respectively positive nodes, negative nodes and cusps. Three of these classes are positive and only one is negative (the one of σ_1^{-2}).

We see in [CLW] that the nodes belong to conjugacy classes $A_1, A_2, A_1^{-1}, A_2^{-1}$ in H , the positive nodes belonging to $A_1 \cup A_2$, and we show, using a representation ρ of a certain subgroup $\tilde{H}$ of ‘liftable’ braids (i.e., braids which centralize the monodromy homomorphism, whence are liftable to the mapping class group of a curve D_0 associated to the monodromy homomorphism) into a symplectic group Σ with $\mathbb{Z}/2$ coefficients, that the classes $\hat{A}_1, \hat{A}_2$ are distinct in $\hat{H}$. Moreover, these are classes in $\mathcal{C}^+$, since these are positive classes in the braid group. We calculate then easily the above function for these two conjugacy classes, i.e., the pair of numbers $(s(\hat{A}_1), s(\hat{A}_2))$.

5. Bidouble covers of the quadric and their symplectic perturbations.

In order to see in more detail how the algebraic considerations of the previous sections apply to our situation, let us go back to our equations

$$\begin{aligned} (***) \quad z_{a,b}^2 &= f_{2a,2b}(x, y) + w_{c,d} \Phi_{2a-c, 2b-d}(x, y) \\ w_{c,d}^2 &= g_{2c,2d}(x, y) + z_{a,b} \Psi_{2c-a, 2d-b}(x, y) \end{aligned}$$

where f, g are bihomogeneous polynomials as before, and instead we shall allow Φ, Ψ , in the case where for instance the degree relative to x is negative, to be antiholomorphic. In other words, we allow Φ, Ψ to be sections of certain line bundles which are dianalytic (holomorphic or antiholomorphic) in each variable x, y .

We can more generally consider the direct sum $\mathbb{V}$ of two complex line bundles $\mathbb{L}_1 \oplus \mathbb{L}_2$ on a compact complex manifold X , and the subset Z of $\mathbb{V}$ defined by the following pair of equations,

$$\begin{aligned} z^2 &= f(x) + w \Phi(x) \\ w^2 &= g(x) + z \Psi(x) \end{aligned}$$

where f, g are respective holomorphic sections of the line bundles $\mathbb{L}_1^{\otimes 2}, \mathbb{L}_2^{\otimes 2}$ (we shall also write $f \in H^0(\mathcal{O}_X(2L_1)), g \in H^0(\mathcal{O}_X(2L_2))$, denoting by L_1, L_2 the associated Cartier divisors), and where Φ is a differentiable section of $\mathbb{L}_1^{\otimes 2} \otimes \mathbb{L}_2^{\otimes -1}$, Ψ is a differentiable section of $\mathbb{L}_2^{\otimes 2} \otimes \mathbb{L}_1^{\otimes -1}$.

The key observation is that, for $\Phi \equiv \Psi \equiv 0$, then $\pi : Z \rightarrow X$ is a Galois cover with group $(\mathbb{Z}/2\mathbb{Z})^2$, while, if Φ, Ψ are not $\equiv 0$, then we obtain a covering whose monodromy group is the symmetric group $\mathfrak{S}_4$.

We have the following Lemmas from [CLW].

Lemma 5.1. *Assume that the two divisors $\{f = 0\}$ and $\{g = 0\}$ are smooth and intersect transversally. Then, for $|\Phi| \ll 1$, $|\Psi| \ll 1$, Z is a smooth submanifold of $\mathbb{V}$, and the projection $\pi : \mathbb{V} \rightarrow X$ induces a finite covering $Z \rightarrow X$ of degree 4, with ramification divisor (i.e., critical set) $R := \{4zw = \Phi\Psi\}$, and with branch divisor (i.e., set of critical values) $\Delta = \{\delta(x) = 0\}$, where*

$$-\frac{1}{16}\delta = -f^2g^2 - \frac{9}{8}fg(\Phi\Psi)^2 + (\Psi)^2f^3 + (\Phi)^2g^3 + \frac{27}{16^2}(\Phi\Psi)^4.$$

We want to analyse now the singularities that Δ has, for a general choice of Φ, Ψ . The singularities which are easily spotted deserve a special definition.

Definition 5.2. *A point x is said to be a **trivial singularity** of Δ if it is either a point where $f = \Phi = 0$, or a point where $g = \Psi = 0$.*

*Z is said to be **mildly general** if*

() at any point where $f = \Phi = 0$, we have $g \cdot \Psi \neq 0$, and symmetrically at any point where $g = \Psi = 0$, we have $f \cdot \Phi \neq 0$*

Besides the trivial singularities, we show in general that the other singularities are located in a neighbourhood of the points with $f(x) = g(x) = 0$.

Lemma 5.3. *If Z is mildly general, and $|\Phi| \ll 1$, $|\Psi| \ll 1$, the singular points of Δ which are not the trivial singularities of Δ occur only for points $p \in Z_x$ (Z_x being the fibre over the point x) of multiplicity 3 which lie over arbitrarily small neighbourhoods of the points x' with $f(x') = g(x') = 0$.*

We can describe more precisely the singularities of Δ in the case where X is a complex surface.

Proposition 5.4. *Assume that X is a compact complex surface, that the two divisors $\{f = 0\}$ and $\{g = 0\}$ are smooth and intersect transversally in a set M of m points. Then, for $|\Phi| \ll 1$, $|\Psi| \ll 1$, and for Z mildly general, Z is a smooth submanifold of $\mathbb{V}$, and the projection $\pi : \mathbb{V} \rightarrow X$ induces a finite covering $Z \rightarrow X$ of degree 4, with smooth orientable ramification divisor $R := \{4zw = \Phi\Psi\}$, and with branch divisor Δ having as singularities precisely*

- 1) $3m$ cusps lying (in triples) in an arbitrarily small neighbourhood of M , and moreover*
- 2) the trivial singularities, which are nodes if the curve $\{f = 0\}$ intersects transversally $\{\Phi = 0\}$, respectively if the curve $\{g = 0\}$ intersects transversally $\{\Psi = 0\}$.*

Since R is orientable, it follows that if the trivial singularities of Δ are nodes, then the intersection number of the two branches is ± 1 ; the local selfintersection number is exactly equal to $+1$ when the two orientations of the two branches combine to yield the natural (complex) orientation of X .

We can calculate the precise number of singularities, and of the positive, respectively negative nodes of Δ , in the special case we are interested in, namely, of a perturbed bidouble cover of the quadric.

Definition 5.5. *Given four integers $a, b, c, d \in \mathbb{N}_{\geq 3}$, the so called **dianalytic perturbations of simple bidouble covers** are the 4-manifolds defined by equations*

$$\begin{aligned} z^2 &= f_{(2a, 2b)}(x_0, x_1; y_0, y_1) + w \Phi_{(2a-c, 2b-d)}(x_0, x_1; y_0, y_1) \\ w^2 &= g_{(2c, 2d)}(x_0, x_1; y_0, y_1) + z \Psi_{(2c-a, 2d-b)}(x_0, x_1; y_0, y_1) \end{aligned}$$

where f, g are bihomogeneous polynomials of respective bidegrees $(2a, 2b)$, $(2c, 2d)$, and where Φ, Ψ are polynomials in the ring

$$\mathbb{C}[x_0, x_1, y_0, y_1, \bar{x}_0, \bar{x}_1, \bar{y}_0, \bar{y}_1],$$

bihomogeneous of respective bidegrees $(2a-c, 2b-d)$, $(2c-a, 2d-b)$ (the ring is bigraded here by setting the degree of $\bar{x}_i$ equal to $(-1, 0)$ and the degree of $\bar{y}_i$ equal to $(0, -1)$).

We choose moreover Φ, Ψ to belong to the subspace where all monomials are either separately holomorphic or antiholomorphic, i.e., they admit no factor of the form $x_i \bar{x}_j$ or of the form $y_i \bar{y}_j$.

Remark 5.6. *Indeed it is convenient to pick Φ (respectively, Ψ) to be a product*

$$\Phi = \Phi_1(x)\Phi_2(y),$$

where Φ_1 is a product of linear forms, either all holomorphic or all antiholomorphic, and similarly for Φ_2 (respectively, for Ψ_1, Ψ_2).

As explained in [Cat08](pages 134-135) to an antiholomorphic homogeneous polynomial $P(\bar{x}_0, \bar{x}_1)$ of degree m we associate a differentiable section p of the tensor power $\mathbb{L}^{\otimes m}$ of the tautological (negative) subbundle, such that $p(x) := P(1, \bar{x})$ inside a big disc $B(0, r)$ in the complex line having centre at the origin and radius r .

We make the following assumptions on the polynomials f, g, Φ, Ψ :

- (1) The algebraic curves $C := \{f = 0\}$ and $D := \{g = 0\}$ are smooth
- (2) C and D intersect transversally at a finite set M contained in the product $B(0, r)^2$ of two big discs in $\mathbb{C}$, and at these points both curves have non vertical tangents
- (3) for both curves C and D the first projection on $\mathbb{P}^1$ is a simple covering and moreover the vertical tangents of C and D are all distinct
- (4) the associated perturbation is sufficiently small and mildly general
- (5) the trivial singularities are nodes with non vertical tangents, contained in $B(0, r)^2$
- (6) the cusps of Δ have non vertical tangent.

Considering then the branch curve Δ of the perturbed bidouble cover, the defining equation δ is bihomogeneous of bidegree $(4(a+c), 4(b+d))$.

We have that Δ has exactly $k := 12(ad+bc)$ cusps, coming from the $m := 4(ad+bc)$ points of the set $M = C \cap D$.

If Φ, Ψ are holomorphic, then Δ has exactly

$$\nu := 4(2ab + 2cd - ad - bc)$$

(positive) nodes.

In general, the above number ν equals the difference $\nu^+ - \nu^-$ between the number of positive and the number of negative nodes.

In fact the nodes of Δ occur only for the trivial singularities.

A trivial singularity $f = \Phi = 0$ yields a node with the same tangent cone as

$$f^2 - \Phi^2 \cdot g(x_0).$$

By remark 5.6 we may assume that there are local holomorphic coordinates $f, \hat{\Phi}$ such that either $\Phi = \text{unit} \cdot \hat{\Phi}$ or $\Phi = \text{unit} \cdot \overline{\hat{\Phi}}$.

In the first case we get the tangent cone of a holomorphic node, hence a **positive** node, in the second case we get the tangent cone of an antiholomorphic node, hence a **negative** node.

We summarize the results in the following

Lemma 5.7. *The number t_f of vertical tangents for the curve C is $t_f = 4(2ab - a)$, while the number t_g of vertical tangents for the curve D is $t_g = 4(2cd - c)$. For a general choice of Φ, Ψ , the number of vertical tangents t of Δ equals $t = 2t_f + 2t_g + m$, where $m := 4(ad + bc)$. The genus of the Riemann surface R equals $g(R) = 1 + 16(a + c)(b + d) - 4(a + b + c + d) - k - \nu$, where $k := 3m = 12(ad + bc)$ is the number of cusps of Δ and $\nu := \nu^+ - \nu^-$ is the number of nodes of Δ , counted with sign.*

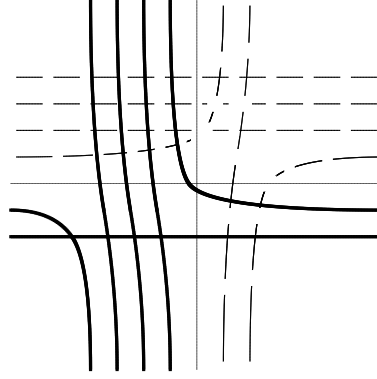
Remark 5.8. 1) Observe that the fact that $p : R \rightarrow \mathbb{P}_{\mathbb{C}}^1$ has a finite number of critical points implies immediately that p is finite, since there is then a finite set in $\mathbb{P}_{\mathbb{C}}^1$ such that over its complement we have a covering space. Thus p is orientation preserving and each non critical point contributes positively to the degree of the map p . Hence some of the above calculations are obtained from Hurwitz' ramification formula.

2) Consider now a real analytic 1-parameter family $Z(\eta)$ such that the perturbation terms $\Phi_\eta, \Psi_\eta \rightarrow 0$. Then we have a family of ramification points $P_i(\eta)$ which tend to the ramification points of $p_0 : R_0 \rightarrow \mathbb{P}_{\mathbb{C}}^1$. For R_0 , a double cover of $C \cup D$, we have $2t_f$ ramification points lying over the t_f vertical tangents of C , $2t_g$ ramification points lying over the t_g vertical tangents of D , whereas the m nodes of R_0 are limits of 4 ramification points, three cusps and a simple vertical tangent, cf. the analysis made in [CW05]. This is the geometric reason why $t = 2t_f + 2t_g + m$; it also tells us where to look for the vertical tangents of Δ .

In practice, in order to effectively compute the braid monodromies, it is convenient to view the curves C and D as small real deformations of reducible real curves with only nodes (with real tangents) as singularities, and to consider real polynomials Φ, Ψ , such that the trivial singularities are also given by real points. By the results of [CW05], from the perturbation of each proper node of $C \cup D$, i.e., of a point of $M = C \cap D$, we shall obtain a real vertical tangent, a real cusp, and two imaginary cusps.

We begin with real polynomials $f^\times$ and $g^\times$ defining nodal curves $C^\times$, respectively $D^\times$.

For the sake of simplicity we replace here bihomogeneous polynomials $f(x_0, x_1; y_0, y_1)$ by their restrictions to the affine open set $x_0 = 1, y_0 = 1$ and write $f(x, y)$ for $f(1, x; 1, y)$.


 FIGURE 1. Real part of Δ_f and Δ_g for $(a, b, c, d) = (1, 2, 2, 1)$

We let then $f^\times := f_1 \cdots f_{2b}$ defining $f_i := y - 2i$ for $i=2, \dots, 2b$ but setting

$$f_1 := (y - 2) \prod_{i=1}^{2a} (x - 2i) + \prod_{i=1}^{2a-1} (x - 2i - 1),$$

and we define similarly $g^\times := g_1 \cdots g_{2d}$ setting $g_i = y + 2i$ for $i=2, \dots, 2d$ except that we set

$$g_1 := (y + 2) \prod_{i=1}^{2c} (x + 2i) - \prod_{i=1}^{2c-1} (x + 2i + 1).$$

Remark 5.9. Note that the equations of f_1 and g_1 are chosen in such a way that their zero sets are graphs of rational functions $\tilde{f}_1$, respectively $\tilde{g}_1$, of x which, regarded as maps from $\mathbb{C}$ to $\mathbb{C}$, preserve the real line. Moreover, $\tilde{f}_1$ preserves both the upper and lower halfplane, while $\tilde{g}_1$ exchanges them.

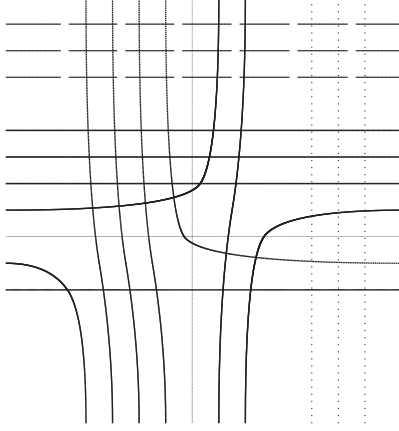
In fact, $\tilde{f}_1$ has no critical points on $\mathbb{P}_{\mathbb{R}}^1$ and

$$\deg(\tilde{f}_1|_{\mathbb{P}_{\mathbb{R}}^1}) : \mathbb{P}_{\mathbb{R}}^1 \rightarrow \mathbb{P}_{\mathbb{R}}^1 = 2a = \deg(\tilde{f}_1).$$

Similarly for $\tilde{g}_1$.

Given $f^\times$, consider the polynomial $f : f^\times + c_f$, where c_f is a small constant; likewise consider $g : g^\times + c_g$. Adding these small constants to the respective equations of $C^\times$ and $D^\times$ we get polynomials f and g which define smooth curves C and D . We have more precisely:

Proposition 5.10. If the constants c_f , c_g , η are chosen sufficiently small, the polynomials $f, g, \eta\Phi, \eta\Psi$ thus obtained satisfy the following list of hypotheses (cf. p. 75), for $r > 8(a + b + c + d)$:

FIGURE 2. Zero sets of Φ (dashed) and Ψ (dotted) added

- (1) *The algebraic curves $C = \{f = 0\}$ and $D = \{g = 0\}$ are smooth*
- (2) *C and D intersect transversally at a finite set M contained in the product $B(0, r)^2$ and at these points both curves have non vertical tangents*
- (3) *for both curves C and D the first projection on $\mathbb{P}^1$ is a simple covering and moreover the vertical tangents of C and D are all distinct*
- (4) *the associated perturbation is mildly general*
- (5) *the trivial singularities of Δ are nodes with non vertical tangents, contained in $B(0, r)^2$*
- (6) *the cusps of Δ have non vertical tangent*

6. Bidouble covers of the quadric and their Braid Monodromy factorization types.

In the previous section we showed how to obtain a symplectic 4-manifold which (we call a dianalytic perturbation and) is canonically symplectomorphic to the bidouble cover S we started with.

Now we have that, for general choice of f, g, Φ, Ψ , the projection π onto $Q = \mathbb{P}^1 \times \mathbb{P}^1$ is **generic** and its branch curve Δ (the locus of the critical values) is a dianalytic curve with nodes and cusps as only singularities. In this way we may however have introduced also negative nodes, i.e., nodes which in local holomorphic coordinates are defined by the equation

$$(y - \bar{x})(y + \bar{x}) = 0.$$

Now, the projection onto the first factor $\mathbb{P}^1$ gives two monodromy factorizations:

Definition 6.1. *The vertical braid monodromy factorization of a bidouble cover is the factorization of the identity in the braid group Br_n of the sphere $\mathbb{P}^1$ associated to the*

first projection onto $\mathbb{P}^1$ of the pair (Q, Δ) . Here $n = 4(b + d)$ is the vertical degree of Δ , and the monodromy factorization describes the movement of $n = 4(b + d)$ points in a fibre $F_0 = \mathbb{P}^1$.

The Braid Monodromy factorization has factors in the braid group of the sphere Br_n :

$$\text{Br}_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \left| \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{if } |i - j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_1 \cdots \sigma_{n-1} \sigma_{n-1} \cdots \sigma_1 = 1 \end{array} \right. \right\rangle$$

The factors are of three following types

- (1) each cusp of Δ (which has non vertical tangent) yields a factor which is a conjugate of σ_1^3
- (2) each vertical tangency (this takes place in smooth points only) yields a factor which is a conjugate of σ_1
- (3) each trivial singularity yields a factor which is a conjugate of σ_1^2 in the case where we have a positive node, and a factor which is a conjugate of σ_1^{-2} in the case where we have a negative node.

We already know, by Kas' theorem, that the differentiable type of the Lefschetz fibration $F : S \rightarrow \mathbb{P}^1$ given by the composition of the perturbed covering $\pi : S \rightarrow Q$ with the first projection onto $\mathbb{P}^1$ is encoded in the M-equivalence class of the factorization of the identity in the Mapping class group Map_g corresponding to the monodromy factorization of this Lefschetz fibration (here g is the genus of the base fibre C_0 of F).

The second factorization is the homomorphic image of the Braid Monodromy factorization.

It is obtained since the braid monodromy factors are contained in a subgroup of the Braid group Br_n , namely, the subgroup of liftable braids, corresponding to diffeomorphisms which lift to the fibre C_0 of the Lefschetz fibration $F : S \rightarrow \mathbb{P}^1$ which lies over F_0 .

More precisely, we have the covering $C_0 \setminus F^{-1}(\Delta) \rightarrow F_0 \setminus \Delta$, and let $\mu : \pi_1(F_0 \setminus \Delta) \rightarrow \mathfrak{S}_4$ be its monodromy homomorphism: then the braid monodromy factors are contained in the subgroup

$$\mathcal{L} := \{\sigma \in \text{Br}_n \mid \mu \sigma_* = \mu\}.$$

And each braid in $\mathcal{L}$ determines a diffeomorphism of the pair $(C_0, C_0 \cap \Delta)$ which in turn gives us an element in Map_g .

The easy but important observation is that the homomorphism

$$\mathcal{L} \rightarrow \text{Map}_g$$

is far from being injective.

In fact, the factors of the Braid Monodromy factorization map to:

- (1) each factor corresponding to a cusp of Δ maps to the identity
- (2) each factor coming from a vertical tangency maps to a Dehn twist which is the lift of a conjugate of σ_1

- (3) each factor coming from a trivial singularity, be it a positive or a negative node, maps to the identity.

Before we state our characterization of the Braid Monodromy factorization, it will be convenient to briefly describe the geometric picture of the degree 4 covering $F : C_0 \rightarrow F_0$.

In the case where $b = d$ we may start with an extremely symmetric picture, taking the bidouble cover F'_0 of $\mathbb{P}^1_{\mathbb{C}}$ of equation

- $z^2 = f(y), w^2 = g(y) := f(-y)$
and of the automorphism ψ given by
- $\psi(y, z, w) := (-y, w, z)$

and we see immediately that the curve C_0 has an automorphism group generated by the previously mentioned automorphism ψ and by the Galois group of the covering, generated by

$$g_1(y, z, w) := (y, -z, w), g_2(y, z, w) := (y, z, -w).$$

This group is indeed generated by g_1 and the order 4 element $g_4(y, z, w) := (-y, w, -z)$; since $g_1 g_4 g_1 g_4 = 1$, it is the dihedral group D_4 .

We recall here once more the role of the automorphism ψ , namely that the following was proven in [CW04] :

Proposition 6.2. *The monodromy of S over the unit circle $\{x \mid |x| = 1\}$ is trivial, and the pair (C_0, ψ) yields the fibre over $x = 1$, considered as a differentiable 2-manifold, together with the isotopy class of the map yielding the twisted fibre sum corresponding to S' .*

In the general case, perturbing the equations of C_0 amounts to making the degree 4 morphism generic. In this context we can view the effect of the perturbation as replacing the branch locus

$$B'_0 := \{f^\times(y) = 0\} \cup \{g^\times(y) = 0\} = \{1, \dots, 2b\} \cup \{-1, \dots, -2d\}$$

with a new ‘doubled’ branch locus

$$B_0 := \{f(y)g(y) = 0\} = \{1 \pm \epsilon\sqrt{-1}, \dots, 2b \pm \epsilon\sqrt{-1}\} \cup \{-1 \pm \epsilon\sqrt{-1}, \dots, -2d \pm \epsilon\sqrt{-1}\}.$$

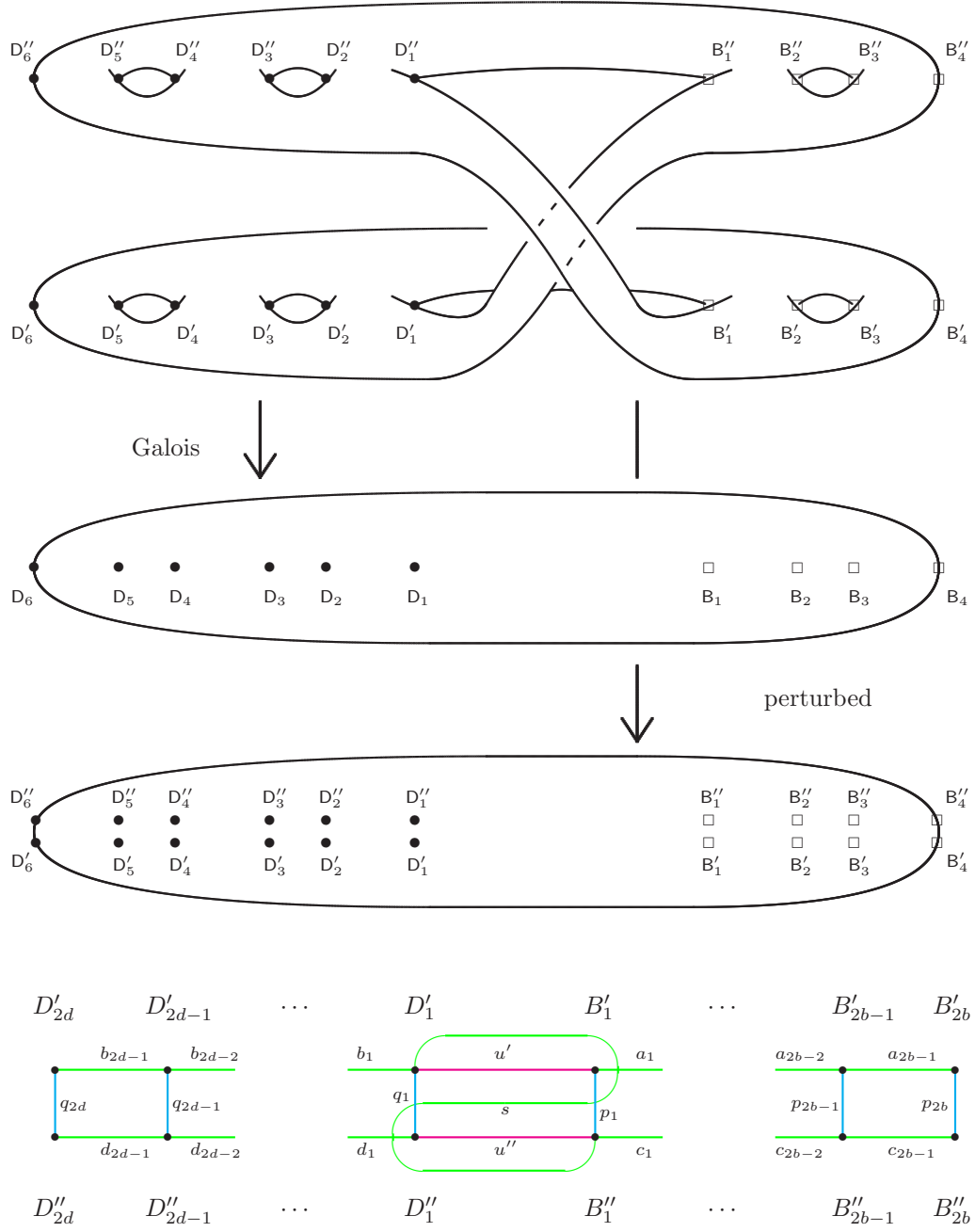
Before giving the full vertical Braid monodromy factorization, we show a simpler byproduct of our description of the Braid monodromy: it makes effective and precise the contents of the above intuitive picture.

The theorem is stated in its most understandable form, namely a pictorial one, recalling that any segment in the plane determines a (non standard) half twist in Br_n , $n = 4(b+d)$.

Theorem 6.3. *There is a braid monodromy factorization of the curve Δ associated to a bidouble cover S of type a, b, c, d whose braid monodromy group $H \subset \text{Br}_{4(b+d)}$ is generated, unless we are in the cases*

$$(I) \ c = 2a \text{ and } d = 2b, \text{ or } (II) \ a = 2c \text{ and } b = 2d$$

by the following powers of half-twists: $\sigma_{a_i}, \sigma_{c_i}$ for $i = 1, \dots, 2b - 1$, $\sigma_{b_i}, \sigma_{d_i}$, for $i = 1, \dots, 2d - 1$, $\sigma_{p_{2b}}^2, \sigma_{q_{2d}}^2, \sigma_s, \sigma_{u'}^3, \sigma_{u''}^3$.



The factorization is such that

- (1) each $(\pm)$ full-twist factor is of type p or q ,
- (2) the weighted count of $(\pm)$ full-twist factors of type p yields $8ab - 2(ad + bc)$,
- (3) the weighted count of $(\pm)$ full-twist factors of type q yields $8cd - 2(ad + bc)$.

The result relies on a complete description of the braid monodromy factorization class associated to Δ : this is given in Theorem 6.6, which proves indeed much more than what we need for the present purposes.

The most important application is the following Main Theorem:

Theorem 6.4. *Assume that S is an $abcd$ -surface, S' is an $a'b'c'd'$ -surface, that S and S' are homeomorphic, and that their vertical Braid monodromy factorizations are stably equivalent. Then, up to swapping the pair (a, b) with the pair (c, d) , we have*

$$a + c = a' + c', b + d = b' + d', ab = a'b', cd = c'd'.$$

The vertical Braid monodromy factorizations associated to an abc -surface S and to an $a'b'c'$ -surface S' are not stably equivalent, except in the trivial cases $a = a', b = b', c = c'$ or $a = c', b = b', c = a'$.

Proof. We sketch only the first calculation, which is not contained in [CLW].

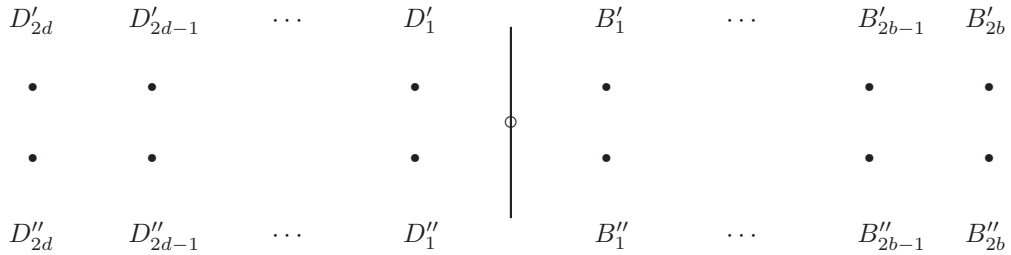
Since S and S' are homeomorphic, they have the same χ and K^2 , thus first of all $ab + cd = a'b' + c'd'$. By the previous theorem 6.3 and the forthcoming proposition (7.6) that the factors of type p are not stably conjugate to the factors of type q , it follows that $ab = a'b', cd = c'd'$.

Then by the same formulae of theorem 6.3 $(a + c)(b + d) = (a' + c')(b' + d')$, hence, since $K_S^2 = (a + c - 2)(b + d - 2)$, we get $(a + c) + (b + d) = (a' + c') + (b' + d')$, and therefore $a + c = a' + c', b + d = b' + d'$, up to swapping the pairs.

Q.E.D.

We start now to prepare the notation for the factorization theorem.

Proposition 6.5. *Up to homotopy in small discs containing pairs D'_i, D''_i or B'_j, B''_j the punctured fibre F_0 is given by the following picture*



and the covering monodromy μ of the perturbed bidouble cover, with respect to the origin,

is given by

$$\begin{array}{ll} \omega_{D'_j} \mapsto (12), & \omega_{B'_j} \mapsto (13), \\ \omega_{D''_j} \mapsto (34), & \omega_{B''_j} \mapsto (24), \end{array}$$

where ω_P is any simple closed path around P not crossing the imaginary axis $i\mathbb{R} \subset F_0$.

Proof. As we remarked before, in the unperturbed Galois cover case we have real points D_i, B_j , only, the D_i 's with positive real coordinate, the B_j 's with negative real coordinates. And the covering's monodromy is (12)(34), resp. (13)(24) for paths which do not cross the imaginary axis.

The deformation then splits each branch point into two, hence the corresponding monodromies (which are transpositions) must be (12) and (34), resp. (13) and (24) (which commute). After a suitable homotopy the points are in the positions given by the picture. $\square$

Referring to the above figures and to the figure below we introduce the following notation for a system of arcs, which are uniquely determined (up to homotopy) by their endpoints and the property that they are monotonous in the real coordinate and do not pass below any puncture; i.e., if they share the real coordinate with a puncture they have larger imaginary coordinate.

$$\begin{array}{ll} p_i : & B'_i, B''_i, \\ a_{ij} : & B'_i, B'_j, \\ c_{ij} : & B''_i, B''_j, \\ u'_{ij} : & B'_i, D'_j \\ q_i : & D'_i, D''_i, \\ b_{ij} : & D'_i, D'_j, \\ d_{ij} : & D''_i, D''_j, \\ u''_{ij} : & B''_i, D''_j \end{array}$$

Further, there are arcs s_{ij} connecting D'_i with B''_j according to a pattern which is shown in Figure 3.

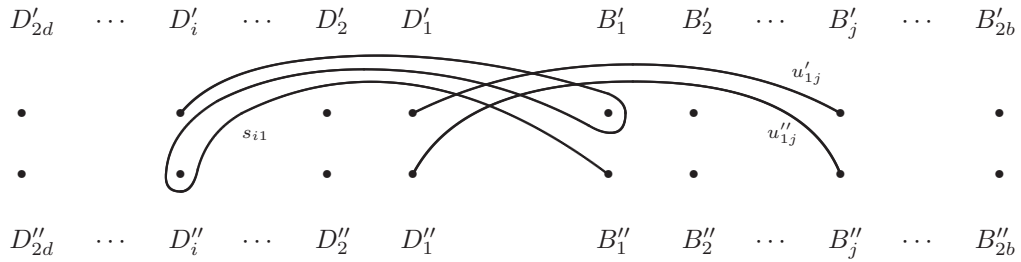


FIGURE 3. arcs for the theorem

Theorem 6.6. *The factorization is a product of factorizations*

$$\underbrace{\left(\beta_f \circ \underbrace{(\sigma_{p_1}^{\pm 2} \circ \dots \circ \sigma_{p_1}^{\pm 2})}_{|2b-d| \text{ factors}} \circ \beta_{fg} \right) \circ \dots \circ \underbrace{(\sigma_{p_1}^{\pm 2} \circ \dots \circ \sigma_{p_{2b}}^{\pm 2})}_{|2a-c| \text{ repetitions}} \circ \dots}_{2a \text{ repetitions}} \\
 \circ \underbrace{(\sigma_{q_1}^{\pm 2} \circ \dots \circ \sigma_{q_{2d}}^{\pm 2})}_{|2c-a| \text{ repetitions}} \circ \dots \circ \underbrace{\left(\beta_g \circ \underbrace{(\sigma_{q_1}^{\pm 2} \circ \dots \circ \sigma_{q_1}^{\pm 2})}_{|2d-b| \text{ factors}} \circ \beta_{gf} \right) \circ \dots}_{2c \text{ repetitions}}$$

where the sign of the exponents 2 is constant inside a pair of brackets and is the sign of the number which determines the number of factors, ie. $(2b-d)$ resp. $(2a-c)$, $(2c-a)$ or $(2d-b)$, and where the β 's further decompose as products of factorizations

$$\begin{aligned}
 \beta_f &= \beta_{f,2} \circ \dots \circ \beta_{f,2b}, & \beta_{fg} &= \beta_{fg,2d} \circ \dots \circ \beta_{fg,1}, \\
 \beta_g &= \beta_{g,2} \circ \dots \circ \beta_{g,2d}, & \beta_{gf} &= \beta_{gf,2b} \circ \dots \circ \beta_{gf,1},
 \end{aligned}$$

based on elementary factorizations each having four factors

$$\begin{aligned}
 \beta_{f,i} &= \sigma_{a_{1i}} \circ \sigma_{p_1}^2 \sigma_{c_{1i}} \sigma_{p_1}^{-2} \circ \sigma_{a_{1i}} \circ \sigma_{p_1}^2 \sigma_{c_{1i}} \sigma_{p_1}^{-2}, & \beta_{fg,j} &= \sigma_{u_{1j}}^3 \circ \sigma_{s_{1j}} \circ \sigma_{u'_{1j}}^3 \circ \sigma_{u''_{1j}}^3, \\
 \beta_{g,j} &= \sigma_{b_{1j}} \circ \sigma_{q_1}^2 \sigma_{d_{1j}} \sigma_{q_1}^{-2} \circ \sigma_{b_{1j}} \circ \sigma_{q_1}^2 \sigma_{d_{1j}} \sigma_{q_1}^{-2}, & \beta_{gf,i} &= \sigma_{u_{i1}}^3 \circ \sigma_{s_{i1}} \circ \sigma_{u'_{i1}}^3 \circ \sigma_{u''_{i1}}^3.
 \end{aligned}$$

The elementary factorizations originate in the regeneration of nodes of the branch curve of the corresponding bidouble Galois-cover.

While it is complicated to explain in detail the above theorem, we would like to point out some other important tools in the local analysis of ‘regenerations’ of the singularities of the branch curve of the bidouble cover.

In the case of a ‘proper node’ (i.e., a point where $f = g = 0$) we get a cusp-cluster, four critical points of which three are cusps and the last is a vertical tangency point.

The braid monodromy for the regeneration into a cusp-cluster had been thoroughly investigated before in [CW05], where the braid monodromy factorization type had been determined up to Hurwitz equivalence and simultaneous conjugation.

In [CLW] we were able to show how one can determine the Hurwitz equivalence class of the factorization from the datum of the product of the factors.

Proposition 6.7. *The braid monodromy factorization of a (regenerated) cusp-cluster with product $\sigma_2^3 \sigma_1 \sigma_3 \sigma_2 \sigma_1^2 \sigma_3^2$ is Hurwitz-equivalent to the factorization*

$$\sigma_2^3 \circ \sigma_1 \sigma_3 \sigma_2 \sigma_3^{-1} \sigma_1^{-1} \circ \sigma_1^3 \circ \sigma_3^3.$$

In the case of an improper node (i.e., a node of $f^\times = 0$ or of $g^\times = 0$) we get a cluster of vertical tangents, four critical points, all of which are vertical tangency points.

Proposition 6.8. *The braid monodromy factorization associated to a cluster of tangents corresponding to an improper node is given by*

$$\sigma_2 \sigma_3 \sigma_2^{-1} \circ \sigma_1 \sigma_2 \sigma_1^{-1} \circ \sigma_2 \sigma_3 \sigma_2^{-1} \circ \sigma_1 \sigma_2 \sigma_1^{-1}$$

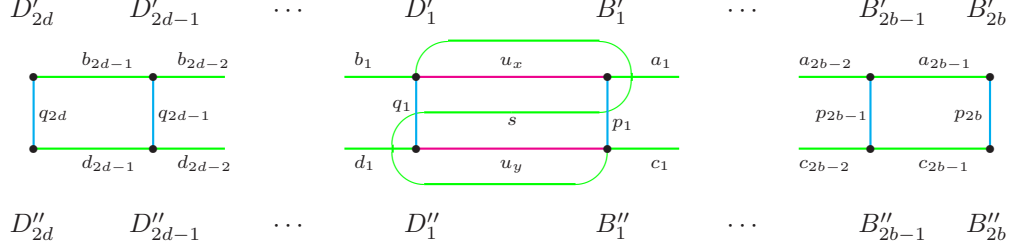


FIGURE 4. isotoped generators

up to Hurwitz equivalence and simultaneous conjugation.

7. Another proof of the non conjugacy theorem

This section is based on the classical correspondence between 4-tuple covers and triple covers, given by the surjection $\mathfrak{S}_4 \rightarrow \mathfrak{S}_3$ whose kernel is the Klein group $(\mathbb{Z}/2)^2$.

This idea was also used in [CLW] to obtain a triple cover $D_0 \rightarrow F_0$ corresponding to a quadruple cover $C_0 \rightarrow F_0$ and a resulting homomorphism of its mapping class group to the symplectic group acting on the $\mathbb{Z}/2$ homology.

In this way in the article [CLW] we gave a proof that the elements σ_p^2 and σ_q^2 in the braid monodromy group H are not conjugate even in the stabilized braid monodromy group $\hat{H}$.

Observe that in the braid group there is a unique square root σ of a full twist: since a full twist determines the homotopy class of an arc between two punctures, and this arc determines a half twist. In the braid group of the sphere, this square root is not unique (as pointed out by a referee), since another square root is also given by $\sigma\Delta^2$, where Δ^2 is a generator of the centre, and has order 2. However, there is a unique square root in the conjugacy class of half-twists.

Hence it suffices to show that there is no h in $\hat{H}$ such that $\sigma_p h = h \sigma_q$.

Our proof there exploited the action of the group $\hat{H}^+$ generated by $\hat{H}$, σ_p and σ_q on the $\mathbb{Z}/2\mathbb{Z}$ homology of a the triple cover D_0 of $F_0 \cong \mathbb{P}^1$.

In the alternative proof we are going to present, we look instead at the action of $\hat{H}^+$ on factorizations in $\mathfrak{S}_4$ of length $n = 4b + 4d$, restriction of the Hurwitz action of the braid group Br_n .

In fact it suffices to consider the $\hat{H}^+$ -orbit $\mathbb{O}$ of the factorization

$$\tau^0 : (12) \circ (34) \circ (12) \circ (34) \cdots (12) \circ (34) \circ (13) \circ (24) \circ (13) \circ (24) \cdots (13) \circ (24),$$

with $2b$ factors equal to (12) , respectively to (34) , and $2d$ factors equal to (13) , respectively to (24) . Note that τ^0 is the factorization sequence corresponding to the covering

monodromy for the 4 : 1 cover given by the restriction of our bidouble cover to the vertical projective line lying over the origin $x = 0$.

The orbit $\mathbb{O}$ is rather small by the following lemma.

Lemma 7.1. *The orbit $\mathbb{O}$ is contained in the set $\hat{\mathbb{O}}$ of factorizations made of transpositions*

$$\tau_1 \circ \tau_2 \circ \cdots \circ \tau_n$$

such that for $i \leq 4b$ either $\tau_i = (12)$ or $\tau_i = (34)$, and for $i > 4b$ either $\tau_i = (13)$ or $\tau_i = (24)$.

Proof. We must show that all elements in $\hat{H}^+$ leave invariant the set given in the claim.

To this purpose we exploit the surjection $\mathfrak{S}_4 \rightarrow \mathfrak{S}_3$. It maps transpositions to transpositions. Naturally the induced map on factorizations commutes with the Hurwitz action.

The upshot is now that the factorizations in the lemma are exactly the factorizations whose factors are transpositions and which map to one and the same factorization in $\mathfrak{S}_3$, namely

$$\bar{\tau}^0 : (12) \circ (12) \circ (12) \circ \cdots \circ (12) \circ (13) \circ (13) \circ (13) \circ \cdots \circ (13).$$

All generators of the braid monodromy subgroup H act trivially on the monodromy factorization in $\mathfrak{S}_4$, hence also on the corresponding factorization $\bar{\tau}^0$ in $\mathfrak{S}_3$. Likewise do the elements σ_p, σ_q : since for instance σ_q is the half twist connecting two neighbouring branch points whose respective local monodromies in $\mathfrak{S}_4$ yield consecutive factors $(1, 2)$ and $(3, 4)$: thus σ_q exchanges these two factors.

It remains to prove that also the additional elements needed to generate $\hat{H}$ act trivially.

By definition of stable equivalence such additional generators are full-twists σ_α^2 which preserve the monodromy factorization τ^0 in $\mathfrak{S}_4$, hence also $\bar{\tau}^0$ and our claim follows. $\square$

We record now in detail how our generators of $\hat{H}^+$ do act.

Lemma 7.2. *The following braids act trivially on all elements in $\hat{\mathbb{O}}$:*

$$\sigma_{u'}^3, \sigma_{u''}^3,$$

and all additional generators of $\hat{H}$ which are not in H .

Proof. Let us use as before the plane lexicographic ordering for the branch points.

Then one can easily verify that the element σ_{4d}^3 acts trivially on all factorizations, since τ_{4d} and τ_{4d+1} are always non commuting transpositions.

Hence both elements $\sigma_{u'}^3, \sigma_{u''}^3$, which are H -conjugate to σ_{4d}^3 , act both trivially.

Consider now an additional generator σ_α^2 . It preserves τ^0 and may be written as $\beta \sigma_1^2 \beta^{-1}$.

Hence σ_1^2 acts trivially on $\bar{\tau}^0 \beta$. Therefore the first two factors of $\bar{\tau}^0 \beta$ must commute. Since both belong to $\mathfrak{S}_3$ they are even equal. The corresponding factors in any lift of $\bar{\tau}^0 \beta$ to a factorization of transpositions in $\mathfrak{S}_4$ are then either equal or disjoint, so σ_1^2 acts trivially also on all such lifts.

We conclude that $\sigma_\alpha^2 = \beta \sigma_1^2 \beta^{-1}$ acts trivially on all elements in $\hat{\mathbb{O}}$. $\square$

Lemma 7.3. *The following types of braids act by the corresponding transpositions on factorizations in $\hat{\mathbb{O}}$:*

$$\sigma_a, \sigma_c, \sigma_b, \sigma_d, \sigma_p, \sigma_q.$$

Proof. We observe that all elements σ_i with $i \neq 4d$ act by the corresponding transposition on the factorizations in $\hat{\mathbb{O}}$, because consecutive transpositions commute except for the transpositions in positions $4d$ and $4d+1$.

Hence the elements $\sigma_a, \sigma_c, \sigma_b, \sigma_d, \sigma_p$ and σ_q , which can be written as a product of factors different from σ_{4d} act by the corresponding transpositions. $\square$

Lemma 7.4. *The action of the ‘snake’ half twist σ_s on factorizations in $\hat{\mathbb{O}}$ is given as follows:*

- (1) *If the composition $\tau_{4d-1}\tau_{4d}\tau_{4d+1}\tau_{4d+2}$ is trivial or equal to $\pi = (14)(23)$, then σ_s acts trivially.*
- (2) *Otherwise σ_s acts by $\tau_i \mapsto \begin{cases} \pi\tau_i\pi & \text{if } 4d-1 \leq i \leq 4d+2 \\ \tau_i & \text{else} \end{cases}$*

Proof. The ‘snake’ half twist σ_s is given in terms of the Artin generators as

$$\sigma_s = \sigma_{4d}(\sigma_{4d-1}^2\sigma_{4d+1}^2)\sigma_{4d}(\sigma_{4d+1}^{-2}\sigma_{4d-1}^{-2})\sigma_{4d}^{-1}$$

Hence the element σ_s acts on the subfactorization $\tau_{4d-1} \circ \tau_{4d} \circ \tau_{4d+1} \circ \tau_{4d+2}$ and leaves the remaining parts unchanged. It thus suffices to know how σ_s acts on all 16 possible subfactorizations.

First we record the action on four specific factorizations. We decompose the action into five steps, each consisting of one or two elementary Hurwitz moves. In the first step we act by σ_{4d} , then by $(\sigma_{4d-1}^2\sigma_{4d+1}^2)$, by σ_{4d} , by $(\sigma_{4d+1}^{-2}\sigma_{4d-1}^{-2})$ and finally by σ_{4d}^{-1} .

$$\begin{array}{ll} (12) \circ (12) \circ (13) \circ (13) & (12) \circ (34) \circ (13) \circ (24) \\ \mapsto (12) \circ (23) \circ (12) \circ (13) & \mapsto (12) \circ (14) \circ (34) \circ (24) \\ \mapsto (23) \circ (13) \circ (13) \circ (23) & \mapsto (14) \circ (24) \circ (24) \circ (23) \\ \mapsto (23) \circ (13) \circ (13) \circ (23) & \mapsto (14) \circ (24) \circ (24) \circ (23) \\ \mapsto (12) \circ (23) \circ (12) \circ (13) & \mapsto (12) \circ (14) \circ (34) \circ (24) \\ \mapsto (12) \circ (12) \circ (13) \circ (13) & \mapsto (12) \circ (34) \circ (13) \circ (24) \end{array}$$

We check that these first two factorizations belong to case (1) of the lemma and remain indeed unchanged.

$$\begin{array}{ll} (12) \circ (34) \circ (13) \circ (13) & (12) \circ (12) \circ (13) \circ (24) \\ \mapsto (12) \circ (14) \circ (34) \circ (13) & \mapsto (12) \circ (23) \circ (12) \circ (24) \\ \mapsto (14) \circ (24) \circ (13) \circ (14) & \mapsto (23) \circ (13) \circ (24) \circ (14) \\ \mapsto (14) \circ (13) \circ (24) \circ (14) & \mapsto (23) \circ (24) \circ (13) \circ (14) \\ \mapsto (34) \circ (14) \circ (12) \circ (24) & \mapsto (34) \circ (23) \circ (34) \circ (13) \\ \mapsto (34) \circ (12) \circ (24) \circ (24) & \mapsto (34) \circ (34) \circ (24) \circ (13) \end{array}$$

The third and fourth factorizations belong to case (2) of the lemma. These subfactorizations are changed in all four positions. Every factor (12) is replaced by (34) and vice versa, and the same occurs with (13) and (24). But this amounts to the same result as conjugation of each factor by π .

The claim can be checked for the remaining 12 subfactorisations by elementary computations of the same kind, or by the observation that they correspond to the cases above under conjugation by π , by (12)(34), or by (13)(24). $\square$

To define a suitable invariant of a factorization in $\hat{\mathbb{O}}$, we observe that each factor of a factorization τ in $\hat{\mathbb{O}}$ coincides either with the corresponding element of the factorization τ^0 or with its conjugate by π .

The product of the factorization is left invariant by the braid group action, whence we claim that

$$\#\{i \mid \tau_i \neq \tau_i^0\}/2 \in \mathbb{N},$$

and thus we get a well-defined parity invariant of the $\hat{H}$ -action.

The claim is easily established in this way: changing one of the factors in the right half amounts (in view of the commutativity of those factors) to multiplication by $(1, 3)(2, 4)$ on the right, while changing one of the factors in the left half amounts to multiplication by $(1, 2)(3, 4)$ on the left. Since $(1, 2)(3, 4)z(1, 3)(2, 4) \neq z \forall z \in \mathfrak{S}_4$ we conclude that the product is left unchanged only if an even number of changes are made.

Lemma 7.5. *The action of $\hat{H}$ on elements in $\hat{\mathbb{O}}$ preserves the parity of the integer*

$$M := \#\{i \mid \tau_i \neq \tau_i^0\}/2 + \#\{i > 4d, i \equiv 0 \pmod{2} \mid \tau_i \neq \tau_i^0\}$$

Proof. All generators of $\hat{H}$ except σ_s act either trivially or by transpositions of factors on the same side (left or right) and at even distance, hence they preserve the first summand and likewise the second summand.

The action of σ_s is different. We have the two possibilities of lemma 7.4. In the first case σ_s acts trivially on τ and thus on M .

In the second case the product $\tau_{4d-1}\tau_{4d}\tau_{4d+1}\tau_{4d+2}$ is equal to (12)(34) or (13)(24) which implies, since in the Klein group $(\cong (\mathbb{Z}/2)^2)$ a product of two non trivial elements is non trivial, that this part of the factorization is one of the following factorizations:

$$\begin{aligned} (12) \circ (34) \circ (13) \circ (13), & \quad (12) \circ (34) \circ (24) \circ (24), \\ (34) \circ (12) \circ (13) \circ (13), & \quad (34) \circ (12) \circ (24) \circ (24), \end{aligned}$$

resp.

$$\begin{aligned} (12) \circ (12) \circ (13) \circ (24), & \quad (12) \circ (12) \circ (24) \circ (13), \\ (34) \circ (34) \circ (13) \circ (24), & \quad (34) \circ (34) \circ (24) \circ (13). \end{aligned}$$

In all cases it can be checked that $\#\{i \in \{4d-1, 4d, 4d+1, 4d+2\} \mid \tau_i \neq \tau_i^0\}$ is either 1 or 3 and $\#\{i = 4d+2 \mid \tau_i \neq \tau_i^0\}$ is either 0 or 1.

Both numbers change when the factorization is conjugated by π , hence the parity of M is preserved, as claimed. $\square$

Finally we can prove the essential step:

Proposition 7.6. *The elements σ_p^2 and σ_q^2 are not conjugate in $\hat{H}$. Hence also σ_p and σ_q are not conjugate in $\hat{H}$.*

Proof. Suppose on the contrary that there exists an $h \in \hat{H}$ such that $\sigma_p h = h \sigma_q$. Then the action of both sides on the factorization τ^0 must give the same factorization. But the invariant M is 2 for $\tau^0 h \sigma_q = \tau^0 \sigma_q$ and odd for $\tau^0 \sigma_p h$, since it is 1 for $\tau_0 \sigma_p$. So the factorizations are different and we get a contradiction.

The second assertion was already shown at the beginning of the present section. $\square$

8. Non equivalence of the horizontal Lefschetz fibrations in some cases.

Consider an abc -surface S and its vertical Lefschetz fibration $F : S \rightarrow \mathbb{P}^1$ given by the composition of the perturbed covering $\pi : S \rightarrow Q$ with the first projection onto $\mathbb{P}^1$; the differentiable class of F is encoded in the M-equivalence class of the factorization of the identity in the Mapping class group $\mathcal{M}ap_g$ corresponding to the monodromy factorization of F (g is the genus of the base fibre C_0 of F).

As already mentioned, we proved in [CW04] that if S' is an (a', b', c') surface, where $b = b'$ and $a + c = a' + c'$ then S and S' are not deformation equivalent but the above defined Lefschetz fibrations are M-equivalent, in particular proving that the surfaces S and S' are diffeomorphic via an orientation preserving diffeomorphism sending the canonical class to the canonical class.

We want to give here another small contribution in the direction of the interesting and difficult question whether the surfaces are symplectomorphic with respect to the canonical symplectic structure induced by the complex structure.

The projections of S and S' to $Q = \mathbb{P}^1 \times \mathbb{P}^1$ have discriminant curves Δ and Δ' . Our computations ([CLW]) of the Braid monodromy factorizations of these curves with respect to the vertical projection (onto the x -axis), that we illustrated previously, show that the braid monodromy factorizations are not stably equivalent. This rises the suspicion that the surfaces may be not canonically symplectomorphic.

We shall in fact point out another difference between the surfaces S and S' . We consider the Lefschetz fibrations corresponding to the horizontal projection (to the y -axis). This time a regular fibre C'_0 is a 4-tuple covering of a horizontal line in Q , with $(4a+4c)$ branch points, hence it has genus $2a+2c-3$. It is identified in a natural way with the curve on Figure 5.

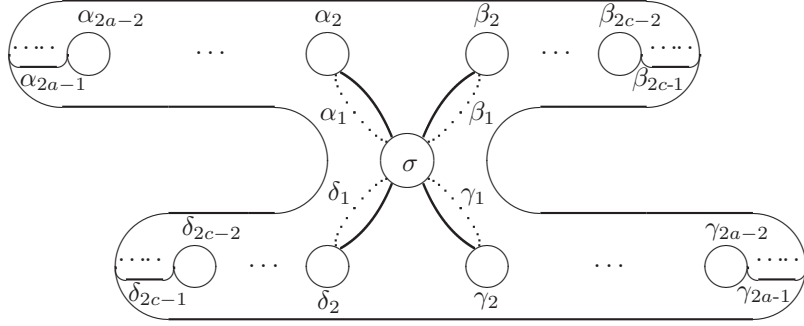
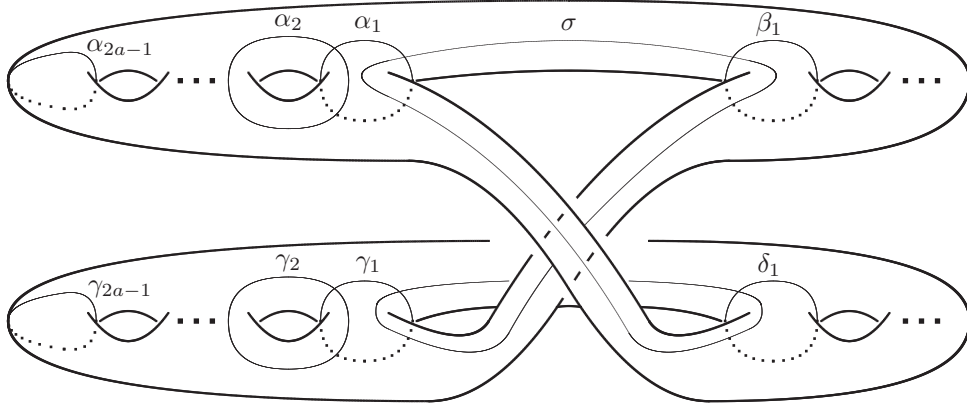
FIGURE 5. The fibre C'_0 of the horizontal Lefschetz fibration

FIGURE 6

Our computation of the Braid monodromy of the discriminant curve, where the roles of the variables x are easily exchanged, yields easily the monodromy factorization corresponding to the horizontal Lefschetz fibration. In particular it shows that the monodromy group of the Lefschetz fibration is generated by Dehn twists with respect to the cycles $\alpha_i, \beta_i, \gamma_i, \delta_i, \sigma$ on Figure 6. We shall prove that if $a + c = a' + c'$ is even and $a - a'$ is odd and $b = b'$ then the monodromy factorizations for the horizontal Lefschetz fibrations are not M-equivalent although the fibrations have the same genus and the surfaces S and S' are diffeomorphic.

We map the cycles in C'_0 into the homology group $H_1(C'_0, \mathbb{Z})$ and further into the vector space $V = H_1(C'_0, \mathbb{Z}/2\mathbb{Z})$ of vectors modulo 2. The intersection form of cycles on

C'_0 induces an alternating symmetric form (u, v) on V . The mapping class group of C'_0 projects onto the symplectic group of V with respect to the form and Dehn twists project to transvections T with respect to cycles: $T_u(v) = v + (u, v)u$.

We shall denote the images of $\alpha_i, \beta_i, \gamma_i, \delta_i, \sigma$ in V by a_i, b_i, c_i, d_i, s respectively.

Considering the intersection matrix of these vectors we see that if we omit the vectors $b_{2c-1}, c_{2a-1}, d_{2c-1}$ then the remaining vectors have a non-singular intersection matrix and thus they form a basis of V .

We define a quadratic function $q : V \rightarrow \mathbb{Z}/2\mathbb{Z}$ using this basis:

q is equal 1 on each basis vector and $q(u + v) = q(u) + q(v) + (u, v)$ for any pair of vectors u, v .

Quadratic functions with an associated bilinear form as above fall into two equivalence classes according to their Arf invariant (modulo 2). In order to compute the Arf invariant we choose a symplectic basis e_i, f_i of V , such that $(e_i, f_i) = 1$ for all i , while all other intersections products of basis vectors are equal 0. Then the Arf invariant of q is equal to $\sum_i q(e_i)q(f_i)$.

We now recall Theorem 4 from [Waj80]:

Theorem 8.1. *Let G be a subgroup of $Sp(V, \mathbb{Z}/2\mathbb{Z})$ generated by transvections, containing the transvections with respect to the vectors u_i of a fixed basis. Let q be the quadratic function defined by $q(u_i) = 1$.*

If G contains a transvection with respect to a vector v satisfying $q(v) = 0$ then $G = Sp(V, \mathbb{Z}/2\mathbb{Z})$.

If this is not the case and the basis is not special, e.g. the intersection diagram of the basis is a tree but it is different from the Dynkin diagrams A_n and D_n , then G is equal to the orthogonal group of the function q , the group of all elements of $Sp(V, \mathbb{Z}/2\mathbb{Z})$ preserving q .

Let us go back to the Lefschetz fibration of the surface S . Let G be the image of the monodromy group, the group generated by the transvections with respect to the vectors a_i, b_i, c_i, d_i, s . Consider the basis described before. The intersection diagram of this basis has the form of a cross, thus the basis is not special.

Consider the corresponding quadratic function q . The vector b_{2c-1} , which is not in the basis, is equal to the sum $v = a_1 + a_3 + a_5 + \dots + a_{2a-1} + b_1 + b_3 + b_5 + \dots + b_{2c-3}$ (both vectors have the same intersection with each basis vector so they are equal).

If $a + c$ is odd then $q(v) = 0$ and the group G generated by the transvections coincides with $Sp(V, \mathbb{Z}/2\mathbb{Z})$.

If $a + c$ is even then $q(b_{2c-1}) = q(c_{2a-1}) = q(d_{2c-1}) = 1$ and the group G is equal to the orthogonal group of q . In this case if we start with any basis v_i satisfying $q(v_i) = 1$ and we define a quadratic function using the basis v_i we get the same function q .

It is easy to compute the Arf invariant of q .

To simplify notation we work out an example with $a = 2, c = 2$.

We can choose the following symplectic basis : $(a_3, a_2), (a_3 + a_1, s), (a_3 + a_1 + b_1, b_2), (a_3 + a_1 + c_1, c_2), (a_3 + a_1 + d_1, d_2)$.

The sum of k pairwise non-intersecting basis vectors has the q -value 1 if k is odd and has the q -value 0 if k is even.

Hence the Arf invariant of q is 0 if a is even and is 1 if a is odd.

We now prove that if $a + c = a' + c'$ is even and $a - a'$ is odd then the monodromy factorizations of the horizontal Lefschetz fibrations of an abc-surface S and of an a'b'c'-surface S' are not M-equivalent. Without loss of generality assume that a is even, and that a' is odd: so that the Arf invariant of q is 0, while the Arf invariant of q' is 1.

Assume the contrary: then, in fact, making an arbitrary identification of the fibers of the fibrations, we may assume that we have two factorizations by Dehn twists which are Hurwitz equivalent.

We map the corresponding cycles to vectors in V . The monodromy factorization maps to a factorization in $Sp(V, \mathbb{Z}/2\mathbb{Z})$ by transvections with respect to some vectors.

This set of vectors contains a basis with the intersection pattern which defines a quadratic function q with the Arf invariant 0. All vectors of the initial set have the q -value 1. The action induced by a Hurwitz move in $Sp(V, \mathbb{Z}/2\mathbb{Z})$ replaces a pair of transvections T_u, T_v by the pair $T_v, T_v^{-1}T_uT_v$. One checks that $T_v^{-1}T_uT_v = T_w$ where $w = u + (u, v)v$. Assume $q(u) = q(v) = 1$: then $q(w) = 1$. So, after the Hurwitz move, we have again a product of transvections with respect to vectors of q -value 1. If we assume that the factorizations are M-equivalent, then we get a factorization by transvections with respect to a set of vectors having q -value 1 and which contains a basis with an intersection pattern such that it defines a quadratic function q' with Arf invariant 1. But, by the previous remark, $q' = q$. This is a contradiction.

One can easily show that the orthogonal groups of q and q' are not conjugate in $Sp(V, \mathbb{Z}/2\mathbb{Z})$ and therefore the monodromy groups of the two fibrations are not conjugate in the mapping class group of a fibre. Probably these monodromy groups are not isomorphic at all, but we do not know at the moment how to prove it.

9. Epilogue

The question of determining whether the Braid Monodromy factorizations of two diffeomorphic surfaces S, S' (an abc-surface, resp. an a'b'c'-surface) are M-equivalent was motivated by the related question of deciding about the existence of a diffeomorphism φ between S, S' commuting with the respective vertical Lefschetz fibrations, and yielding a canonical symplectomorphism.

Our main result that these factorizations are not stably equivalent suggests that such a diffeomorphism cannot exist.

The relation of stable equivalence (the authors call it indeed m-equivalence) was introduced by Auroux and Katzarkov in [A-K00] in order to obtain invariants of symplectic 4-manifolds.

The reason to introduce it, allowing not only Hurwitz equivalence and simultaneous conjugation, but also creation/cancellation of admissible pairs of a positive and of a negative node (here a node, and its corresponding full twist β , is said to be admissible if the inverse image of the node inside the ramification divisor consists of two disjoint

smooth branches) has to do with the fact that this phenomenon occurs when considering isotopies of approximately holomorphic maps.

Our result (6.3) shows that the braid monodromy group H depends only upon the numbers b and d , provided for instance that we have nonvanishing of the respective numbers $8ab - 2(ad + bc)$, $8cd - 2(ad + bc)$, or provided that we are in the case $b = d$.

A fortiori if the groups H are the same for different choices of (a, b, c, d) then the fundamental groups $\pi_1(Q \setminus \Delta)$ are isomorphic.

Our new method of distinguishing factorizations, and not only associated fundamental groups of complements of branch curves (by the Zariski-van Kampen method), leads to the above results representing the first positive step towards the realization of a more general program set up by Moishezon ([Moi81], [Moi83]) in order to produce braid monodromy invariants which should distinguish the connected components of a moduli space $\mathfrak{M}_{\chi, K^2}$.

Moishezon's program is based on the consideration (assume here for simplicity that K_S is ample) of a general projection $\Psi_m : S \rightarrow \mathbb{P}^2$ of a pluricanonical embedding $\Phi_m : S \rightarrow \mathbb{P}^{P_m-1}$, and of the braid monodromy factorization corresponding to the (cuspidal) branch curve B_m of Ψ_m .

An invariant of the connected component is here given by the M-equivalence class (i.e., for Hurwitz equivalence plus simultaneous conjugation) of this braid monodromy factorization. Moishezon, and later Moishezon-Teicher, calculated a coarser invariant, namely the fundamental group $\pi_1(\mathbb{P}^2 - B_m)$.

This group turned out to be not overly complicated, and in fact, as shown in many cases in [ADKY04], it tends to give no extra information beyond the one given by the topological invariants of S (such as χ, K^2).

Auroux and Katzarkov showed that, for $m \gg 0$, the stable-equivalence class of the above braid monodromy factorization determines the canonical symplectomorphism class of S , and conversely.

The work by Auroux, Katzarkov adapted Donaldson's techniques for proving the existence of symplectic Lefschetz fibrations ([Don96-2], [Don99]) in order to show that each integral symplectic 4-manifold is in a natural way 'asymptotically' realized by a generic symplectic covering of $\mathbb{P}_{\mathbb{C}}^2$, given by approximately holomorphic sections of a high multiple $L^{\otimes m}$ of a complex line bundle L whose class is the one of the given integral symplectic form.

The methods of Donaldson on one side, Auroux and Katzarkov on the other, use algebro geometric methods in order to produce invariants of symplectic 4-manifolds.

For instance, in the case of a generic symplectic covering of the plane, we get a corresponding branch curve Δ_m which is a symplectic submanifold with singularities only nodes and cusps.

To Δ_m corresponds then a factorization in the braid group, called m -th braid monodromy factorization: it contains, as in the present paper, only factors which are conjugates of σ_1^j , not only with $j = 1, 2, 3$ as in the complex algebraic case, but also with $j = -2$.

The factorization is not unique because it may happen that a pair of two consecutive nodes, one positive and one negative, may be created, or disappear, and this is the reason to consider its stable equivalence class which the authors show, for $m \gg 1$, to be an invariant of the integral symplectic manifold.

As we have already remarked, in the case of abc-surfaces the vertical Braid monodromy groups H are determined by b (up to conjugation), hence the fundamental groups $\pi_1(Q \setminus \Delta)$ are isomorphic for a fixed value of b .

So, Moishezon's technique produces no invariants.

Our result indicates that one should try to go all the way in the direction of understanding stable-equivalence classes of pluricanonical braid monodromy factorizations. Let us describe here how this could be done.

Define $p : S \rightarrow \mathbb{P}^2$ as the morphism p obtained as the composition of $\pi : S \rightarrow Q$ with the embedding $Q \hookrightarrow \mathbb{P}^3$ followed by a general projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$.

In the even more special case of abc-surfaces such that $a + c = 2b$, the m -th pluricanonical mapping $\Phi_m : S \rightarrow \mathbb{P}^{P_m-1}$ has a (non generic) projection given by the composition of p with a Fermat type map $\nu_r : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ (given by $\nu_r(x_0, x_1, x_2) = (x_0^r, x_1^r, x_2^r)$ in a suitable linear coordinate system), where $r := m(2b - 2)$.

Let B be the branch curve of a generic perturbation of p : then the braid monodromy factorization corresponding to B can be calculated from the braid monodromy factorization corresponding to Δ .

We hope, in a sequel to this paper, to be able to determine whether these braid monodromy factorizations are equivalent, respectively stably-equivalent, for abc-surfaces such that $a + c = 2b$.

The problem of calculating the braid monodromy factorization corresponding to the (cuspidal) branch curve B_m starting from the braid monodromy factorization of B has been addressed, in the special case $m = 2$, by Auroux and Katzarkov ([A-K08]). Iteration of their formulae should lead to the calculation of the braid monodromy factorization corresponding to the (cuspidal) branch curve B_m in the case, sufficient for applications, where m is a sufficiently large power of 2.

Whether it will be possible to decide about the stable equivalence of the factorizations which these formidable calculations will yield, is a quite open question.

In both directions the result would be extremely interesting, leading either to

- i) a counterexample to the speculation $\text{DEF} = \text{CAN}$. SYMPL also in the simply connected case, or to
- ii) examples of diffeomorphic but not canonically symplectomorphic simply connected algebraic surfaces.

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