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Construction of symplectic cohomology $S^2 \times S^2$

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Abstract. In this article, we present symplectic 4-manifolds with the same integral cohomology as $S^2 \times S^2$. A generalization of this construction is given as well, an infinite family of symplectic 4-manifolds cohomology equivalent to $\#_{(2g-1)}(S^2 \times S^2)$ for any $g \geq 2$. We also compute the Seiberg-Witten invariants of the 4-manifolds we construct.

1. Introduction

The aim of this article is to construct examples of symplectic 4-manifolds with the same integral cohomology as $S^2 \times S^2$. Similar problems have been studied in the algebro-geometric category, i.e. existence of algebraic surfaces homology equivalent but not isomorphic to $\mathbf{P^2}$ (or $\mathbf{P^1} \times \mathbf{P^1}$) as an algebraic variety. D. Mumford [Mu] and R. Pardini [P] gave the constructions of such fake $\mathbf{P^2}$ and $\mathbf{P^1} \times \mathbf{P^1}$.

We study this problem in the symplectic category. Our main results are the following two theorems.

Theorem 1.1. *Let K be a genus one fibered knot in S^3. Then there exist a minimal symplectic 4-manifold X_K cohomology equivalent to $S^2 \times S^2$.*

Theorem 1.2. *Let K be a genus one and K' be any genus $g \geq 2$ fibered knot in S^3. Then there exist an infinite family of minimal symplectic 4-manifolds $V_{KK'}$ that is cohomology equivalent to $\#_{(2g-1)}(S^2 \times S^2)$.*

This paper is organized as follows: Section 2 contains the basic definitions and formulas that will be important throughout this paper. Section 3 gives a quick introduction to Seiberg-Witten invariants. The remaining two sections are devoted to the construction of family of symplectic 4-manifolds cohomology equivalent to $\#_{(2g-1)}(S^2 \times S^2)$ and the fundamental group computation for our examples.

2. Preliminaries

2.1. Fibered Knots

In this section, we give a short introduction to fibered knots and state a few facts that will be needed in our construction. We refer the reader to Section 10.H [R] for a more complete treatment.

Definition 2.1. Let p and q be relatively prime positive integers. The knot which wraps around the solid torus in the longitudinal direction p times and in the meridianal direction q times is called the (p, q) *torus knot* and denoted as $T_{p,q}$.

Lemma 2.1. [S] *a) The group of the torus knot $T_{p,q}$ can be represented as follows:*

$$\pi_1(S^3 \setminus T_{p,q}) \quad = \quad \langle u, v \mid u^p = v^q \rangle$$

b) The elements $m = u^s v^r$, $l = u^p m^{-pq}$, where $pr + qs = 1$, describe meridian and longitude of the $T_{p,q}$ for a suitable chosen basepoint.

All torus knots belong to the larger category of fibered knots.

Definition 2.2. A knot K in S^3 is called *fibered* if there is fibration $f : S^3 \setminus K \longrightarrow S^1$ behaving "nicely" near K. This means that K has a neighbourhood framed as $S^1 \times D^2$, with $K \cong S^1 \times 0$ and restriction of the map f to $S^1 \times (D^2 - 0)$ is the map to S^1 given by $(t, x) \to x/|x|$.

It follows from the definition that a preimage for each point $p \in S^1$ is the Seifert surface for the given knot. The genus of this Seifert surface will be called the genus of the given fibered knot.

The fibered knots form a large class among the all classical knots. Below we state two theorems that can be used to detect if the given knot is fibered or not.

Theorem 2.2. [S] *The knot $K \subset S^3$ is a fibered knot of genus g if and only if the commutator subgroup of its knot group $\pi_1(S^3 \backslash K)$ is finitely-generated and free group of rank $2g$.*

Theorem 2.3. [BZ] *The Alexander polynomial $\Delta_K(t)$ of a fibered knot in S^3 is monic, i.e. the first and the last non-zero coefficients of $\Delta_K(t)$ are ± 1.*

If a genus one knot is fibered, then it can be shown by above theorems and also by explicit construction [BZ] that it must be either the trefoil or the figure eight knot. Also, one can construct infinitely many fibered knots for a fixed genus $g \geq 2$.

2.2. Generalized fiber sum

Definition 2.3. Let X and Y be closed, oriented, smooth 4-manifolds each containing a smoothly embedded surface Σ of genus $g \geq 1$. Assume Σ represents a homology class of infinite order and has self-intersection zero in X and Y, so that there exists a product tubular neighborhood, say $\nu\Sigma \cong \Sigma \times D^2$, in both X and Y. Using an orientation-reversing and fiber-preserving diffeomorphism $\psi : \Sigma \times S^1 \to \Sigma \times S^1$, we can glue $X \setminus \nu\Sigma$ and $Y \setminus \nu\Sigma$ along the boundary $\partial(\nu\Sigma) \cong \Sigma \times S^1$. The resulting closed oriented smooth 4-manifold, denoted by $X \#_\psi Y$, is called a *generalized fiber sum* of X and Y along Σ, determined by ψ.

Definition 2.4. Let $e(X)$ and $\sigma(X)$ denote the Euler characteristic and the signature of a closed oriented smooth 4-manifold X, respectively. We define

$$c_1^2(X) := 2e(X) + 3\sigma(X), \quad \chi_h(X) := \frac{e(X) + \sigma(X)}{4}.$$

Lemma 2.4. *Let X and Y be closed, oriented, smooth 4-manifolds containing an embedded surface Σ of self-intersection 0. Then*

$$\begin{aligned} c_1^2(X \#_\psi Y) &= c_1^2(X) + c_1^2(Y) + 8(g-1), \\ \chi_h(X \#_\psi Y) &= \chi_h(X) + \chi_h(Y) + (g-1), \end{aligned}$$

where g is the genus of the surface Σ.

Proof. The above simply follow from the well-known formulas

$$e(X \#_\psi Y) = e(X) + e(Y) - 2e(\Sigma), \quad \sigma(X \#_\psi Y) = \sigma(X) + \sigma(Y). \qquad \square$$

If X and Y are symplectic 4-manifolds and Σ is a symplectic submanifold in both, then according to a theorem of Gompf [Go], $X \#_\psi Y$ admits a symplectic structure. In such a case, we will call $X \#_\psi Y$ a *symplectic sum*.

To show the minimality of our symplectic manifolds, we use the following theorem of M. Usher [U]. In order to state his theorem, we slightly abuse the notation for the fiber sum above.

Theorem 2.5. [U] **(Minimality of Symplectic Sums)** *Let $Z = X_1 \#_{F_1 = F_2} X_2$ be sympletic fiber sum of manifolds X_1 and X_2. Then:*

(i) If either $X_1 \backslash F_1$ or $X_2 \backslash F_2$ contains an embedded sympletic sphere of square -1, then Z is not minimal.

(ii) If one of the summands X_i (say X_1) admits the structure of an S^2-bundle over a surface of genus g such that F_i is a section of this fiber bundle, then Z is minimal if and only if X_2 is minimal.

(iii) In all other cases, Z is minimal.

3. Seiberg-Witten Invariants

In this section, we review the basics of Seiberg-Witten invariants introduced by Seiberg and Witten. Let us recall that the Seiberg-Witten invariant of a smooth closed oriented

4-manifold X with $b_2^+(X) > 1$ is an integer valued function which is defined on the set of *spin*^{c} structures over X [W]. For simplicity, we assume that $H_1(X,\mathbb{Z})$ has no 2-torsion. Then there is a natural identification of the *spin*^{c} structures of X with the characteristic elements of $H^2(X,\mathbb{Z})$ as follows: to each *spin*^{c} structure $\mathfrak{s}$ over X corresponds a bundle of positive spinors $W_{\mathfrak{s}}^+$ over X. Let $c(\mathfrak{s}) \in H_2(X)$ denote the Poincaré dual of $c_1(W_{\mathfrak{s}}^+)$. Each $c(\mathfrak{s})$ is a characteristic element of $H_2(X,\mathbb{Z})$ (i.e. its Poincaré dual $\hat{c}(\mathfrak{s}) = c_1(W_{\mathfrak{s}}^+)$ reduces mod 2 to $w_2(X)$).

In this set up, we can view the Seiberg-Witten invariant as an integer valued function

$$\mathrm{SW}_X : \{K \in H^2(X,\mathbb{Z}) | K \equiv w_2(TX) \pmod 2\} \to \mathbb{Z}.$$

If $\mathrm{SW}_X(\beta) \neq 0$, then we call β a *basic class* of X. It is a fundamental fact that the set of basic classes is finite. Furthermore, if β is a basic class, then so is $-\beta$ with

$$\mathrm{SW}_X(-\beta) = (-1)^{(e+\sigma)(X)/4}\,\mathrm{SW}_X(\beta)$$

where $e(X)$ is the Euler characteristic and $\sigma(X)$ is the signature of X.

When $b_2^+(X) > 1$, then Seiberg-Witten invariant is a diffeomorphism invariant. It does not depend on the choice of generic metric on X or a generic perturbation of Seiberg-Witten equations.

If $b_2^+(X) = 1$, then the Seiberg-Witten invariant depends on the chosen metric and perturbation of Seiberg-Witten equations. Let us recall that the perturbed Seiberg-Witten moduli space $\mathcal{M}_X(\beta, g, h)$ is defined as the solutions of the Seiberg-Witten equations

$$F_A^+ = q(\psi) + ih, \;\; D_A\psi = 0$$

divided by the action of the gauge group, where A is a connection on the line bundle L with $c_1(L) = \beta$, g is Riemannian metric on X, ψ is the section of the positive spin bundle corresponding to the *spin*^{c} structure determined by β, F_A^+ is a self-dual part of the curvature F_A, D_A is the twisted Dirac operator, q is a quadratic function, and h is self-dual 2-form on X. If $b_2^+(X) \geq 1$ and h is generic metric, then Seiberg-Witten moduli space $\mathcal{M}_X(\beta, g, h)$ is a closed manifold with dimension $d = (\beta^2 - 2e(X) - 3\sigma(X))/4$. The Seiberg-Witten invariant is defined as follows:

$$\begin{cases} \mathrm{SW}_X(\beta) = \langle [\mathcal{M}_X(\beta, g, h)], \mu^{d/2} \rangle & \text{if } d \geq 0 \text{ and even} \\ \mathrm{SW}_X(\beta) = 0 & \text{otherwise} \end{cases}$$

where $\mu \in H^2(\mathcal{M}_X(\beta, g, h), \mathbb{Z})$ is the Euler class of the base fibration.

When $b_2^+(X) = 1$, the Seiberg-Witten invariant $SW_X(\beta, g, h)$ depends on g and h. Because of this, there are two types of Seiberg-Witten invariants: SW_X^+ and SW_X^-.

Theorem 3.1. [KM], [OS] *Let X be closed, oriented, smooth 4-manifold with $b_1(X) = 0$ and $b_2^+(X) = 1$. Fix a homology orientation of $H_+^2(X, \mathbb{R})$. For given Riemannian metric g let ω_+^g be the unique g-harmonic self-dual 2-form that has norm 1 and is compatible with the orientation of $H_+^2(X, \mathbb{R})$. Then for each characteristic element β with $(\beta^2 - 2e(X) - 3\sigma(X))/4 \geq 0$ the following holds: If $(2\pi\beta + h_1) \cdot \omega_+^{g_1}$ and $(2\pi\beta + h_2) \cdot \omega_+^{g_2}$ are not zero and have same sign, then $SW_X(\beta, g_1, h_1) = SW_X(\beta, g_2, h_2)$.*

Definition 3.1. If $(2\pi\beta + h) \cdot \omega_+^g > 0$, then write $\mathrm{SW}_X^+(\beta)$ for $\mathrm{SW}_X(\beta, g, h)$. If $(2\pi\beta + h) \cdot \omega_+^g < 0$, then write $\mathrm{SW}_X^-(\beta)$ for $\mathrm{SW}_X(\beta, g, h)$

Theorem 3.2. [Sz] *Let X be closed, oriented, smooth 4-manifold with $b_1(X) = 0$ and $b_2^+(X) = 1$ and $b_2^- \leq 9$. Then for each characteristic element β, pair of Riemannian metrics g_1, g_2, and small perturbing 2-forms h_1, h_2, we have $SW_X(\beta, g_1, h_1) = SW_X(\beta, g_2, h_2)$.*

Proof. . Let β be a characteristic element for which $d \geq 0$. Then $2e(X) + 3\sigma(X) = 4 + 5b_2^+ - b_2^- \geq 0$, which as implies $\beta^2 \geq 0$. It follows that $\beta \cdot \omega_+^{g_1}$ and $\beta \cdot \omega_+^{g_2}$ are both non-zero and have same signs. Now using the Theorem 3.1, the result follows.

□

Theorem 3.3. [LL] **(Wall crossing formula)** *Assume that X is a closed, oriented, smooth 4-manifold with $b_1(X) = 0$ and $b_2^+(X) = 1$. Then for each characteristic line bundle L on X such that the formal dimension of the Seiberg-Witten moduli space is non-negative even integer $2m$, then $SW_X^+(L) - SW_X^-(L) = -(-1)^m$.*

Note that when $b_2^- \leq 9$, it follows from the above result that there is well defined Seiberg-Witten invariant which will be denoted as $\mathrm{SW}_X^o(X)$.

Theorem 3.4. [T] *Suppose that X is closed symplectic 4-manifold with $b_2^+(X) > 1$ ($b_2^+(X) = 1$). If K_X is a canonical class of X, then $SW_X(\pm K_X) = \pm 1$ ($SW_X^-(-K_X) = \pm 1$).*

Definition 3.2. The 4-manifold X is of simple type if each basic class β satisfies the equation $\beta^2 = {c_1}^2(X) = 3\sigma(X) + 2e(X)$.

Theorem 3.5. [KM], [OS] **(Generalized adjunction formula for $b_2^+ > 1$)** *Assume that $\Sigma \subset X$ is an embedded, oriented, connected surface of genus $g(\Sigma)$ with self-intersection $|\Sigma|^2 \geq 0$ and represents nontrivial homology class. Then for every Seiberg-Witten basic class β, $2g(\Sigma) - 2 \geq |\Sigma|^2 + |\beta(|\Sigma|)|$. If X is of simple type and $g(\Sigma) > 0$, then the same inequality holds for Σ with arbitrary self-intersection.*

Theorem 3.6. [LL] **(Generalized adjunction formula for $b_2^+ = 1$)** *Let $\Sigma \subset X$ is an embedded, oriented, connected surface of genus $g(\Sigma)$ with self-intersection $|\Sigma|^2 \geq 0$ and $[\Sigma]$ represents nontrivial homology class. Then any characteristic class β with $SW_X{}^o(\beta) \neq 0$ satisfies $2g(\Sigma) - 2 \geq \Sigma^2 + |\beta(|\Sigma|)|$.*

4. Symplectic manifolds cohomology equivalent to $S^2 \times S^2$

To construct our manifolds, we will start with well known symplectic 4-manifolds described below. By applying Gompf's symplectic fiber sum operation along genus one and then genus two surfaces, we will obtain our manifolds X_K.

Let K be a genus one fibered knot (i.e. the trefoil or the figure eight knot) in S^3 and m a meridianal circle to K. Let M_K denote 3-manifold obtained as the result of 0-framed Dehn surgery on K. The manifold M_K has the same integral homology as $S^2 \times S^1$, where the class of m generates $H_1(M_K)$. Since the knot K has genus one and is fibered, it follows that the manifold $M_K \times S^1$ is a torus bundle over a torus which is homology equivalent to $T^2 \times S^2$. Since K is a fibered knot, $M_K \times S^1$ admits a symplectic structure. Note that $m \times S^1 = m \times x = T_m$ is a section of this fibration. Both the torus fiber and the torus section are symplectically embedded and have a self-intersection zero. The first homology of $M_K \times S^1$ is generated by the standard first homology generators m and x of the torus section. The generators γ_1 and γ_2 of the fiber F, coming from Seifert surface of knot K, are trivial in homology.

The intermediate building block in our construction will be a twisted fiber sum of two copies of the manifold $M_K \times S^1$, where we identify a fiber of first fibration with a section of the second one. Let Y_K denote the mentioned twisted fiber sum $Y_K = M_K \times S^1 \#_{F=T_m} M_K \times S^1$. It follows from Gompf's theorem [Go] that Y_K is symplectic. Notice that the manifold Y_K is obtained by knot surgery operation of Fintushel-Stern [FS1] from manifold $M_K \times S^1$. Thus $Y_K = (M_K \times S^1)_K$. In this step we could have also chosen a different genus 1 fibered knot.

Let T_1 be the section of the first copy of $M_K \times S^1$ and T_2 be the fiber of the second copy. Note that $T_1 \# T_2$ symplectically embeds into Y_K. Now suppose that $\Sigma_2 = T_1 \# T_2$ is the genus two symplectic submanifold of self-intersection zero sitting inside of Y_K. Let $(m, x, \gamma_1, \gamma_2)$ be the generators of $H_1(\Sigma_2)$ under the inclusion-induced homomorphism (here we use the same letters γ_1 and γ_2 to denote the generators of the fiber coming from the second copy of $M_K \times S^1$). We choose the involution diffeomorphism $\phi : T_1 \# T_2 \longrightarrow T_1 \# T_2$ of the Σ_2 which induces the map on first homology according to the following rule: $\phi_*(m') = \gamma_1$, $\phi_*(\gamma_1{}') = m$, $\phi_*(x') = \gamma_2$ and $\phi_*(\gamma_2{}') = x$. Next, we take the fiber sum of two copies of Y_K along the genus two surface Σ_2 and denote the new symplectic manifold as X_K, i.e. $X_K = Y_K \#_\phi Y_K$. We will show that the new manifold X_K has trivial first homology and has the same integral cohomology as $S^2 \times S^2$. Consequently, $e(X_K) = 4$, $\sigma(X_K) = 0$, $c_1{}^2(X_K) = 8$, and $\chi_h(X_K) = 1$. We will compute $H_1(X_K, \mathbb{Z})$

(also $H_2(X_K, \mathbb{Z})$) by using Mayer-Vietoris sequence and then by directly computing the fundamental group of X_K.

Lemma 4.1. *$H_1(X_K, \mathbb{Z}) = 0$ and $H_2(X_K, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. We use Mayer-Vietoris sequence to compute the homology of $X_K = Y_K \#_\phi\ Y_K$. Let $Y_1 = Y_2 = Y_K \setminus \nu\Sigma_2$. By applying the reduced Mayer-Vietoris sequence to the triple (X_K, Y_1, Y_2), we have the following long exact sequence

$$\begin{aligned} \cdots \quad \longrightarrow \quad & H_2(S^1 \times \Sigma_2, \mathbb{Z}) \longrightarrow H_2(Y_1, \mathbb{Z}) \oplus H_2(Y_2, \mathbb{Z}) \longrightarrow H_2(X_K, \mathbb{Z}) \\ \longrightarrow \quad & H_1(S^1 \times \Sigma_2, \mathbb{Z}) \longrightarrow H_1(Y_1, \mathbb{Z}) \oplus H_1(Y_2, \mathbb{Z}) \longrightarrow H_1(X_K, \mathbb{Z}) \longrightarrow 0 \end{aligned}$$

The simple computation by Kunneth formula yields $H_1(\Sigma_2 \times S^1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = < \lambda > \oplus < m > \oplus < x > \oplus < \gamma_1 > \oplus < \gamma_2 >$. Also, we have $H_1(Y_1, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} = < m > \oplus < x >$ and $H_1(Y_2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} = < m' > \oplus < x' >$.

Let ϕ_* and δ denote the last two arrows in the long exact sequence above. Because the way the gluing map ϕ is defined, the essential homology generators will map to the trivial ones. Thus, we have $\phi_*(m) = \phi_*(x) = \phi_*(m') = \phi_*(x') = 0$. Since $Im(\phi_*) = Ker(\delta)$, we conclude that $H_1(X_K) = Ker(\delta) = Im(\phi_*) = 0$

Next, by using the facts that $b_1 = b_3 = 0$, $b_0 = b_4 = 1$, and the symplectic sum formula for Euler characteristics, we compute $b_2 = e(Y_K) + e(Y_K) + 2 = 0 + 0 + 2 = 2$.

We conclude that $H_2(X_K, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. A basis for the second homology consists of classes of $\Sigma_2 = S$ and the new genus two surface T resulting from the second fiber sum operation (i.e. two punctured genus one surfaces glues to form a genus two surface), where $S^2 = T^2 = 0$ and $S \cdot T = 1$. Thus, the manifolds obtained by the above construction have intersection form H, so they are spin 4-manifolds. □

Lemma 4.2. *$e(X_K) = 4$, $\sigma(X_K) = 0$, ${c_1}^2(X_K) = 8$, and $\chi_h(X_K) = 1$.*

Proof. Using the lemma 2.8, we have $e(X_K) = 2e(Y_K) + 4$, $\sigma(X_K) = 2\sigma(Y_K)$, ${c_1}^2(X_K) = 2{c_1}^2(Y_K) + 8$, and $\chi_h(X_K) = 2\chi_h(Y_K) + 1$. Since $e(Y_K) = 0$, $\sigma(Y_K) = 0$, ${c_1}^2(Y_K) = 0$ and $\chi_h(Y_K) = 0$, the result follows. □

Since our basic building block $M_K \times S^1$ is a minimal symplectic 4-manifold, it follows from Usher's Theorem that the symplectic manifolds Y_K and X_K are both minimal.

4.1. Fundamental Group Computation for Trefoil

4.1.1. Step 1: Fundamental Group of $M_K \times S^1$

In this section we will assume that K is the trefoil. The case when K is the figure eight can be treated similarly.

Let a and b denote the Wirtinger generators of the trefoil knot. Then the group of K has the following presentation

$$\pi_1(S^3 \setminus \nu K) \ = \langle a, b \mid aba = bab \rangle \ = \langle u, v \mid u^2 = v^3 \rangle$$

where $u = bab$ and $v = ab$. By Lemma 2.2, the homotopy classes of the meridian and the longitude of the trefoil are given as follows: $m = uv^{-1} = b$ and $l = u^2(uv^{-1})^{-6} = ab^2ab^{-4}$. Notice that $\gamma_1 = a^{-1}b$ and $\gamma_2 = b^{-1}aba^{-1}$ generate the image of the fundamental group of the Seifert surface of K under the inclusion-induced homomorphism. Let M_K denote the result of 0-surgery on K.

Lemma 4.3.

$$\begin{aligned} \pi_1(M_K \times S^1) &= \pi_1(M_K) \oplus \mathbb{Z} \\ &= \langle a, b, x \mid aba = bab,\ ab^2ab^{-4} = 1,\ [a, x] = [b, x] = 1 \rangle. \end{aligned}$$

Proof. Notice that the fundamental group of M_K is obtained from the knot group of the trefoil by adjoining the relation $l = u^2(uv^{-1})^{-6} = ab^2ab^{-4} = 1$. Thus, we have the presentation given above.

□

4.1.2. Step 2: Fundamental Group of Y_K

Next, we take the two copies of the manifold $M_K \times S^1$. In the first copy, take a tubular neighborhood of the torus section $b \times x$, remove it from $M_K \times S^1$ and denote the resulting manifold as C_S. In the second copy, we remove a tubular neighborhood of the fiber F and denote the complement by C_F.

Lemma 4.4. *Let C_S be the complement of a neighborhood of a section in $M_K \times S^1$. Then we have*

$$\pi_1(C_S) = \langle a, b, x \mid aba = bab,\ [a, x] = [b, x] = 1 \rangle.$$

Proof. Note that $C_S = (M_K \setminus \nu(b)) \times S^1 = (S^3 \setminus \nu K) \times S^1$. □

Lemma 4.5. *Let C_F be the complement of a neighborhood of a fiber in $M_K \times S^1$. Then we have*

$$\begin{aligned} \pi_1(C_F) = \langle \gamma_1', \gamma_2', d, y \quad \mid \quad & [\gamma_1', \gamma_2'] = [y, \gamma_1'] = [y, \gamma_2'] = 1, \\ & d\gamma_1' d^{-1} = \gamma_1'\gamma_2',\ d\gamma_2' d^{-1} = (\gamma_1')^{-1} \rangle. \end{aligned}$$

Proof. To compute the fundamental group of C_F, we will use the following observation: C_F is homotopy equivalent to a torus bundle over a wedge of two circles. The generators d and y do not commute in the fundamental group of C_F. Also, the monodromy along the circle y is trivial whereas the monodromy along the circle d is the same as the monodromy of M_K. Notice that $\gamma_1' = c^{-1}d$, $\gamma_2' = d^{-1}cdc^{-1}$ and $cdc = dcd$. Thus, we have

$d\gamma_1'd^{-1} = dc^{-1}dd^{-1} = dc^{-1} = \gamma_1'\gamma_2'$, and $d\gamma_2'd^{-1} = dd^{-1}cdc^{-1}d^{-1} = d^{-1}c = (\gamma_1')^{-1}$. □

Lemma 4.6. *Let Y_K be the symplectic sum of two copies of $M_K \times S^1$, identifying a section in one copy with a fiber in the other copy. If the gluing map ψ satisfies $\psi_*(x) = \gamma_1'$ and $\psi_*(b) = \gamma_2'$, then*

$$\begin{aligned}\pi_1(Y_K) &= \langle a, b, x, \gamma_1', \gamma_2', d, y \mid aba = bab,\ [x,a] = [x,b] = 1,\\ &\quad [\gamma_1', \gamma_2'] = [y, \gamma_1'] = [y, \gamma_2'] = 1,\ d\gamma_1'd^{-1} = \gamma_1'\gamma_2',\ d\gamma_2'd^{-1} = (\gamma_1')^{-1},\\ &\quad x = \gamma_1',\ b = \gamma_2',\ ab^2ab^{-4} = [d,y]\rangle\\ &= \langle a, b, x, d, y \mid aba = bab,\ [x,a] = [x,b] = 1,\\ &\quad [y,x] = [y,b] = 1,\ dxd^{-1} = xb,\ dbd^{-1} = x^{-1},\ ab^2ab^{-4} = [d,y]\rangle.\end{aligned}$$

Proof. By Van Kampen's Theorem, $\pi_1(Y_K) = \pi_1(C_S) * \pi_1(C_F)/\pi_1(T^3)$. One circle factor of T^3 is identified with the longitude of K on one side and the meridian of the torus fiber in $M_K \times S^1$ on the other side. This gives the last relation. □

Inside Y_K, we can find a genus 2 symplectic submanifold Σ_2 which is the internal sum of a punctured fiber in C_S and a punctured section in C_F. The inclusion-induced homomorphism maps the standard generators of $\pi_1(\Sigma_2)$ to $a^{-1}b$, $b^{-1}aba^{-1}$, d and y.

Lemma 4.7. *There are nonnegative integers m and n such that*

$$\begin{aligned}\pi_1(Y_K \setminus \nu\Sigma_2) &= \langle a, b, x, d, y;\ g_1, \ldots, g_m \mid aba = bab,\\ &\quad [y,x] = [y,b] = 1,\ dxd^{-1} = xb,\ dbd^{-1} = x^{-1},\\ &\quad ab^2ab^{-4} = [d,y],\ r_1 = \cdots = r_n = 1,\ r_{n+1} = 1\rangle,\end{aligned} \tag{1}$$

where the generators $g_1, \ldots, g_m$ and relators $r_1, \ldots, r_n$ all lie in the normal subgroup N generated by the element $[x,b]$, and the relator r_{n+1} is a word in x, a and elements of N. Moreover, if we add an extra relation $[x,b] = 1$ to (1), then the relation $r_{n+1} = 1$ simplifies to $[x,a] = 1$.

Proof. This follows from Van Kampen's Theorem. Note that $[x,b]$ is a meridian of Σ_2 in Y_K. Hence setting $[x,b] = 1$ should turn $\pi_1(Y_K \setminus \nu\Sigma_2)$ into $\pi_1(Y_K)$. Also note that $[x,a]$ is the boundary of a punctured section in $C_S \setminus \nu(\text{fiber})$, and is no longer trivial in $\pi_1(Y_K \setminus \nu\Sigma_2)$. By setting $[x,b] = 1$, the relation $r_{n+1} = 1$ is to turn into $[x,a] = 1$.

It remains to check that the relations in $\pi_1(Y_K)$ other than $[x,a] = [x,b] = 1$ remain the same in $\pi_1(Y_K \setminus \nu\Sigma_2)$. By choosing a suitable point $\theta \in S^1$ away from the image of the fiber that forms part of Σ_2, we obtain an embedding of the knot complement

$(S^3 \setminus \nu K) \times \{\theta\} \hookrightarrow C_S \setminus \nu(\text{fiber})$. This means that $aba = bab$ holds in $\pi_1(Y_K \setminus \nu\Sigma_2)$. Since $[\Sigma_2]^2 = 0$, there exists a parallel copy of Σ_2 outside $\nu\Sigma_2$, wherein the identity $ab^2ab^{-4} = [d, y]$ still holds. The other remaining relations in $\pi_1(Y_K)$ are coming from the monodromy of the torus bundle over a torus. Since these relations will now describe the monodromy of the punctured torus bundle over a punctured torus, they hold true in $\pi_1(Y_K \setminus \nu\Sigma_2)$. □

4.1.3. Step 3: Fundamental Group of X_K

Now take two copies of Y_K. Suppose that the fundamental group of the second copy has e, f, z, s, t as generators, and the inclusion-induced homomorphism in the second copy maps the generators of $\pi_1(\Sigma_2)$ to $e^{-1}f$, $f^{-1}efe^{-1}$, s and t. Let X_K denote the symplectic sum of two copies of Y_K along Σ_2, where the gluing map ψ maps the generators as follows:

$$\psi_*(a^{-1}b) = s,\ \psi_*(b^{-1}aba^{-1}) = t,\ \psi_*(d) = e^{-1}f,\ \psi_*(y) = f^{-1}efe^{-1}.$$

Lemma 4.8. *There are nonnegative integers m and n such that*

$$\begin{aligned}
\pi_1(X_K) \quad = \quad & \langle a, b, x, d, y;\ e, f, z, s, t;\ g_1, \dots, g_m;\ h_1, \dots, h_m \mid \\
& aba = bab,\ [y, x] = [y, b] = 1, \\
& dxd^{-1} = xb,\ dbd^{-1} = x^{-1},\ ab^2ab^{-4} = [d, y], \\
& r_1 = \cdots = r_n = r_{n+1} = 1,\ r'_1 = \cdots = r'_n = r'_{n+1} = 1, \\
& efe = fef,\ [t, z] = [t, f] = 1, \\
& szs^{-1} = zf,\ sfs^{-1} = z^{-1},\ ef^2ef^{-4} = [s, t], \\
& d = e^{-1}f,\ y = f^{-1}efe^{-1},\ a^{-1}b = s,\ b^{-1}aba^{-1} = t, [x, b] = [z, f]\rangle,
\end{aligned}$$

where g_i, h_i $(i = 1, \dots, m)$ and r_j, r'_j $(j = 1, \dots, n)$ all lie in the normal subgroup N generated by $[x, b] = [z, f]$. Moreover, r_{n+1} is a word in x, a and elements of N, and r'_{n+1} is a word in z, e and elements of N.

Proof. This is just a straightforward application of Van Kampen's Theorem and Lemma 4.7. □

Remark 1. Notice that the abelianization of $\pi_1(X_K)$ is trivial. Also, different symplectic cohomology $S^2 \times S^2$'s can be obtained if we use other genus 1 fibered knot, the figure eight knot, or a combination of both (the trefoil or the figure eight knot) in our construction. Using computational software [GAP], we show that these manifolds have different fundamental groups. In addition, considering an infinite family of non-fibered genus one twist knots, we also obtain an infinite family of cohomology $S^2 \times S^2$. This family of cohomology $S^2 \times S^2$'s will not be symplectic anymore and can be distinguished by their Seiberg-Witten invariants. Also, using [GAP], it is not hard to verify that their fundamental groups are different as well.

Question 1. Are there two distinct genus 1 knots K and $K^{'}$ such that $\pi_1(X_K)$ is isomorphic to $\pi_1(X_{K'})$?

4.2. Seiberg-Witten invariants for manifold X_K

Let C be a basic class of the manifold X_K. We can write C as a linear combination of S and T, i.e. $C = aS + bT$. X_K is a symplectic manifold and has simple type. So for any basic class C, $C^2 = 3\sigma(X_K) + 2e(X_K) = 8$. It follows that $2ab = 8$. Next we apply the adjunction inequality to S and T to get $2g(S) - 2 \geq [S]^2 + |C(S)|$ and $2g(T) - 2 \geq [T]^2 + |C(T)|$. These gives us two more restriction on a and b : $2 \geq |b|$ and $2 \geq |a|$. Thus, it follows that $C = \pm(2S + 2T)$, which is $\pm$ the canonical class $K_{X_K} = 2S + 2T$ of X_K. Notice that, since $b_2^-(X_K) \leq 9$ and $(-K_{X_K}) \cdot \omega < 0$, we have a well defined $\mathrm{SW}^o_{X_K}$. Now it follows from the theorems of Section 3 that $\mathrm{SW}^0_{X_K}(-K_{X_K}) = \mathrm{SW}^-_{X_K}(-K_{X_K}) = \pm 1$.

5. Symplectic manifolds cohomology equivalent to $\#_{(2g-1)}(S^2 \times S^2)$

In this section, we will modify the construction above to get an infinite family of symplectic 4-manifolds cohomology equivalent to $\#_{(2g-1)}(S^2 \times S^2)$ for any $g \geq 2$.

Let K' denote a genus g fibered knot in S^3 and m a meridianal circle to K'. We first perform 0-framed surgery on K' and denote the resulting 3-manifold by $Z_{K'}$. The 4-manifold $Z_{K'} \times S^1$ is a Σ_g bundle over the torus and has the same integral homology as $T^2 \times S^2$. Since K' is a fibered knot, $Z_{K'} \times S^1$ is a symplectic manifold. Again, there is a torus section $m \times S^1 = T_m$ of this fibration. The first homology of $Z_{K'} \times S^1$ is generated by the standard first homology generators of this torus section. The standard homology generators of the fiber F, which we denote as $\alpha_2, \beta_2, \cdots, \alpha_{g+1}$, and β_{g+1} of the given bundle is trivial in the homology. The section T_m has zero self-intersection and the its neighborhood in $Z_{K'} \times S^1$ has a canonical identification with $T_m \times D^2$.

We form a twisted fiber sum of the manifold $M_K \times S^1$ with $Z_{K'} \times S^1$ where we identify the fiber of the first fibration to the section of other. Let $W_{KK'}$ denote the new manifold $W_{KK'} = M_K \times S^1 \#_{F=T_{m'}}\ Z_{K'} \times S^1$. Let us remark that the manifold $W_{KK'}$ is obtained from $M_K \times S^1$ by knot surgery of Fintushel-Stern [FS1] along knot $K^{'}$. Again, it follows from Gompf's theorem [Go] that $W_{KK'}$ is symplectic.

Let T_1 be the section of the $M_K \times S^1$ which we discussed earlier and Σ_g be the fiber of the $Z_{K'} \times S^1$. Then $T_1 \# \Sigma_g$ embeds in $W_{KK'}$ and has self-intersection zero. Now suppose that $W_{KK'}$ is the symplectic 4-manifold given, and $\Sigma_{g+1} = T_1 \# \Sigma_g$ is the genus $g+1$ symplectic submanifold of self-intersection 0 sitting inside of $W_{KK'}$. Let α_1, β_1, α_2, β_2, $\ldots$, α_{g+1}, and β_{g+1} be the generators of the first homology of the surface $T_1 \# \Sigma_g = \Sigma_{g+1}$. Let $\psi : T^2 \# \Sigma_g \to \Sigma_g \# T^2$ be any diffeomorphism of the genus $g+1$ surface that changes the generators of the first homology according to the following rule: $\psi_*(\alpha_1) = \alpha'_{g+1}$, $\psi_*(\beta_1) = \beta'_{g+1}$, $\psi_*(\alpha_{g+1}) = \alpha'_1$, and $\psi_*(\beta_{g+1}) = \beta'_1$. Take the fiber sum along this genus $g+1$ surface Σ_{g+1} and denote the resulting symplectic manifold as $V_{KK'}$. The new manifold $V_{KK'} = W_{KK'} \#_{\psi}\ W_{KK'}$ has trivial first homology and has the same homology of $\#_{(2g-1)} S^2 \times S^2$.

Lemma 5.1. *$H_1(V_{KK'}, \mathbb{Z}) = 0$ and $H_2(V_{KK'}, \mathbb{Z}) = \oplus_{2(2g-1)} \mathbb{Z}$.*

Proof. The proof is similar to genus one case and can be obtained by applying Mayer-Vietoris sequence.

□

Note that $H_2(V_{KK'}, \mathbb{Z})$ has rank $4g-2$. A basis for the second homology consist of the $[S]$ and $[\Sigma]$, where S is the genus $g+1$ surface Σ_{g+1} and Σ is the genus two surface Σ_2 resulting from the last fiber sum operation, $(2g-2)$ Lagrangian tori R_i and $(2g-2)$ associated dual Lagrangian tori V_i. In the manifold $V_{KK'}$ these classes contribute $2g-2$ new hyperbolic pairs. Thus, the manifolds obtained by the above construction have intersection form $\oplus_{2g-1} H$. Notice that $b_2^+(V_{KK'}) = b_2^-(V_{KK'}) = 2g-1$

Lemma 5.2. *$e(V_{KK'}) = 4g$, $\sigma(V_{KK'}) = 0$, ${c_1}^2(V_{KK'}) = 8g$, and $\chi_h(V_{KK'}) = g$.*

Proof. Using the lemma 2.8, we have $e(V_{KK'}) = 2e(W_{KK'}) + 4g$, $\sigma(V_{KK'}) = 2\sigma(W_{KK'})$, ${c_1}^2(V_{KK'}) = 2{c_1}^2(W_{KK'}) + 8g$ and $\chi_h(V_{KK'}) = 2\chi_h(W_{KK'}) + g$. Since $e(W_{KK'}) = 0$ and $\sigma(W_{KK'}) = 0$, our result follows. □

Remark 2. Using Lemma 2.2 and applying steps of Section 4.1, one can compute $\pi_1(V_{KK'})$ when K is the trefoil and K' is the (p,q) torus knot.

Question 2. Are there two distinct genus $g \geq 2$ knots K' and K'' such that $\pi_1(V_{KK'})$ is isomorphic to $\pi_1(V_{KK''})$?

5.1. Seiberg-Witten invariants for manifold $V_{KK'}$

Let C be a basic class of the manifold $V_{KK'}$. Let us write C as a linear combination of S, Σ, tori R_i, $i = 1, ..., 2g-2$ and the dual tori V_i, $i = 1, \cdots, 2g-2$, $C = aS + b\Sigma + \sum_{i=1}^{2g-2} u_i R_i + v_i V_i$. By adjunction inequality, the intersection number of any basic class with tori R_i is zero, i.e. $C \cdot R_i = 0$, $i = 1, \cdots, 2g-2$. It follows that $Q^T v = 0$ where Q is the intersection matrix and $v = (v_1, \cdots, v_{2g-2})$. Using the fact that Q is invertible, we obtain $v_1 = \cdots = v_{2g-2} = 0$. Next, by applying the adjunction inequality again, we have $V_i \cdot C = 0$ for $i = 1, \cdots, 2g-2$. This gives rise to the system $Qu = 0$, which implies $u_1 = \cdots = u_{2g-2} = 0$. This shows that any basic class has form $C = aS + b\Sigma$. Since $V_{KK'}$ is a symplectic manifold and has simple type. So for any basic class C, $C^2 = 3\sigma(V_{KK'}) + 2e(V_{KK'}) = 8g$. It follows that $2ab = 8g$. Next we apply the adjunction inequality to S and Σ to get $2g(S) - 2 \geq [S]^2 + |C(S)|$ and $2g(\Sigma) - 2 \geq [\Sigma]^2 + |C(\Sigma)|$. These gives two more restrictions on a and b : $2g \geq |b|$ and $2 \geq |a|$. It follows that $C = \pm(2S + 2g\Sigma)$, which are $\pm$ the canonical class of the given manifold. By applying the Theorem 3.5, we conclude that the value of Seiberg-Witten invariants of these classes is ± 1.

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