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Ricci-flat deformations of asymptotically cylindrical Calabi–Yau manifolds

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Dedicated to the memory of Raoul Bott.

Abstract. We study a class of asymptotically cylindrical Ricci-flat Kähler metrics arising on quasiprojective manifolds. Using the Calabi–Yau geometry and analysis and the Kodaira–Kuranishi–Spencer theory and building up on results of N.Koiso, we show that under rather general hypotheses any local asymptotically cylindrical Ricci-flat deformations of such metrics are again Kähler, possibly with respect to a perturbed complex structure. We also find the dimension of the moduli space for these local deformations. In the class of asymptotically cylindrical Ricci-flat metrics on $2n$-manifolds, the holonomy reduction to $SU(n)$ is an open condition.

Let M be a compact smooth manifold with integrable complex structure J and g a Ricci-flat Kähler metric with respect to J. A theorem due to N.Koiso [10] asserts that if the deformations of the complex structure of M are unobstructed then the Ricci-flat Kähler metrics corresponding to the nearby complex structures and Kähler classes fill in an open neighbourhood in the moduli space of Ricci-flat metrics on M. The proof of this result relies on Hodge theory and Kodaira–Spencer–Kuranishi theory and Koiso also found the dimension of the moduli space.

The purpose of this paper is to extend the above result to a class of complete Ricci-flat Kähler manifolds with asymptotically cylindrical ends (see §1 for precise definitions). A suitable version of Hodge theory was developed as part of elliptic theory for asymptotically cylindrical manifolds in [13, 14, 15, 16]. A complex manifold underlying an asymptotically cylindrical Ricci-flat Kähler manifold admits a compactification by adding a 'divisor at infinity'. There is an extension of Kodaira–Spencer–Kuranishi theory for this class of non-compact complex manifolds using the cohomology of logarithmic sheaves [8]. On the other hand, manifolds with asymptotically cylindrical ends appear as an essential step in the gluing constructions of compact manifolds endowed with special Riemannian structures. In particular, the Ricci-flat Kähler asymptotically cylindrical manifolds were prominent in [11] in the construction of compact 7-dimensional Ricci-flat manifolds with special holonomy G_2.

We introduce the class of Ricci-flat Kähler asymptotically cylindrical manifolds in §1, where we also state our first main Theorem 1.3 and give interpretation in terms of special

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holonomy. We review basic facts about the Ricci-flat deformations in §2. §§3–5 contain the proof of Theorem 1.3 and our second main result Theorem 5.1 on the dimension of the moduli space for the Ricci-flat asymptotically cylindrical deformations of a Ricci-flat Kähler asymptotically cylindrical manifold. Some examples (motivated by [11]) are considered in §6.

1. Asymptotically cylindrical manifolds

A non-compact Riemannian manifold (M, g) is called *asymptotically cylindrical* with cross-section Y if

(1) M can be decomposed as a union $M = M_{\mathrm{cpt}} \cup_Y M_e$ of a compact manifold M_{cpt} with boundary Y and an end M_e diffeomorphic to half-cylinder $[1, \infty) \times Y$, the two pieces attached via $\partial M_{\mathrm{cpt}} \cong \{1\} \times Y$, and

(2) The metric g on M is asymptotic, along the end, to a product cylindrical metric $g_0 = dt^2 + g_Y$ on $[1, \infty) \times Y$,

$$\lim_{t\to\infty} (g - g_0) = 0, \qquad \lim_{t\to\infty} \nabla_0^k g = 0, \quad k = 1, 2, \dots,$$

where t is the coordinate on $[1, \infty)$ and ∇_0 denotes the Levi–Civita connection of g_0.

Note that the cross-section Y is always a compact manifold. We shall sometimes assume that t is extended to a smooth function defined on all of M, so that $t \geq 1$ on the end and $0 \leq t \leq 1$ on the compact piece of M_{cpt}.

Remark 1.1. Setting $x = e^{-t}$, one can attach to M a copy of Y corresponding to $x = 0$ and obtain a compactification $\mathbf{M} = M \cap Y$ 'with boundary at infinity'. Then x defines a normal coordinate near the boundary of $\mathbf{M}$. The metric g is defined on the interior of $\mathbf{M}$ and blows up in a particular way at the boundary,

$$g = \left(\frac{dx}{x}\right)^2 + \tilde{g}, \tag{1}$$

for some semi-positive definite symmetric bilinear $\tilde{g}$ smooth on M and continuous on $\mathbf{M}$, such that $\tilde{g}|_{x=0} = g_Y$. Metrics of this latter type are called 'exact b-metrics' and are studied in [16].

Our main result concerns a Kähler version of the asymptotically cylindrical Riemannian manifolds which we now define. Suppose that M has an integrable complex structure J and write Z for the resulting complex manifold. The basic idea is to replace a real parameter t along the cylindrical end by a complex parameter, $t + i\theta$ say, where $\theta \in S^1$. Thus in the complex setting the asymptotic model for a cylindrical end of Z takes a slightly special form $\mathbb{R}_{>0} \times S^1 \times D$, for some compact complex manifold D. Respectively, the normal coordinate $x = e^{-t}$ becomes the real part of a holomorphic local coordinate $z = e^{-t-i\theta}$ taking values in the punctured unit disc $\Delta^* = \{0 < |z| < 1\} \subset \mathbb{C}$. It follows that the complex structure on the cylindrical end is asymptotic to the product $\Delta^* \times D$ and the complex manifold Z is *compactifiable*, $Z = \overline{Z} \setminus D$, where $\overline{Z}$ is a compact complex manifold of the same dimension as Z and D is a complex submanifold of codimension 1 in $\overline{Z}$ with holomorphically trivial normal bundle $N_{D/\,\overline{Z}}$.

The local complex coordinate z on $\overline{Z}$ vanishes to order one precisely on D and a tubular neighbourhood $\overline{Z}_e = \{|z| < 1\}$ is a local deformation family for D,

$$\pi : \overline{Z}_e \to \Delta, \qquad D = \pi^{-1}(0), \tag{2}$$

where π denotes the holomorphic map defining the coordinate z. Note that the cylindrical end $Z_e = \overline{Z}_e \smallsetminus D$ is diffeomorphic (as a real manifold) but not in general biholomorphic to $\mathbb{R}_{>0} \times S^1 \times D$ as the complex structure on the fibre $\pi^{-1}\{z\}$ depends on z.

Remark 1.2. If $H^{0,1}(\overline{Z}) = 0$ then the local map (2) extends to a holomorphic fibration $\overline{Z} \to \mathbb{C}P^1$ (cf. [6, pp.34–35]).

A product Kähler metric, with respect to a product complex structure on $\mathbb{R} \times S^1 \times D$, has Kähler form $a^2 dt \wedge d\theta + \omega_D$, where ω_D is a Kähler form on D and a is a positive function of t, θ. We shall be interested in the situation when the product Kähler metric is *Ricci-flat*; then a is a constant and can be absorbed by rescaling the variable t.

We say that a Kähler metric on Z is *asymptotically cylindrical* if its Kähler form ω can be expressed on the end $Z_e = \overline{Z}_e \smallsetminus D \subset Z$ as

$$\omega|_{Z_e} = \omega_D + dt \wedge d\theta + \varepsilon,$$

for some closed form $\varepsilon \in \Omega^2(Z_e)$ decaying, with all derivatives, to zero uniformly on $S^1 \times D$ as $t \to \infty$. An asymptotically cylindrical Kähler metric defines an asymptotically cylindrical Riemannian metric on the underlying real manifold.

We shall sometimes refer to Kähler metrics by their Kähler forms.

Proposition 1.1. *Let Z be a compactifiable complex manifold as defined above. If ω is an asymptotically cylindrical Kähler metric on M then the decaying term on Z_e is* exact,

$$\omega|_{Z_e} = \omega_D + dt \wedge d\theta + d\psi. \tag{3}$$

Proof. We can write $\varepsilon = \varepsilon_0(t) + dt \wedge \varepsilon_1(t)$, where $\varepsilon_0(t), \varepsilon_1(t)$ are 1-parameter families of, respectively, 2-forms and 1-forms on the cross-section $S^1 \times D$. As ε is closed, $\varepsilon_0(t)$ must be closed for each t and $\frac{\partial}{\partial t}\varepsilon_0(t) = d_{S^1\times D}\varepsilon_1(t)$. As ε_1 decays exponentially fast, we have $\varepsilon_0(t) = \int_\infty^t d_{S^1\times D}\varepsilon_1(s)ds$ and the integral converges absolutely. So we can write

$$\varepsilon = d_{S^1\times D} \int_\infty^t \varepsilon_1(s)ds + dt \wedge \varepsilon_1(t)$$

which is an exact differential of a 1-form $\psi = -\int_t^\infty \varepsilon_1(s)ds$ on Z_e. □

Recall that by Yau's solution of the Calabi conjecture a compact Kähler manifold admits Ricci-flat Kähler metrics if and only if its first Chern class vanishes [21]. Moreover, the Ricci-flat Kähler metric is uniquely determined by the cohomology class of its Kähler form. Ricci-flat Kähler manifolds are sometimes called Calabi–Yau manifolds.

Remark 1.3. There is an alternative way to define the Calabi–Yau manifolds using the holonomy reduction. The holonomy group of a Riemannian $2n$-manifold is the group of isometries of a tangent space generated by parallel transport using the Levi–Civita connection over closed paths based at a point. The holonomy group can be identified with a subgroup of $SO(2n)$ if the manifold is orientable. If the holonomy of a Riemannian $2n$-manifold is contained in $SU(n) \subset SO(2n)$ then the manifold has an integrable complex structure J, so that with respect to J the metric is Ricci-flat Kähler. The converse is in general not true unless the manifold is simply-connected.

A version of the Calabi conjecture for asymptotically cylindrical Kähler manifolds is proved in [20, Thm. 5.1] and [11, §§2–3]. It can be stated as the following.

Theorem 1.2. (cf. [11, Thms. 2.4 and 2.7]) *Suppose that $Z = \overline{Z} \setminus D$ is a compactifiable complex n-fold as defined above, such that D is an anticanonical divisor on $\overline{Z}$ and the normal bundle of D is holomorphically trivial and $b^1(\overline{Z}) = 0$. Let $\overline{g}$ be a Kähler metric on $\overline{Z}$ and denote by g_D the Ricci-flat Kähler metric on D in the Kähler class defined by the embedding in $\overline{Z}$.*

Then $Z = \overline{Z} \setminus D$ admits a complete Ricci-flat Kähler metric g_Z. The Kähler form of g_Z can be written, on the cylindrical end of Z, as in (3) with ω_D the Kähler form of g_D.

If, in addition, $\overline{Z}$ and D are simply-connected and there is a closed real 2-dimensional submanifold of $\overline{Z}$ meeting D transversely with non-zero intersection number then the holonomy of g is $SU(n)$.

Note that an anticanonical divisor D admits Ricci-flat Kähler metrics as $c_1(D) = 0$ by the adjunction formula. The result in [11] is stated for threefolds, but the proof generalizes to an arbitrary dimension by a change of notation. We consider examples arising by application of the above theorem in §6. A consequence of the arguments in [11] is that if an asymptotically cylindrical Kähler metric ω is Ricci-flat then the 1-form $\psi \in \Omega^1(M_e)$ in (3) can be taken to be decaying, with all derivatives, at an *exponential* rate $O(e^{-\lambda t})$ as $t \to \infty$, for some $0 < \lambda < 1$ depending on g_D. Furthermore, if ω and $\tilde{\omega}$ are asymptotically cylindrical Ricci-flat metrics on Z such that $\tilde{\omega} = \omega + i\partial\bar{\partial}u$ for some $u \in C^\infty(Z)$ decaying to zero as $t \to \infty$ then $\omega = \tilde{\omega}$ [11, Prop. 3.11].

Let (M, g) be an asymptotically cylindrical Riemannian manifold. A local deformation $g + h$ of g is given by a field of symmetric bilinear forms satisfying $|h|_g < 1$ at each point, so that $g + h$ is a well-defined metric. Suppose that $g + h$ is asymptotically cylindrical. Then there is a well-defined symmetric bilinear form h_Y on Y obtained as the limit of h as $t \to \infty$ and h_Y is a deformation of the limit g_Y of g, in particular $|h_Y|_{g_Y} < 1$. The h_Y defines via the obvious projection $\mathbb{R} \times Y \to Y$ a t-independent symmetric bilinear form on the cylinder, still denoted by h_Y. Let $\rho : \mathbb{R} \to [0, 1]$ denote a smooth function, such that $\rho(t) = 1$, for $t \geq 2$, and $\rho(t) = 0$, for $t \leq 1$. In view of the remarks in the previous paragraph we shall be interested in the class of metrics which are asymptotically cylindrical at an exponential rate and deformations h satisfying $h - \alpha h_Y = e^{-\mu t}\tilde{h}$ for some $\mu > 0$ and a bounded $\tilde{h}$. Given an exponentially asymptotically cylindrical metric g, a deformation $g + h$ 'sufficiently close' to g is understood in the sense of sufficiently small Sobolev norms of $\tilde{h}$ and h_Y, where the Sobolev norms are chosen to dominate the uniform norms on M and Y, respectively.

We now state our first main result in this paper.

Theorem 1.3. *Let $W = \overline{W} \setminus D$, where $\overline{W}$ is a compact complex manifold and D is a smooth anticanonical divisor on $\overline{W}$ with holomorphically trivial normal bundle. Let g be an asymptotically cylindrical Ricci-flat Kähler metric on W. Suppose that all the compactifiable infinitesimal deformations of the complex manifold W are integrable (arise as tangent vectors to paths of deformations).*

Then any Ricci-flat asymptotically cylindrical metric on W sufficiently close to g is Kähler with respect to some compactifiable deformation of the complex structure on W.

The additional conditions for the holonomy reduction given in Theorem 1.2 are topological and we deduce from Theorem 1.3.

Corollary 1.4. *Assume that $W = \overline{W} \setminus D$ satisfies the hypotheses of Theorem 1.3. Suppose further that W, $\overline{W}$, and D are simply-connected and so the metric g has holonomy $SU(n)$, $n = \dim_{\mathbb{C}} W$. Then any Ricci-flat asymptotically cylindrical metric on W close to g also has holonomy $SU(n)$.*

Our second main result determines the dimension of the moduli space of the asymptotically cylindrical Ricci-flat Kähler metrics and is given by Theorem 5.1 below.

2. Infinitesimal Ricci-flat deformations

Before dealing with the moduli of asymptotically cylindrical Ricci-flat metrics we recall, in summary, some results on the moduli problem for the Ricci-flat metrics on a compact manifold. The case of a compact manifold is standard and further details can be found in [3, Ch. 12] and references therein.

A natural symmetry group of the equation $\mathrm{Ric}(g) = 0$ for a metric g on a compact manifold X is the group $\mathsf{Diff}X$ of diffeomorphisms of X. It is also customary to identify a metric g with $a^2 g$, for any positive constant a. This is equivalent to considering only the metrics such that X has volume 1. The moduli space of Ricci-flat metrics on X is defined as the space of orbits of all the solutions g of $\mathrm{Ric}(g) = 0$ in the action of $\mathsf{Diff}(X) \times \mathbb{R}_{>0}$,

$$g \mapsto a^2 \phi^* g, \qquad \phi \in \mathsf{Diff}X,\ a > 0,$$

or, equivalently, the space of all $(\mathsf{Diff}X)$-orbits of the solutions of $\mathrm{Ric}(g) = 0$ such that $\mathrm{vol}_g(X) = 1$. The tangent space at g to an orbit of g under the action of $\mathsf{Diff}X$ is the image of the first order linear differential operator

$$\delta_g^* : V^\flat \in \Omega^1(X) \to \tfrac{1}{2}\mathcal{L}_V g \in \mathrm{Sym}^2 T^* X, \tag{4}$$

where $\mathcal{L}$ denotes the Lie derivative. The operator δ_g^* may be equivalently expressed as the symmetric component of the Levi–Civita covariant derivative $\nabla_g : \Omega^1(X) \to \Omega^1 \otimes \Omega^1(X)$, for the metric g,

$$\nabla_g \eta = \delta_g^* \eta + \frac{1}{2} d\eta, \quad \eta \in \Omega^1(X). \tag{5}$$

The L^2 formal adjoint of δ_g^* is therefore given by

$$\delta_g : h \in \mathrm{Sym}^2 T^*X \to \nabla_g^* h \in \Omega^1(X).$$

The operator δ_g^* is overdetermined-elliptic with finite-dimensional kernel and closed image and there is an L^2-orthogonal decomposition

$$\mathrm{Sym}^2 T^*X = \mathrm{Ker}\,\delta_g \oplus \mathrm{Im}\,\delta_g^*.$$

The equation $\delta_g h = 0$ defines a local transverse slice for the action of $\mathsf{Diff}(X)$.

The infinitesimal Ricci-flat deformations h of a Ricci-flat g preserving the volume are obtained by linearizing the equation $\mathrm{Ric}(g+h) = 0$ at $h = 0$, imposing an additional condition $\int_X \mathrm{tr}_g\, h\, \nu_g = 0$, where ν_g is the volume form of g. By a theorem of Berger and Ebin, the space of infinitesimal Ricci-flat deformations of g is given by a system of linear PDEs

$$(\nabla_g^*\nabla_g - 2\overset{\circ}{R}_g)h = 0, \qquad \delta_g h = 0, \qquad \mathrm{tr}_g\, h = 0. \tag{6}$$

Here $\overset{\circ}{R}_g$ is a linear map induced by the Riemann curvature and acting on symmetric bilinear forms, $\overset{\circ}{R}_g h(X,Y) = \sum_i h(R_g(X,e_i)Y, e_i)$ (e_i is an orthonormal basis). The first equation in (6) is elliptic and so the solutions of (6) form a finite-dimensional space.

Suppose that every infinitesimal deformation h satisfying (6) arises as the tangent vector to a path of Ricci-flat metrics. Then it turns out that a neighbourhood of g in the moduli space of Ricci-flat metrics on X is diffeomorphic to the quotient of the solutions space of (6) by a finite group. This finite group depends on the isometry group of g and the moduli space is an orbifold of dimension equal to the dimension of the solution space of (6).

Now suppose that the manifold X has an integrable complex structure, J say, and the Ricci-flat metric g on X is Kähler, with respect to J. Then any deformation h of g may be written as a sum $h = h_+ + h_-$ of Hermitian form h_+ and skew-Hermitian form h_- defined by the conditions $h_\pm(Jx, Jy) = \pm h(x,y)$. Furthermore, the operator $\nabla_g^*\nabla_g - 2\overset{\circ}{R}_g$ preserves the subspaces of Hermitian and skew-Hermitian forms.

The skew-Hermitian forms h_- may be identified, via

$$g(x, Iy) = h_-(x, Jy), \tag{7}$$

with the symmetric real endomorphisms I satisfying $IJ + JI = 0$. Thus $J + I$ is an almost complex structure and the endomorphism I may be regarded as a $(0,1)$-form with values in the holomorphic tangent bundle $T^{1,0}X$. Then one has

$$\delta_g h_- = -J \circ (\bar\partial^* I). \tag{8}$$

In particular, $\delta h_- = 0$ if and only if I defines an class in $H^1(X, T^{1,0}X)$, that is I defines an infinitesimal deformation of the complex manifold (X, J) (see [9]). With the help of

Weitzenböck formula one can replace $\nabla_g^*\nabla_g - 2\overset{\circ}{R}_g$ by the complex Laplacian for $(0,q)$-forms with values in $T^{1,0}X$

$$((\nabla_g^*\nabla_g - 2\overset{\circ}{R}_g)h_-)(\cdot, J\cdot) = g(\cdot, (\Delta_{\bar\partial} I)\cdot).$$

Thus $(\nabla_g^*\nabla_g - 2\overset{\circ}{R}_g)h_- = 0$ precisely when $I \in \Omega^{0,1}(T^{1,0}X)$ is harmonic.

Hermitian forms h_+ are equivalent, with the help of the complex structure, to the real differential (1,1)-forms

$$\psi(\cdot,\cdot) = h_+(\cdot, J\cdot). \tag{9}$$

The Weitzenböck formula yields

$$((\nabla_g^*\nabla_g - 2\overset{\circ}{R}_g)h_+)(\cdot, J\cdot) = \Delta\psi,$$

for a Ricci-flat metric g, thus h_+ satisfies the first equation in (6) if and only if $\psi \in \Omega^{1,1}$ is harmonic. The other two equations in (6) become

$$\operatorname{tr}_g h_+ = \langle \psi, \omega\rangle_g, \quad \text{and} \quad \delta_g h_+ = -d^*\psi, \tag{10}$$

where ω denotes is the Kähler form of g.

3. The moduli problem and a transverse slice

We want to extend the set-up of the moduli space for Ricci-flat metrics outlined in §2 to the case when (M,g) is an asymptotically cylindrical Ricci-flat manifold. For this, we require a Banach space completion for sections of vector bundles associated to the tangent bundle of M and we use Sobolev spaces with exponential weights. A weighted Sobolev space $e^{-\mu t}L^p_k(M)$ is, by definition, the space of all functions $e^{-\mu t}f$ such that $f \in L^p_k(M)$. The norm of $e^{-\mu t}f$ in $e^{-\mu t}L^p_k(M)$ is defined to be the L^p_k-norm of f. The definition generalizes in the usual way to vector fields, differential forms, and, more generally, tensor fields on M with the help of the Levi–Civita connection. Note that if $k - \dim M/p > \ell$, for some integer $\ell \geq 0$, then there is a bounded inclusion map between Banach spaces $L^p_k(M) \to C^\ell(M)$ because (M,g) is complete and has bounded curvature [2, §2.7].

The weighted Sobolev spaces $e^{-\mu t}L^p_k(M)$ are not quite convenient for working with bounded sections that are asymptotically t-independent but not necessarily decaying to zero on the end of M. We shall use slightly larger spaces which we call, following a prototype in [1], the *extended weighted Sobolev spaces*, denoted $W^p_{k,\mu}(M)$.

As before, use Y to denote the cross-section of the end of M. Fix once and for all a smooth cut-off function $\rho(t)$ such that $0 \leq \rho(t) \leq 1$, $\rho(t) = 0$ for $t \leq 1$, and $\rho(t) = 1$ for $t \geq 2$. Define

$$W^p_{k,\mu}(M) = e^{\mu t}L^p_k(M) + \rho(t)L^p_k(Y)$$

where, by abuse of notation, $L^p_k(Y)$ in the above formula is understood as a space of t-independent functions on the cylinder $\mathbb{R}\times Y$ pulled back from Y. Elements in $\rho(t)L^p_k(Y)$ are well-defined as functions supported on the end of M. The norm of $f + \rho(t)f_Y$ in $W^p_{k,\mu}(M)$ is defined as the sum of the $e^{\mu t}L^p_k(M)$-norm of f and the L^p_k-norm of f_Y (where f_Y is interchangeably considered as a function on Y). More generally, the extended weighted Sobolev space of $W^p_{k,\mu}$ sections of a bundle associated to TM is defined in a similar manner using parallel transport in the t direction defined by the Levi–Civita connection.

We shall need some results of the elliptic theory and Hodge theory for an asymptotically cylindrical manifold (M, g). The Hodge Laplacian Δ on M is an instance of an *asymptotically translation invariant* elliptic operator. That it, Δ can be written locally on the end of M in the form $a(t, y, \partial_t, \partial_y)$, where a is smooth in $(t, y) \in \mathbb{R} \times Y$ and polynomial in ∂_t, ∂_y. The coefficients $a(t, y, \partial_t, \partial_y)$ have a t-independent asymptotic model $a_0(y, \partial_t, \partial_y)$ on the cylinder $\mathbb{R} \times Y$, so that $a(t, y, \partial_t, \partial_y) - a_0(y, \partial_t, \partial_y)$ decays to zero, together with all derivatives, as $t \to \infty$.

Proposition 3.1. *Let (M, g) be an oriented asymptotically cylindrical manifold with Y a cross-section of M and let Δ denote the Hodge Laplacian on M. Then there exists $\mu_1 > 0$ such that for $0 < \mu < \mu_1$ the following holds.*

(i) The Hodge Laplacian defines bounded Fredholm linear operators

$$\Delta_{\pm\mu} : e^{\pm\mu t} L^p_{k+2} \Omega^r(M) \to e^{\pm\mu t} L^p_k \Omega^r(M)$$

with index, respectively, $\pm(b^r(Y) + b^{r-1}(Y))$. The image of $\Delta_{\pm\mu}$ is, respectively, the subspace of the forms in $e^{\pm\mu t} L^p_k \Omega^r(M)$ which are L^2-orthogonal to the kernel of $\Delta_{\mp\mu}$.

(ii) Any r-form $\eta \in \operatorname{Ker} \Delta_g \cap e^{\mu t} L^p_{k+2} \Omega^r(M)$ is smooth and can be written on the end $\mathbb{R}_+ \times Y$ of M as

$$\eta|_{\mathbb{R} \times Y} = \eta_{00} + t\eta_{10} + dt \wedge (\eta_{01} + t\eta_{11}) + \eta', \tag{11}$$

where η_{ij} are harmonic forms on Y of degree $r - j$ and the r-form η' is $O(e^{-\mu_1 t})$ with all derivatives. In particular, any L^2 harmonic form on M is $O(e^{-\mu_1 t})$. The harmonic form η is closed and co-closed precisely when $\eta_{10} = 0$ and $\eta_{11} = 0$, i.e. when η is bounded.

Proof. For (i) see [13] or [16]. In particular, the last claim is just a Fredholm alternative for elliptic operators on weighted Sobolev spaces.

The clause (ii) is an application of [15, Theorem 6.2]. Cf. also [16, Prop. 5.61 and 6.14] proved with an assumption that the b-metric corresponding to g is smooth up to and on the boundary of M at infinity. The last claim is verified by the standard integration by parts argument. □

Corollary 3.2. *Assume the hypotheses and notation of Proposition 3.1. Suppose also that the metric g on M is asymptotic to a product cylindrical metric on $\mathbb{R}_+ \times Y$ at an* exponential *rate $O(e^{-\mu_1 t})$. Then for $\xi \in e^{-\mu t} L^p_k \Omega^r(M)$, the equation $\Delta\eta = \xi$ has a solution $\eta \in e^{-\mu t} L^p_{k+2} \Omega^r(M) + \rho(t)(\eta_{00} + dt \wedge \eta_{01})$ if and only if ξ is L^2-orthogonal to $\mathcal{H}^r_{\mathrm{bd}}(M)$, where $0 < \mu < \mu_1$ and $\mathcal{H}^r_{\mathrm{bd}}(M)$ denotes the space of bounded harmonic r-forms on M.*

Proof. The hypotheses on g and μ implies that the Laplacian defines a Fredholm operator

$$e^{-\mu t} L^p_{k+2} \Omega^r(M) + \rho(t)(\eta_{00} + dt \wedge \eta_{01}) \to e^{-\mu t} L^p_{k+2} \Omega^r(M). \tag{12}$$

It follows from Proposition 3.1 that the index of (12) is zero and the kernel is $\mathcal{H}^r_{\mathrm{bd}}(M)$. Further, if $\xi \in e^{-\mu t} L^p_{k+2} \Omega^r(M) + \rho(t)(\eta_{00} + dt \wedge \eta_{01})$ then we find from (11) that $d\xi$ and

$d^*\xi$ decay to zero as $t \to \infty$. Recall from Proposition 3.1 that any bounded harmonic form is closed and co-closed and then the standard Hodge theory argument using integration by parts is valid and shows that the image of (12) is L^2-orthogonal to $\xi \in \mathcal{H}^r_{\mathrm{bd}}(M)$. But as the codimension of the image of (12) is equal to $\dim \mathcal{H}^r_{\mathrm{bd}}(M)$ the image must be precisely the L^2-orthogonal complement of $\mathcal{H}^r_{\mathrm{bd}}(M)$ in $e^{-\mu t}L^p_{k+2}\Omega^r(M)$. □

For an asymptotically cylindrical n-dimensional manifold (M, g), let $\mathsf{Diff}_{p,k,\mu}M$ (where $k - n/p > 1$, $0 < \mu < \mu_1$) denote the group of locally L^p_k diffeomorphisms of M generated by $\exp_V$, for all vector fields V on M that can be written as $V = V_0 + \rho(t)V_Y$, where $V_0 \in e^{-\mu t}L^p_k$ and a t-independent V_Y is defined by a Killing field for g_Y. Respectively, $V_Y^\flat$ is defined by a harmonic 1-form on Y (cf. (14) below). Also require that V has a sufficiently small C^1 norm on M, so that that $\exp_V$ is a well-defined diffeomorphism. Denote by $\mathsf{Metr}_{p,k,\mu}(g)$ (where $k - n/p > 0$, $0 < \mu < \mu_1$) the space of deformations $h + \rho(t)h_Y$ of g where $h \in e^{-\mu t}L^p_k$, $|h|_g < 1$ at every point of M, and $\delta_{g_Y} h_Y = 0$, $\mathrm{tr}_{g_Y} h_Y = 0$. ($g_Y$ is the limit of g as defined in §1.) Then $\mathsf{Diff}_{p,k+1,\mu}M$ acts on $\mathsf{Metr}_{p,k,\mu}(g)$ by pull-backs and the linearization of the action is given by the operator δ^*_g on weighted Sobolev spaces,

$$\delta^*_g : e^{-\mu t}L^p_{k+1}\Omega^1(M) + \rho(t)\mathcal{H}^1(Y) \to e^{-\mu t}L^p_k \operatorname{Sym}^2 T^*M. \tag{13}$$

It will be convenient to replace the last two equations in (6) and instead use another local slice equation for the action of $\mathsf{Diff}_{p,k,\mu}M$

$$\delta_g h + \frac{1}{2} d\,\mathrm{tr}_g\, h = 0.$$

A transverse slice defined by the operator $\delta_g + \frac{1}{2}d\,\mathrm{tr}_g$ was previously used for different classes of complete non-compact manifolds in [4, I.1.C and I.4.B]. The operator $\delta_g + \frac{1}{2}d\,\mathrm{tr}_g$ satisfies a useful relation:

$$(2\delta_g + d\,\mathrm{tr}_g)\delta^*_g = 2\nabla^*_g\nabla_g - \nabla^*_g d + d\,\mathrm{tr}_g\,\delta^*_g = 2\nabla^*_g\nabla_g - d^*_g d - dd^*_g = \Delta_g, \tag{14}$$

where Δ_g is the Hodge Laplacian and we used the Weitzenböck formula for 1-forms on a Ricci-flat manifold in the last equality.

Proposition 3.3. *Assume that Y is connected and that $k - \dim M/p > 1$, $0 < \mu < \mu_1$, where μ_1 is defined in Prop. 3.1 for the Laplacian on differential forms on M. Then there is a direct sum decomposition into closed subspaces*

$$\mathsf{Metr}_{p,k,\mu}(g) = \delta^*_g(e^{-\mu t}L^p_{k+1}\Omega^1(M) + \rho(t)\mathcal{H}^1(Y)) \oplus \big(\mathrm{Ker}(\delta_g + \frac{1}{2}d\,\mathrm{tr}_g) \cap \mathsf{Metr}_{p,k,\mu}(g)\big). \tag{15}$$

Proof. Any bounded harmonic 1-form on M is in $e^{-\mu t}L^p_{k+1}\Omega^1(M) + \rho(t)\mathcal{H}^1(Y)$ by [16, Prop. 6.16 and 6.18] (see also Prop. 4.3 below) and because Y is connected. For any $\eta \in e^{-\mu t}L^p_{k+1}\Omega^1(M) + \rho(t)\mathcal{H}^1(Y)$, $\nabla_g\eta$ decays on the end of M, so the standard integration by parts applies to show that the bounded harmonic 1-forms on M are parallel with respect to g. As the bounded harmonic 1-forms on M are closed we obtain using (5) and (14) that these are in the kernel of δ^*. It follows that the two subspaces in (15) have trivial intersection.

By the definition of $\mathsf{Metr}_{p,k,\mu}(g)$ the 'constant term' h_Y of h satisfies $\delta_Y h_Y = 0$ and $\operatorname{tr}_g h_Y = 0$. If $\eta \in \mathcal{H}^1_{\mathrm{bd}}(M)$ and $h \in \mathsf{Metr}_{p,k,\mu}(g)$ then the 1-form $\langle \eta, h\rangle_g$ decays along the end of M and we can integrate by parts

$$\langle \eta, \delta_g h + \frac{1}{2} d \operatorname{tr}_g h\rangle_{L^2} = \langle \delta^*_g \eta, h\rangle_{L^2} + \frac{1}{2}\langle d^*\eta, \operatorname{tr}_g h\rangle_{L^2} = 0.$$

Thus the image $(\delta_g + \frac{1}{2} d\operatorname{tr}_g)\mathsf{Metr}_{p,k,\mu}(g)$ is L^2-orthogonal to $\mathcal{H}^1_{\mathrm{bd}}$. By Corollary 3.2 the equation $\Delta\eta = (\delta + \frac{1}{2} d\operatorname{tr}_g)h$ has a solution η in $e^{-\mu t} L^p_{k+1,\mu}\Omega^1(M) + \rho(t)\mathcal{H}^1_Y$ and so

$$\delta^*_g\eta - h \in \operatorname{Ker}(\delta_g + \frac{1}{2} d\operatorname{tr}_g) \cap \mathsf{Metr}_{p,k,\mu}(g)$$

which gives the required decomposition $h = \delta^*_g\eta + (\delta^*_g\eta - h)$. □

Proposition 3.4. *Assume that p, k, μ are as in Proposition 3.3. Let $\tilde{g}$ an asymptotically cylindrical deformation of g. If $\tilde{h} = \tilde{g} - g \in \mathsf{Metr}_{pk,\mu}(g)$ is sufficiently small in $W^p_{k,\mu} \operatorname{Sym}^2 T^*M$ then there exists $\phi \in \mathsf{Diff}_{p,k+1,\mu}M$ such that $\phi^*\tilde{g} = g + h$, for some $h \in \mathsf{Metr}_{p,k,\mu}(g)$ with $(\delta_g + \frac{1}{2} d\operatorname{tr}_g)h = 0$.*

Proof. If the desired ϕ is close to the identity then $\phi = \exp_V$ for a vector field V on M with small $e^{-\mu t}L^p_k$ norm. We want to show that the map

$$\mathsf{Diff}_{p,k+1,\mu} \times \{h \in \mathsf{Metr}_{p,k,\mu}(g) : (\delta_g + \frac{1}{2} d\operatorname{tr}_g)h = 0\} \to \mathsf{Metr}_{p,k,\mu}(g)$$

defined by

$$(V, h) \mapsto \exp^*_V(g + h) - g$$

is a onto a neighbourhood of $(0, 0)$. The linearization of $(DF)_{(0,0)}$ is given by $(V, h) \mapsto \delta^*_g(V^\flat) + h$ and is surjective by (15). By the implicit function theorem for Banach spaces, a solution (V, h) of $F(V, h) = \tilde{g}$ exists, whenever $\tilde{g} - g$ is sufficiently small. □

Finally, we obtain the system of linear PDEs describing the infinitesimal Ricci-flat deformations of an asymptotically cylindrical metric transverse to the action of the diffeomorphism group on the asymptotically cylindrical metrics.

Theorem 3.5. *Suppose that (M, g) is a Ricci-flat asymptotically cylindrical Riemannian manifold, but not a cylinder $\mathbb{R} \times Y$, and $g(s)$, $|s| < \varepsilon$ ($\varepsilon > 0$) is a smooth path of asymptotically cylindrical Ricci-flat metrics on M with $g(0) = g$. Suppose also that $g(s) - g \in \mathsf{Metr}_{p,k,\mu}(g)$, with p, k, μ as in Proposition 3.3. Then there is a smooth path $\psi(s) \in \mathsf{Diff}_{p,k,\mu}M$, so that $h = \frac{d}{ds}|_{s=0}\big[\psi(s)^*g(s)\big]$ satisfies the equations*

$$(\nabla^*_g\nabla_g - 2\overset{\circ}{R}_g)h = 0, \tag{16a}$$

$$(\delta_g + \frac{1}{2} d\operatorname{tr}_g)h = 0, \tag{16b}$$

Furthermore, if every bounded solution h of (16) is the tangent vector at g to a path of Ricci-flat asymptotically cylindrical metrics on M then the moduli space is an orbifold.

The dimension of this orbifold is equal to the dimension of the space of solutions of (16) *that are bounded on* M*.*

Proof. Applying Proposition 3.4 for each $g(s)$, we can find a path of diffeomorphisms in $\psi(s) \in \mathsf{Diff}_{p,k,\mu}M$ so that the slice equation (16b) holds for h.

The linearization of $\mathrm{Ric}(g+h)=0$ in h is $\nabla_g^*\nabla_g h - 2\delta_g^*\delta_g h - \nabla_g d\,\mathrm{tr}_g\,h - 2\overset{\circ}{R}_g h = 0$. which becomes equivalent to $(\nabla_g^*\nabla_g - 2\overset{\circ}{R}_g)h = 0$ in view of of (16b) and (5).

The last claim follows similarly to the case of a compact base manifold, cf. [3, 12.C]. It can be shown using Proposition 3.3 that the infinitesimal action of the identity component of the group $I(M,g)$ of isometries of g in $\mathsf{Diff}^p_{k,\mu}M$ is trivial on the slice $(\delta_g + \frac{1}{2}d\,\mathrm{tr}_g)h = 0$. As M is not a cylinder, it has only one end [18] and we show in Lemma 3.6 below that $I(M,g)$ is compact. It follows that a neighbourhood of the orbit of g in the orbit space $\mathsf{Metr}_{p,k,\mu}(g)/\mathsf{Diff}_{p,k,\mu}M$ is homeomorphic to a finite quotient of the kernel of $\delta_g + \frac{1}{2}d\,\mathrm{tr}_g$. □

Lemma 3.6. *Let* (M,g) *be an asymptotically cylindrical manifold with a connected cross-section* Y *(that is,* M *has only one end). Then the group* $I(M,g)$ *of isometries of* M *is compact.*

Proof. It is a well-known result the isometry group $I(M,g)$ of any Riemannian manifold (M,g) is a finite-dimensional Lie group and if a sequence $T_k \in I(M,g)$ is such that, for some $P \in M$, $T_k(P)$ is convergent then T_k has a convergent subsequence [17].

For an asymptotically cylindrical (M,g), it is not difficult to check that there is a choice of point P_0 on the end of M and $r>0$, so that $M_{0,r} = \{P \in M : \mathrm{dist}(P_0,P) > r\}$ is connected but for any P_1 such that $\mathrm{dist}(P_0,P_1) > 3r$ the set $M_{1,r} = \{P \in M : \mathrm{dist}(P_1,P) > r\}$ is not connected. It follows that for any sequence $\tilde{T}_k \in I(M,g)$ we must have $\mathrm{dist}(P_0,\tilde{T}_k(P_0)) \le 3r$ and hence $\tilde{T}_k$ has a convergent subsequence. □

4. Infinitesimal Ricci-flat deformations of asymptotically cylindrical Kähler manifolds

We now specialize to the *Kähler* Ricci-flat metrics. It is known [10] that if an infinitesimal deformation h of a Ricci-flat Kähler metric on a *compact* manifold satisfies the Berger–Ebin equations (6) then the Hermitian and skew-Hermitian components h_+ and h_- of h also satisfy (6). In this section we prove a version of this result for the asymptotically cylindrical manifolds.

Proposition 4.1. *Let* W *be a compactifiable complex manifold with* g *an asymptotically cylindrical Ricci-flat Kähler metric on* W*, as defined in §1. Suppose that an asymptotically cylindrical deformation* $h \in \mathsf{Metr}$ *of* g *satisfies* (16)*. Then the skew-Hermitian component* h_- *of* h *also satisfies* (16)*.*

Proof. The proof is uses the same ideas as in the case of for a compact manifold ([10, §7] or [3, Lemma 12.94]). The operator $\nabla_g^*\nabla_g - 2\overset{\circ}{R}_g$, for a Kähler metric g, preserves the subspaces of Hermitian and skew-Hermitian forms, so $(\nabla_g^*\nabla_g - 2\mathring{R}_g)h_- = 0$. Recall from §2 that the latter equation implies that the form $I \in \Omega^{0,1}(T^{1,0})$ corresponding to h_- via (7) is harmonic, $\Delta_{\bar\partial} I = 0$. An argument similar to that of Proposition 3.1 shows that a bounded harmonic section I satisfies $\bar\partial I = 0$ which implies $\delta_g h_- = 0$ by (8) and, further, $\delta_g - \frac{1}{2} d\,\mathrm{tr}_g\, h_- = 0$ as a skew-Hermitian deformation h_- is automatically trace-free. □

Proposition 4.2. *Any infinitesimal Ricci-flat asymptotically cylindrical deformation $h \in \mathsf{Metr}_{p,k,\mu}(g)$ of a Ricci-flat Kähler asymptotically cylindrical metric g on W is the sum of a Hermitian and a skew-Hermitian infinitesimal deformation.*

The space of skew-Hermitian infinitesimal Ricci-flat asymptotically cylindrical deformations of g is isomorphic to the space of bounded harmonic $(0,1)$-forms on W with values in $T^{1,0}(W)$.

The space of Hermitian infinitesimal Ricci-flat asymptotically cylindrical deformations of g is isomorphic to the orthogonal complement of the Kähler form of g in the space of bounded harmonic real $(1,1)$-forms on W.

Proof. Only the last statement requires justification. Let ω denote the Kähler form of g.

Recall from §2 that the equation $(\nabla_g^*\nabla_g - 2\mathring{R}_g)h_+ = 0$ satisfied by a Hermitian infinitesimal Ricci-flat asymptotically cylindrical deformations h_+ is equivalent to the condition that $\psi \in \Omega^{1,1}(W)$ defined in (9) is harmonic $\Delta\psi = 0$. Hence $\delta_g h_+ = 0$ by (10) and Proposition 3.1 and so the second equation in (16) tells us that $\langle\psi,\omega\rangle_g = \text{const}$, in view of (10). Considering the limit as $t \to \infty$ and the definition of $\mathsf{Metr}_{p,k,\mu}(g)$ we find that the latter constant must be zero. □

Thus in order to find the dimension of the space of infinitesimal Ricci-flat deformations of an asymptotically cylindrical Kähler metric, we may consider the Hermitian and a skew-Hermitian cases separately. This is done in the next subsection.

4.1. Bounded harmonic forms and logarithmic sheaves

It is well-known that harmonic forms on a compact manifold are identified with the de Rham cohomology classes via Hodge theorem. On a non-compact manifold W one can consider the usual de Rham cohomology $H^*(W)$ and also the de Rham cohomology $H_c^*(W)$ *with compact support*. The latter is the cohomology of the de Rham complex of compactly supported differential forms. We shall write $b^r(W) = \dim H^r(W)$ and $b_c^r(W) = \dim H_c^r(W)$, for the respective Betti numbers. There is a natural inclusion homomorphism $H_c^r(W) \to H^r(W)$ whose image is the subspace of the de Rham cohomology classes representable by closed forms with compact support; the dimension of this subspace will be denoted by $b_0^r(W)$.

Proposition 4.3. *Let (W,g) be an oriented asymptotically cylindrical manifold. Then the space $\mathcal{H}_{L^2}(W)$ of L^2 harmonic r-forms on W has dimension $b_0^r(W)$. The space $\mathcal{H}_{\mathrm{bd}}(W)$ of bounded harmonic r-forms on W has dimension $b^r(W) + b_c^r(W) - b_0^r(W)$.*

Proof. For the claim on L^2 harmonic forms see [1, Prop. 4.9] or [14, §7]. In the case when an asymptotically cylindrical metric g corresponds to an exact b-metric smooth up to the boundary at infinity (see Remark 1.1), the dimension of bounded harmonic forms is a direct consequence of [16, Prop. 6.18] identifying a Hodge-theoretic version of the long exact sequence

$$\ldots \to H^{r-1}(Y) \to H^r_c(W) \to H^r(W) \to h^r(Y) \to \ldots. \tag{17}$$

The argument of [16, Prop. 6.18] can be adapted for arbitrary asymptotically cylindrical metrics; the details will appear in [12]. □

If W is an asymptotically cylindrical Kähler manifold then there is a well-defined subspace $\mathcal{H}^{1,1}_{\mathrm{bd},\mathbb{R}}(W) \subset \mathcal{H}^2_{\mathrm{bd}}(W)$ of bounded harmonic real forms of type $(1,1)$. The bounded harmonic 2-forms in the orthogonal complement of $\mathcal{H}^{1,1}_{\mathrm{bd},\mathbb{R}}(W)$ are the real and imaginary parts of bounded harmonic $(0,2)$-forms. We shall denote the complex vector space of bounded harmonic $(0,2)$-forms on W by $\mathcal{H}^{0,2}_{\mathrm{bd}}(W)$.

The space of bounded harmonic real (1,1)-forms on W orthogonal to the Kähler form ω therefore has dimension $b^r(W) + b^r_c(W) - b^r_0(W) - 1 - 2\dim_{\mathbb{C}} \mathcal{H}^{0,2}_{\mathrm{bd}}(W)$.

Now for the skew-Hermitian infinitesimal deformations. Recall from §1 that the definition of an asymptotically cylindrical Ricci-flat Kähler manifold (M, J, ω) includes the condition that a complex manifold $W = (M, J)$ is compactifiable. That is, there exist a compact complex n-fold $\overline{W}$ and a compact complex $(n-1)$-dimensional submanifold D in $\overline{W}$, so that W is isomorphic to $\overline{W} \setminus D$. We saw in Proposition 4.1 that any skew-Hermitian Ricci-flat asymptotically translation-invariant deformation of ω can be expressed as a $\bar\partial$- and $\bar\partial^*$-closed symmetric $(0,1)$-form I with values in the holomorphic tangent bundle of W. A $\bar\partial$- and $\bar\partial^*$-closed such I, not necessarily symmetric, defines an infinitesimal deformation $J + I$ of the integrable complex structure J on W. The deformations given by skew-symmetric such forms I correspond to the bounded harmonic $(2,0)$-forms on W.

Let z denote a complex coordinate on $\overline{W}$ so that D is defined by the equation $z = 0$, as in §1. Let $T_{\overline{W}}$ denote the sheaf of holomorphic local vector fields on $\overline{W}$. The subsheaf of the holomorphic local vector fields whose restrictions to D are tangent to D is denoted by $T_{\overline{W}}(\log D)$ and called the *logarithmic tangent sheaf.* The form I in general has a simple pole precisely along D and defines a class in the Čech cohomology $H^1(T_{\overline{W}}(\log D))$. The classical Kodaira–Spencer–Kuranishi theory of deformations of the holomorphic structures on compact manifolds [9] has an extension for the compactifiable complex manifolds; the details can be found in [8]. In this latter theory, the cohomology groups $H^i(T_{\overline{W}}(\log D))$ have the same role as the cohomology of tangent sheaves for the compact manifolds. In particular, the isomorphisms classes of infinitesimal deformations of W are canonically parameterized by classes in $H^1(T_{\overline{W}}(\log D))$. These classes arise from the actual deformations of W is the obstruction space $H^2(T_{\overline{W}}(\log D))$ vanishes.

Thus the space of the skew-Hermitian Ricci-flat asymptotically cylindrical deformations I of the Ricci-flat Kähler asymptotically cylindrical metric g on W is identified as a subspace of the infinitesimal compactifiable deformations of W. The real dimension of this subspace is $2(\dim_{\mathbb{C}} H^1(T_{\overline{W}}(\log D)) - \dim_{\mathbb{C}} \mathcal{H}^{0,2}_{\mathrm{bd}}(W))$.

5. The asymptotically cylindrical Ricci-flat deformations

In this section, we show that every infinitesimal Ricci-flat deformation of an asymptotically cylindrical Ricci-flat Kähler manifold is tangent to a genuine deformation.

Theorem 5.1. *Let (W, g) be as in Theorem 1.3. Then every bounded solution h of* (16) *arises as $h = \frac{d}{ds}|_{s=0} g(s)$ for some path of asymptotically cylindrical Ricci-flat metrics on W with $g(0) = g$. The moduli space of asymptotically cylindrical Ricci-flat deformations of g is an orbifold of real dimension*

$$2\dim_{\mathbb{C}} H^1(T_W(\log D)) + b^2(W) + b^2_c(W) - b^2_0(W) - 1 - 4\dim_{\mathbb{C}} \mathcal{H}^{2,0}_{\mathrm{bd}}(W).$$

Proof. By the hypotheses of Theorem 1.3, there is a manifold $\mathcal{M}$ of small compactifiable deformations of W, so that $H^1(T_{\overline{W}}(\log D))$ is the tangent space to $\mathcal{M}$ at W. The data of the compactifiable deformations of W includes the deformations of $\overline{W}$ [8]. Let ω' be a Kähler metric on $\overline{W}$. By the results of Kodaira and Spencer [9], for a family of sufficiently small deformations $\overline{I}$ of a compact complex manifold $\overline{W}$, there is a family of forms $\omega'(\overline{J}+\overline{I})$ on $\overline{W}$ depending smoothly on $\overline{I}$ and such that $\omega'(\overline{J}) = \omega'$ and $\omega'(\overline{J}+\overline{I})$ defines a Kähler metric with respect to a perturbed complex structure $\overline{J}+\overline{I}$. Using the methods of [11, §3], we can construct from $\omega'(\overline{J}+\overline{I})$ a smooth family $\omega(J+I)$ of asymptotically cylindrical Kähler metrics (not necessarily Ricci-flat) on the respective deformations of $W = \overline{W} \setminus D$.

Consider a vector bundle $\mathcal{V}$ over $\mathcal{M}$ whose fibre over $\overline{I} \in \mathcal{M}_{\overline{W}}$ is the space of bounded harmonic (1,1)-forms with respect to the Kähler metric $\omega(\overline{J}+\overline{I})$. The task of integrating an infinitesimal Ricci-flat deformation of the given asymptotically cylindrical Kähler metric ω on W is expressed by the complex Monge–Ampére equation (with parameters) for a function u on W

$$(\omega(J+I) + \beta + i\partial\bar{\partial}u)^n - e^{f_{I,\beta}}(\omega(J+I) + \beta)^n = 0, \tag{18}$$

where $n = \dim_{\mathbb{C}} W$ and $\beta \in \mathcal{V}$ is a bounded harmonic real (1,1)-form with respect to the Kähler metric $\omega(J+I)$ and orthogonal to $\omega(J+I)$. The operators $\partial,\bar{\partial}$ in (18) are those defined by $J+I$.

If $I = 0$ and $\beta = 0$ then $u = 0$ is a solution of (18) as the metric ω is Ricci-flat. Consider the right-hand side of (18) as a function $f(I, \beta, u)$ where the domain of u is a version of extended weighted Sobolev space $E^p_{k,\mu}(W) = e^{-\mu t} L^p_{k+2}(W) + \{\rho(t)(at+b) \mid a, b \in \mathbb{R}\}$ for a sufficiently small $\mu > 0$ ($Y = S^1 \times D$ is the cross-section of W in the present case and D is connected). The linearization of f in u at $u = 0$ is the Laplacian for functions on the asymptotically cylindrical Kähler manifold (W, ω). A dimension counting argument similar to that in Corollary 3.2 shows that this latter Laplacian defines a *surjective* linear map $E^p_{k,\mu}(W) \to e^{-\mu t} L^p_k(W)$. The Laplacian has a one-dimensional kernel given by the constant functions on W, so we reduce the domain for u by taking the L^2 orthogonal complement of the constants. Then the implicit function theorem applies to $f(I, \beta, u)$

and defines a smooth family $u = u(I, \beta)$ so that $f(I, \beta, u(I, \beta)) = 0$ for every small I, β in the respective spaces of bounded harmonic forms. This defines a smooth family of Ricci-flat metrics $\omega(J + I) + \beta + i\partial\bar{\partial}u(I, \beta)$ tangent to the infinitesimal deformations identified in the previous section. $\square$

6. Examples

In this section, we consider some examples of asymptotically cylindrical Ricci-flat Kähler manifolds arising by application of Theorem 1.2 and compute the dimension of the moduli space for their asymptotically cylindrical Ricci-flat deformations. This is done by considering appropriate long exact sequences and applying vanishing theorems to determine the dimensions of cohomology groups appearing in Theorem 5.1.

6.1. Rational elliptic surfaces

An elliptic curve $C = \mathbb{C}/\Lambda$ embeds in the complex projective plane as a cubic curve in the anticanonical class. Choosing another non-singular elliptic curve C' in $\mathbb{C}P^2$ we obtain a pencil $aC + bC'$, $a : b \in \mathbb{C}P^1$. Assuming that C' is chosen generically and blowing up the 9 intersection points $C \cap C'$ we obtain an algebraic surface $\tilde{S}$ so that the proper transform $\tilde{C}$ of C is in the anticanonical class, $\tilde{C} \in |-K_{\tilde{S}}|$, and $\tilde{C}$ has a holomorphically trivial normal bundle, in particular $\tilde{C} \cdot \tilde{C} = 0$. Then, by Theorem 1.2, the quasiprojective surface $S = \tilde{S} \smallsetminus \tilde{C}$ has a complete Ricci-flat Kähler metric asymptotic to the flat metric on the half-cylinder $\mathbb{R}_{>0} \times S^1 \times \mathbb{C}/\Lambda$ with cross-section a 3-dimensional torus. Although in this example the divisor at infinity is not simply-connected it can be easily checked that S is simply-connected and the asymptotically cylindrical Ricci-flat Kähler metric on S has holonomy $SU(2)$ (cf. [11, Theorem 2.7]). It is well-known that a Ricci-flat Kähler metric on a complex surface is hyper-Kähler.

Furthermore, S is topologically a 'half of the K3 surface' in the sense that there is an embedding of a 3-torus T^3 in the K3 surface so that the complement of this T^3 consists of two components, each homeomorphic to S. From the arising Mayer–Vietoris exact sequence, we find that $b^2(S) = b^2_c(S) = 11$ using also the Poincaré duality. The long exact sequence (17) with $W = S$ and $Y = T^3$ yields $b^2_0 = 8$.

As S is simply-connected with holonomy $SU(2)$ there is a nowhere-vanishing parallel (hence holomorphic) $(2,0)$-form Ω on S. Any other $(2,0)$-form on S can be written as $f\Omega$ for some complex function f and $f\Omega$ will be a bounded harmonic form if and only if the real and imaginary parts of f are bounded harmonic functions, hence constants by the maximum principle. Thus $\dim_{\mathbb{C}} \mathcal{H}^{2,0}_{\mathrm{bd}}(S) = 1$.

The dimensions of $H^1(T_S(\log \tilde{C}))$ and $H^2(T_S(\log \tilde{C}))$ are obtained by taking the cohomology of the exact sequences

$$0 \to T_{\tilde{S}}(-C) \to T_{\tilde{S}}(\log \tilde{C}) \to T_{\tilde{C}} \to 0$$

and

$$0 \to T_{\tilde{S}}(\log \tilde{C}) \to T_{\tilde{S}} \to N_{\tilde{C}/\tilde{S}} \to 0$$

(see [8]). Using Serre duality [6] we find that $H^2(T_{\tilde S}(-C)) = H^0(\Omega^1_{\tilde S}) = H^{0,1}(S) = 0$, hence $H^2(T_S(\log \tilde C))$ vanishes and the compactifiable deformations of S are unobstructed. Note that any small deformation of $\tilde S$ is the blow-up of a small deformation of the cubic $\tilde C$ in $\mathbb{C}P^2$ ([5] or [7, Theorem 9.1]). Therefore, $\dim_{\mathbb{C}} H^1_{\tilde S} = 10$ and we deduce that $\dim_{\mathbb{C}} H^1(T_S(\log \tilde C)) = 10$.

Now by Theorem 5.1 the moduli space of asymptotically cylindrical Ricci-flat deformations of S has dimension 29. All these deformations are hyper-Kähler with holonomy $SU(2)$.

6.2. Blow-ups of Fano threefolds

A family of examples of asymptotically cylindrical Ricci-flat Kähler threefolds is constructed in [11, §6] using Fano threefolds. A Fano threefold is a non-singular complex threefold V with $c_1(V) > 0$. Any Fano threefold is necessarily projective and simply-connected. A generically chosen anticanonical divisor D_0 in V is a K3 surface [19]. Let $D_1 \in |-K_V|$ be another K3 surface such that $D_0 \cap D_1 = C$ is a smooth curve.

The blow-up of V along C is a Kähler complex threefold $(\overline{W}, \omega')$ and the proper transform $D \subset \overline{W}$ of D_0 is an anticanonical divisor on $\overline{W}$ with the normal bundle of D holomorphically trivial. The complement $W = \overline{W} \smallsetminus D$ is simply-connected.

Thus W is topologically a manifold with a cylindrical end $\mathbb{R}_{>0} \times S^1 \times D$. By Theorem 1.2 W admits a complete Ricci-flat Kähler metric ω, with holonomy $SU(3)$. The metric ω is asymptotic on the end of W to the product of the standard flat metric on $\mathbb{R}_{>0} \times S^1$ and a Yau's hyper-Kähler metric on D.

By the Weitzenböck formula, the Hodge Laplacian Δ for the $(2,0)$-forms on a Ricci-flat Kähler manifold can be expressed as $\Delta = \nabla_g^* \nabla_g$. The quantity $\langle \nabla\eta, \eta\rangle_g$ for a bounded harmonic form η decays on the end of W, so we can integrate by parts to show that a bounded harmonic $(2,0)$-form is parallel. But the holonomy of the metric ω is $SU(3)$ which has no invariant elements in $\Lambda^{2,0}\mathbb{C}^3$. Therefore, W admits no parallel $(2,0)$-forms and thus no bounded harmonic $(2,0)$-forms.

The dimension of the moduli space for asymptotically cylindrical Ricci-flat deformations of ω then becomes $2\dim_{\mathbb{C}} H^1(T_W(\log D)) + b^2(W) + b^2_c(W) - b^2_0(W) - 1$.

The dimensions of $H^i(T_W(\log D))$, $i = 1, 2$, are obtained from the two long exact sequences similar to §6.1. To verify that the compactifiable deformations of W are unobstructed note that $H^2(T_{\overline{W}}) = H^1(\Omega^1_{\overline{W}}(-D)) = 0$ by the Kodaira vanishing theorem and $H^1(N_{D/\overline{W}}) = H^{1,0}(D) = 0$. It is shown in [11, §8] that $b^2(W) = \rho(V)$ and $h^{2,1}(\overline{W}) = h^{2,1}(V) + g(V)$, where $g(V) = -K_V^3/2 + 1$ is the genus of V and $\rho(V)$ is the Picard number. Taking the cohomology of $0 \to T_{\overline{W}}(-D) \to T_{\overline{W}}(\log D) \to T_D \to 0$ we obtain $\dim_{\mathbb{C}} H^1(T_W(\log D)) = 20 + h^{2,1}(V) + g(V) - \rho(V)$. From the long exact sequence (17) we find that $b^2(W) + b^2_c(W) - b^2_0(W) = b^2(W) + 1$.

Thus the dimension of the moduli space for W in this example is given by

$$b^3(V) + 2g(V) - \rho(V) + 40$$

in terms of standard invariants of the Fano threefold.

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