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Resolution of singularities of analytic spaces

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Abstract. Building upon work of Villamayor Bierstone-Milman and our recent paper we give a proof of the canonical Hironaka principalization and desingularization of analytic spaces. Though the inductive scheme of the proof is the same as in algebraic case there is a number of technical differences between analytic and algebraic situation.

1. Introduction

In the present paper we give a short proof of the Hironaka theorem on resolution of singularities of analytic spaces. The structure of the proof and its organization is very similar with the one given in the paper [38].

The strategy of the proof we formulate here is essentially the same as the one found by Hironaka and simplified by Bierstone-Milman and Villamayor ([8], [9], [10]), ([35], [36], [37]). In particular we apply here one of Villamayor's key simplifications, eliminating the use of the Hilbert-Samuel function and the notion of normal flatness (see [13]).

The main idea of the algorithm is to control the resolution procedure by two simple invariants: order of the weak transform of the ideal sheaf $\mathcal{I}$ and the dimension of the ambient manifold M. The process of dropping the order starts from the isolating the "worst singularity locus" -the set where the order is maximal $\mathrm{ord}_x(\mathcal{I}) = \mu$. This leads to considerations of ideal sheaves with assigned order $(\mathcal{I}, \mu)$.

Eliminating "worst singularity locus" $\mathrm{supp}(\mathcal{I}, \mu)$ builds upon reduction of the dimension of the ambient variety. It was observed by Abhyankhar and successfully implemented by Hironaka that $\mathrm{supp}(\mathcal{I}, \mu)$ is contained in a certain smooth hypersurface M' of M. The concept of hypersurface of maximal contact can be expressed nicely by using Giraud approach with derivations.

The blow-ups used for eliminating $\mathrm{supp}(\mathcal{I}, \mu)$ are performed only at centers which are contained in $\mathrm{supp}(\mathcal{I}, \mu)$. This has two major consequences:

1. The outside of the locus $\mathrm{supp}(\mathcal{I}, \mu)$ can be ignored in the process. Thus $(\mathcal{I}, \mu)$ can be considered as a "part of the ideal sheaf of $\mathcal{I}$ where the order is $\geq \mu$". Solving of $(\mathcal{I}, \mu)$ is merely eliminating $\mathrm{supp}(\mathcal{I}, \mu)$.

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2. The total transform of ideal is divisible by μ-power of exceptional divisor. Thus the transformation of the ideal $\mathcal{I}$ can be described by explicit formula:

$$\sigma^c(\mathcal{I},\mu) = \mathcal{I}(E)^{-\mu}\sigma^*(\mathcal{I}).$$

This makes a basis for the reduction to the hypersuface of maximal contact. Although it is not possible to restrict $\mathcal{I}$ directly to $M' \subset M$ we can find an ideal sheaf $(\mathcal{I}',\mu')$, called "coefficient ideal ", which lives on M', and which is related to $(\mathcal{I},\mu)$ by the equality

$$\mathrm{supp}(\mathcal{I},\mu) = \mathrm{supp}(\mathcal{I}',\mu').$$

Now the problem of eliminating "bad locus" $\mathrm{supp}(\mathcal{I},\mu)$ is reduced to the lower dimension where we proceed by induction.

This approach has a major flaw. The procedure of restricting $\mathcal{I}$ to the hypersurface of maximal contact is not canonical and is defined locally. In fact for two different hypersurfaces of maximal contact we get two different objects which are loosely related. In order to resolve this issue Hironaka used the following approach: The local resolutions can be encoded by a certain invariant. Each single operation used in the above mentioned induction leaves its "trace" which is a single entry of the invariant. As a result the invariant is a sequence of the numbers occuring in local resolutions. The invariant is upper semicontinuous and defines a stratification of the ambient space. This invariant drops after the blow-up of the maximal stratum. It determines the centers of the resolution and allows one to patch up local desingularizations to a global one. What adds to the complexity is that the invariant is defined within some rich inductive scheme encoding the desingularization and assuring its canonicity (Bierstone-Milman's towers of local blow-ups with *admissible centers* and Villamayor's *general basic objects*) (see also Encinas-Hauser [17]).

Instead of considering the invariant as the key notion of the algorithm, in [38] we proposed a different approach. It is based upon two simple observations.

(1) The resolution process defined as a sequence of suitable blow-ups of ambient spaces can be applied simultaneously not only to the given singularities but rather to a class of equivalent singularities obtained by simple arithmetical modifications. This means that we can "tune" singularities before resolving them.
(2) In the equivalence class we can choose a convenient representative given by the *homogenized ideals* introduced in the paper. The restrictions of homogenized ideals to different hypersurfaces of maximal contact define locally analytically isomorphic singularities. Moreover the local isomorphism of hypersurfaces of maximal contact is defined by a local analytic automorphism of the ambient space preserving all the relevant resolutions.

"Homogenization" of the ideal makes the operation of restriction to hypersurface of maximal contact canonical- independent of any choices. In particular there is no necessity of describing and comparing local algorithms. The inductive structure of the process is reduced to the existence of a canonical functorial resolution in lower dimensions. This approach puts much less emphasis on the invariant. In fact as was observed by Kollár by mere allowing reducible algebraic varieties (or analytic spaces) in the inductive scheme one eliminates the "long" invariant completely ([31]). What is left is a "bare" two- step induction.

In **Step 2** of the proof, given an ideal $(\mathcal{I}, \mu)$ we assign to it the worst singularity order μ'. Instead of dealing with $(\mathcal{I}, \mu)$ directly we form an auxiliary ideal (companion ideal) which is roughly $(\mathcal{I}, \mu')$. Its resolution determines the drop of the maximal order of the weak transform (nonmonomial part) of $\mathcal{I}$. By repeating this process sufficiently many times the weak transform of $(\mathcal{I}, \mu)$ disappear and $(\mathcal{I}, \mu)$ becomes principal monomial thus, easy to solve directly. The procedure in Step 2 uses the fact that companion ideals and, in general, all ideals $(\mathcal{I}, \mu')$, where $\mu' = \max\{\mathrm{ord}_x(\mathcal{I}) \mid x \in M\}$ are possible to solve by reduction to the hypersurface of maximal contact. This is done in **Step 1** of the proof. That's where the operation of tuning comes handy. The "tuning" of ideals has two aspects. First, homogenization gives us the canonicity of resolution and solves the glueing problem. Second, we can view a coefficient ideal as a part of the tuning too. In this approach coefficient ideal $\mathcal{C}(\mathcal{I}, \mu)$ lives on M and is equivalent to $\mathcal{I}$ but its "restricts well" not only to the hypersurface of maximal contact but to any smooth subvariety $Z \subset M$, that is,

$$\mathrm{supp}(\mathcal{C}(\mathcal{I}), \mu) \cap Z = \mathrm{supp}(\mathcal{C}(\mathcal{I})_{|Z}, \mu')$$

In the analytic situation, considered in the paper, in the algorithm of resolution of $(\mathcal{I}, \mu)$ the compactness condition is essential. In particular isolating "the worst singularity" locus is possible only under the assumption of compactness. Even if we start our considerations from ideal sheaves on compact manifolds the operation of local restriction to hypersurface of maximal contact leads to noncompact submanifolds. That is why in the analytic case it is natural to consider not manilfolds or compact manifolds but rather germs of manifolds at compact subsets. After establishing a few technical differences between analytic and algebraic case we can carry the inductive algorithm essentially in the same way as in the algebraic case. As a result we construct a resolution which is locally but not globally a sequence of blow-ups at smooth centers.

The presented proof is elementary, constructive and self-contained.

The paper is organized as follows. In section 1 we formulate three main theorems: the theorem of canonical principalization (Hironaka's "Desingularization II"), the theorem of canonical embedded resolution (a slightly weaker version of Hironaka's "Desingularization I") and the theorem of canonical resolution. In section 2 we introduce basic notions we are going to use throughout the paper. In section 3 we formulate the theorem of canonical resolution of marked ideals and show how it implies three main theorems (Hironaka's resolution principle). Section 4 gives important technical ingredients. In particular we introduce here the notion of homogenized ideals. In section 5 we formulate the resolution algorithm and prove the theorem of canonical resolution of marked ideals. In section 6 we make final conclusions from the proof.

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2. Formulation of the main theorems

All analytic spaces in this paper are defined over a ground field $\mathbf{K} = \mathbb{C}$ or $\mathbb{R}$. We give a proof of the following Hironaka Theorems (see [26]):

Canonical resolution of singularities

Theorem 2.0.1. *Let Y be an analytic space. There exists a canonical desingularization of Y that is a manifold $\widetilde{Y}$ together with a proper bimeromorphic morphism $\mathrm{res}_Y : \widetilde{Y} \to Y$ such that*

(1) $\mathrm{res}_Y : \widetilde{Y} \to Y$ *is an isomorphism over the nonsingular part Y_{ns} of Y.*
(2) *The inverse image of the singular locus $\mathrm{res}_Y^{-1}(Y_{\mathrm{sing}})$ is a simple normal crossing divisor.*
(3) res_Y *is functorial with respect to local analytic isomorphisms. For any local analytic isomorphism $\phi : Y' \to Y$ there is a natural lifting $\widetilde{\phi} : \widetilde{Y'} \to \widetilde{Y}$ which is a local analytic isomorphism.*

Locally finite embedded desingularization

Theorem 2.0.2. *Let Y be an analytic subspace of an analytic manifold M. There exists a manifold $\widetilde{M}$, a simple normal crossing locally finite divisor E on $\widetilde{M}$, and a bimeromorphic proper morphism*

$$\mathrm{res}_{Y,M} : \widetilde{M} \to M$$

such that the strict transform $\widetilde{Y} \subset \widetilde{M}$ is smooth and have simple normal crossings with the divisor E. The support of the divisor E is the the exceptional locus of $\mathrm{res}_{Y,M}$. The morphism $\mathrm{res}_{Y,M}$ locally factors into a sequence of blow-ups at smooth centers. That is, for any compact set $Z \subset Y$ there is an open subset $U \subset M$ and $\widetilde{U} = \mathrm{res}_{Y,M}^{-1}(U) \subset \widetilde{M}$ and a sequence

$$U_0 = U \stackrel{\sigma_{U1}}{\longleftarrow} U_1 \stackrel{\sigma_{U2}}{\longleftarrow} U_2 \longleftarrow \ldots \longleftarrow U_i \longleftarrow \ldots \longleftarrow U_r = \widetilde{U} \quad (*)$$

of blow-ups $\sigma_{Ui} : U_{i-1} \longleftarrow U_i$ with smooth closed centers $C_{i-1} \subset U_{i-1}$ such that

(1) *The exceptional divisor E_{Ui} of the induced morphism $\sigma_U^i = \sigma_{U1} \circ \ldots \circ \sigma_{Ui} : U_i \to U$ has only simple normal crossings and C_i has simple normal crossings with E_i.*
(2) *Let $Y_{Ui} := Y \cap U_i$ be the strict transform of Y. All centers C_i are disjoint from the set $\mathrm{Reg}(Y) \subset Y_i$ of points where Y (not Y_i) is smooth (and are not necessarily contained in Y_i).*

(3) *The strict transform* $Y_{Ur} = \widetilde{Y} \cap U_r$ *of* $Y_U := Y \cap U$ *is smooth and has only simple normal crossings with the exceptional divisor* E_r*.*
(4) *The morphism* $\mathrm{res}_{Y,M} : (M, Y) \leftarrow (\widetilde{M}, \widetilde{Y})$ *defined by the embedded desingularization commutes with local analytic isomorphisms, embeddings of ambient varieties.*
(5) *For any compact sets* $Z_1 \subset Z_2$ *and corresponding open neighborhoods* $U_1 \subset U_2$ *the restriction of the factorization (*) of* $\mathrm{res}_{Y,M|\widetilde{U_2}} : \widetilde{U}_2 \to U_2$ *to* $\widetilde{U_1}$ *determines the factorization of* $\mathrm{res}_{Y,M|\widetilde{U_1}} : \widetilde{U}_1 \to U_1$*.*
(6) *(Strengthening of Bravo-Villamayor* [13]*)*

$$\sigma^*(\mathcal{I}_Y) = \mathcal{I}_{\widetilde{Y}} \mathcal{I}_{\widetilde{E}},$$

where $\mathcal{I}_{\widetilde{Y}}$ *is the sheaf of ideals of the subvariety* $\widetilde{Y} \subset \widetilde{M}$ *and* $\mathcal{I}_{\widetilde{E}}$ *is the sheaf of ideals of a simple normal crossing divisor* $\widetilde{E}$ *which is a locally finite combination of the irreducible components of the divisor* E_{Ur}*.*

Locally finite principalization of sheaves of ideals

Theorem 2.0.3. *Let* $\mathcal{I}$ *be a sheaf of ideals on a analytic manifold* M *(not necessarily compact). There exists a locally finite principalization of* $\mathcal{I}$*, that is, a manifold* $\widetilde{M}$*, a proper morphism* $\mathrm{prin}_{\mathcal{I}} : \widetilde{M} \to M$*, and a sheaf of ideals* $\widetilde{\mathcal{I}}$ *on* M *such that*

(1) *For any compact set* $Z \subset M$*, there is an open neighborhoods* $U \supset Z$ *and* $\widetilde{U} := \mathrm{prin}_{\mathcal{I}}^{-1}(U) \subset \widetilde{M}$ *for which the restriction* $\mathrm{prin}_{\mathcal{I}|\widetilde{U}} : \widetilde{U} \to U$ *splits into a finite sequence of blow-ups*

$$U = U_0 \overset{\sigma_{U1}}{\longleftarrow} U_1 \overset{\sigma_{U2}}{\longleftarrow} U_2 \longleftarrow \ldots \longleftarrow U_i \longleftarrow \ldots \longleftarrow U_r = \widetilde{U} \qquad (*)$$

of blow-ups $\sigma_{Ui} : U_{i-1} \leftarrow U_i$ *with smooth centers* $C_{i-1} \subset U_{i-1}$ *such that*
(2) *The exceptional divisor* E_{Ui} *of the induced morphism* $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : U_i \to U$ *has only simple normal crossings and* C_i *has simple normal crossings with* E_i*.*
(3) *The total transform* $\mathrm{prin}^*_{\mathcal{I}|\widetilde{U}}(\mathcal{I}) = \sigma^{r*}(\mathcal{I})$ *is the ideal of a simple normal crossing divisor* $\widetilde{E}_U$ *which is a locally finite combination of the irreducible components of the divisor* E_{Ur}*.*
(4) *For any compact sets* $Z_1 \subset Z_2$ *and corresponding open neighborhoods* $U_1 \subset U_2$ *the restriction of the factorization (*) of* $\mathrm{prin}_{\mathcal{I}|\widetilde{U_2}} : \widetilde{U}_2 \to U_2$ *to* $\widetilde{U_1}$ *determines the factorization of* $\mathrm{prin}_{\mathcal{I}|\widetilde{U_1}} : \widetilde{U}_1 \to U_1$*.*

The morphism $\mathrm{prin} : (\widetilde{M}, \widetilde{\mathcal{I}}) \to (M, \mathcal{I})$ *commutes with local analytic isomorphisms, embeddings of ambient varieties.*

Remarks. (1) By the exceptional divisor of the blow-up $\sigma : M' \to M$ with a smooth center C we mean the inverse image $E := \sigma^{-1}(C)$ of the center C. By the exceptional divisor of the composite of blow-ups σ_i with smooth centers C_{i-1} we mean the union of the strict transforms of the exceptional divisors of σ_i. This definition coincides with the standard definition of the exceptional set of points of the bimeromorphic morphism in the case when $\mathrm{codim}(C_i) \geq 2$ (as in Theorem 2.0.2). If $\mathrm{codim}(C_{i-1}) = 1$ the blow-up of C_{i-1} is an identical isomorphism and defines a formal operation of converting a subvariety $C_{i-1} \subset M_{i-1}$ into a component of the exceptional divisor E_i on M_i. This formalism is convenient for the proofs. In particular it indicates that C_{i-1} identified via σ_i with a component of E_i has simple normal crossings with other components of E_i.

(2) In the Theorem 2.0.2 we blow up centers of codimension ≥ 2 and both definitions coincide.

(3) Given a closed embedding of manifolds $i : M \hookrightarrow M'$, the coherent sheaf of ideals $\mathcal{I}$ on M defines a coherent subsheaf $i_*(\mathcal{I}) \subset i_*(\mathcal{O}_M)$ of $\mathcal{O}_{M'}$-module $i_*(\mathcal{O}_M)$. Let $i^\sharp : O_{M'} \to i_*(\mathcal{O}_M)$ be the natural surjection of $\mathcal{O}_{M'}$-modules. The inverse image $\mathcal{I}' = (i^\sharp)^{-1}(i_*(\mathcal{I}))$ defines a coherent sheaf of ideals on M'. By abuse of notation $\mathcal{I}'$ will be denoted as $i_*(\mathcal{I}) \cdot O_{M'}$.

3. Preliminaries

3.1. Germs of analytic spaces at compact subsets

Definition 3.1.1. Let M be an analytic space and $Z \subset M$ be a compact subset. By *a representative of germ* M_Z of M at Z we mean a pair (U, Z) where $U \subset M$ is any open subset of M containing Z. We say that for any two open subsets U, U' of M containing Z the representative of germs (U, Z), and (U', Z) define the same *germ* M_Z. We write $M_Z = (U, Z)$ and call U a *neighborhood* of a germ M_Z. By a *morphism* $f : M_Z \to M'_{Z'}$ we mean a morphism $f_U : U \to U'$ between some neighborhoods of M_Z and $M'_{Z'}$ such that $f(Z) \subset Z'$. The morphism f is proper, projective, (resp. is an open or closed inclusion) if f_U has this property for the corresponding neighborhoods U, U'.

We introduce the operation of union and intersection of germs : If $U, U' \subset M$ then

$$(U, Z) \cup (U', Z') := (U \cup U', Z \cup Z'), \quad (U, Z) \cap (U', Z') := (U \cap U', Z \cap Z')$$

Then $(U, Z) \to (U, Z) \cup (U', Z')$ and $(U, Z) \cap (U', Z') \to (U, Z)$ are open inclusions.

3.2. Resolution of marked ideals

We shall consider ideal sheaves and divisors on germs M_Z. If $U \subset M$ is a smooth open subset containing Z then we call the germ $M_Z = (U, Z)$ *smooth.* A sheaf of ideal on M_Z is a sheaf $\mathcal{I}$ on some neighborhood U of M_Z. For any sheaf of ideals $\mathcal{I}$ on a smooth germ $M_Z = (U, Z)$ and any point $x \in U$ we denote by

$$\mathrm{ord}_x(\mathcal{I}) := \max\{i \mid \mathcal{I}_x \subset m_x^i\}$$

the *order* of $\mathcal{I}$ at x. (Here m_x denotes the maximal ideal of x.)

Definition 3.2.1. (Hironaka [26], [28], Bierstone-Milman [8], Villamayor [35]) A *marked ideal* is a collection $(M_Z, \mathcal{I}, E, \mu)$, where M_Z is a smooth germ, $\mathcal{I}$ is a sheaf of ideals on M_Z, μ is a nonnegative integer and E is a totally ordered collection of divisors on M_Z whose irreducible components are pairwise disjoint and all have multiplicity one. Moreover the irreducible components of divisors in E have simultaneously simple normal crossings.

Let $(M_Z, \mathcal{I}, E, \mu)$ be a marked ideal such that the ideal sheaf $\mathcal{I}$ is defined on an open neighborhood U of M_Z. One can show that the set

$$\mathrm{supp}_Z(M_Z, \mathcal{I}, E, \mu) := \{x \in Z \mid \mathrm{ord}_x(\mathcal{I}) \geq \mu\}$$

is compact. On the other hand the set

$$\mathrm{supp}_U(M_Z, \mathcal{I}, E, \mu) := \{x \in U \mid \mathrm{ord}_x(\mathcal{I}) \geq \mu\}$$

defines a closed analytic subspace of U. (see Lemma 5.2.2).

Definition 3.2.2. (Hironaka [26], [28], Bierstone-Milman [8], Villamayor [35]) By the *support* (originally *singular locus*) of $(M_Z, \mathcal{I}, E, \mu)$ we mean the germ of analytic space

$$\mathrm{supp}(M_Z, \mathcal{I}, E, \mu) := (\mathrm{supp}_U(M_Z, \mathcal{I}, E, \mu), \mathrm{supp}_Z(M_Z, \mathcal{I}, E, \mu)),$$

Remarks.
(1) The ideals with assigned orders or functions with assigned multiplicities and their supports are key objects in the proofs of Hironaka, Villamayor and Bierstone-Milman. In particular Hironaka introduced the notion of *idealistic exponent*.
(2) To simplify notation we often write marked ideals $(M_Z, \mathcal{I}, E, \mu)$ as couples $(\mathcal{I}, \mu)$ or even ideals $\mathcal{I}$.
(3) For any sheaf of ideals $\mathcal{I}$ on $M_Z = (U, Z)$ we have

$$\mathrm{supp}(\mathcal{I}, 1) = V(\mathcal{I}) := \{x \in U \mid f(x) = 0, \quad \text{for any } f \in \mathcal{I}\}.$$

Definition 3.2.3. Let M_Z be a germ of an analytic manifold M. Let $C \subset U$ be a smooth closed subspace of a neighborhood $U \subset Z$. Let $\sigma_U : U' \to U$ denote the blow-up of a smooth center C. Set $Z' := \sigma_U^{-1}(Z)$, $M'_{Z'} := (U', Z')$. The germ of σ_U is a bimeromorphic morphism $\sigma : M'_{Z'} \to M_Z$ which is called a *blow-up* of M_Z at the center $C \subset M_Z$.

Definition 3.2.4. (Hironaka [26], [28], Bierstone-Milman [8], Villamayor [35]) By a *resolution* of $(M_Z, \mathcal{I}, E, \mu)$ we mean a sequence of blow-ups $\sigma_i : M_{i,Z_i} \to M_{i-1,Z_{i-1}}$ of disjoint unions of smooth centers $C_{i-1} \subset M_{i-1}$,

$$M_{0,Z_0} \stackrel{\sigma_1}{\longleftarrow} M_{1,Z_1} \stackrel{\sigma_2}{\longleftarrow} M_{2,Z_2} \stackrel{\sigma_3}{\longleftarrow} \ldots M_{i,Z_i} \longleftarrow \ldots \stackrel{\sigma_r}{\longleftarrow} M_{r,Z_r},$$

which defines a sequence of marked ideals $(M_{i,Z_i}, \mathcal{I}_i, E_i, \mu)$ where
(1) $C_i \subset \mathrm{supp}(M_{i,Z_i}, \mathcal{I}_i, E_i, \mu)$.
(2) C_i has simple normal crossings with E_i.
(3) $\mathcal{I}_i = \mathcal{I}(D_i)^{-\mu}\sigma_i^*(\mathcal{I}_{i-1})$, where $\mathcal{I}(D_i)$ is the ideal of the exceptional divisor D_i of σ_i.
(4) $E_i = \sigma_i^{\mathrm{c}}(E_{i-1}) \cup \{D_i\}$, where $\sigma_i^{\mathrm{c}}(E_{i-1})$ is the set of strict transforms of divisors in E_{i-1}.

(5) The order on $\sigma_i^{\rm c}(E_{i-1})$ is defined by the order on E_{i-1} while D_i is the maximal element of E_i.
(6) $\operatorname{supp}(M_{r,Z_r}, \mathcal{I}_r, E_r, \mu) = \emptyset$.

Remark. Note that the resolution of $(M_Z, \mathcal{I}, E, \mu)$ coincides with the resolution of $(M_{Z'}, \mathcal{I}, E, \mu)$, where $Z' := Z \cap \operatorname{supp}(\mathcal{I}, \mu)$ so we can assume that

$$Z \subset \operatorname{supp}(\mathcal{I}, \mu).$$

Definition 3.2.5. The sequence of morphisms which are either isomorphisms or blow-ups satisfying conditions (1)-(5) is called a *multiple test blow-up.* The number of morphisms in a multiple test blow-up will be called its *length.*

Definition 3.2.6. An *extension* of a sequence of blow-ups $(M_{iZ_i})_{0\leq i\leq m}$ is a sequence $(M'_{jZ_j})_{0\leq j\leq m'}$ of blow-ups and isomorphisms $M'_{0Z_0} = M'_{j_0Z_{j_0}} = \ldots = M'_{j_1-1,Z_{j_1-1}} \leftarrow M'_{j_1} = \ldots = M'_{j_2-1,Z_{j_2-1}} \leftarrow \ldots M'_{j_m,Z_{j_m}} = \ldots = M'_{m'}$, where $M'_{j_iZ_{j_i}} = M_{iZ_i}$.

In particular we shall consider *extensions of multiple test blow-ups.*

Remarks. (1) The definition of extension arises naturally when we pass to open subsets of the considered ambient manifold M.
(2) The notion of a *multiple test blow-up* is analogous to the notions of *test* or *admissible* blow-ups considered by Hironaka, Bierstone-Milman and Villamayor.

3.3. Transforms of marked ideals and controlled transforms of functions

In the setting of the above definition we shall call

$$(\mathcal{I}_i, \mu) := \sigma_i^{\rm c}(\mathcal{I}_{i-1}, \mu)$$

a *transform of the marked ideal* or *controlled transform* of $(\mathcal{I}, \mu)$. It makes sense for a single blow-up in a multiple test blow-up as well as for a multiple test blow-up. Let $\sigma^i := \sigma_1 \circ \ldots \circ \sigma_i : M_i \to M$ be a composition of consecutive morphisms of a multiple test blow-up. Then in the above setting

$$(\mathcal{I}_i, \mu) = (\sigma^i)^{\rm c}(\mathcal{I}, \mu).$$

We shall also denote the controlled transform $(\sigma^i)^{\rm c}(\mathcal{I}, \mu)$ by $(\mathcal{I}, \mu)_i$ or $[\mathcal{I}, \mu]_i$.

The controlled transform can also be defined for local sections $f \in \mathcal{I}(U)$. Let $\sigma : M \leftarrow M'$ be a blow-up with a smooth center $C \subset \operatorname{supp}(\mathcal{I}, \mu)$ defining a transformation of marked ideals $\sigma^{\rm c}(\mathcal{I}, \mu) = (\mathcal{I}', \mu)$. Let $f \in \mathcal{I}(U)$ be a section of a sheaf of ideals. Let $U' \subseteq \sigma^{-1}(U)$ be an open subset for which the sheaf of ideals of the exceptional divisor is generated by a function y. The function

$$g = y^{-\mu}(f \circ \sigma) \in \mathcal{I}(U')$$

is a *controlled transform* of f on U' (defined up to an invertible function). As before we extend it to any multiple test blow-up.

The following lemma shows that the notion of controlled transform is well defined.

Lemma 3.3.1. *Let $C \subset \operatorname{supp}(\mathcal{I}, \mu)$ be a smooth center of the blow-up $\sigma : M \leftarrow M'$ and let D denote the exceptional divisor. Let $\mathcal{I}_C$ denote the sheaf of ideals defined by C. Then*

(1) $\mathcal{I} \subset \mathcal{I}_C^{\mu}$.
(2) $\sigma^*(\mathcal{I}) \subset (\mathcal{I}_D)^{\mu}$.

Proof. (1) We can assume that the ambient manifold M is isomorphic to an open ball in $\mathring{A}^n$. Let $u_1, \ldots, u_k$ be coordinates generating $\mathcal{I}_C$. Suppose $f \in \mathcal{I} \setminus \mathcal{I}_C^{\mu}$. Then we can write $f = \sum_\alpha c_\alpha u^\alpha$, where either $|\alpha| \geq \mu$ or $|\alpha| < \mu$ and $c_\alpha \notin \mathcal{I}_C$. By assumption there is α with $|\alpha| < \mu$ such that $c_\alpha \notin \mathcal{I}_C$. Take α with the smallest $|\alpha|$. There is a point $x \in C$ for which $c_\alpha(x) \neq 0$ and in the Taylor expansion of f at x there is a term $c_\alpha(x)u^\alpha$. Thus $\operatorname{ord}_x(\mathcal{I}) < \mu$. This contradicts the assumption $C \subset \operatorname{supp}(\mathcal{I}, \mu)$.

(2) $\sigma^*(\mathcal{I}) \subset \sigma^*(\mathcal{I}_C)^\mu = (\mathcal{I}_D)^\mu$. □

3.4. Functorial properties of multiple test blow-ups

We can define the fiber products for the germs of analytic spaces

$$(X, Z_X) \times_{(Y,Z_Y)} (\overline{X}, \overline{Z_X}) := (X \times_Y \overline{X}, Z_X \times_{Z_Y} \overline{Z_X}).$$

Proposition 3.4.1. *Let M_{iZ_i} be a multiple test blow-up of a marked ideal $(M_Z, \mathcal{I}, E, \mu)$ defining a sequence of marked ideals $(M_{iZ_i}, \mathcal{I}_i, E_i, \mu)$. Given a local analytic isomophism $\phi : M'_{Z'} \to M_Z$, the induced sequence $M'_{iZ_i} := M' \times_{M_Z} M_{i,Z_i}$ is a multiple test blow-up of $(M'_{Z'}, \mathcal{I}', E', \mu)$ such that*

(1) *ϕ lifts to local analytic isomorphisms $\phi_{iZ_i} : M'_{iZ_i} \to M_{iZ_i}$.*
(2) *$(M'_{iZ'_i})$ defines a sequence of marked ideals $(M'_{Z'_i}, \mathcal{I}'_i, E'_i, \mu)$ where $\mathcal{I}'_i = \phi_i^*(\mathcal{I}_i)$, the divisors in E'_i are the inverse images of the divisors in E_i and the order on E'_i is defined by the order on E_i.*
(3) *If (M_{iZ_i}) is a resolution of $(M_Z, \mathcal{I}, E, \mu)$ then $(M'_{iZ'_i})$ is an extension of a resolution of $(M'_{Z'}, \mathcal{I}', E', \mu)$.*

Proof Follows from definition. □

Definition 3.4.2. We say that the above multiple test blow-up $(M'_{iZ'_i})$ is *induced* via ϕ_i by M_{iZ_i}. We shall denote $(M'_{iZ'_i})$ and the corresponding marked ideals $(M'_{iZ'_i}, \mathcal{I}', E', \mu)$ by

$$\phi^*(M_{iZ_i}) := M'_{iZ'_i}, \quad \phi^*(M_{iZ_i}, \mathcal{I}_i, E_i, \mu) := (M'_{iZ'_i}, \mathcal{I}'_i, E'_i, \mu).$$

The above proposition and definition generalize to any sequence of blow-ups with smooth centers.

Proposition 3.4.3. *Let M_{iZ_i} be a sequence blow-ups with smooth centers having simple normal crossings with exceptional divisors.*

(1) *Given a surjective local analytic isomophism $\phi : M'_{Z'} \to M_Z$, the induced sequence $M'_{i,Z'_i} := M'_{Z'} \times_{M_Z} M_{iZ_i}$ is a sequence of blow-ups with smooth centers having simple normal crossings with exceptional divisors.* □

(2) *Given a local analytic isomophism* $\phi : M'_{Z'} \to M_Z$*, the induced sequence* $M'_{i,Z'_i} := M'_{Z'} \times_{M_Z} M_{iZ_i}$ *is an extension of a sequence of blow-ups with smooth centers having simple normal crossings with exceptional divisors.*

3.5. Canonical resolution of marked ideals

Theorem 3.5.1. *With any marked ideal* $(M_Z, \mathcal{I}, E, \mu)$ *there is associated a resolution* (M_{iZ_i}) *called canonical such that*

(1) *For any surjective local analytic isomorphism* $\phi : M'_{Z'} \to M_Z$ *the induced resolution* $\phi^*(M_{iZ_i})$ *is the canonical resolution of* $\phi^*(M_Z, \mathcal{I}, E, \mu)$.
(2) *For any local analytic isomorphism* $\phi : M'_{Z'} \to M_Z$ *the induced resolution* $\phi^*(M_{iZ_i})$ *is an extension of the canonical resolution of* $\phi^*(M_Z, \mathcal{I}, E, \mu)$.
(3) *If* $E = \emptyset$ *then* (M_i) *commutes with closed embeddings of the ambient manifolds* $M_Z \hookrightarrow M'_{Z'}$*, that is, the canonical resolution* (M_{iZ_i}) *of* $(M_Z, \mathcal{I}, \emptyset, \mu)$ *with centers* C_i *defines the canonical resolution* $(M'_{iZ'_i})$ *of* $(M'_{Z'}, \mathcal{I}', \emptyset, \mu)$*, where* $\mathcal{I}' = i_*(\mathcal{I}) \cdot \mathcal{O}_{M'}$*, with the centers* $i(C_i)$.

3.6. Canonical principalization of germs of ideals

Theorem 3.6.1. *Let* $\mathcal{I}$ *be a sheaf of ideals on a germ* M_Z *of an analytic manifold* M*. There exists a principalization of* $\mathcal{I}$*, that is, a projective morphism* $\mathrm{prin}(\mathcal{I}) : \widetilde{M}_{\widetilde{Z}} \to M_Z$ *a finite sequence*

$$M_Z = M_{0,Z_0} \xleftarrow{\sigma_1} M_{1Z_1} \xleftarrow{\sigma_2} M_{2,Z_2} \longleftarrow \ldots \longleftarrow M_{i,Z_i} \longleftarrow \ldots \longleftarrow M_{r,Z_r} = \widetilde{M}_{\widetilde{Z}}$$

of blow-ups with smooth centers $C_{i-1} \subset M_{i-1,Z_{i-1}}$ *such that*

(1) *The exceptional divisor* E_i *of the induced morphism* $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : U_i \to U$ *has only simple normal crossings and* C_i *has simple normal crossings with* E_i.
(2) *The total transform* $\mathrm{prin}^*_{|\widetilde{U}}(\mathcal{I}) = \sigma^{r*}(\mathcal{I})$ *is the ideal of a simple normal crossing divisor* $\widetilde{E}$ *which is a natural combination of the irreducible components of the divisor* E_r.

The morphism $\mathrm{prin} : (\widetilde{M}, \widetilde{\mathcal{I}}) \to (M, \mathcal{I})$ *commutes with local analytic isomorphisms, embeddings of ambient manifolds.*

3.7. Canonical embedded desingularization of germs of analytic spaces

Theorem 3.7.1. *Let* M_Z *be a germ of an analytic manifold and* Y_Z *be a germ of analytic subspace of a germ* M_Z*. There exists an embedded desingularization of* $Y_Z \subset M_Z$ *that is, a finite sequence*

$$M_Z = M_{0,Z_0} \xleftarrow{\sigma_1} M_{1Z_1} \xleftarrow{\sigma_2} M_{2,Z_2} \longleftarrow \ldots \longleftarrow M_{i,Z_i} \longleftarrow \ldots \longleftarrow M_{r,Z_r} = \widetilde{M}_{\widetilde{Z}}$$

of blow-ups with smooth centers $C_{i-1} \subset M_{i-1,Z_{i-1}}$ *such that*

(1) *The exceptional divisor* E_i *of the induced morphism* $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : U_i \to U$ *has only simple normal crossings and* C_i *has simple normal crossings with* E_i.

(2) *The strict transform* $\widetilde{Y}_{\widetilde{Z}} := Y_{r,Z_r}$ *of* Y_Z *is smooth and has only simple normal crossings with the exceptional divisor* E_r*.*
(3) *The morphism* $(M_Z, Y_Z) \leftarrow (\widetilde{M}_{\widetilde{Z}}, \widetilde{Y}_{\widetilde{Z}})$ *defined by the embedded desingularization commutes with local analytic isomorphisms, embeddings of ambient manifolds.*

3.8. Canonical desingularization of germs of analytic spaces

Theorem 3.8.1. *Let* Y *be an analytic space and* $Z \subset Y$ *be a compact subset. There exists a canonical desingularization of* Y_Z *that is a germ of a manifold* $\widetilde{Y}_{\widetilde{Z}}$ *together with a proper bimeromorphic morphism* $\mathrm{res}_{Y_Z} : \widetilde{Y}_{\widetilde{Z}} \to Y_Z$ *such that*

(1) $\widetilde{Z} = \mathrm{res}_{Y_Z}^{-1}(Z)$.
(2) $\mathrm{res}_{Y_Z} : \widetilde{Y}_{\widetilde{Z}} \to Y_Z$ *is an isomorphism over the nonsingular part* Y_{ns} *of* Y*.*
(3) *The inverse image of the singular locus* $\mathrm{res}_{Y_Z}^{-1}(Y_{Z\mathrm{sing}})$ *is a simple normal crossing divisor.*
(4) res_{Y_Z} *is functorial with respect to local analytic isomorphisms. For any local analytic isomorphism* $\phi : Y'_{Z'} \to Y_Z$ *there is a natural lifting* $\widetilde{\phi} : \widetilde{Y'}_{\widetilde{Z}'} \to \widetilde{Y}_{\widetilde{Z}}$ *which is a local analytic isomorphism.*

4. Hironaka resolution principle

Our proof is based upon the following principle which can be traced back to Hironaka and was used by Villamayor in his simplification of Hironaka's algorithm:

Proposition 4.0.2. *The following implications hold true:*

Canonical resolution of germs of marked ideals $(M_Z, \mathcal{I}, E, \mu)$ (1)

$\Downarrow$

Canonical principalization of germs of sheaves $\mathcal{I}$ ***on manifolds*** M (2)

$\Downarrow$

Canonical embedded desingularization of germs $Y_Z \subset M_Z$ (3)

$\Downarrow$

Canonical desingularization of germs of analytic spaces (4)

Proof (1)⇒(2) **Canonical principalization**

Let $\sigma : M_Z \leftarrow \widetilde{M_Z}$ denote the morphism defined by the canonical resolution $M_Z = M_{0,Z_0} \leftarrow M_{1,Z_1} \leftarrow M_{2,Z_2} \leftarrow \ldots \leftarrow M_{k,Z_k} = \widetilde{M_Z}$ of $(M_Z, \mathcal{I}, \emptyset, 1)$. The controlled transform $(\widetilde{\mathcal{I}}, 1) = (\mathcal{I}_k, 1) = \sigma^{\mathrm{c}}(\mathcal{I}, 1)$ has empty support. Consequently, $V(\widetilde{\mathcal{I}}) = V(\mathcal{I}_k) = \emptyset$, which implies $\widetilde{\mathcal{I}}_{\widetilde{Z}} = \mathcal{I}_k = \mathcal{O}_{\widetilde{M}_{\widetilde{Z}}}$. By definition for $i = 1, \ldots, k$, we have $(\mathcal{I}_i, 1) = \sigma_i^{\mathrm{c}}(\mathcal{I}_{i-1}) = \mathcal{I}(D_i)^{-1}\sigma^*(\mathcal{I}_{i-1})$, and thus

$$\sigma_i^*(\mathcal{I}_{i-1}) = \mathcal{I}_i \cdot \mathcal{I}(D_i).$$

Note that if $\mathcal{I}(D) = \mathcal{O}(-D)$ is the sheaf of ideals of a simple normal crossing divisor D on a smooth M_Z and $\sigma : M'_{Z'} \to M_Z$ is the blow-up with a smooth center C which has only simple normal crossings with D then $\sigma^*(\mathcal{I}(D)) = \mathcal{I}(\sigma^*(D))$ is the sheaf of ideals of the divisor with simple normal crossings. The components of the induced Cartier divisors $\sigma^*(D)$ are either the strict transforms of the components of D or the components of the exceptional divisors. (The local equation $y_1^{a_1} \cdot \ldots \cdot y_l^{a_k}$ of D is transformed by the blow-up $(y_1, \ldots, y_n) \to (y_1, y_1y_2, y_1y_3, \ldots, y_1y_l, y_{l+1}, \ldots, y_n)$ into the equation $y_1^{a_1+\ldots a_l} y_2^{a_2} \ldots y_n^{a_n}$.) This implies by induction on i that

$$\sigma_i^* \sigma_{i-1}^* \ldots \sigma_2^* \sigma_1^*(\mathcal{I}_0) = \mathcal{I}_i \cdot \mathcal{I}(E_i)$$

where E_i is an exceptional divisor with simple normal crossings constructed inductively as

$$\mathcal{I}(E_i) = \sigma^*(\mathcal{I}(E_{i-1}))\mathcal{I}(D_i).$$

Finally the full transform $\sigma_k^*(\mathcal{I}) = \mathcal{I}_k \cdot \mathcal{I}(E_k) = \mathcal{O}_{\widetilde{M}} \cdot \mathcal{I}(E_k) = \mathcal{I}(E_k)$ is principal and generated by the sheaf of ideals of a divisor whose components are the exceptional divisors. The canonicity conditions for principalization follow from the canonicity of resolution of marked ideals.

(2)⇒(3) **Canonical embedded desingularization of germs of analytic spaces**

Lemma 4.0.3. *The canonical principalization of $\mathcal{I}$ on M_Z defines an isomorphism over $M_Z \setminus V(\mathcal{I})$.*

Proof. Let $p = 0 \in \mathbf{A}^n$ denote the origin of the affine space $\mathbf{A}^n$. The canonical principalization of the germ $(\mathbf{A}^n_{\{p\}}, \mathcal{O}_{\mathbf{A}^n})$ is an isomorphism over generic points in a neighborhood of p and is equivariant with respect to Gl(n) action , thus it is an isomorphism. The restriction of the canonical principalization $(\widetilde{M}_{\widetilde{Z}}, \widetilde{\mathcal{I}})$ of $(M_Z, \mathcal{I})$ to an open subset $U_{Z_U} \subset M_Z$ determines the canonical principalization of $(U_{Z_U}, \mathcal{I}_{|U_{Z_U}})$. Let $\widetilde{M}_{\widetilde{Z}} \to M_Z$ be the canonical principalization of $(M_Z, \mathcal{O}_{M_Z})$ and $x \in Z \setminus V(\mathcal{I})$. Locally we find an open subset $U_{\{x\}} \subset M_Z \setminus V(\mathcal{I})$ isomorphic to $(\mathbf{A}^n_{\{p\}}, \mathcal{O}_{\mathbf{A}^n})$. The canonical principalization of $(U_{\{x\}}, \mathcal{I}_U) = ((U_{\{x\}}, \mathcal{O}_U) \simeq (\mathbf{A}^n_{\{p\}}, \mathcal{O}_{\mathbf{A}^n})$ is an isomorphism. □

Let $Y_Z \subset M_Z$ be a germ of a closed analytic subspace $Y \subset M$. Let $M_Z = M_{0,Z_0} \leftarrow M_{1,Z_1} \leftarrow M_{2,Z_2} \leftarrow \ldots \leftarrow M_{k,Z_k} = \widetilde{M_Z}$ be the canonical principalization of germs sheaves of ideals $\mathcal{I}_Y$. It defines a sequence of blow-ups $U_0 \leftarrow U_k$ which is a principalization of $\mathcal{I}_{U_0}$ for a suitable open neighborhood U_0 of Z.

Suppose all centers C_{i-1} of the blow-ups $\sigma_i : U_{i-1} \leftarrow U_i$ are disjoint from the generic points of strict transforms Y_{i-1} of $Y_0 = Y \cap U_0$. Then $\widetilde{\sigma}$ is an isomorphism over the generic points y of Y_0 and $\widetilde{\sigma}^*(\mathcal{I})_y = \sigma^*(\mathcal{I})_y$. Moreover no exceptional divisor pass through y. This contradicts the condition $\widetilde{\sigma}^*(\mathcal{I}) = \mathcal{I}_{\widetilde{E}}$. Thus there is a smallest $i_{\rm res}$ with the property that $C_{i_{\rm res}}$ contains the strict transform $Y_{i_{\rm res}}$ and all centers C_j for $j < i_{\rm res}$ are disjoint from the generic points of strict transforms Y_j. Let $y \in Y_{i_{\rm res}}$ be a generic point for which $U_{i_{\rm res}} \to U_0$ is an isomorphism. Find an open set $U \subset U_0$ intersecting Y such that

$U_{i_{\rm res}} \to U_0$ is an isomorphism over U. Then $Y_{i_{\rm res}} \cap U = Y \cap U$ and $C_{i_{\rm res}} \cap U \supseteq Y_{i_{\rm res}} \cap U$ by the definition of $Y_{i_{\rm res}}$. On the other hand, by the previous lemma $C_{i_{\rm res}} \cap U \subseteq Y_{i_{\rm res}} \cap U$, which gives $C_{i_{\rm res}} \cap U = Y_{i_{\rm res}} \cap U$. Finally, $Y_{i_{\rm res}}$ is an irreducible component of a smooth (possibly reducible) center C_i. This implies that $Y_{i_{\rm res}}$ is smooth and has simple normal crossings with the exceptional divisors. We define the canonical embedded resolution of (M_Z, Y_Z) to be

$$(M_Z, Y_Z) = (U_{0Z}, Y_{0Z}) \leftarrow (U_{1Z_1}, Y_{1Z_1}) \leftarrow (U_{2Z_2}, Y_{2Z_2}) \leftarrow \ldots \leftarrow (U_{i_{\rm res}, Z_{i_{\rm res}}}, Y_{i_{\rm res}Z_{i_{\rm res}}}).$$

It is independent of the choice of U. If $(M'_{Z'}, Y'_Z) \to (M_Z, Y_Z)$ is a local analytic isomophism then the induced sequence of blow-ups $(U'_{iZ_i})_{0\le i\le k} = (U'_{Z'}, \times_{M_Z} U_{iZ_i})_{0\le i\le k}$ is an extension of the canonical principalization $(U'_{j,Z'_j})_{0\le j\le k'}$ of $(U'_{0Z'_0}, \mathcal{I}_{Y'|U'_0})$. Moreover $U'_{j_{\rm res}} = U'_{i_{\rm res}}$ and $(U'_i)_{0\le i\le i_{\rm res}}$ is is an extension of the canonical resolution $(U'_j)_{0\le j\le j_{\rm res}}$ of $(M'_{Z'}, Y'_Z)$. Commutativity with closed embeddings for embedded desingularizations follows from the commutativity with closed embeddings for principalizations.

(3)⇒(4) **Canonical desingularization of germs**

Let Y be an analytic space. Every point of $y \in Y$ has a neighborhood V which is locally isomorphic to a closed analytic subset of an open ball $U \subset \mathbb{C}^n$. The coordinates $u_1, \ldots u_n$ on Y define a minimal embedding $Y \supset V \to U$ into an open subset U of $\mathbb{C}^n$. Let $Z \subset V = Y \cap U$ be a compact set. Then Y_Z can be identified with V_Z. Consider the canonical embedded desingularization $(\widetilde{U_Z}, \widetilde{Y_Z}) \to (U_Z, Y_Z)$. Then we define the canonical desingularization of Y_Z to be $\widetilde{Y_Z} \to Y_Z$. Two minimal embeddings $\phi_1 : Z \subset V_1 \to U_1 \supset Z_1 = \phi_1(Z)$ and $\phi_2 : Z \subset V_2 \to U_2 \supset Z_2 = \phi_2(Z)$ of two different open subsets V_1, V_2 containing Z are defined by two different sets of coordinates $u_1, \ldots u_n$ and $u'_1, \ldots u'_n$ differ by an isomorphism

$$\psi := \phi_2^{-1}\phi_1 : (U_{1Z_1}, (\phi_1(V_1)_{Z_1}) \to (U_{2Z_2}, (\phi_2(V_2)_{Z_2})$$

mapping coordinates $x_1, \ldots, x_n$ to $x'_1, \ldots, x'_n$. Note that both $\phi_1(V_1)_{Z_1}$ and $\phi_2(V_1)_{Z_2}$ can be identified with $\widetilde{Y}_{\widetilde{Z}}$. The isomorphism ψ, by canonicity, lifts to the isomorphisms between embedded desingularizations $\widetilde{\psi} : (\widetilde{U_{1\widetilde{Z}_1}}, \widetilde{Y_{1\widetilde{Z}}}) \to (\widetilde{U_{2Z}}, \widetilde{Y_{2Z}})$ and nonembedded desingularizations $\widetilde{Y_{1Z}} \to \widetilde{Y_{2Z}}$. The latter shows that $\widetilde{Y_Z} \to Y_Z$ is independent of the choice of ambient manifold U. Observe that if $Y_Z \subset Y'_{Z'}$ is an open embedding then it extends to an open embedding $U_Z \subset U'_{Z'}$ and it defines an open embeddings of desingularizations $\widetilde{Y_Z} \subset \widetilde{Y'_{Z'}}$.

Let Y_Z denote the analytic germ of Y at Z. Consider an open cover of Z with the open subsets $V_i \subset W_i \subset U_i$ of Y, such that $\overline{V_i} \subset W_i$ and $\overline{V_i} \subset U_i$ are compact and U_i is isomorphic to an open balls as above. Set $S_i := \overline{V_i}$, $Z_i := \overline{W_i} \cap Z$.

The desingularization of $Y_{S_i} = U_{iS_i}$ determines the desingularization $\widetilde{U'_i}$ of an open neighborhood U'_i of Y_{S_i} and thus the desingularization $\widetilde{V_i} \to V_i$ of $V_i \subset U'_i$.

For each i, j, the embedding $Y_{Z_i \cap Z_j} \to Y_{Z_i}$ lifts to embeddings of nonembedded desingularizations of germs $\widetilde{Y_{Z_i \cap Z_j}} \to \widetilde{Y_{Z_i}}$. Note that the open embedding $V_i \cap V_j \to V_i$ is the restriction of $Y_{Z_i \cap Z_j} \to Y_{Z_i}$. It defines an embedding of desingularizations $(V_i \cap V_j)\tilde{} \to \widetilde{V}_i$.

Let $\widetilde{V}$ be a manifold obtained by gluing V_i along $V_i \cap V_j$. The desingularization morphism $\mathrm{des}_V : \widetilde{V} \to V$ is bimeromorphic and proper. Let $\widetilde{Z} := \mathrm{des}_V^{-1}(Z)$. Note that $Y_Z = \bigcup Y_{Z_i} = \bigcup (V_i)_{Z_i}$. We define the canonical desingularization of Y_Z to be

$$\widetilde{Y_Z} := \widetilde{V}_{\widetilde{Z}} = \bigcup \widetilde{V_{iZ_i}}.$$

It follows from the definition that it commutes with local analytic isomorphisms. $\square$

4.1. Canonical principalization of ideal sheaves on analytic spaces

Let $\mathcal{I}$ be an ideal sheaf on a manifold M. Consider an open cover $\{U_i\}_{i \in I}$ of M, such that $Z_i := \overline{U_i}$ are compact. For every i let $\mathrm{prin}_i : (\widetilde{Y}_{Z_i}, \widetilde{\mathcal{I}}_{Z_i}) \to (Y_{Z_i}, \mathcal{I}_{Z_i})$ be a canonical principalization of $\mathcal{I}$ on Y_{Z_i}. Let $\widetilde{U}_i := \mathrm{prin}_i^{-1}(U_i) \to (U_i, \mathcal{I}_{|U_i})$ be its restriction. By canonicity, $\mathrm{prin}_i : \mathrm{prin}_i^{-1}(Y_{Z_i} \cap Y_{Z_j})$ is isomorphic over $Y_{Z_i} \cap Y_{Z_j}$ to $\widetilde{Y}_{Z_i \cap Z_j}$. Thus the meromorphic map

$$\widetilde{U}_{ij} := \mathrm{prin}_i^{-1}(U_i \cap U_j) \simeq \widetilde{U}_{ji} := \mathrm{prin}_j^{-1}(U_i \cap U_j)$$

is an isomorphism. We define $\widetilde{M}$ to be a manifold obtained by gluing $\widetilde{U}_i$ along $\widetilde{U}_{ij}$. Then $\mathrm{prin} : \widetilde{M} \to M$ is a proper bimeromorphic morphism. Moreover for any compact $Z \subset M$, $(\widetilde{M}_{\widetilde{Z}}, \widetilde{\mathcal{I}}_{\widetilde{Z}}) \to (M_Z, \mathcal{I}_Z)$ is a canonical principalization of $\mathcal{I}$ on the germ M_Z.

4.2. Canonical embedded desingularization of analytic spaces

Let $Y \subset M$ be an analytic subspace of a manifold. Consider an open cover $\{U_i\}_{i \in I}$ of M, such that $Z_i := \overline{U_i}$ are compact. For every i let $\mathrm{des}_i : (\widetilde{M}_{Z_i}, \widetilde{Y}_{Z_i}) \to (M_{Z_i}, Y_{Z_i})$ be the canonical desingularization of Y_{Z_i}. Let $(\widetilde{U}_i, \widetilde{U}_i^Y) := \mathrm{des}_i^{-1}(U_i, Y \cap U_i) \to (U_i, Y \cap U_i)$ be its restriction. As before we define $\widetilde{M}$ to be a manifold obtained by gluing $\widetilde{U}_i$ along $\widetilde{U}_{ij}$. A subspace $\widetilde{Y} \subset \widetilde{M}$ is a manifold obtained by gluing $\widetilde{U^Y}_i$ along $\widetilde{U^Y}_{ij}$. Then $\mathrm{des} : (\widetilde{M}, \widetilde{Y}) \to (M, Y)$ is a proper bimeromorphic morphism. Moreover for any compact $Z \subset M$, $(\widetilde{M}_{\widetilde{Z}} \widetilde{Y}_{\widetilde{Z}}) \to (M_Z, Y_Z)$ is a canonical embedded desingularization of the germ $Y_Z \subset M_Z$.

4.3. Canonical desingularization of analytic spaces

Let Y be an analytic space. Consider an open cover $\{U_i\}_{i \in I}$ of Y, such that $Z_i := \overline{U_i}$ are compact. For every i let $\mathrm{des}_i : \widetilde{Y}_{Z_i} \to Y_{Z_i}$ be the canonical desingularization of the germ Y_{Z_i}. Let $\widetilde{U}_i := \mathrm{des}_i^{-1}(U_i) \to U_i$ be its restriction. As before we define $\widetilde{Y}$ to be a manifold obtained by gluing $\widetilde{U}_i$ along $\widetilde{U}_{ij}$. Then $\mathrm{des} : \widetilde{Y} \to Y$ is a proper bimeromorphic morphism. Moreover for any compact $Z \subset Y$, $\widetilde{Y}_{\widetilde{Z}} \to Y_Z$ is a canonical desingularization of germ Y_Z.

5. Marked ideals

5.1. Equivalence relation for marked ideals

Let us introduce the following equivalence relation for marked ideals:

Definition 5.1.1. Let $(M_Z, \mathcal{I}, E_{\mathcal{I}}, \mu_{\mathcal{I}})$ and $(M_Z, \mathcal{J}, E_{\mathcal{J}}, \mu_{\mathcal{J}})$ be two marked ideals on the manifold M_Z. Then $(M_Z, \mathcal{I}, E_{\mathcal{I}}, \mu_{\mathcal{I}}) \simeq (M_Z, \mathcal{J}, E_{\mathcal{J}}, \mu_{\mathcal{J}})$ if

(1) $E_{\mathcal{I}} = E_{\mathcal{J}}$ and the orders on $E_{\mathcal{I}}$ and on $E_{\mathcal{J}}$ coincide.
(2) $\mathrm{supp}(M_Z, \mathcal{I}, E_{\mathcal{I}}, \mu_{\mathcal{I}}) = \mathrm{supp}(M_Z, \mathcal{J}, E_{\mathcal{J}}, \mu_{\mathcal{J}})$.
(3) All the multiple test blow-ups $M_{Z0} = M_Z \xleftarrow{\sigma_1} M_{1Z_1} \xleftarrow{\sigma_2} \ldots \longleftarrow M_{iZ_i} \xleftarrow{\sigma_r} \ldots \longleftarrow M_{rZ_r}$ of $(M_Z, \mathcal{I}, E_{\mathcal{I}}, \mu_{\mathcal{I}})$ are exactly the multiple test blow-ups of $(M_Z, \mathcal{J}, E_{\mathcal{J}}, \mu_{\mathcal{J}})$ and moreover we have
$$\mathrm{supp}(M_{iZ_i}, \mathcal{I}_i, E_i, \mu_{\mathcal{I}}) = \mathrm{supp}(M_{iZ_i}, \mathcal{J}_i, E_i, \mu_{\mathcal{J}}).$$

It is easy to show the lemma:

Lemma 5.1.2. *For any $k \in \mathbf{N}$, $(\mathcal{I}, \mu) \simeq (\mathcal{I}^k, k\mu)$.*

Remark. The marked ideals considered in this paper satisfy a stronger equivalence condition: For any local analytic isomorphisms $\phi : M'_Z \to M_Z$, $\phi^*(\mathcal{I}, \mu) \simeq \phi^*(\mathcal{J}, \mu)$. This condition will follow and is not added in the definition.

5.2. Ideals of derivatives

Ideals of derivatives were first introduced and studied in the resolution context by Giraud. Villamayor developed and applied this language to his *basic objects*.

Definition 5.2.1. (Giraud, Villamayor) Let $\mathcal{I}$ be a coherent sheaf of ideals on a germ of manifold M_Z. By the *first derivative* (originally *extension*) $\mathcal{D}_{M_Z}(\mathcal{I})$ of $\mathcal{I}$ (or simply $\mathcal{D}(\mathcal{I})$) we mean the coherent sheaf of ideals generated by all functions $f \in \mathcal{I}$ with their first derivatives. Then the *i-th derivative* $\mathcal{D}^i(\mathcal{I})$ is defined to be $\mathcal{D}(\mathcal{D}^{i-1}(\mathcal{I}))$. If $(\mathcal{I}, \mu)$ is a marked ideal and $i \leq \mu$ then we define
$$\mathcal{D}^i(\mathcal{I}, \mu) := (\mathcal{D}^i(\mathcal{I}), \mu - i).$$

Recall that on a manifold M there is a locally free sheaf of differentials $\Omega_{M/K}$ generated locally by $du_1, \ldots, du_n$ for a set of local coordinates $u_1, \ldots, u_n$. The dual sheaf of derivations $\mathrm{Der}_K(\mathcal{O}_M)$ is locally generated by the derivations $\frac{\partial}{\partial u_i}$. Immediately from the definition we observe that $\mathcal{D}(\mathcal{I})$ is a coherent sheaf defined locally by generators f_j of $\mathcal{I}$ and all their partial derivatives $\frac{\partial f_j}{\partial u_i}$. We see by induction that $\mathcal{D}^i(\mathcal{I})$ is a coherent sheaf defined locally by the generators f_j of $\mathcal{I}$ and their derivatives $\frac{\partial^{|\alpha|} f_j}{\partial u^\alpha}$ for all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$, where $|\alpha| := \alpha_1 + \ldots + \alpha_n \leq i$.

Lemma 5.2.2. *(Giraud, Villamayor) For any $i \leq \mu - 1$,*
$$\mathrm{supp}(\mathcal{I}, \mu) = \mathrm{supp}(\mathcal{D}^i(\mathcal{I}), \mu - i)).$$

In particular $\mathrm{supp}(\mathcal{I},\mu)=\mathrm{supp}(\mathcal{D}^{\mu-1}(\mathcal{I}),1)=V(\mathcal{D}^{\mu-1}(\mathcal{I}))$ *is a closed set* $(i=\mu-1)$*.*

Proof. It suffices to prove the lemma for $i=1$. If $x\in\mathrm{supp}(\mathcal{I},\mu)$ then for any $f\in\mathcal{I}$ we have $\mathrm{ord}_x(f)\geq\mu$. This implies $\mathrm{ord}_x(Df)\geq\mu-1$ for any derivative D and consequently $x\in\mathrm{supp}(\mathcal{D}(\mathcal{I}),\mu-1))$. Now, let $x\in\mathrm{supp}(\mathcal{D}(\mathcal{I}),\mu-1))$. Then for any $f\in\mathcal{I}$ we have $\mathrm{ord}_x(f)\geq\mu-1$. Suppose $\mathrm{ord}_x(f)=\mu-1$ for some $f\in\mathcal{I}$. Then $f=\sum_{|\alpha|\geq\mu-1}c_\alpha x^\alpha$ and there is α such that $\alpha=\mu-1$ and $c_\alpha\neq 0$. We find $\frac{\partial}{\partial x_i}$ for which $\mathrm{ord}_x(\frac{\partial x^\alpha}{\partial x_i})=\mu-2$ and thus $\mathrm{ord}_x(\frac{\partial f}{\partial x_i})=\mu-2$ and $x\notin\mathrm{supp}(\mathcal{D}(\mathcal{I}),\mu-1))$. □

We write $(\mathcal{I},\mu)\subset(\mathcal{J},\mu)$ if $\mathcal{I}\subset\mathcal{J}$.

Lemma 5.2.3. *(Giraud, Villamayor) Let* $(\mathcal{I},\mu)$ *be a marked ideal and* $C\subset\mathrm{supp}(\mathcal{I},\mu)$ *be a smooth center and* $r\leq\mu$*. Let* $\sigma: M_Z\leftarrow M'_Z$ *be a blow-up at* C*. Then*

$$\sigma^{\mathrm{c}}(\mathcal{D}^r_{M_Z}(\mathcal{I},\mu))\subseteq\mathcal{D}^r_{M_Z{}'}(\sigma^{\mathrm{c}}(\mathcal{I},\mu)).$$

Proof. First assume that $r=1$. Let $u_1,\ldots,u_n$ denote the local coordinates at $x\in C$ such that C is a coordinate subspace. Then the local coordinates at $x'\in\sigma^{-1}(x)$ are of the form $u'_i=\frac{u_i}{u_m}$ for $i<m$ and $u'_i=u_i$ for $i\geq m$, where $u_m=u'_m=y$ denotes the local equation of the exceptional divisor.

The derivations $\frac{\partial}{\partial u_i}$ of $\mathcal{O}_{x,M}$ extend to derivations of the rational field $K(\mathcal{O}_{x,M})$. Note also that

$$\frac{\partial u'_j}{\partial u_i}=\frac{\delta_{ij}}{u_m},\quad i<m,1\leq j\leq n;\qquad \frac{\partial u'_j}{\partial u_m}=-\frac{1}{u_m^2}u_j,\quad j<m;\qquad \frac{\partial u'_m}{\partial u_m}=1;$$
$$\frac{\partial u'_j}{\partial u_m}=0, j>m;\qquad \frac{\partial u'_i}{\partial u_j}=\delta_{ij},\quad i\geq m.$$

This gives

$$\frac{\partial}{\partial u_i}=\frac{1}{u_m}\frac{\partial}{\partial u'_i}=\frac{1}{y}\frac{\partial}{\partial u'_i},\quad 1\leq i<m;\qquad \frac{\partial}{\partial u'_i}=\frac{\partial}{\partial u_i},\quad m<i\leq n,$$
$$\frac{\partial}{\partial u_m}=-\frac{1}{y}(u'_1\frac{\partial}{\partial u'_1}+\ldots+u'_{m-1}\frac{\partial}{\partial u'_{m-1}}-u'_m\frac{\partial}{\partial u'_m}).$$

We see that any derivation D of $\mathcal{O}_{x,M}$ induces a derivation $y\sigma^*(D)$ of $\mathcal{O}_{x',M'}$. Let E be the exceptional divisor $\mathcal{I}(E)$ be its ideal sheaf (locally generated by y). Thus the sheaf of derivations $\mathcal{I}(E)\sigma^*(\mathrm{Der}_K(\mathcal{O}_M))$ is a subsheaf of $\mathrm{Der}_K(\mathcal{O}_{M'})$ locally generated by

$$\frac{\partial}{\partial u'_i}, i<m;\quad y\frac{\partial}{\partial y},\quad \text{and}\quad y\frac{\partial}{\partial u'_i}, i>m.$$

In particular $\mathcal{I}(E)\sigma^*(\mathcal{D}_M(\mathcal{I}))\subset\mathcal{D}_{M'}(\sigma^*(\mathcal{I}))$. For any sheaf of ideals $\mathcal{J}$ on M' denote by $\mathcal{I}(E)\sigma^*(\mathcal{D}_M)(\mathcal{J})\subset\mathcal{D}_{M'}(\mathcal{J})$ the ideal generated by $\mathcal{J}$ and the derivatives $D'(f)$, where $f\in\mathcal{J}$ and $D'\in\mathcal{I}(E)\sigma^*(\mathrm{Der}_K(\mathcal{O}_M))$. Note that for a neighborhood $U'\ni x'$ and any $f\in\mathcal{J}(U')$ and $D'\in y\sigma^*(\mathrm{Der}_K(\mathcal{O}_M))$, y divides $D'(y)$ and

$$D'(yf)=yD'(f)+D'(y)f\quad\in\quad y\sigma^*(\mathcal{D}_M)(\mathcal{J})+y\mathcal{J}=y\sigma^*(\mathcal{D}_M)(\mathcal{J}).$$

Consequently, $y\sigma^*(\mathcal{D}_M)(y\mathcal{J}) \subseteq yy\sigma^*(\mathcal{D}_M)(\mathcal{J})$ and more generally $y\sigma^*(\mathcal{D}_M)(y^\mu\mathcal{J}) \subseteq y^\mu y\sigma^*(\mathcal{D}_{M'})(\mathcal{J})$. Then

$$\begin{aligned} y\sigma^*(\mathcal{D}_M(\mathcal{I})) \subseteq y\sigma^*(\mathcal{D}_M)(\sigma^*(\mathcal{I})) &= y\sigma^*(\mathcal{D}_M)(y^\mu\sigma^{\mathrm{c}}(\mathcal{I})) \\ &\subseteq y^\mu y\sigma^*(\mathcal{D}_M)(\sigma^{\mathrm{c}}(\mathcal{I})) \subseteq y^\mu\mathcal{D}_{M'}(\sigma^{\mathrm{c}}(\mathcal{I})). \end{aligned}$$

Then

$$\sigma^{\mathrm{c}}(\mathcal{D}_M(\mathcal{I})) = y^{-\mu+1}\sigma^*(\mathcal{D}_M(\mathcal{I})) \subseteq \mathcal{D}_{M'}(\sigma^{\mathrm{c}}(\mathcal{I})).$$

Assume now that r is arbitrary. Then $C \subset \mathrm{supp}(\mathcal{I},\mu) = \mathrm{supp}(\mathcal{D}^i_M(\mathcal{I},\mu))$ for $i \leq r$ and by induction on r,

$$\sigma^{\mathrm{c}}(\mathcal{D}^r_M\mathcal{I}) = \sigma^{\mathrm{c}}(\mathcal{D}_M(\mathcal{D}^{r-1}_M(\mathcal{I}))) \subseteq \mathcal{D}_{M'}(\sigma^{\mathrm{c}}\mathcal{D}^{r-1}_M(\mathcal{I})) \subseteq \mathcal{D}^r_{M'}(\sigma^{\mathrm{c}}(\mathcal{I})).$$

□

As a corollary from Lemma 5.2.3 we prove the following

Lemma 5.2.4. *A multiple test blow-up $(M_i)_{0\leq i\leq k}$ of $(\mathcal{I},\mu)$ is a multiple test blow-up of $\mathcal{D}^j(\mathcal{I},\mu)$ for $0 \leq j \leq \mu$ and*

$$[\mathcal{D}^j(\mathcal{I},\mu)]_k \subset \mathcal{D}^j(\mathcal{I}_k,\mu).$$

Proof. Induction on k. For $k = 0$ evident. Let $\sigma_{k+1} : M_k \leftarrow M_{k+1}$ denote the blow-up with a center $C_k \subseteq \mathrm{supp}(\mathcal{I}_k,\mu) = \mathrm{supp}(\mathcal{D}^j(\mathcal{I}_k,\mu)) \subseteq \mathrm{supp}([\mathcal{D}^j(\mathcal{I},\mu)]_k)$. Then by induction $[\mathcal{D}^j(\mathcal{I},\mu)]_{k+1} = \sigma^{\mathrm{c}}_{k+1}[\mathcal{D}^j(\mathcal{I},\mu)]_k \subseteq \sigma^{\mathrm{c}}_{k+1}(\mathcal{D}^j(\mathcal{I}_k,\mu))$. Lemma 5.2.3 gives $\sigma^{\mathrm{c}}_{k+1}(\mathcal{D}^j(\mathcal{I}_k,\mu)) \subseteq \mathcal{D}^j\sigma^{\mathrm{c}}_{k+1}(\mathcal{I}_k,\mu) = \mathcal{D}^j(\mathcal{I}_{k+1},\mu)$. □

5.3. Hypersurfaces of maximal contact

The concept of the *hypersurfaces of maximal contact* is one of the key points of this proof. It was originated by Hironaka, Abhyankhar and Giraud and developed in the papers of Bierstone-Milman and Villamayor.

In our terminology we are looking for a smooth hypersurface containing the supports of marked ideals and whose strict transforms under multiple test blow-ups contain the supports of the induced marked ideals. Existence of such hypersurfaces allows a reduction of the resolution problem to codimension 1.

First we introduce marked ideals which locally admit hypersurfaces of maximal contact.

Definition 5.3.1. (Villamayor [35]) We say that $(M_Z,\mathcal{I},E,\mu)$ be a marked ideal of *maximal order* (originally *simple basic object*) if there exists an open neighborhood U of $M_Z = (U,Z)$ such that $\mathcal{I}$ is defined on $U \supset Z$ and $\max\{\mathrm{ord}_x(\mathcal{I}) \mid x \in U\} \leq \mu$ or equivalently $\mathcal{D}^\mu(\mathcal{I}) = \mathcal{O}_{M_Z}$.

Lemma 5.3.2. *(Villamayor* [35]*) Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order and $C \subset \mathrm{supp}(\mathcal{I},\mu)$ be a smooth center. Let $\sigma : M_Z \leftarrow M'_{Z'}$ be a blow-up at $C \subset \mathrm{supp}(\mathcal{I},\mu)$. Then $\sigma^{\mathrm{c}}(\mathcal{I},\mu)$ is of maximal order.*

Proof. If $(\mathcal{I},\mu)$ is a marked ideal of maximal order then $\mathcal{D}^\mu(\mathcal{I}) = \mathcal{O}_{M_Z}$. Then by Lemma 5.2.3, $\mathcal{D}^\mu(\sigma^{\mathrm{c}}(\mathcal{I},\mu)) \supset \sigma^{\mathrm{c}}(\mathcal{D}^\mu(\mathcal{I}),0) = \mathcal{O}_{M_Z}$. □

Lemma 5.3.3. *(Villamayor* [35]*) If $(\mathcal{I},\mu)$ is a marked ideal of maximal order and $0 \leq i \leq \mu$ then $\mathcal{D}^i(\mathcal{I},\mu)$ is of maximal order.*

Proof. $\mathcal{D}^{\mu-i}(\mathcal{D}^i(\mathcal{I},\mu)) = \mathcal{D}^\mu(\mathcal{I},\mu) = \mathcal{O}_{M_Z}$. □

In particular $(\mathcal{D}^{\mu-1}(\mathcal{I}),1)$ is a marked ideal of maximal order.

Lemma 5.3.4. *(Giraud) Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order and let $\sigma : M_Z \leftarrow M'_{Z'}$ be a blow-up at a smooth center $C \subset \mathrm{supp}(\mathcal{I},\mu)$. Let $u \in \mathcal{D}^{\mu-1}(\mathcal{I},\mu)(U)$ be a function of multiplicity one on U, that is, for any $x \in V(u)$, $\mathrm{ord}_x(u) = 1$. In particular $\mathrm{supp}(\mathcal{I},\mu) \cap U \subset V(u)$. Let $U' \subset \sigma^{-1}(U) \subset M'_{Z'}$ be an open set where the exceptional divisor is described by y. Let $u' := \sigma^{\mathrm{c}}(u) = y^{-1}\sigma^*(u)$ be the controlled transform of u. Then*

1. *$u' \in \mathcal{D}^{\mu-1}(\sigma^{\mathrm{c}}(\mathcal{I}_{|U'},\mu))$.*
2. *u' is a function of multiplicity one on U'.*
3. *$V(u')$ is the restriction of the strict transform of $V(u)$ to U'.*

Proof. (1) $u' = \sigma^{\mathrm{c}}(u) = u/y \in \sigma^{\mathrm{c}}(\mathcal{D}^{\mu-1}(\mathcal{I})) \subset \mathcal{D}^{\mu-1}(\sigma^{\mathrm{c}}(\mathcal{I}))$.

(2) Since u was one of the local coordinates describing the center of blow-ups, $u' = u/y$ is a parameter, that is, a function of order one.

(3) follows from (2). □

Definition 5.3.5. We shall call a function

$$u \in T(\mathcal{I})(U) := \mathcal{D}^{\mu-1}(\mathcal{I}(U))$$

of multiplicity one a *tangent direction* of $(\mathcal{I},\mu)$ on U.

As a corollary from the above we obtain the following lemma:

Lemma 5.3.6. *(Giraud) Let $u \in T(\mathcal{I})(U)$ be a tangent direction of $(\mathcal{I},\mu)$ on U. Then for any multiple test blow-up (U_i) of $(\mathcal{I}_{|U},\mu)$ all the supports of the induced marked ideals $\mathrm{supp}(\mathcal{I}_i,\mu)$ are contained in the strict transforms $V(u)_i$ of $V(u)$.* □

Remarks.
1. Tangent directions are functions defining locally hypersurfaces of maximal contact.
2. The main problem leading to complexity of the proofs is that of noncanonical choice of the tangent directions. We overcome this difficulty by introducing *homogenized ideals.*

Lemma 5.3.7. *(Villamayor) Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order whose support is of codimension 1. Then all codimension one components of $\mathrm{supp}(\mathcal{I},\mu)$ are smooth and isolated. After the blow-up $\sigma : M_Z \leftarrow M'_{Z'}$ at such a component $C \subset \mathrm{supp}(\mathcal{I},\mu)$ the induced support $\mathrm{supp}(\mathcal{I}',\mu)$ does not intersect the exceptional divisor of σ.*

Proof. By the previous lemma there is a tangent direction $u \in \mathcal{D}^{\mu-1}(\mathcal{I})$ whose zero set is smooth and contains $\mathrm{supp}(\mathcal{I},\mu)$. Then $\mathcal{D}^{\mu-1}(\mathcal{I}) = (u)$ and $\mathcal{I}$ is locally described as $\mathcal{I} = (u^\mu)$. Suppose there is $g \in \mathcal{I}$ written as $g = c_\mu(x,u)u^\mu + c_{\mu-1}(x)u^{\mu-1} + \ldots + c_0(x)$, where at least one function $c_i(x) \neq 0$ for $0 \le i \le \mu-1$. Then there is a multiindex α such $|\alpha| = \mu - i - 1$ and $\frac{\partial^{|\alpha|}c_i}{\partial x^\alpha}$ is not the zero function. Then the derivative $\frac{\partial^{\mu-1}g}{\partial u^i \partial x^\alpha} \in \mathcal{D}^{\mu-1}(\mathcal{I})$ does not belong to the ideal (u).

The blow-up at the component C locally defined by u transforms $(\mathcal{I},\mu)=((u^\mu),\mu)$ to $(\mathcal{I}',\mu)$, where $\sigma^*(\mathcal{I})=y^\mu\mathcal{O}_M$, and $\mathcal{I}'=\sigma^{\mathrm{c}}(\mathcal{I})=y^{-\mu}\sigma^*(\mathcal{I})=\mathcal{O}_M$, where $y=u$ describes the exceptional divisor. □

Remark. Note that the blow-up of codimension one components is an isomorphism. However it defines a nontrivial transformation of marked ideals. In the actual desingularization process this kind of blow-up may occur for some marked ideals induced on subvarieties of ambient varieties. Though they define isomorphisms of those subvarieties they determine blow-ups of ambient varieties which are not isomorphisms.

5.4. Arithmetical operations on marked ideals

In this section all marked ideals are defined for the germ of the manifold M and the same set of exceptional divisors E. Define the following operations of addition and multiplication of marked ideals:

(1) $(\mathcal{I},\mu_{\mathcal{I}})+(\mathcal{J},\mu_{\mathcal{J}}):=(\mathcal{I}^{\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}})/\mu_{\mathcal{I}}}+\mathcal{J}^{\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}})/\mu_{\mathcal{J}}},\mathrm{lcm}(\mu_{\mathcal{I}},\mu_{\mathcal{J}}))$
or more generally (the operation of addition is not associative)
$$(\mathcal{I}_1,\mu_1)+\ldots+(\mathcal{I}_m,\mu_m):=(\mathcal{I}_1^{\mathrm{lcm}(\mu_1,\ldots,\mu_m)/\mu_1}+\mathcal{I}_2^{\mathrm{lcm}(\mu_1,\ldots,\mu_m)/\mu_2}+\ldots+\mathcal{I}_m^{\mathrm{lcm}(\mu_1,\ldots,\mu_m)/\mu_m},\mathrm{lcm}(\mu_1,\ldots,\mu_m)).$$

(2) $(\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}}):=(\mathcal{I}\cdot J,\mu_{\mathcal{I}}+\mu_{\mathcal{J}})$.

Lemma 5.4.1. (1) $\mathrm{supp}((\mathcal{I}_1,\mu_1)+\ldots+(\mathcal{I}_m,\mu_m))=\mathrm{supp}(\mathcal{I}_1,\mu_1)\cap\ldots\cap\mathrm{supp}(\mathcal{I}_m,\mu_m)$. *Moreover multiple test blow-ups (M_k) of $(\mathcal{I}_1,\mu_1)+\ldots+(\mathcal{I}_m,\mu_m)$ are exactly those which are simultaneous multiple test blow-ups for all $(\mathcal{I}_j,\mu_j)$ and for any k we have the equality for the controlled transforms $(\mathcal{I}_j,\mu_{\mathcal{I}})_k$*
$$(\mathcal{I}_1,\mu_1)_k+\ldots+(\mathcal{I}_m,\mu_m)_k=[(\mathcal{I}_1,\mu_1)+\ldots+(\mathcal{I}_m,\mu_m)]_k$$

(2)
$$\mathrm{supp}(\mathcal{I},\mu_{\mathcal{I}})\cap\mathrm{supp}(\mathcal{J},\mu_{\mathcal{J}})\subseteq\mathrm{supp}((\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}})).$$
Moreover any simultaneous multiple test blow-up M_i of both ideals $(\mathcal{I},\mu_{\mathcal{I}})$ and $(\mathcal{J},\mu_{\mathcal{J}})$ is a multiple test blow-up for $(\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}})$, and for the controlled transforms $(\mathcal{I}_k,\mu_{\mathcal{I}})$ and $(\mathcal{J}_k,\mu_{\mathcal{J}})$ we have the equality
$$(\mathcal{I}_k,\mu_{\mathcal{I}})\cdot(\mathcal{J}_k,\mu_{\mathcal{J}})=[(\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}})]_k.$$

Proof.

(1) Follows from two simple observations:

(i) $(\mathcal{I},\mu)\simeq(\mathcal{I}^k,k\mu)$

(ii) $\mathrm{supp}(\mathcal{I},\mu)\cap\mathrm{supp}(\mathcal{I}',\mu)=\mathrm{supp}(\mathcal{I}+\mathcal{I}',\mu)$ and the property is persistent for controlled transforms.

(2) Follows from the following fact:

If $\mathrm{ord}_x(\mathcal{I})\geq\mu_{\mathcal{I}}$ and $\mathrm{ord}_x(\mathcal{J})\geq\mu_{\mathcal{J}}$ then $\mathrm{ord}_x(\mathcal{I}\cdot\mathcal{J})\geq\mu_{\mathcal{I}}+\mu_{\mathcal{J}}$. This implies that $\mathrm{supp}(\mathcal{I},\mu_{\mathcal{I}})\cap\mathrm{supp}(\mathcal{J},\mu_{\mathcal{J}})\subseteq\mathrm{supp}((\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}}))$. Then by induction we have the equality:
$$(\mathcal{I}_k,\mu_{\mathcal{I}})\cdot(\mathcal{J}_k,\mu_{\mathcal{J}})=[(\mathcal{I},\mu_{\mathcal{I}})\cdot(\mathcal{J},\mu_{\mathcal{J}})]_k.$$

□

5.5. Homogenized ideals and tangent directions

Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order. Set $T(\mathcal{I}) := \mathcal{D}^{\mu-1}\mathcal{I}$. By the *homogenized ideal* we mean

$$\mathcal{H}(\mathcal{I},\mu) := (\mathcal{H}(\mathcal{I}),\mu) = (\mathcal{I} + \mathcal{D}\mathcal{I}\cdot T(\mathcal{I}) + \ldots + \mathcal{D}^i\mathcal{I}\cdot T(\mathcal{I})^i + \ldots + \mathcal{D}^{\mu-1}\mathcal{I}\cdot T(\mathcal{I})^{\mu-1},\mu).$$

Lemma 5.5.1. *Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order.*

1. *If $\mu = 1$, then $(\mathcal{H}(\mathcal{I}),1) = (\mathcal{I},1)$.*
2. $\mathcal{H}(\mathcal{I}) = \mathcal{I}+\mathcal{D}\mathcal{I}\cdot T(\mathcal{I})+\ldots+\mathcal{D}^i\mathcal{I}\cdot T(\mathcal{I})^i+\ldots+\mathcal{D}^{\mu-1}\mathcal{I}\cdot T(\mathcal{I})^{\mu-1}+\mathcal{D}^{\mu}\mathcal{I}\cdot T(\mathcal{I})^{\mu}+\ldots$.
3. $(\mathcal{H}(\mathcal{I}),\mu) = (\mathcal{I},\mu) + \mathcal{D}(\mathcal{I},\mu)\cdot(T(\mathcal{I}),1) + \ldots + \mathcal{D}^i(\mathcal{I},\mu)\cdot(T(\mathcal{I}),1)^i$
$+ \ldots + \mathcal{D}^{\mu-1}(\mathcal{I},\mu)\cdot(T(\mathcal{I}),1)^{\mu-1}$.
4. *If $\mu > 1$ then $\mathcal{D}(\mathcal{H}(\mathcal{I},\mu)) \subseteq \mathcal{H}(\mathcal{D}(\mathcal{I},\mu))$.*
5. $T(\mathcal{H}(\mathcal{I},\mu)) = T(\mathcal{I},\mu)$.

Proof. (1) $T(\mathcal{I}) = \mathcal{I}$ and $\mathcal{D}^i(\mathcal{I})T(\mathcal{I})^i \subseteq \mathcal{I}$. (2) $\mathcal{D}^{\mu-1}(\mathcal{I})T(\mathcal{I}) = T(\mathcal{I})^{\mu}$ and $\mathcal{D}^i(\mathcal{I})T(\mathcal{I})^i \subset T(\mathcal{I})^{\mu}$ for $i \geq \mu$. (3) By definition. (4) Note that $T(\mathcal{D}(\mathcal{I})) = T(\mathcal{I})$ and $\mathcal{D}(\mathcal{D}^i(\mathcal{I})T^i(\mathcal{I})^i) \subseteq \mathcal{D}^i(\mathcal{D}(\mathcal{I}))T(\mathcal{D}(\mathcal{I})) + \mathcal{D}^{i-1}(\mathcal{D}\mathcal{I})T(\mathcal{D}(\mathcal{I}))^{i-1} \subseteq \mathcal{H}(\mathcal{D}(\mathcal{I},\mu))$. (5) $T(\mathcal{I}) = \mathcal{D}^{\mu-1}(\mathcal{I}) \subseteq \mathcal{D}^{\mu-1}(\mathcal{H}(\mathcal{I})) \subseteq \mathcal{H}(\mathcal{D}^{\mu-1}(\mathcal{I})) = \mathcal{H}(T(\mathcal{I})) = T(\mathcal{I})$. □

Remark. A homogenized ideal features two important properties:

1. It is equivalent to the given ideal.
2. It "looks the same" from all possible tangent directions.

By the first property we can use the homogenized ideal to construct resolution via the Giraud Lemma 5.3.6. By the second property such a construction does not depend on the choice of tangent directions.

Lemma 5.5.2. *Let $(\mathcal{I},\mu)$ be a marked ideal of maximal order. Then*

1. $(\mathcal{I},\mu) \simeq (\mathcal{H}(\mathcal{I}),\mu)$.
2. *For any multiple test blow-up (M_k) of $(\mathcal{I},\mu)$,*
$(\mathcal{H}(\mathcal{I}),\mu)_k = (\mathcal{I},\mu)_k + [\mathcal{D}(\mathcal{I},\mu)]_k\cdot[(T(\mathcal{I}),1)]_k + \ldots + [\mathcal{D}^{\mu-1}(\mathcal{I},\mu)]_k\cdot[(T(\mathcal{I}),1)]_k^{\mu-1}$.

Proof. Since $\mathcal{H}(\mathcal{I}) \supset \mathcal{I}$, every multiple test blow-up of $\mathcal{H}(\mathcal{I},\mu)$ is a multiple test blow-up of $(\mathcal{I},\mu)$. By Lemma 5.2.4, every multiple test blow-up of $(\mathcal{I},\mu)$ is a multiple test blow-up for all $\mathcal{D}^i(\mathcal{I},\mu)$ and consequently, by Lemma 5.4.1 it is a simultaneous multiple test blow-up of all $(\mathcal{D}^i(\mathcal{I})\cdot T(\mathcal{I})^i,\mu) = (\mathcal{D}^i(\mathcal{I}),\mu-i)\cdot(T(\mathcal{I})^i,i)$ and

$$\begin{aligned} \mathrm{supp}(\mathcal{H}(\mathcal{I},\mu)_k) &= \textstyle\bigcap_{i=0}^{\mu-1}\mathrm{supp}(\mathcal{D}^i(\mathcal{I})\cdot T(\mathcal{I})^i,\mu)_k \\ &= \textstyle\bigcap_{i=0}^{\mu-1}\mathrm{supp}(\mathcal{D}^i(\mathcal{I}),\mu-i)_k\cdot(T(\mathcal{I})^i,i)_k \\ &\supseteq \textstyle\bigcap_{i=0}^{\mu-1}\mathrm{supp}(\mathcal{D}^i(\mathcal{I},\mu))_k = \mathrm{supp}(\mathcal{I}_k,\mu). \end{aligned}$$

Therefore every multiple test blow-up of $(\mathcal{I},\mu)$ is a multiple test blow-up of $\mathcal{H}(\mathcal{I},\mu)$ and by Lemmas 5.5.1(3) and 5.4.1 we get (2). □

Although the following Lemma 5.5.3 are used in this paper only in the case $E = \emptyset$ we formulate them in slightly more general versions.

Lemma 5.5.3. *(* **Glueing Lemma** *) Let $(M_Z, \mathcal{I}, E, \mu)$ be a marked ideal of maximal order. Assume there exist tangent directions $u, v \in T(\mathcal{I}, \mu)_x = \mathcal{D}^{\mu-1}(\mathcal{I}, \mu)$ at $x \in \mathrm{supp}(\mathcal{I}, \mu)$ which are transversal to E. Then there exists an open neighborhood V of x such that $\overline{V}$ is compact and an automorphism ϕ_{uv} of M_S where $S := Z \cap \overline{V}$ such that*

(1) $\phi^*_{uv}(\mathcal{HI})_{|M_S} = \mathcal{HI}_{|M_S}$.
(2) $\phi^*_{uv}(E) = E$.
(3) $\phi^*_{uv}(u) = v$.
(4) $\mathrm{supp}(\mathcal{I}, \mu) := V(T(\mathcal{I}, \mu))$ *is contained in the fixed point set of* ϕ.
(5) *Any test resolution M_{iS_i} of $(M_S, \mathcal{I}, E, \mu)$ is equivariant with respect to ϕ_{uv} and moreover the properties (1)-(4) are satisfied for the lifting $\phi_{uvi} : M_{iS_i} \to M_{iS_i}$ of $\phi_{uv} : M_S \to M_S$ and the induced marked ideal $\mathcal{HI})_i$.*

Proof. (0) **Construction of the automorphism ϕ_{uv}.**

Find coordinates $u_2, \ldots, u_n$ transversal to u and v such that $u = u_1, u_2, \ldots, u_n$ and $v, u_2, \ldots, u_n$ form two sets of coordinates at x and divisors in E are described by some coordinates u_i where $i \geq 2$. Set

$$\phi_{uv}(u_1) = v, \quad \phi_{uv}(u_i) = u_i \quad \text{for} \quad i > 1.$$

The morphism $\phi_{uv} : U \to U'$ defines an open embedding from some neighborhood U of x to another neighborhood U' of x.

(1) Let $h := v - u \in T(\mathcal{I})$. For any $f \in \mathcal{I}$,

$$\phi^*_{uv}(f) = f(u_1 + h, u_2, \ldots, u_n) = f(u_1, \ldots, u_n) + \frac{\partial f}{\partial u_1} \cdot h + \frac{1}{2!}\frac{\partial^2 f}{\partial u_1^2} \cdot h^2 + \ldots + \frac{1}{i!}\frac{\partial^i f}{\partial u_1^i} \cdot h^i + \ldots$$

The latter element belongs to

$$\mathcal{I} + \mathcal{DI} \cdot T(\mathcal{I}) + \ldots + \mathcal{D}^i\mathcal{I} \cdot T(\mathcal{I})^i + \ldots + \mathcal{D}^{\mu-1}\mathcal{I} \cdot T(\mathcal{I})^{\mu-1} = \mathcal{HI}.$$

Hence $\phi^*_{uv}(\mathcal{I}) \subset \mathcal{HI}$. Analogously $\phi^*_{uv}(\mathcal{D}^i\mathcal{I}) \subset \mathcal{D}^i\mathcal{I} + \mathcal{D}^{i+1}\mathcal{I} \cdot T(\mathcal{I}) + \ldots + \mathcal{D}^{\mu-1}\mathcal{I} \cdot T(\mathcal{I})^{\mu-i-1} = \mathcal{HD}^i I$. In particular by Lemma 5.5.1, $\phi^*_{uv}(T(\mathcal{I}), 1) \subset \mathcal{H}(T(\mathcal{I}), 1) = (T(\mathcal{I}), 1)$. This gives

$$\phi^*_{uv}(\mathcal{D}^i\mathcal{I} \cdot T(\mathcal{I})^i) \subset \mathcal{D}^i\mathcal{I} \cdot T(\mathcal{I})^i + \ldots + \mathcal{D}^{\mu-1}\mathcal{I} \cdot T(\mathcal{I})^{\mu-1} \subset \mathcal{HI}.$$

By the above $\phi^*_{uv}(\mathcal{HI})_x \subset (\mathcal{HI})_x$ and since the scheme is noetherian, $\phi^*_{uv}(\mathcal{HI})_x = (\mathcal{HI})_x$. Consequently $\phi^*_{uv}(\mathcal{HI})_y = (\mathcal{HI})_y$ for all points y in some neighborhood $V \subset U$ of x. We can assume that $\overline{V} \subset U$ is compact.

(2)(3) Follow from the construction.

(4) The fixed point set of ϕ_{uv} is defined by $u_i = \phi^*_{uv}(u_i)$, $i = 1, \ldots, n$, that is, $h = 0$. But $h \in \mathcal{D}^{\mu-1}(\mathcal{I})$ is 0 on $\mathrm{supp}(\mathcal{I}, \mu)$. In particular ϕ_{uv} defines an automorphism of M_S identical on $S = \overline{V} \cap M$.

(5) Let $C_0 \subset \mathrm{supp}(\mathcal{I}, \mu)$ be the center of σ_0. Then we can find coordinates $u'_1, u'_2, \ldots, u'_n$ transversal to $u = u'_1$ and $v = u + h$ such that C_0 is described by coordinates $u'_1 = u'_2 =$

$\ldots = u'_m = 0$ for some $m \geq 0$ or equivalently $v = u'_2 = \ldots = u'_m = 0$. By (4), the automorphism ϕ_{uv} is described by

$$\phi_{uv}(u'_i) = u'_i + h'_i, \quad \text{where} \quad h'_i \in (h) \in T(\mathcal{I}) \subset \mathcal{D}^{\mu-1}\mathcal{I}.$$

By (3), C is invariant with respect to ϕ_{uv} and it lifts to an automorphism ϕ_{uv1} of M_1. Note also that at any point $p \in \sigma_0^{-1}(x) \cap \mathrm{supp}(\mathcal{I}_1, \mu)$ there is a set of coordinates $u''_1, u''_2, \ldots, u''_n$ where $u''_i = \frac{u'_i}{u'_m}$, $u''_i = u'_i$ for $i > m$. Then the form of ϕ_{uv1} is the same as ϕ_{uv}.

$$\phi_{uv1}(u''_i) = u''_1 + h''_i, \quad \text{where} \quad h'' \in T(\mathcal{I})_1 \subset \mathcal{D}^{\mu-1}\mathcal{I}_1$$

The fixed point set of ϕ_{uv} is defined by $h'' = 0$ in a neighborhood U_p of p and it contains $\mathrm{supp}(\mathcal{I}_1, \mu) \cap U_p$. In particular all points $p \in \mathrm{supp}(\mathcal{I}_1, \mu) \cap (\sigma_1)^{-1}(x)$ are fixed under ϕ_{uv1}. Thus ϕ_{uv1} defines an automorphism of $M_{1,S_1} = \sigma_1^{-1}(M_S)$. We continue the reasoning by induction. □

5.6. Coefficient ideals and Giraud Lemma

The idea of coefficient ideals was originated by Hironaka and then developed in papers of Villamayor and Bierstone-Milman. The following definition modifies and generalizes the definition of Villamayor.

Definition 5.6.1. Let $(\mathcal{I}, \mu)$ be a marked ideal of maximal order. By the *coefficient ideal* we mean

$$\mathcal{C}(\mathcal{I}, \mu) = (\mathcal{I}, \mu) + (\mathcal{D}\mathcal{I}, \mu - 1) + \ldots + (\mathcal{D}^{\mu-1}\mathcal{I}, 1).$$

Remark. The coefficient ideals $\mathcal{C}(\mathcal{I})$ feature two important properties.

1. $\mathcal{C}(\mathcal{I})$ is equivalent to $\mathcal{I}$.
2. The intersection of the support of $(\mathcal{I}, \mu)$ with any submanifold S is the support of the restriction of $\mathcal{C}(\mathcal{I})$ to S:
$$\mathrm{supp}(\mathcal{I}) \cap S = \mathrm{supp}(\mathcal{C}(\mathcal{I})_{|S}).$$
Moreover this condition is persistent under relevant multiple test blow-ups.

These properties allow one to control and modify the part of support of $(\mathcal{I}, \mu)$ contained in S by applying multiple test blow-ups of $\mathcal{C}(\mathcal{I})_{|S}$.

Lemma 5.6.2. $\mathcal{C}(\mathcal{I}, \mu) \simeq (\mathcal{I}, \mu)$.

Proof. By Lemma 5.4.1 multiple test blow-ups of $\mathcal{C}(\mathcal{I}, \mu)$ are simultaneous multiple test blow-ups of $\mathcal{D}^i(\mathcal{I}, \mu)$ for $0 \leq i \leq \mu - 1$. By Lemma 5.2.4 multiple test blow-ups of $(\mathcal{I}, \mu)$ define a multiple test blow-up of all $\mathcal{D}^i(\mathcal{I}, \mu)$. Thus multiple test blow-ups of $(\mathcal{I}, \mu)$ and $\mathcal{C}(\mathcal{I}, \mu)$ are the same and $\mathrm{supp}(\mathcal{C}(\mathcal{I}, \mu))_k = \bigcap \mathrm{supp}(\mathcal{D}^i\mathcal{I}, \mu - i)_k = \mathrm{supp}(\mathcal{I}_k, \mu)$. □

Lemma 5.6.3. *Let $(M_Z, \mathcal{I}, E, \mu)$ be a marked ideal of maximal order whose support* $\mathrm{supp}(\mathcal{I}, \mu)$ *does not contain a submanifold S of M_Z. Assume that S has only simple normal crossings with E. Then*

$$\mathrm{supp}(\mathcal{I}, \mu) \cap S \subseteq \mathrm{supp}((\mathcal{I}, \mu)_{|S}).$$

Proof. The order of an ideal does not drop but may rise after restriction to a submanifold. □

Proposition 5.6.4. *Let $(M_Z, \mathcal{I}, E, \mu)$ be a marked ideal of maximal order whose support* $\operatorname{supp}(\mathcal{I}, \mu)$ *does not contain the germ of a submanifold S_T of M_Z. Assume that S has only simple normal crossings with E and $T := Z \cap S$. Let $E' \subset E$ be the set of divisors transversal to S. Set $E'_{|S} := \{D \cap S \mid D \in E'\}$, $\mu_c := \operatorname{lcm}(1, 2, \ldots, \mu)$, and consider the marked ideal $\mathcal{C}(\mathcal{I}, \mu)_{|S} = (S, \mathcal{C}(\mathcal{I}, \mu)_{|S}, E'_{|S}, \mu_c)$. Then*

$$\operatorname{supp}(\mathcal{I}, \mu) \cap S = \operatorname{supp}(\mathcal{C}(\mathcal{I}, \mu)_{|S}).$$

Moreover let (M_{iZ_i}) be a multiple test blow-up with centers C_i contained in the strict transforms $S_i \subset M_i$ of S. Then

1. *The restrictions $\sigma_{i|S_i} : S_{iT_i} \to S_{i-1T_{i-1}}$ of the morphisms $\sigma_i : M_{iZ_i} \to M_{i-1Z_{i-1}}$ define a multiple test blow-up (S_{iT_i}) of $\mathcal{C}(\mathcal{I}, \mu)_{|S_T}$ (where $T_i := Z_i \cap S_i$.)*
2. $\operatorname{supp}(\mathcal{I}_i, \mu) \cap S_i = \operatorname{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|S}]_i$.
3. *Every multiple test blow-up (S_{iT_i}) of $\mathcal{C}(\mathcal{I}, \mu)_{|S}$ defines a multiple test blow-up (M_{iZ_i}) of $(\mathcal{I}, \mu)$ with centers C_i contained in the strict transforms $S_{iT_i} \subset M_{iZ_i}$ of $S_T \subset M_T$.*

Proof. By Lemmas 5.6.2 and 5.6.3, $\operatorname{supp}(\mathcal{I}, \mu) \cap S = \operatorname{supp}(\mathcal{C}(\mathcal{I}, \mu)) \cap S \subseteq \operatorname{supp}(\mathcal{C}(\mathcal{I}, \mu)_{|S})$.

Let $x_1, \ldots, x_k, y_1, \ldots, y_{n-k}$ be local coordinates at p such that $\{x_1 = 0, \ldots, x_k = 0\}$ describes S. Then write a function $f \in \mathcal{I}$ can be written as

$$f = \sum c_{\alpha f}(y) x^\alpha.$$

Now $x \in \operatorname{supp}(\mathcal{I}, \mu) \cap S$ iff $\operatorname{ord}_x(c_{\alpha,f}) \geq \mu - |\alpha|$ for all $f \in \mathcal{I}$ and $0 \leq |\alpha| < \mu$. Note that

$$c_{\alpha f|S} = \left(\frac{1}{\alpha!} \frac{\partial^{|\alpha|}(f)}{\partial x^\alpha}\right)_{|S} \in \mathcal{D}^{|\alpha|}(\mathcal{I})_{|S}$$

and hence $\operatorname{supp}(\mathcal{I}, \mu) \cap S = \bigcap_{f \in \mathcal{I}, |\alpha| \leq \mu} \operatorname{supp}(c_{\alpha f|S}, \mu - |\alpha|) \supseteq \bigcap_{0 \leq i < \mu} \operatorname{supp}((\mathcal{D}^i \mathcal{I})_{|S}) = \operatorname{supp}(\mathcal{C}(\mathcal{I}, \mu)_{|S})$.

Assume that all multiple test blow-ups of $(\mathcal{I}, \mu)$ of length k with centers $C_i \subset S_i$ are defined by multiple test blow-ups of $\mathcal{C}(\mathcal{I}, \mu)_{|S}$ and moreover for $i \leq k$,

$$\operatorname{supp}(\mathcal{I}_i, \mu) \cap S_i = \operatorname{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|S}]_i.$$

For any $f \in \mathcal{I}$ define $f = f_0 \in \mathcal{I}$ and $f_{i+1} = \sigma_i^{\mathrm{c}}(f_i) = y_i^{-\mu} \sigma^*(f_i) \in \mathcal{I}_{i+1}$. Assume that

$$f_k = \sum c_{\alpha f k}(y) x^\alpha,$$

where $c_{\alpha f k|S_k} \in (\sigma^k_{|S_k})^{\mathrm{c}}(\mathcal{D}^{\mu - |\alpha|}(\mathcal{I})_{|S})$. Consider the effect of the blow-up of C_k at a point p_{k+1} in the strict transform $S_{k+1} \subset M_{k+1}$. By Lemmas 5.6.2 and 5.6.3,

$$\begin{aligned} \operatorname{supp}(\mathcal{I}_{k+1}, \mu) \cap S_{k+1} &= \operatorname{supp}[\mathcal{C}(\mathcal{I}, \mu)]_{k+1} \cap S_{k+1} \\ &\subseteq \operatorname{supp}[\mathcal{C}(\mathcal{I}, \mu)]_{k+1|S_{k+1}} = \operatorname{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|S}]_{k+1} \end{aligned}$$

Let $x_1, \ldots, x_k$ describe the submanifold S_k of M_k. We can find coordinates $x_1, \ldots, x_k$, $y_1, \ldots, y_{n-k}$ at the point p_k, by taking if necessary linear combinations of $y_1, \ldots, y_{n-k}$, such that the center of the blow-up is described by $x_1, \ldots, x_k, y_1, \ldots, y_m$ and the coordinates at p_{k+1} are given by

$$x'_1 = x_1/y_m, \ldots, x'_k = x_k/y_m, y'_1 = y_1/y_m, \ldots, y'_m = y_m, y'_{m+1} = y_{m+1}, \ldots, y'_n = y_n.$$

Note that replacing $y_1, \ldots, y_{n-k}$ with their linear combinations does not modify the form $f_k = \sum c_{\alpha f k}(y) x^\alpha$. Then the function $f_{k+1} = \sigma^{\mathrm{c}}(f_k)$ can be written as

$$f_{k+1} = \sum c_{\alpha f, k+1}(y) {x'}^\alpha,$$

where $c_{\alpha f k+1} = y_m^{-\mu+|\alpha|} \sigma^*_{k+1}(c_{\alpha f k})$. Thus

$$c_{\alpha f k+1|S_{k+1}} = (\sigma_{k+1|S_{k+1}})^{\mathrm{c}}(c_{\alpha f k|S_k}) \in (\sigma^{k+1}_{|S_{k+1}})^{\mathrm{c}}(\mathcal{D}^{\mu-|\alpha|}(\mathcal{I})_{|S}) = (\sigma^{k+1})^{\mathrm{c}}(\mathcal{D}^{\mu-|\alpha|}(\mathcal{I}))_{|S_{k+1}}$$

and consequently

$$\begin{aligned} \mathrm{supp}(\mathcal{I}_{k+1}, \mu) \cap S_{k+1} &= \bigcap_{f \in \mathcal{I}, |\alpha| \leq \mu} \mathrm{supp}(c_{\alpha f k+1|S_{k+1}}, \mu - |\alpha|) \\ &\supseteq \mathrm{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|S}]_{k+1} = \mathrm{supp}(\mathcal{C}(\mathcal{I}, \mu)_{k+1})_{|S_{k+1}}. \end{aligned}$$

□

As a simple consequence of Lemma 5.6.4 we formulate the following refinement of the Giraud Lemma.

Lemma 5.6.5. *Let $(M_Z, \mathcal{I}, \emptyset, \mu)$ be a marked ideal of maximal order whose support* $\mathrm{supp}(\mathcal{I}, \mu)$ *has codimension at least 2 at some point x. Let $U \ni x$ be an open subset for which there is a tangent direction $u \in T(\mathcal{I})$ and such that* $\mathrm{supp}(\mathcal{I}, \mu) \cap U$ *is of codimension at least 2. Let $V(u)$ be the regular subscheme of U defined by u. Then for any multiple test blow-up (M_{iZ_i}) of M_Z,*

1. $\mathrm{supp}(\mathcal{I}_i, \mu)$ *is contained in the strict transform $V(u)_{iT_i}$ of $V(u)_T$ as a proper subset (where $T = Z \cap V(u)$ and $T_i = Z_i \cap V(u)_i$).*
2. *The sequence $(V(u)_{iT_i})$ is a multiple test blow-up of $\mathcal{C}(\mathcal{I}, \mu)_{|V(u)_T}$.*
3. $\mathrm{supp}(\mathcal{I}_i, \mu) \cap V(u)_{iT_i} = \mathrm{supp}[\mathcal{C}(\mathcal{I}, \mu)_{|V(u)_T}]_i$.
4. *Every multiple test blow-up $(V(u)_{iT_i})$ of $\mathcal{C}(\mathcal{I}, \mu)_{|V(u)_T}$ defines a multiple test blow-up (M_{iZ_i}) of $(\mathcal{I}, \mu)$.*

□

6. Algorithm for canonical resolution of marked ideals

The presentation of the following resolution algorithm builds upon Villamayor's and Bierstone-Milman's proofs.

Theorem 6.0.6. *For any marked ideal $(M_Z, \mathcal{I}, E, \mu)$ such that $\mathcal{I} \neq 0$ there is an associated resolution $(M_{iZ_i}))_{0 \leq i \leq m_M}$, called canonical, satisfying the following conditions:*

1. *For any surjective local analytic isomophism $\phi : M'_{Z'} \to M_Z$ the induced sequence $(M'_{iZ'_i}) = \phi^*(M_{iZ_i})$ is the canonical resolution of M'.*
2. *For any local analytic isomophism $\phi : M' \to M$ the induced sequence $(M'_{iZ_i}) = \phi^*(M_{iZ_i})$ is an extension of the canonical resolution of $M'_{Z'}$.*

Remarks. (1) In Step 2 we resolve general marked ideals by reducing the algorithm to resolving some marked ideals of maximal order (companion ideals).

(2) In Step 1 we resolve marked ideals of maximal order. It is the heart part of the algorthm.
(3) The main idea of the algorithm of resolving marked ideals of maximal order in Step 1 is to reduce the procedure to the hypersurface of maximal contact (Step 1b).
(4) By Lemma 5.3.4 hypersurfaces of maximal contact can be constructed locally. They are in general not transversal to E and can not be used for the reduction procedure. We think of E and its strict transforms as an obstacle to existence of a hypersurface of maximal contact (transversal to E). These divisors are often referred to as "old" ones.
(5) In Step 1a we move "old" divisors apart from the support of the marked ideal. In this process we create "new" divisors but these divisors are "born" from centers lying in the hypersurface of maximal contact. The "new" divisors are transversal to hypersurfaces of maximal contact. After eliminating "old" divisors from the support in Step 1a all divisors are "new" and we may reduce the resolving procedure to hypersurfaces of maximal contact (Step 1b).

Proof. Induction on the dimension of M_Z. If M is 0-dimensional, $\mathcal{I} \neq 0$ and $\mu > 0$ then $\mathrm{supp}(M, \mathcal{I}, \mu) = \emptyset$ and all resolutions are trivial.

Step 1. Resolving a marked ideal $(M_Z, \mathcal{J}, E, \mu)$ of maximal order.

Before we start our resolution algorithm for the marked ideal $(\mathcal{J}, \mu)$ of maximal order we shall replace it with the equivalent homogenized ideal $\mathcal{C}(\mathcal{H}(\mathcal{J}, \mu))$. Resolving the ideal $\mathcal{C}(\mathcal{H}(\mathcal{J}, \mu))$ defines a resolution of $(\mathcal{J}, \mu)$ at this step. To simplify notation we shall denote $\mathcal{C}(\mathcal{H}(\mathcal{J}, \mu))$ by $(\overline{\mathcal{J}}, \overline{\mu})$.

Step 1a. Reduction to the nonboundary case. For any multiple test blow-up (M_{iZ_i}) of $(M_Z, \overline{\mathcal{J}}, E, \overline{\mu})$ we shall identify (for simplicity) strict transforms of E on M_{iZ_i} with E. For any $x \in Z_i$, let $s(x)$ denote the number of divisors in E through x and set

$$s_i = \max\{s(x) \mid x \in \mathrm{supp}(\overline{\mathcal{J}}_i) \cap Z_i\}.$$

Let $s = s_0$. By assumption the intersections of any $s > s_0$ components of the exceptional divisors are disjoint from $\mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})$. Each intersection of divisors in E on M_Z is locally defined by intersection of some irreducible components of these divisors. Find all intersections $H^s_\alpha \subset M_Z, \alpha \in A$, of s irreducible components of divisors E such that $\mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu}) \cap H^s_\alpha \cap Z \neq \emptyset$. By the maximality of s, the supports $\mathrm{supp}(\overline{\mathcal{J}}_{|H^s_\alpha}) \subset H^s_\alpha$ are disjoint from $H^s_{\alpha'}$ (in a neighborhood of Z), where $\alpha' \neq \alpha$.

Step 1aa. Eliminating the components H^s_α contained in $\mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})$.

Let $H^s_\alpha \subset \mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})$ (in a neighborhood of Z). If $s \geq 2$ then by blowing up $C = H^s_a$ we separate divisors contributing to H^s_a, thus creating new points all with $s(x) < s$. If $s = 1$ then by Lemma 5.3.7, $H^s_\alpha \subset \mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})$ is a codimension one component and by blowing up H^s_α we create all new points off $\mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})$.

Note that all $H^s_\alpha \subset \mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})$ will be blown up first and we reduce the situation to the case where no H^s_α is contained in $\mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})$.

Step 1ab. Moving $\mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})$ and H^s_α apart.

After the blow-up in Step 1aa we arrive at M_{pZ_p} for which no H^s_α is contained in $\mathrm{supp}(\overline{\mathcal{J}}_p,\overline{\mu})$ (in a neighborhood of Z), where $p=0$ if there were no such components and $p=1$ if there were some. Let $U^s_\alpha := M_p \setminus \bigcup_{\beta\neq\alpha} H^s_\beta$ $\quad Z^s_\alpha := Z\cap H^s_\alpha \cap \mathrm{supp}(\overline{\mathcal{J}}_p,\overline{\mu})$. Note that by the maximality condition for s all $H^s_\alpha \cap \mathrm{supp}(\overline{\mathcal{J}}_p,\overline{\mu})$ are disjoint for two different $\alpha\in A_s$. By definition $Z^s_\alpha \subset \mathrm{supp}(\overline{\mathcal{J}}_p,\overline{\mu})\cap H^s_\alpha \subset U^s_\alpha$ is compact. Set

$$\widetilde{Z}^s = \coprod Z^s_\alpha \quad Z^s = \bigcup Z^s_\alpha = Z\cap \mathrm{supp}(\overline{\mathcal{J}}_p,\overline{\mu}) \quad \widetilde{M}_p := \coprod U^s_\alpha \qquad \widetilde{H^s} := \coprod H^s_\alpha \cap U^s_\alpha$$

Consider the surjective local analytic isomorphism $\phi : \widetilde{M_p} := \coprod U^s_\alpha \to M_p$. Note that Z^s_α is disjoint from $U^s_{\alpha'}$, where $\alpha'\neq\alpha$. The morphism ϕ defines a morphisms of germs $\phi_Z : \widetilde{M}_{\widetilde{Z^s}} \to M_{Z^s}$ which is locally an isomorphism

$$\widetilde{M}_{\widetilde{Z^s}} \supseteq \phi^{-1}(U^s_{\alpha Z^s_\alpha}) \simeq U^s_{\alpha Z^s_\alpha} \subseteq M_{Z^s}.$$

Denote by $\widetilde{J}$ the pull back of the ideal sheaf $\mathcal{J}$ via ϕ_Z. The closed embeddings $H^s_\alpha \cap U^s_\alpha \subset U^s_\alpha$ define the closed embedding $\widetilde{H^s}\subset\widetilde{M}$. Let $Z_H := Z\cap H$.

Construct by the inductive assumption the canonical resolution $(\widetilde{H}^{s_i}_{iZ_{Hi}})$ of $\widetilde{\mathcal{J}}_{p|\widetilde{H^s}}$. By Proposition 5.6.4 such a resolution defines a multiple test blow-up $(\widetilde{M}_{iZ_i})$ of $(\widetilde{\mathcal{J}}_p,\overline{\mu})$ (and of $(\overline{\mathcal{J}},\overline{\mu})$). By Proposition 5.6.4,

$$\mathrm{supp}((\widetilde{\mathcal{J}}_i,\overline{\mu})_{|\widetilde{H}^s}) = \mathrm{supp}(\widetilde{\mathcal{J}}_i,\overline{\mu})\cap \widetilde{H}^s.$$

Descending the multiple test blow-up to M_{Z^s}, defines a multiple test blow-up of $(\overline{\mathcal{J}}_i,\overline{\mu})$ such that

$$\mathrm{supp}((\overline{\mathcal{J}}_i,\overline{\mu})_{|H^s_\alpha}) = \mathrm{supp}(\overline{\mathcal{J}}_i,\overline{\mu})\cap H^s_\alpha.$$

This creates a marked ideal $(\overline{\mathcal{J}}_{j_1},\overline{\mu})$ with support disjoint from all H^s_α.

Conclusion of the algorithm in Step 1a. After performing the blow-ups in Steps 1aa and 1ab for the marked ideal $(\overline{\mathcal{J}},\overline{\mu})$ we arrive at a marked ideal $(\overline{\mathcal{J}}_{j_1},\overline{\mu})$ with $s_{j_1} < s_0$. Now we put $s=s_{j_1}$ and repeat the procedure of Steps 1aa and 1ab for $(\overline{\mathcal{J}}_{j_1},\overline{\mu})$. Note that any $H^s_{\alpha j_1}$ on M_{j_1} is the strict transform of some intersection $H^{s_{j_1}}_\alpha$ of $s=s_{j_1}$ divisors in E on M. Moreover by the maximality condition for all s_i, where $i\leq j_1$ and $\alpha\neq\alpha'$, the set $\mathrm{supp}(\overline{\mathcal{J}}_i,\overline{\mu})\cap H^{s_i}_{\alpha' i}$ is either disjoint from $H^{s_{j_1}}_{\alpha i}$ or contained in it. Thus for $0\leq i\leq j_1$, all centers C_i have components either contained in $H^{s_{j_1}}_{\alpha i} = H^s_{\alpha i}$ or disjoint from them and by Proposition 5.6.4,

$$\mathrm{supp}((\overline{\mathcal{J}}_i,\overline{\mu})_{|H^s_{\alpha i}}) = \mathrm{supp}(\overline{\mathcal{J}}_i,\overline{\mu})\cap H^s_{\alpha i}.$$

Moreover if we repeat the procedure in Steps 1aa and 1ab the above property will still be satisfied until either $(\overline{\mathcal{J}}_i,\overline{\mu})_{|H^s_\alpha}$ are resolved as in Step 1ab or H^s_α disappear as in Step 1aa.

We continue the above process till $s_{j_k} = s_r = 0$. Then $(M_j)_{0\leq j\leq r}$ is a multiple test blow-up of $(M,\overline{\mathcal{J}},E,\overline{\mu})$ such that $\mathrm{supp}(\overline{\mathcal{J}}_r,\overline{\mu})$ does not intersect any divisor in E. Therefore $(M_j)_{0\leq j\leq r}$ and further longer multiple test blow-ups $(M_j)_{0\leq j\leq r_0}$ for any $r\leq r_0$ can be considered as multiple test blow-ups of $(M,\overline{\mathcal{J}},\emptyset,\overline{\mu})$ since starting from M_r the strict transforms of E play no further role in the resolution process since they do not intersect $\mathrm{supp}(\overline{\mathcal{J}}_j, \overline{\mu})$ for $j \geq r$.

Step 1b. **Nonboundary case**

Let $(M_{jZ_j})_{0\leq j\leq r}$ be the multiple test blow-up of $(M, \overline{\mathcal{J}}, \emptyset, \overline{\mu})$ defined in Step 1a.

Step 1ba. **Eliminating the codimension one components of** $\mathrm{supp}(\overline{\mathcal{J}}_r, \overline{\mu})$**.**

If $\mathrm{supp}(\overline{\mathcal{J}}_r, \overline{\mu})$ is of codimension 1 then by Lemma 5.3.7 all its codimension 1 components are smooth and disjoint from the other components of $\mathrm{supp}(\overline{\mathcal{J}}_r, \overline{\mu})$. These components are strict transforms of the codimension 1 components of $\mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})$. Moreover the irreducible components of the centers of blow-ups were either contained in the strict transforms or disjoint from them. Therefore E_r will be transversal to all the codimension 1 components. Let $\mathrm{codim}(1)(\mathrm{supp}(\overline{\mathcal{J}}_i, \overline{\mu}))$ be the union of all components of $\mathrm{supp}(\overline{\mathcal{J}}_i, \overline{\mu}))$ of codimension 1. By Lemma 5.3.7 blowing up the components reduces the situation to the case when $\mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})$ is of codimension ≥ 2.

Step 1bb. **Eliminating the codimension** ≥ 2 **components of** $\mathrm{supp}(\overline{\mathcal{J}}_r, \overline{\mu})$**.**

For any $x \in Z \cap \mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu}) \setminus \mathrm{codim}(1)(\mathrm{supp}(\overline{\mathcal{J}}, \overline{\mu})) \subset M_Z$ find a tangent direction $u_\alpha \in \mathcal{D}^{\overline{\mu}-1}(\overline{\mathcal{J}})$ on some neighborhood U_α of x. Then $H_\alpha := V(u_\alpha) \subset U_\alpha$ is a hypersurface of maximal contact. Take a finite open covers (U_α) and (V_α) of Z such that the ideal sheaf is defined on each U_α, $\overline{V_\alpha} \subset U_\alpha$ is compact , and U_α satisfies the property of Glueing Lemma. Let $Z_\alpha := Z \cap \overline{V_\alpha}$ and $Z_{V,\alpha} \subset Z_\alpha$ be any compact set contained in $Z \cap V_\alpha$. Set

$$\widetilde{V} := \coprod \overline{V}_\alpha \qquad \widetilde{Z}_V := \coprod Z_{V\alpha} \quad \widetilde{M} := \coprod U_\alpha \qquad \widetilde{Z} := \coprod Z_\alpha \quad \widetilde{H} := \coprod H_\alpha \subseteq \widetilde{M}$$

The closed embeddings $H_\alpha \subseteq U_\alpha$ define the closed embedding $\widetilde{H} \subset \widetilde{M}$ of a hypersurface of maximal contact $\widetilde{H}$.

Consider the surjective local analytic isomorphism

$$\phi_U : \widetilde{M} := \coprod U^s_\alpha \to M.$$

It defines a morphism of germs $\phi_{Z_V} : \widetilde{M}_{\widetilde{Z}} \to M_{\overline{Z}}$. Denote by $\widetilde{J}$ the pull back of the ideal sheaf $\overline{\mathcal{J}}$ via ϕ_U. The multiple test blow-up $(M_{iZ_i})_{0\leq i\leq p}$ of $\overline{J}$ defines a multiple test blow-up $(\widetilde{M}_{\widetilde{Z}_i i})_{0\leq i\leq p}$ of $\widetilde{J}$ and a multiple test blow-up $(\widetilde{H}_i)_{0\leq i\leq p}$ of $\widetilde{J}_H$.

Let $U_{\alpha,i} \subset M_i$ be the inverse image of U_α and let $H_{\alpha i} \subset U_{\alpha i}$ denote the strict transform of H_α. By Lemma 5.6.5, $(H_{\alpha i})_{0\leq i\leq p}$ is a multiple test blow-up of $(H_\alpha, \overline{\mathcal{J}}_{|H_\alpha}, \emptyset, \overline{\mu})$. In particular the induced marked ideal for $i = p$ is equal to

$$\overline{\mathcal{J}}_{p|H_{\alpha p}} = (H_{\alpha p}, \overline{\mathcal{J}}_{p|H_{\alpha p}}, (E_p \setminus E)_{|H_{\alpha p}}, \overline{\mu}).$$

Construct the canonical resolution of $(\widetilde{H}_{iZ_i})_{p\leq i\leq m_u}$ of the marked ideal $\widetilde{\mathcal{J}}_{p|\widetilde{H}_p}$ on $\widetilde{H}_{\widetilde{Z}}$. It defines, by Lemma 5.6.5, a resolution $(\widetilde{M}_{i\widetilde{Z}_i})_{p\leq i\leq m}$ of $\widetilde{\mathcal{J}}_p$ and thus also a resolution $(\widetilde{M}_{i\widetilde{Z}_i})_{0\leq i\leq m}$ of $(\widetilde{M_Z}, \widetilde{\mathcal{J}}, \emptyset, \overline{\mu})$. Moreover both resolutions are related by the property

$$\mathrm{supp}(\widetilde{\mathcal{J}}_i) = \mathrm{supp}(\widetilde{\mathcal{J}}_{i|\widetilde{H}_i}).$$

The resolution $(\widetilde{M}_{i\widetilde{Z}_i})_{0\leq i\leq m}$ defines the canonical resolution $(\widetilde{V}_{i\widetilde{Z}_{Vi}})_{0\leq i\leq m}$ for any compact $Z_V \subset \widetilde{V}$.

Consider the surjective local analytic isomorphism

$$\phi_V : \widetilde{V} := \coprod V_\alpha \to M.$$

We have to show that the resolution $(\widetilde{V}_{i\widetilde{Z_{Vi}}})_{0\leq i\leq m}$ descends to the resolution $(M_{iZ_i})_{0\leq i\leq m}$ which is independent of the choice of local hypersurfaces of maximal contact and $\widetilde{M}$. We show by induction that there exists a resolution $(M_{iZ_{Vi}}))_{0\leq i\leq m}$ such that its restriction $((H_\alpha)_{iZ_{V}{}^{\alpha}_{i}}))_{k\leq i\leq m}$ is an extension of the part of the canonical resolution.

Consider the inverse image

$$\phi_j^{-1}(V_{\beta,i}) = \coprod V_{\beta,j} \cap V_{\alpha,j}.$$

Let $\widetilde{C}_j$ be the center of the blow-up $\widetilde{\sigma}_j : \widetilde{V}_{j+1} \to \widetilde{V}_j$. If $\widetilde{C}_j \cap V_{\beta,j} \cap V_{\alpha,j} \neq \emptyset$ then $\widetilde{C}_j \cap V_{\beta,j}$ defines the center of an extension of the part of the canonical resolution $((H_{\beta j})_{Z_{V\beta j}})_{p\leq j\leq m}$. By the canonicity the intersection $\widetilde{C}_j \cap V_{\beta,j} \cap V_{\alpha,j}$ defines the center of an extension of the part of the canonical resolution $((H_{\beta j} \cap V_{\alpha j})_{Z_{V\beta j}\cap Z_{V\beta j}})_{p\leq j\leq m}$.

By Glueing Lemma 5.5.3 for the tangent directions u_α and u_β we find an automorphism $\phi_{i\alpha\beta}$ of $(U_{\beta i} \cap U_{\alpha i})_{Z_{\beta j}\cap Z_{\beta j}}$ and its restriction to $(V_\alpha \cap V_\beta)_{Z_{V\beta j}\cap Z_{V\beta j}})_{p\leq j\leq m}$.

such that

(1) $(\phi_{i\alpha\beta})(H_{\alpha i}) = H_{\beta i}$.
(2) $\phi_{\alpha\beta i}$ is the identity for $\mathrm{supp}(\overline{\mathcal{J}}_i)$
(3) $\phi_{\alpha\beta i}$ preserves the marked ideal $\overline{J}_i$
(4) $\phi_{i\alpha\beta}(\overline{J}_{i|H_{\alpha i}}) = \overline{J}_{i|H_{\beta i}}$

Its restriction to $(V_\alpha \cap V_\beta)_{Z_{V\alpha}\cap Z_{V\beta}}$ defines an automoprhism for any compact $Z_{V\alpha} \subset Z \cap V_\alpha$ and $Z_{V\beta} \subset Z \cap V_\beta$. By the above $\widetilde{C}_j \cap (V_{\beta,j} \cap V_{\alpha,i})_{Z_{V\beta j}\cap Z_{V\beta j}}$ is the center of the canonical resolution of $\overline{J}_{i|H_{\beta i}}$ and of $\overline{J}_{i|H_{\alpha i}}$. Thus the restriction of the natural embedding $\widetilde{C}_j \cap (V_{\beta,j} \cap V_{\alpha,i})_{Z_{V\alpha j}\cap Z_{V\beta j}} \subset (\widetilde{C}_j \cap V_{\alpha,i})_{Z_{V\alpha i}}$ is an open embedding and $\widetilde{C}_j$ descends to a smooth center $C_j := \bigcup \widetilde{C}_j \cap V_{\alpha,jZ_{V\alpha j}} \subset \bigcup V_{\alpha j Z_{V\alpha j}} = M_{jZ_j}$.

Step 2. Resolving marked ideals $(M_Z, \mathcal{I}, E, \mu)$**.**

For any marked ideal $(M_Z, \mathcal{I}, E, \mu)$ write

$$I = \mathcal{M}(\mathcal{I})\mathcal{N}(\mathcal{I}),$$

where $\mathcal{M}(\mathcal{I})$ is the *monomial part* of $\mathcal{I}$, that is, the product of the principal ideals defining the irreducible components of the divisors in E, and $\mathcal{N}(\mathcal{I})$ is a *nonmonomial part* which is not divisible by any ideal of a divisor in E. Let

$$\mathrm{ord}_{\mathcal{N}(\mathcal{I})} := \max\{\mathrm{ord}_x(\mathcal{N}(\mathcal{I})) \mid x \in Z \cap \mathrm{supp}(\mathcal{I}, \mu)\}.$$

Definition 6.0.7. (Hironaka, Bierstone-Milman, Villamayor, Encinas-Hauser) By the *companion ideal* of $(\mathcal{I},\mu)$ where $I=\mathcal{N}(\mathcal{I})\mathcal{M}(\mathcal{I})$ we mean the marked ideal of maximal order

$$O(\mathcal{I},\mu)=\begin{cases}(\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})})+(\mathcal{M}(\mathcal{I}),\mu-\mathrm{ord}_{\mathcal{N}(\mathcal{I})}) & \text{if } \mathrm{ord}_{\mathcal{N}(\mathcal{I})}<\mu,\\ (\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}) & \text{if } \mathrm{ord}_{\mathcal{N}(\mathcal{I})}\geq\mu.\end{cases}$$

Step 2a. **Reduction to the monomial case by using companion ideals.**

By Step 1 we can resolve the marked ideal of maximal order $(\mathcal{J},\mu_{\mathcal{J}}):=O(\mathcal{I},\mu)$. By Lemma 5.4.1, for any multiple test blow-up of $O(\mathcal{I},\mu)$,

$$\begin{aligned}\mathrm{supp}(O(\mathcal{I},\mu))_i&=\mathrm{supp}[\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}]_i\ \cap\ \mathrm{supp}[M(\mathcal{I}),\mu-\mathrm{ord}_{\mathcal{N}(H\mathcal{I})}]_i\\&=\mathrm{supp}[\mathcal{N}(\mathcal{I}),\mathrm{ord}_{\mathcal{N}(\mathcal{I})}]_i\ \cap\ \mathrm{supp}(\mathcal{I}_i,\mu).\end{aligned}$$

Consequently, such a resolution leads to the ideal $(\mathcal{I}_{r_1},\mu)$ such that $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_1})}<\mathrm{ord}_{\mathcal{N}(\mathcal{I})}$.Then we repeat the procedure for $(\mathcal{I}_{r_1},\mu)$. We find marked ideals $(\mathcal{I}_{r_0},\mu)=(\mathcal{I},\mu),(\mathcal{I}_{r_1},\mu),\ldots,(\mathcal{I}_{r_m},\mu)$ such that $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_0)}>\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_1})}>\ldots>\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_m})}$. The procedure terminates after a finite number of steps when we arrive at the ideal $(\mathcal{I}_{r_m},\mu)$ with $\mathrm{ord}_{\mathcal{N}(\mathcal{I}_{r_m})}=0$ or with $\mathrm{supp}(\mathcal{I}_{r_m},\mu)=\emptyset$. In the second case we get the resolution. In the first case $\mathcal{I}_{r_m}=\mathcal{M}(\mathcal{I}_{r_m})$ is monomial.

Step 2b. **Monomial case $\mathcal{I}=\mathcal{M}(\mathcal{I})$.**

Let $\mathrm{Sub}(E_i)$ denote the set of all subsets of E_i. For any subset in $\mathrm{Sub}(E_i)$ write a sequence $(D_1,D_2,\ldots,0,\ldots)$ consisting of all elements of the subset in increasing order followed by an infinite sequence of zeros. We shall assume that $0\leq D$ for any $D\in E_i$. Consider the lexicographic order $\leq$ on the set of such sequences. Then for any two subsets $A_1=\{D^1_i\}_{i\in I}$ and $A_2=\{D^2_j\}_{j\in J}$ we write

$$A_1\leq A_2$$

if for the corresponding sequences $(D^1_1,D^1_2,\ldots,0,\ldots)\leq(D^2_1,D^2_2,\ldots,0,\ldots)$.

Let $x_1,\ldots,x_k$ define equations of the components $D^x_1,\ldots,D^x_k\ \in\ E$ through $x\in\mathrm{supp}(M_Z,\mathcal{I},E,\mu)$ and $\mathcal{I}$ be generated by the monomial $x^{a_1,\ldots,a_k}$ at x. Note that $\mathrm{ord}_x(\mathcal{I})=a_1+\ldots+a_k$.

Let $\rho(x)=\{D_{i_1},\ldots,D_{i_l}\}\in\mathrm{Sub}(E)$ be the maximal subset satisfying the properties

(1) $a_{i_1}+\ldots+a_{i_l}\geq\mu$.
(2) For any $j=1,\ldots,l$, $a_{i_1}+\ldots+\check{a}_{i_j}+\ldots+a_{i_l}<\mu$.

Let $R(x)$ denote the subsets in $\mathrm{Sub}(E)$ satisfying the properties (1) and (2). The maximal components of $\mathrm{supp}(\mathcal{I},\mu)$ through x are described by the intersections $\bigcap_{D\in A}D$ where $A\in R(x)$. The maximal locus of ρ determines at most one maximal component of $\mathrm{supp}(\mathcal{I},\mu)$ through each x.

After the blow-up at the maximal locus $C=\{x_{i_1}=\ldots=x_{i_l}=0\}$ of ρ, the ideal $\mathcal{I}=(x^{a_1,\ldots,a_k})$ is equal to $\mathcal{I}'=(x'^{a_1,\ldots,a_{i_j-1},a,a_{i_j+1},\ldots,a_k})$ in the neighborhood corresponding to x_{i_j}, where $a=a_{i_1}+\ldots+a_{i_l}-\mu<a_{i_j}$. In particular the invariant ν drops for all points of some maximal components of $\mathrm{supp}(\mathcal{I},\mu)$. Thus the maximal value of ν on the maximal components of $\mathrm{supp}(\mathcal{I},\mu)$ which were blown up is bigger than the maximal value

of $\mathrm{ord}_x(\mathcal{I})$ on the new maximal components of $\mathrm{supp}(\mathcal{I},\mu)$. It follows that the algorithm terminates after a finite number of steps. □

Remarks. (1) (*) The ideal $\mathcal{J}$ is replaced with $\mathcal{H}(\mathcal{J})$ to ensure that the algorithm in Step 1b is independent of the choice of the tangent direction u. We replace $\mathcal{H}(\mathcal{J})$ with $\mathcal{C}(\mathcal{H}(\mathcal{J}))$ to ensure the equalities $\mathrm{supp}(\mathcal{J}_{|S}) = \mathrm{supp}(\mathcal{J}) \cap S$, where $S = H^s_\alpha$ in Step 1a and $S = V(u)$ in Step 1b.

(2) If $\mu = 1$ the companion ideal is equal to $O(\mathcal{I},1) = (\mathcal{N}(\mathcal{I}), \mu_{\mathcal{N}(\mathcal{I})})$ so the general strategy of the resolution of $\mathcal{I},\mu$ is to decrease the order of the nonmonomial part and then to resolve the monomial part.

(3) In particular if we desingularize Y we put $\mu = 1$ and $\mathcal{I} = \mathcal{I}_Y$ to be equal to the sheaf of the submanifold Y and we resolve the marked ideal $(M_Z, \mathcal{I}, \emptyset, \mu)$. The nonmonomial part $\mathcal{N}(\mathcal{I}_i)$ is nothing but the weak transform $(\sigma^i)^{\mathrm{w}}(\mathcal{I})$ of $\mathcal{I}$.

7. Conclusion of the resolution algorithm

7.1. Commutativity of resolving marked ideals $(M_Z, \mathcal{I}, \emptyset, 1)$ with embeddings of ambient manifolds

Let $(M_Z, \mathcal{I}, \emptyset, 1)$ be a marked ideal and $\phi : M_Z \hookrightarrow M'_Z$ be a closed embedding of germs of manifolds. Then ϕ defines the marked ideal $(M'_Z, \mathcal{I}', \emptyset, 1)$, where $\mathcal{I}' = \phi_*(\mathcal{I}) \cdot \mathcal{O}_{M'_Z}$ (see remark after Theorem 2.0.1). We may assume that M_Z is a germ of the submanifold M of M' which is locally generated by coordinates $u_1, \ldots, u_k$. Then $u_1, \ldots, u_k$ in $\mathcal{I}'(U') = T(\mathcal{I})(U')$ define tangent directions on some open $U' \subset M'_Z$. We run Steps 2a and 1bb of our algorithm. That is, we pass to the hypersurface $V(u_1)$ and replace $\mathcal{I}$ with its restriction. By Step 1bb resolving $(M'_Z, \mathcal{I}', \emptyset, \mu)$ is locally equivalent to resolving $(V(u_1)_Z, \mathcal{I}'_{|V(u_1)}, \emptyset, \mu)$.

By repeating the procedure k times and restricting to the tangent directions $u_1, \ldots, u_k$ of the marked ideal $\mathcal{I}$ on M_Z we obtain:

Resolving $(M'_Z, \mathcal{I}', \emptyset, \mu)$ is equivalent to resolving $(M_Z, \mathcal{I}, \emptyset, \mu)$.

7.2. Principalization

Resolving the marked ideal $(M_Z, \mathcal{I}, \emptyset, 1)$ determines a principalization commuting with local analytic isomorphisms and embeddings of the ambient manifolds.

The principalization is often reached at an earlier stage upon transformation to the monomial case (Step 2b) (However the latter procedure does not commute with embeddings of ambient manifolds)

7.3. Weak embedded desingularization

Let Y be a closed subspace of the germ M_Z. Consider the marked ideal $(M_Z, \mathcal{I}_Y, \emptyset, 1)$. Its support $\mathrm{supp}(\mathcal{I}_Y, 1)$ is equal to Y. In the resolution process of $(M_Z, \mathcal{I}_Y, \emptyset, 1)$, the strict transform of Y is blown up. Otherwise the generic points of Y would be transformed isomorphically, which contradicts the resolution of $(M_Z, \mathcal{I}_Y, \emptyset, 1)$.

7.4. Bravo-Villamayor strengthening of the weak embedded desingularization

Theorem 7.4.1. *(Bravo-Villamayor* [13], [11]*) Let Y be a closed subspace of a manifold M and $Y = \bigcup Y_i$ be its decomposition into the union of irreducible components. There is a canonical locally finite resolution of a subspace $Y \subset M$, subject to the conditions from Theorem 2.0.2 such that the strict transforms $\widetilde{Y}_i$ of Y_i are smooth and disjoint. Moreover the full transform of Y is of the form*

$$(\widetilde{\sigma})^*(\mathcal{I}_Y) = \mathcal{M}((\widetilde{\sigma})^*(\mathcal{I}_Y)) \cdot \mathcal{I}_{\widetilde{Y}},$$

where $\widetilde{Y} := \bigcup \widetilde{Y}_i \subset \widetilde{M_Z}$ is a disjoint union of the strict transforms $\widetilde{Y}_i$ of Y_i, $\mathcal{I}_{\widetilde{Y}}$ is the sheaf of ideals of $\widetilde{Y}$ and $\mathcal{M}((\widetilde{\sigma})^*(\mathcal{I}_Y))$ is the monomial part of $(\widetilde{\sigma})^*(\mathcal{I}_Y)$.

Proof. Let $\mathcal{I} := \mathcal{I}_Y$ be the ideal sheaf of Y. Fix any compact set $Z \subset M$.

We use the following:

Modified algorithm in Step 2. We run the algorithm of resolving $(M_Z, \mathcal{I}, \emptyset, 1)$ as before until we drop the $\max\{\mathrm{ord}_x(\mathcal{N}(\mathcal{I})) : x \in \mathrm{supp}(\mathcal{I})\}$ to 1. The control transform of $(\mathcal{I}, 1)$ becomes equal to $(\mathcal{M}(\mathcal{I})\mathcal{N}(\mathcal{I}), 1)$. At this point algorithm is altered. We resolve the monomial ideal $(\mathcal{M}(\mathcal{I}), 1)$. The blow-ups are performed at exceptional divisors for which $\rho(x)$ is maximal. We arrive at the purely nonmonomial case $\mathcal{I}' = \mathcal{N}(\mathcal{I}')$, where $\max\{\mathrm{ord}_x(\mathcal{I}) : x \in \mathrm{supp}(\mathcal{I}') \cap Z\} = 1$. This concludes the altered procedure in Step 2. At this point we perform the altered Step 1 described below.

Modified algorithm in Step 1. In Step 1a we move the "old divisors" E as before. In Step 1b we consider two possibilities. If $\mathcal{I}' = (u)$ is the ideal of smooth hypersurface of (maximal contact) as in Step 1ba) the algorithm is stopped. Otherwise we restrict $(\mathcal{I}', 1) = \mathcal{C}(\mathcal{H}(\mathcal{I}))$ to a hypersurface of maximal contact $V(u_1)$.

The modified algorithm in Step 2 and Step 1 is then repeated for the restriction $\mathcal{I}'_{|}V(u_1)$.

We continue this procedure until it terminates. Then the resulting controlled transform of $(\mathcal{I}, 1)$ is locally equal to $\mathcal{I}'' = (u_1, \ldots, u_k)$, where u_i are coordinates transversal to exceptional divisors. The sheaf $\mathcal{I}''$ desribes the germ of submanifold which is a union of disjoint irreducible components. Some of them are the strict transforms of Y_i. Other components are possible strict transforms of embedding components occuring the process. At the end we blow-up all the irreducible components which are not strict transforms of Y_i. The procedure is canonical. It is defined for germs of analytic subspace at compact sets and it glues to the algorthm for whole subspace of manifolds.

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