

L^2 harmonic theory, Seiberg-Witten theory and asymptotics of differential forms

Tsuyoshi Kato

ABSTRACT. We present a pair of open smooth 4-manifolds that are mutually homeomorphic. One of them admits a Riemannian metric that possesses quasi-cylindricity, and positivity of scalar curvature, and of dimension of certain L^2 harmonic forms. By contrast, for the other manifold, no Riemannian metric can simultaneously satisfy these properties. Our method uses Seiberg-Witten theory on compact 4-manifolds and applies L^2 harmonic theory on non-compact, complete Riemannian 4-manifolds. We introduce a new argument to apply Gauge theory, which arises from a discovery of an asymptotic property of the range of the differential.

1. Introduction

It is a basic question to ask how a smooth structure influences the global Riemannian structure of a smooth manifold X . The de Rham cohomology group is given a priori by using a smooth structure on X , and is actually a topological invariant. If we set a Riemannian metric g on X , where X is compact, then each element admits a harmonic representative.

If X is non-compact, we obtain (un-)reduced L^2 cohomology groups by using g . In contrast to the compact case, these cohomology groups depend on the choice of complete Riemannian metrics. It is well known that the reduced L^2 cohomology group of (X, g) is isomorphic to the space of L^2 harmonic forms. So it would be interesting to ask how a particular choice of smooth structure on X influences the structure of L^2 harmonic forms on X . Let us say that a closed differential form $u \in \Omega^*(X)$ is L^p exact at infinity, if there is a compact subset $K \subset X$ and a differential form $\alpha \in \Omega^{*-1}(X \setminus K)$ such that it is exact outside K , with finite L^p norm $\|\alpha\|_{L^p(X \setminus K)} < \infty$.

$$u|_{X \setminus K} = d\alpha$$

The L^p -exactness was originally introduced by Gromov to study the Singer conjecture [G]. In fact, it has been verified that if a compact Kähler manifold (M, ω) satisfies the property that the lift $\tilde{\omega}$ of the Kähler form on the universal covering space $X := \tilde{M}$ is exact $\tilde{\omega} = d\alpha$ with $\|\alpha\|_{L^\infty(X)} < \infty$, then the L^2 -Betti numbers all vanish except the middle degree. Moreover the L^2 -Betti number at the middle degree does not vanish. In particular the Hopf conjecture holds, that states $(-1)^m \chi(M) > 0$, where χ is the Euler characteristic and $\dim_{\mathbb{C}} M = m$.

The Singer conjecture has been applied to study of 4-dimensional differential topology through Gauge theory. Furuta has verified 10/8-type inequality for a compact spin 4-manifold [Fu]. Combining of a covering version of the Furuta's 10/8-type inequality with the Singer conjecture leads us to the aspherical 10/8-type inequality, that replaces the self-dual Betti number in the Furuta's inequality by the Euler characteristic (Section 1 in [K2]). The covering version of the Furuta's 10/8-type inequality is satisfied for compact spin 4-manifolds with residually finite fundamental groups.

Let (X, g) be a complete Riemannian manifold, and take an exhaustion

$$K_1 \subset\subset K_2 \subset\subset \cdots \subset\subset X$$

by compact subsets, where $K \subset\subset L$ means that the interior \mathring{L} contains K . Let us say that the family $\{K_i\}_i$ is isometric-pasting, if there is $\epsilon > 0$ and diffeomorphisms

$$\phi_i : K_i \cong K_{i+1}$$

such that the restrictions $\phi_i : N_\epsilon(\partial K_i) \cong N_\epsilon(\partial K_{i+1})$ are isometric, where $N_\epsilon(\partial K_i) \subset K_i$ is an ϵ neighborhood.

Definition 1.1. (X, g) is quasi-cylindrical-end, if it admits an isometric-pasting family.

We have the following basic example.

Lemma 1.2. *A Riemannian manifold with cylindrical-end is quasi-cylindrical-end.*

Proof. X is isometric to a cylindrical-end manifold of the form $X_0 \cup Y \times [0, \infty)$. Then, we set $K_i := X_0 \cup Y \times [0, i + \epsilon]$ with $N_\epsilon(\partial K_i) = Y \times [i, i + \epsilon]$ for $i \geq 1$. Let $f_i : [0, i + \epsilon] \rightarrow [0, i + 1 + \epsilon]$ be a diffeomorphism with $f_i(t) = t$ for $t \in [0, \frac{1}{2}]$ and $f_i(t) = t + 1$ for $t \in [i, i + \epsilon]$. Then f_i extends to the desired diffeomorphism $\phi_i : K_i \cong K_{i+1}$. \square

Scalar curvature is another basic invariant of complete Riemannian manifolds (X, g) . In particular, in the non-compact case, uniform positivity of the scalar curvature allows us to construct a Fredholm theory of Dirac operators, and apply it to study the topology of manifolds [GL2]. Note that there is a difference between the existence of positive and flat scalar curvatures. For example any torus can admit a flat metric, but cannot admit any metric of positive scalar curvature. In this paper, we treat an intermediate class that consists of complete Riemannian manifolds with a positive scalar curvature that are not assumed to be uniform. In our non-uniform case, we cannot expect to obtain a Fredholm theory as above.

Let us use $(*)$ to denote if (X, g) satisfies the following conditions:

$$(*) \left\{ \begin{array}{l} (X, g) \text{ is quasi-cylindrical-end and has positive scalar curvature,} \\ \dim \mathcal{H}_e^+(X, g) > 0 \text{ is positive} \end{array} \right.$$

where $\mathcal{H}_e^+(X, g)$ is the space of self-dual L^2 harmonic forms that are L^2 exact at infinity.

In this paper we present a pair of smooth 4-dimensional open manifolds which have the following characteristics.

Theorem 1.3. *There is a pair of oriented smooth 4-dimensional open manifolds X and X' with the following properties:*

- (1) X and X' are mutually homeomorphic.
- (2) X' admits a complete Riemannian metric with $(*)$.
- (3) X cannot admit any complete Riemannian metric with $(*)$.

Our proof is based on a new approach to Seiberg-Witten theory based on the theory of L^2 harmonic forms over complete Riemannian 4-manifolds. Among the three conditions $(*)$ as defined above, both quasi-cylindricity and positivity of scalar curvature are used to conclude that a SW solution at the limit of metric deformation on X consists of a zero spinor section. It allows us to apply L^2 harmonic theory on complete Riemannian manifolds.

Remark 1.4. Quasi-cylindricity is a differential-topological condition, and it is not known whether X above admits such a structure. Note that quasi-cylindricity condition on X does not involve smooth embedding $X \subset M$. On the other hand in our case in Theorem 1.3, there is a smooth embedding $X \subset M$ into a compact 4-manifold. Quasi-cylindricity is used to guarantee two metric properties that (1) the scalar curvatures satisfy a uniformly lower bound from below, and that (2) volumes of compact subsets $M \setminus X$ are uniformly bounded from above, during metric deformation. Actually we can replace the condition of quasi-cylindricity by the conclusions of metric properties in Lemma 4.1, if we focus on an open 4-manifold X with a smooth embedding into a compact smooth 4-manifold M .

We believe that our method could still work without the above two conditions. We conjecture that Theorem 1.3 can still hold (for the same example X and X' above), if we replace the condition $(*)$ by

$$(*)' \quad \dim \mathcal{H}_e^+(X, g) > 0 \text{ is positive.}$$

To follow a parallel argument without such conditions, one will have to construct Seiberg-Witten moduli theory over X . So far Gauge theory over end-periodic manifolds has been extensively developed [T]. Our main result has been known under the stronger assumption of end-periodic metrics. The end-periodic condition allows one to apply the analytic method of the Taubes-Fourier transform by attaching the boundaries of the building-block of the periodic-end, that consists of a compact 4-manifold. The analytic setting gives a Fredholm theory of the linearized operators of the SW equations with respect to the end-periodic metric. Even though a small perturbation can be applied to the original end-periodic metric and still the Fredholm property is preserved, the analytic mechanism essentially requires existence of an end-periodic metric. By contrast, in our condition of quasi-periodicity, we have used two properties of metrics that touch essentially on uniformity of estimates (see Remark 1.4). Hence, it would be quite difficult to extend the techniques of end-periodic case, and apply it to our case. In particular, it is not easy to construct moduli theory for any wider classes of open Riemannian 4-manifolds such as

the quasi-cylindrical-end case. In such situations, the de Rham differentials do not have closed range in general, and so the standard Fredholm property breaks. There has been some partial progress in this direction [K3]. The quasi-cylindricity concerns existence of asymptotically smooth exhaustion by a compact building-block subsets, whose overlap widths are isometric. It would be of interest for us to ask whether an exotic R^4 could admit the asymptotically smooth exhaustion by a compact building-block subsets in our sense.

Our main analytic tool is given by the following Theorem. Let (X, g) be an oriented complete Riemannian 4-manifold, and take an exhaustion $K_1 \subset\subset K_2 \subset\subset \cdots \subset\subset X$ by compact subsets. We say that an element $u \in L^2(X; \Lambda^*)$ is an L^2 harmonic form, if it satisfies the equations $du = d^*u = 0$. Corollary 2.10 verifies the following property.

Theorem 1.5. *Suppose a non-zero L^2 harmonic self-dual 2 form $0 \neq u \in \mathcal{H}_e^+(X; \mathbb{R})$ exists, which is L^2 exact at infinity. Then there is no family $a_i \in \Omega^1(K_i)$ such that*

- (1) *the convergence $d^+(a_i) \rightarrow u$ holds in L^2 on each compact subset, and*
- (2) *the uniform bound $\|d(a_i)\|_{L^2(K_i)} \leq C < \infty$ holds.*

Let us consider a basic case where (X, g) is a Riemannian 4-manifold with cylindrical-end so that there is an isometry $\text{end}(X) \cong Y \times [0, \infty)$, where Y is a compact oriented Riemannian three-manifold. The following Lemma is well known (see Proposition 2.12).

Lemma 1.6. *Assume that Y is a rational homology sphere. Then the following spaces of (X, g) are all isomorphic:*

- *The unreduced self-dual L^2 cohomology group.*
- *The reduced self-dual L^2 cohomology group.*
- *The space of self-dual L^2 harmonic forms.*
- *The self-dual de Rham cohomology group.*

See Definition 2.3 below. Recall that for a compact oriented Riemannian 4-manifold M , the self-dual de Rham cohomology group is defined by the cokernel of $d : \Omega^1(M) \rightarrow \Omega^+(M)$ in $\Omega^+(M)$, where $\Omega^+(M)$ is the space of self-dual smooth 2-forms and $\Omega^1(M)$ is the space of smooth 1-forms on M .

Our example of the pair (X, X') in Theorem 1.3 satisfies the following properties.

- Both X and X' can be smoothly embedded into a compact smooth 4-manifold $S := S^2 \times S^2 \# S^2 \times S^2 \# S^2 \times S^2$.
- X' is given by the complement of one point $X' := S \setminus \text{pt}$.
- There is a closed set that is homeomorphic to the 4-dimensional closed disc D with $X := S \setminus D$.

Let us equip X' above with a cylindrical-end metric g' , and verify that (X', g') satisfies the required properties in Theorem 1.3.

Lemma 1.7. [GL1] *Let N, N' be compact manifolds of dimension $n \geq 3$. Assume they admit metrics of positive scalar curvature. Then,*

- (1) *their connected sum $N \# N'$ also admits a metric of positive scalar curvature, and*
- (2) *$N \setminus \text{pt}$ also admits a cylindrical-end metric of positive scalar curvature.*

Proof. See [GL1] pages 425 – 429. □

$S^2 \times S^2$ admits a metric of positive scalar curvature. Thus $S = S^2 \times S^2 \# S^2 \times S^2 \# S^2 \times S^2$ also admits a metric of positive scalar curvature by Lemma 1.7 (1). Then $X' := S \setminus \text{pt}$ admits a cylindrical-end metric g' of positive scalar curvature by Lemma 1.7 (2).

Since the self-dual de Rham cohomology group on S is non-zero, the self-dual L^2 cohomology group on X' is also non zero by Lemma 1.6. It follows from Proposition 2.12 that any self-dual L^2 harmonic form on a cylindrical-end 4-manifold is L^2 exact at infinity, if the cross section is a rational homology sphere. Thus with Lemma 1.2, we have verified that (X', g') possesses the required properties in Theorem 1.3.

Let us roughly describe our strategy for the rest of the proof of Theorem 1.3. It is well known that the Seiberg-Witten invariant is invariant under any choice of generic Riemannian metrics. In particular, a solution exists for any metric, if the invariant is non-zero. Let M be the $K3$ surface. It satisfies two remarkable properties:

- (1) It admits a spin structure and the SW invariant is non-zero with respect to the spin structure (see [M]).
- (2) M has an open subset $X \subset M$ that is diffeomorphic to $S \setminus D$ as above (see [FU]).

The second property arises from Casson-Freedman theory [Fr], which has a very different aspect from the former one. Our argument uses a family of Riemannian metrics on M that converges to a complete Riemannian metric g on X on each compact subset. There is a family of perturbed SW solutions with respect to these metrics, and we study the asymptotic behavior of this family of solutions. We apply the following idea. Let us choose an exhaustion $K_0 \subset \subset K_1 \subset \subset \dots \subset \subset X$ by compact subsets with a family of Riemannian metrics h_i on M with $h_i|_{K_i} = g|_{K_i}$. Since the SW invariant is non zero, there are solutions to the perturbed SW equation with respect to h_i . Passing through a limiting procedure, one should be able to obtain a solution to the perturbed SW equation over (X, g) . However L^2 harmonic theory excludes such a situation.

Because our argument is quite general, we can obtain more examples which satisfy the conclusion of Theorem 1.3 for any simply connected spin 4-manifold M with a non-zero Seiberg-Witten invariant with respect to the spin structure.

The prototype of the argument of such a limiting process was given for the class of manifolds with cylindrical-end. In particular, one can verify the fact that a $K3$ surface does not admit smoothly connected-sum decomposition in which the homology of one side corresponds to the E_8 -summand of $H_2(M; \mathbb{Z})$ [DK]. This result is based on the construction of moduli theory over cylindrical-end 4-manifolds.

If one tries to apply the same argument for more general classes of open Riemannian 4-manifolds, a striking difficulty appears that at limit, the solution is generally far from L^2 . This essentially comes from the fact that the L^2 de Rham differential does not have closed range in general. However, as far as we know, our result is first for a metric property in the situation where even linear Fredholm theory cannot be applied.

2. L^2 harmonic forms

Let (X, g) be a complete Riemannian 4-manifold.

2.1. De Rham differential

We start by observing the following basic property. For simplicity of the argument, we assume that $\text{end}(X)$ is homeomorphic to $[0, \infty) \times S^3$. Let $H_c^*(X; \mathbb{R})$ be the de Rham cohomology with compact support. We also use the notation $\Omega^*(X) := C^\infty(X; \Lambda^*)$. If X_0 is a manifold with boundary, then $\Omega_c^p(X_0)$ is the space of compactly supported smooth p -forms that vanish on the boundary.

Lemma 2.1. *Suppose that an element $[u] \in H_c^2(X; \mathbb{R})$ satisfies the positivity condition $\int_X u \wedge u > 0$. Then there are no families $a_l \in \Omega^1(X)$ such that the convergence*

$$d(a_l) \rightarrow u$$

holds in C^∞ on each compact subset.

Proof. Consider an embedded Riemann surface $\Sigma \subset X$ which represents a Poincaré dual class to u (see [BoT], page 44). Suppose such a family $\{a_l\}_l$ exists. Then by Stokes' theorem, the convergence

$$\begin{aligned} 0 &< \int_X u \wedge u = \int_\Sigma u \\ &= \int_\Sigma (u - d(a_l)) + \int_\Sigma d(a_l) = \int_\Sigma (u - d(a_l)) \rightarrow 0 \end{aligned}$$

must hold, which cannot happen. □

Let

$$d^+ : L_1^2(X; \Lambda^1) \rightarrow L^2(X; \Lambda^+)$$

be the composition of the differential with the projection of two forms to the self-dual part. We refer to this as the self-dual differential. The above argument heavily depends on the Stokes theorem, and it cannot be directly applied to the self-dual differential in general. However, a parallel argument can still work for a certain L^2 harmonic form. An element $u \in L^2(X; \Lambda^+)$ is called an L^2 harmonic self-dual 2 form, if it satisfies the equations

$$du = d^*u = 0.$$

One can obtain L^2 harmonic self-dual 2-forms in the following way.

Lemma 2.2. *Let $k \geq 1$. Suppose $d^+ : L_k^2(X; \Lambda^1) \rightarrow L_{k-1}^2(X; \Lambda^+)$ has closed range. Then, any element in the co-kernel space can be represented by an L^2 harmonic self-dual 2-form.*

Note that d^+ does not always have closed range if (X, g) is non-compact.

Definition 2.3. (1) The unreduced self-dual L^2 cohomology group is given by

$$L^2(X; \Lambda^+)/d^+(L_1^2(X; \Lambda^1)).$$

(2) The reduced self-dual L^2 cohomology group is given by

$$H^+(X, g) := L^2(X; \Lambda^+)/\overline{d^+(L_1^2(X; \Lambda^1))}$$

where the notation $\overline{}$ denotes the closure.

(3) We denote by $\mathcal{H}^+(X, g)$ the space of L^2 harmonic self-dual 2-forms.

Lemma 2.4. *The inclusion $\mathcal{H}^+(X, g) \hookrightarrow L^2(X; \Lambda^+)$ induces an isomorphism $\mathcal{H}^+(X, g) \cong H^+(X, g)$.*

Proof. This is well known. □

2.2. Asymptotics of the differential image

Let us introduce a method of cut-off function, whose idea has appeared in [G]. The author is thankful to M. Furuta for discussion on how to use a family of a cut-off functions, instead of boundary integrals. Let $K_i \subset\subset K_{i+1} \subset\subset \cdots \subset\subset X$ be an exhaustion by compact subsets, and take cut-off functions

$$\chi_i : X \rightarrow [0, 1]$$

with $\chi_i|_{K_{i-1}} \equiv 1$ and $\chi_i|(K_i)^c \equiv 0$ such that

$$\lim_{i \rightarrow \infty} \|d\chi_i\|_{L^\infty(X)} = 0$$

holds. Such a family of cut-off functions exists when (X, g) is non-compact and complete.

Lemma 2.5. *Suppose a non-zero L^2 harmonic self-dual 2-form*

$$0 \neq u \in \mathcal{H}^+(X; \mathbb{R})$$

exists. Then, there is no sequence $a_i \in \Omega^1(K_i)$ with uniform bound

$$\|a_i\|_{L_1^2(K_i)} \leq c < \infty$$

such that the convergence

$$d^+(a_i) \rightarrow u = u^+$$

holds in L^2 norm on each compact subset.

Remark 2.6. One can replace $a_i \in \Omega^1(K_i)$ by $a_i \in \Omega_c^1(X)$ by using suitable cut-off functions, and the same conclusion holds under the same conditions. This is also the case in Theorem 2.9 and Corollary 2.10

Proof. Step 1: Suppose the sequence exists. For any $\delta > 0$, there is a compact subset $K \subset X$ such that $\|u\|_{L^2(K_i \setminus K)} \leq \|u\|_{L^2(X \setminus K)} < \delta$ hold for all large i .

By contrast, there is i_0 such that for any $i \geq i_0$,

$$\|u - d^+ a_i\|_{L^2(K)} < \delta$$

also holds. Then, the following equalities hold:

$$\begin{aligned} \int_{K_i} u \wedge d^+ a_i &= \int_K u \wedge d^+ a_i + \int_{K_i \setminus K} u \wedge d^+ a_i \\ &= \int_K u \wedge (d^+ a_i - u) + \int_K u \wedge u + \int_{K_i \setminus K} u \wedge d^+ a_i \\ &= \int_K u \wedge (d^+ a_i - u) + \|u\|_{L^2(K)}^2 + \int_{K_i \setminus K} u \wedge d^+ a_i. \end{aligned}$$

By the Cauchy-Schwartz inequality, both the estimates

$$\begin{aligned} \left| \int_K u \wedge (d^+ a_i - u) \right| &\leq \delta \|u\|_{L^2(K)}, \\ \left| \int_{K_i \setminus K} u \wedge d^+ a_i \right| &\leq \delta \|d^+(a_i)\|_{L^2(K_i \setminus K)} \end{aligned}$$

hold. Hence the following statement holds: for any $\delta > 0$, there is i_0 and a compact subset $K \subset X$ such that for all $i \geq i_0$, the estimates

$$\begin{aligned} \left| \int_K u \wedge d^+ a_i - \|u\|_{L^2(X)}^2 \right| &< \delta, \\ \left| \int_{K_i \setminus K} u \wedge d^+ a_i \right| &< \delta \end{aligned}$$

hold. Hence uniform positivity holds:

$$\int_{K_i} u \wedge d^+ a_i > \|u\|_{L^2(K)}^2 - 2\delta > 0.$$

Step 2: One may assume $K \subset K_{i-1}$ by choosing large i . Then consider the equalities

$$\begin{aligned} \int_{K_i} u \wedge d^+ a_i &= \int_{K_i} u \wedge da_i = \int_{K_i} u \wedge d(\chi_i a_i) + \int_{K_i} u \wedge d(1 - \chi_i) a_i \\ &= \int_{K_i} d(u \wedge \chi_i a_i) + \int_{K_i \setminus K_{i-1}} u \wedge d(1 - \chi_i) a_i \\ &= \int_{K_i \setminus K_{i-1}} u \wedge d(1 - \chi_i) a_i. \end{aligned}$$

Then the estimates hold:

$$\begin{aligned} \left| \int_{K_i \setminus K_{i-1}} u \wedge d(1 - \chi_i) a_i \right| &\leq \|u\|_{L^2(K_i \setminus K_{i-1})} \|a_i\|_{L^2_1(K_i \setminus K_{i-1})} \\ &\leq c \|u\|_{L^2(K_i \setminus K_{i-1})}. \end{aligned}$$

Right-hand side can be arbitrarily small as $u \in L^2(X; \Lambda^+)$. This contradicts Step 1. \square

Remark 2.7. The condition on a_i is too strong for our later purpose, and in Corollary 2.10 below, we use a weaker condition on a_i assuming a stronger one on u .

Lemma 2.8. *Suppose an L^2 harmonic self-dual 2-form $u \in \mathcal{H}^+(X; \mathbb{R})$ exists, which is exact at infinity so that $u = d\alpha$ holds on the complement of a compact subset $K \subset X$ for some $\alpha \in \Omega^1(X \setminus K)$. Then any $a \in \Omega^1_c(X \setminus K)$ satisfies vanishing*

$$\int_X u \wedge d^+ a = 0.$$

Proof. We have the equality

$$\int_X u \wedge d^+ a = \int_X u \wedge da$$

since u is self-dual. By the assumption,

$$u|_{X \setminus K} = d\alpha$$

holds for some $\alpha \in \Omega^1(X \setminus K)$. Then,

$$\int_X u \wedge da = \int_{X \setminus K} d\alpha \wedge da.$$

Choose a compactly supported cut-off function $\varphi : X \rightarrow [0, 1]$ with

$$\varphi|_K \equiv 0, \quad \varphi|_{\text{supp } a} \equiv 1.$$

Then, we have the equalities

$$\int_{X \setminus K} d\alpha \wedge da = \int_{X \setminus K} d(\varphi\alpha) \wedge da = \int_X d(\varphi\alpha) \wedge da = \int_X d(\varphi\alpha \wedge a) = 0.$$

These equalities are combined to obtain the conclusion. \square

The following Theorem requires no uniform bound on the values of the L^2 norm of a_i .

Theorem 2.9. *Fix $1 \leq p, q \leq \infty$ with $p^{-1} + q^{-1} = 1$. Suppose $u \in \mathcal{H}^+(X; \mathbb{R})$ is a non-zero L^2 harmonic self-dual 2-form that is also in $L^p \cap L^q$ and is L^p exact at infinity. Then there is no sequence $a_i \in \Omega^p(K_i)$ such that*

- (1) *uniform bound $\|d(a_i)\|_{L^q(K_i)} \leq C < \infty$ holds, and*
- (2) *convergence $d^+(a_i) \rightarrow u = u^+$ holds in L^q norm on each compact subset.*

Proof. Step 1: Suppose such a sequence exists. Let us fix i_0 and choose arbitrarily small $\delta > 0$. Then we obtain the estimates

$$\begin{aligned} \int_{K_{i_0}} u \wedge d^+(a_i) &= \|u\|_{L^2(K_{i_0})}^2 + \int_{K_{i_0}} u \wedge (d^+(a_i) - u) \\ &\geq \|u\|_{L^2(K_{i_0})}^2 - \|u\|_{L^p(K_{i_0})} \|d^+(a_i) - u\|_{L^q(K_{i_0})} \\ &\geq \|u\|_{L^2(K_{i_0})}^2 - \delta > 0. \end{aligned}$$

if $i > i_0$ is sufficiently large.

Step 2: Since the estimates

$$\begin{aligned} \left| \int_{K_i \setminus K_{i_0}} \chi_i u \wedge d^+ a_i \right| &\leq \|\chi_i u\|_{L^p(K_i \setminus K_{i_0})} \|d^+ a_i\|_{L^q(K_i \setminus K_{i_0})} \\ &< \delta \|d^+ a_i\|_{L^q(K_i \setminus K_{i_0})} \leq C\delta \end{aligned}$$

hold, we obtain positivity

$$\int_{K_i} \chi_i u \wedge d^+ a_i = \int_{K_{i_0}} u \wedge d^+ a_i + \int_{K_i \setminus K_{i_0}} \chi_i u \wedge d^+ a_i > \int_{K_{i_0}} u \wedge d^+ a_i - C\delta > 0$$

by Step 1. On the other hand consider the equalities

$$\begin{aligned} \int_{K_i} \chi_i u \wedge d^+ a_i &= \int_X \chi_i u \wedge da_i = \int_X d(\chi_i u \wedge a_i) - \int_X d\chi_i \wedge u \wedge a_i \\ &= - \int_{\text{Supp } d\chi_i} d\chi_i \wedge u \wedge a_i = - \int_{\text{Supp } d\chi_i} d\chi_i \wedge d\alpha \wedge a_i \\ &= \int_{\text{Supp } d\chi_i} d(d\chi_i \wedge \alpha \wedge a_i) - \int_{\text{Supp } d\chi_i} d\chi_i \wedge \alpha \wedge da_i \\ &= - \int_{\text{Supp } d\chi_i} d\chi_i \wedge \alpha \wedge da_i \end{aligned}$$

by Stokes' theorem. Then, we have the estimates

$$\left| \int_{\text{supp } d\chi_i} d\chi_i \wedge \alpha \wedge da_i \right| \leq \|d\chi_i\|_{L^\infty(X)} \|\alpha\|_{L^p(\text{supp } d\chi_i)} \|da_i\|_{L^q(\text{supp } d\chi_i)}$$

which is arbitrarily small for large i . This is a contradiction. \square

In particular, we have the following Corollary by setting $p = q = 2$.

Corollary 2.10. *Suppose $u \in \mathcal{H}_e^+(X; \mathbb{R})$ is a non-zero L^2 harmonic self-dual 2-form that is L^2 exact at infinity. Then there is no sequence $a_i \in \Omega^1(K_i)$ such that*

- (1) *the convergence $d^+(a_i) \rightarrow u$ holds in the L^2 norm on each compact subset, and*
- (2) *there is a uniform bound $\|d(a_i)\|_{L^2(K_i)} \leq C < \infty$.*

Corollary 2.11. *Suppose $u \in \mathcal{H}_e^+(X; \mathbb{R})$ is a non-zero L^2 harmonic self-dual 2 form that is L^2 exact at infinity with $u = d\alpha$ outside of $K \subset\subset X$. Then there exists a compactly supported 2-form $v \in \Omega_c^2(X)$ such that the following property holds. There is no sequence $a_i \in \Omega^1(K_i)$ such that*

(1) *the convergence $d^+(a_i) \rightarrow v^+$ holds in L^2 norm on each compact subset, where v^+ is the projection to the self-dual part of v , and*

(2) *a uniform bound $\|d(a_i)\|_{L^2(K_i)} \leq C < \infty$ holds.*

Proof. Let $\chi \in C^\infty(X)$ be a cut-off function which is 1 near infinity and vanishes on K . Then $\alpha' := \chi \cdot \alpha \in L^2(X; \Lambda^1) \cap \Omega^1(X)$ satisfies $d\alpha' \equiv u$ on a complement of a compact subset. Then we can conclude that there is no family $a'_i \in \Omega^1(K_i)$ with uniformly bounded norms $\|da'_i\|_{L^2(K_i)} \leq C$ such that the convergence

$$d^+(a'_i) \rightarrow v^+ := \text{pr}_+(u - d\alpha') = u - d^+\alpha'$$

holds in the L^2 norm on each compact subset, where pr_+ is the projection to the self-dual part. If there were such a family, then

$$a_i := a'_i + \alpha'$$

would satisfy the conditions (1) and (2) in Corollary 2.10. \square

2.3. Atiyah-Hitchin-Singer complexes over cylindrical-end manifolds

The Atiyah-Hitchin-Singer (AHS) complex is an elliptic differential complex over a Riemannian 4-manifold X

$$0 \longrightarrow L_{k+1}^2(X, g) \xrightarrow{d} L_k^2((X, g); \Lambda^1) \xrightarrow{d^+} L_{k-1}^2((X, g); \Lambda_+^2) \longrightarrow 0$$

between Sobolev spaces, where d^+ is the composition of the differential with the projection to the self-dual 2-forms. Here $k \geq 1$. Note that $H^0 = 0$ always holds when X is connected and non-compact. Recall that an element in the second reduced L^2 cohomology group admits a harmonic representative by Lemma 2.4.

Suppose $\text{end}(X)$ is isometric to the product $Y \times [0, \infty)$ so that $g = g' + dt^2$ on the end, where (Y, g') is a closed Riemannian three-manifold. Such a space is called a cylindrical-end manifold.

Let us fix a small and positive $\mu > 0$. Then we set

$$\tau : Y \times [0, \infty) \mapsto [0, \infty), \quad \tau(m, t) = \mu t$$

and extend it to a function $\tau : X \rightarrow [0, \infty)$ that coincides with $\tau(m, t)$ on $\text{end}(X)$. Then, we define the weighted Sobolev k norm on X by

$$\|u\|_{L_{k, \mu}^2} = \left(\sum_{l \leq k} \int_X \exp(\tau) |\nabla^l u|^2 \right)^{\frac{1}{2}}.$$

We can denote by $L_{k,\mu}^2$ the completion of $C_c^\infty(X)$ with respect to the norm, because the isomorphism class of the function space depends only on $\mu > 0$, rather than τ itself. Then we have the weighted AHS complex

$$0 \longrightarrow L_{k+1,\mu}^2(X) \xrightarrow{d} L_{k,\mu}^2(X; \Lambda^1) \xrightarrow{d^+} L_{k-1,\mu}^2(X; \Lambda_+^2) \longrightarrow 0.$$

Let us identify the orthogonal complement of the image of d^+ with the space of the cokernel, and take an element $u \in L_{k-1,\mu}^2(X; \Lambda^+)$ in the cokernel of d^+ . Then u satisfies the equality

$$0 = (d^+)_\tau^*(u) := \exp(-\tau)(d^+)^*(\exp(\tau)u),$$

and hence $(d^+)^*(\exp(\tau)u) = 0$ holds.

Note that the de Rham cohomology $H^2(\text{end}(X); \mathbb{R}) = 0$ vanishes on the end, if and only if Y is a rational homology sphere. The following property is well known.

Proposition 2.12. [K1] *Suppose Y is a rational homology sphere. Then for any small $\mu > 0$,*

$$\exp(\tau)u \in L^2(X; \Lambda^+)$$

holds for any element $u \in L_{k-1,\mu}^2(X; \Lambda^+)$ in the orthogonal complement of the image of d^+ . Moreover, $\exp(\tau)u$ is L^2 exact at infinity.

See also [K3].

3. Seiberg-Witten theory and scalar curvature

Let us quickly review Seiberg-Witten theory over compact 4-manifolds [M].

3.1. Seiberg-Witten map over compact 4-manifolds

Let V be a real 4-dimensional Euclidean space, and consider the \mathbb{Z}_2 -graded Clifford algebra $Cl(V) = Cl_0(V) \oplus Cl_1(V)$. Let S be the unique complex 4-dimensional irreducible representation of $Cl(V)$. Then, for any vector $v \in S^+$, we set

$$\sigma(v) \equiv v \otimes v^* - \frac{|v|^2}{2} \text{id} \in \wedge_+^2(V) \otimes i\mathbb{R}.$$

One can apply it to the cotangent bundle of a compact Riemannian 4-manifold on each fiber. Let (M, h) be an oriented compact Riemannian 4-manifold equipped with a spin^c structure \mathfrak{L} . Let S^\pm and L be the spinor bundles and the determinant bundle respectively. Let A_0 be a smooth $U(1)$ connection on L . With a Riemannian metric on M , A_0 induces a spin^c connection and the associated Dirac operator D_{A_0} on S^\pm . Fix a large $k \geq 2$, and consider the configuration space

$$\mathfrak{D} = \{(A_0 + a, \psi) \mid a \in L_k^2(M; \Lambda^1 \otimes i\mathbb{R}), \psi \in L_k^2(M; S^+)\}.$$

Let $u \in C^\infty(M; \Lambda^+)$ be a smooth self-dual 2-form. Then we have the perturbed Seiberg-Witten map

$$\begin{aligned} SW_u : \mathfrak{D} &\rightarrow L_{k-1}^2(M; S^- \oplus \Lambda_+^2 \otimes i\mathbb{R}), \\ (A_0 + a, \psi) &\rightarrow (D_{A_0+a}(\psi), F_{A_0+a}^+ - \sigma(\psi) - iu). \end{aligned}$$

Note that the space of connections is independent of choice of A_0 as long as M is compact. Let $* \in M$ be any fixed point, and $\mathfrak{G}_*(M) := L_{k+1}^2(M; S^1)_*$ be the L_{k+1}^2 -completion of

$$\{u \in C^\infty(M, S^1) \mid u(*) = 1\},$$

which acts on both \mathfrak{D} and $L_{k-1}^2(M; S^- \oplus \Lambda_+^2 \otimes i\mathbb{R})$. The action of the gauge transformation $u \in \mathfrak{G}_*(M)$ on the spinors are the complex multiplication, and on a 1-form is given by

$$a \rightarrow a - 2u^{-1}du.$$

The action is trivial on self-dual 2-forms. The map SW_u is equivariant with respect to the $\mathfrak{G}_*(M)$ actions, and hence the gauge group acts on the zero set

$$\tilde{\mathfrak{M}}((M, h), u) := \{(A_0 + a, \psi) \in \mathfrak{D} \mid SW_u(A_0 + a, \psi) = 0\}.$$

Moreover the quotient space $\mathfrak{B}^0 \equiv \mathfrak{D}/\mathfrak{G}_*(M)$ is Hausdorff. The based and perturbed Seiberg-Witten moduli space is given by the quotient space

$$\mathfrak{M}_*((M, h), u) := \tilde{\mathfrak{M}}((M, h), u)/\mathfrak{G}_*(M).$$

A connection $A_0 + a$ with $a \in L_k^2(M; \Lambda^1 \otimes i\mathbb{R})$ can be gauge transformed so that it satisfies $\text{Ker } d^*(a) = 0$. Such a gauge transformation is unique, since it is based. Then the slice map is given by the restriction

$$SW_u : (A_0 + \text{Ker } d^*) \times L_k^2(M; S^+) \rightarrow L_{k-1}^2(M; S^- \oplus \Lambda_+^2 \otimes i\mathbb{R}).$$

We consider the zero set

$$\mathfrak{M}^0((M, h), u) := SW_u^{-1}(0) \cap \{(A_0 + \text{Ker } d^*) \times L_k^2(M; S^+)\}.$$

The inclusion of the slice into the configuration space descends to an S^1 -equivariant homeomorphism from the slice version $\mathfrak{M}^0((M, h), u)$ to the quotient version $\mathfrak{M}_*((M, h), u)$.

Definition 3.1. The Seiberg-Witten invariant is defined by counting the algebraic number of the oriented space

$$SW(M, \mathfrak{L}) := \# \mathfrak{M}((M, h), u) \in \mathbb{Z}$$

for a generic choice of perturbation.

It is independent of choice of perturbation and Riemannian metric, and hence is a smooth invariant.

3.2. Scalar curvature

Let M be a compact spin 4-manifold, and h be a Riemannian metric on M . Then take a solution $(\phi, A = \nabla + a)$ to the SW equation perturbed by v^+ with respect to (M, h) .

Proposition 3.2. *Given constants $C, \delta > 0$, there is a constant c such that the following holds: Suppose there is a compact subset $K \subset M$ such that the scalar curvature κ on (M, h) is bounded from below as*

$$\kappa \geq -C.$$

(1) *If non-negativity*

$$\kappa|_{M \setminus K} \geq 0$$

holds on the complement of K , then there is a constant $c > 0$ determined by v^+ , C and $\text{vol } K$ such that the following uniform bound holds:

$$\|\phi\|_{L^4(M, h)}, \|da\|_{L^2(M, h)} \leq c.$$

(2) *If the uniform positivity*

$$\kappa|_{M \setminus K} \geq \delta > 0$$

holds on the complement of K , then there is a constant $c > 0$ determined by C, δ, v^+ and $\text{vol } K$ such that the uniform bound

$$\|\phi\|_{L^2(M, h)} \leq c$$

holds, in addition to the estimates in (1).

Let $X \subset M$ be an open subset. Then by restriction, one obtains the estimates

$$\|\phi\|_{L^4(X, h)}, \|da\|_{L^2(X, h)} \leq c$$

and

$$\|\phi\|_{L^2(X, h)} \leq c$$

respectively.

Proof. One may assume that support of v^+ lies in K , by replacing K with $K \cup \text{supp } v^+$, if necessarily. It follows from the Weitzenböck formula

$$D_A^2(\phi) = \nabla_A^* \nabla_A(\phi) + \frac{\kappa}{4} \phi + \frac{F_A}{2} \phi$$

that the equality

$$0 = \|\nabla_A(\phi)\|_{L^2(M)}^2 + \int_M \frac{\kappa}{4} |\phi|^2 \text{vol} + \int_M \langle \frac{F_A}{2} \phi, \phi \rangle \text{vol}$$

holds. From the SW equations, we have the equalities

$$\begin{aligned} \langle F_A \phi, \phi \rangle &= \langle F_A^+ \phi, \phi \rangle \\ &= \langle (F_A^+ - \sigma(\phi) - iv^+) \cdot \phi + (\sigma(\phi) + iv^+) \cdot \phi, \phi \rangle \\ &= \frac{|\phi|^4}{2} + \langle iv^+ \cdot \phi, \phi \rangle. \end{aligned}$$

Then, we have the estimate

$$0 \geq \int_K \frac{\kappa}{4} |\phi|^2 \text{vol} + \frac{1}{2} \int_K \langle iv^+ \cdot \phi, \phi \rangle + \frac{1}{4} \int_K |\phi|^4 \text{vol} \\ + \int_{M \setminus K} \frac{\kappa}{4} |\phi|^2 \text{vol} + \frac{1}{4} \int_{M \setminus K} |\phi|^4 \text{vol}.$$

We have the estimate

$$| \int_K \langle iv^+ \cdot \phi, \phi \rangle | \leq \sqrt{\int_K |\phi|^4} \cdot \|v^+\|_{L^2(K)} \leq c \sqrt{\int_K |\phi|^4}.$$

Hence

$$- \int_K (\kappa + |\phi|^2) \frac{|\phi|^2}{4} \text{vol} + c \sqrt{\int_K |\phi|^4} \geq \int_{M \setminus K} (\kappa + |\phi|^2) \frac{|\phi|^2}{4} \text{vol} \geq 0 \quad (*)$$

By the assumption with (*) above, we have the estimates

$$C \int_K |\phi|^2 \text{vol} \geq \int_K -\frac{\kappa}{4} |\phi|^2 \text{vol} \geq \frac{1}{4} \int_K |\phi|^4 \text{vol} - c \sqrt{\int_K |\phi|^4}.$$

Note the estimate

$$\int_K |\phi|^4 \text{vol} \geq \text{vol}(K)^{-1} \left(\int_K |\phi|^2 \text{vol} \right)^2$$

by Cauchy-Schwartz. Then, for $x^2 = \frac{1}{4} \int_K |\phi|^4 \text{vol}$,

$$x^2 - c_K x \leq 0$$

holds for some $c_K > 0$. Hence, we obtain the estimate

$$c_K^2 \geq \frac{1}{4} \int_K |\phi|^4 \text{vol}.$$

Combining these estimates, we obtain the estimate

$$c_K \sqrt{4 \text{vol } K} \geq \int_K |\phi|^2 \text{vol}.$$

Hence the left hand side of (*) is bounded by some C_K , and so we have the bound

$$C_K \geq \int_{M \setminus K} \left(\frac{\kappa}{4} |\phi|^2 + |\phi|^4 \right) \text{vol}.$$

Combining these estimates, we obtain the uniform bound

$$\int_M |\phi|^4 \text{vol} \leq c'_K$$

in the case of (1). For (2), we also obtain the uniform bound $\int_M |\phi|^2 \text{vol} \leq c'_K$. Now the uniform bound

$$\|d^+(a)\|_{L^2(M)}^2 = \|F_A^+\|_{L^2(M)}^2 \leq \|\phi\|_{L^4(M)}^4 + \|v^+\|_{L^2(K)}^2 \leq C'_K$$

holds by the equality $F_A^+ = \sigma(\phi) + \sqrt{-1}v^+$. Consider the topological invariant

$$0 = 4\pi^2 c_1(L)^2 = \int_M F_A \wedge F_A \text{ vol} = \int_M |F_A^+|^2 \text{ vol} - \int_M |F_A^-|^2 \text{ vol}.$$

Thus the following bound also holds:

$$\|d^-(a)\|_{L^2(M)}^2 = \int_M |F_A^-|^2 \text{ vol} = \int_M |F_A^+|^2 \text{ vol} \leq C'_K.$$

Combining with the above, we obtain the bound $\|da\|_{L^2(M)} \leq c_K$. \square

Remark 3.3. (1) We have not assumed that the solution is gauge fixed; hence, we have freedom of choice of solutions in its gauge equivalent class. (2) Later, we will apply Proposition 3.2 with a family of Riemannian metrics h_λ on M such that their restrictions $h_\lambda|_U$ coincide with each other on an open subset $U \subset X \subset M$ (see Lemma 4.1 later). Moreover, we choose perturbation v^+ by a self-dual 2-form that is smooth and supported inside U (see Corollary 2.11). Then, we can take $K = (M \setminus U) \cup \text{supp } v^+$.

From an analytical perspective, we have the following Lemma in the case of uniformly positive scalar curvature.

Lemma 3.4. [GL2] *Suppose X is spin with a complete Riemannian metric (X, g) . If the scalar curvature κ is uniformly positive*

$$\kappa|_{X \setminus K} \geq \delta > 0$$

on the complement of a compact subset K , then the Dirac operator D is Fredholm.

In our non-uniform case, we cannot expect to obtain such a conclusion. In fact, ultimately, we will not use Fredholm theory over a non compact manifold. Our use of positivity is to guarantee vanishing of an L^4 spinor section on a complete Riemannian 4-manifold (see Lemma 4.3 below).

4. Convergent process

4.1. Preparation

Let M be a compact oriented smooth 4-manifold, and $X \subset M$ be an open subset equipped with a complete Riemannian metric g on X . Choose an exhaustion

$$K_0 \subset\subset K_1 \subset\subset \cdots \subset\subset K_{i+1} \subset\subset \cdots \subset\subset X$$

by compact subsets. We will later assume that X is simply connected and simply connected at infinity. One may assume that the inclusion $I_i : K_i \subset K_{i+1}$ induces null homomorphism on the fundamental groups

$$(I_i)_* = 0 : \pi_1(K_i) \rightarrow \pi_1(K_{i+1})$$

by replacing $\{K_i\}$ by its subset $\{K_{l_i}\}_i$ for some subindices $\{l_i\}_i$, if necessarily. This property is used when we apply Corollary 4.4 below.

Note that the quasi-cylindrical-end condition requires isometric-pasting condition (see Definition 1.1). The latter condition is preserved, if one takes a subset $\{K_{l_i}\}_i$ as above. Hence in Lemma 4.1 below, one can assume that the exhaustion $\{K_i\}_i$ simultaneously satisfy the condition that the inclusions of the K_i induce null homomorphisms on fundamental groups.

Lemma 4.1. *Suppose g is quasi-cylindrical-end with respect to the exhaustion above. Then there is a family of Riemannian metrics $\{h_i\}_{i \geq 0}$ on M such that the following properties hold for any i :*

- (1) $h_i|_{K_i} \equiv g|_{K_i}$,
- (2) $\text{vol}(M \setminus K_i, h_i) \leq c$ is uniformly bounded, and
- (3) $\{h_i\}_{i \geq 0}$ is a family of Riemannian metrics on M such that their scalar curvatures are uniformly bounded from below $\kappa_{h_i} \geq -C$.

Remark 4.2. (1) Note that if a Riemannian manifold (M, g) has positive scalar curvature, then it is uniformly positive if M is compact. The same thing holds for a non-compact Riemannian manifold, if g is cylindrical-end, or more generally end-periodic. However, this property does not hold for the quasi-cylindrical-end case in general. (2) It follows from the construction of the family of Riemannian metrics $\{h_i\}_{i \geq 0}$ on M that the restriction $(M \setminus K_0, h_i)$ is isometric to $(M \setminus K_0, h_0)$.

Proof. Recall the notations in Definition 1.1 with the data $\epsilon > 0$ and $\{\phi_i : K_i \cong K_{i+1}\}_{i \geq 0}$. We consider the isometries

$$\Psi_i := \phi_{i-1} \circ \cdots \circ \phi_0 : N_\epsilon(\partial K_0) \cong N_\epsilon(\partial K_i).$$

For $K'_0 := K_0 \setminus N_\epsilon(\partial K_0)$, we glue the disjoint union

$$M_i := (M \setminus K'_0) \sqcup_{\Psi_i} K_i$$

through the isometry Ψ_i . Note that Ψ_i is extended as a diffeomorphism

$$\Psi_i = \phi_{i-1} \circ \cdots \circ \phi_0 : K_0 \cong K_i.$$

Then, there is a diffeomorphism $\Psi_i : M \cong M_i$ by setting

$$\Psi_i(m) := \begin{cases} \Psi_i(m), & m \in K_0, \\ m, & m \in M \setminus K_0. \end{cases}$$

Then, we define

$$h_i(x) := \begin{cases} g(x), & x \in K_i, \\ h_0(x), & x \in M \setminus K_0. \end{cases}$$

□

4.2. Positivity of scalar curvature

Suppose X is spin with a complete Riemannian metric (X, g) , and let ∇ be the spin connection with the Dirac operator D .

Lemma 4.3. *Let (X, g) be a quasi-cylindrical-end manifold and assume that the scalar curvature is (not necessarily uniformly) positive $\kappa > 0$. Let (A, ϕ) be a solution to the SW equations perturbed by a self-dual 2-form $u \in \Omega_c^+(K_0)$ with sufficiently small L^∞ norm $\|u\|_{L^\infty} \ll 1$. Then ϕ is actually zero, if $\phi \in L^4((X, g); S^+) \cap L_{1,loc}^2$.*

Proof. This is well known if $\phi \in L_1^2((X, g); S^+)$. Let us use the same notations as above. Since each $N_\epsilon(\partial K_i)$ is isometric to $N_\epsilon(\partial K_0)$, for any $\delta > 0$, there is some i_0 such that

$$\|\phi\|_{L^4(N_\epsilon(\partial K_i))} < \delta$$

holds for any $i \geq i_0$. By Cauchy-Schwartz, the following estimates hold:

$$\|\phi\|_{L^2(N_\epsilon(\partial K_i))} \leq \text{Vol}(N_\epsilon(\partial K_i))^{\frac{1}{4}} \|\phi\|_{L^4(N_\epsilon(\partial K_i))} < \text{Vol}(N_\epsilon(\partial K_i))^{\frac{1}{4}} \delta.$$

Let $\chi \in C_c^\infty(K_0)$ be a cut-off function which vanishes near the boundary. Then, we define $\chi_i \in C_c^\infty(X)$ by

$$\chi_i(x) = \begin{cases} 0 & x \in X \setminus K_i, \\ (\Psi_i^{-1})^*(\chi)(x) & x \in N_\epsilon(K_i), \\ 1 & x \in K_i \setminus N_\epsilon(K_i). \end{cases}$$

Since $D_A(\phi) = 0$, we have the equality

$$D_A(\chi_i \phi) = d\chi_i \cdot \phi + \chi_i D_A(\phi) = d\chi_i \cdot \phi.$$

Hence

$$\|D_A(\chi_i \phi)\|_{L^2(X)} \leq C \|\phi\|_{L^2(N_\epsilon(K_i))} \rightarrow 0$$

holds as $i \rightarrow \infty$. Then, by Weitzenböck formula we have the following equality:

$$\begin{aligned} \|D_A(\chi_i \phi)\|_{L^2(X)}^2 &= \langle D_A^2(\chi_i \phi), \chi_i \phi \rangle > \\ &= \langle \nabla^* \nabla(\chi_i \phi), \chi_i \phi \rangle + \frac{\kappa}{4} \langle \chi_i \phi, \chi_i \phi \rangle + \int_X \frac{\chi_i^2 |\phi|^4}{4} + \langle u \cdot \chi_i \phi, \chi_i \phi \rangle \\ &= \|\nabla(\chi_i \phi)\|_{L^2(X)}^2 + \int_X \frac{\kappa}{4} |\chi_i \phi|^2 + \int_X \frac{\chi_i^2 |\phi|^4}{4} + \langle u \cdot \chi_i \phi, \chi_i \phi \rangle \\ &\geq \|\nabla(\chi_i \phi)\|_{L^2(X)}^2 + \int_X \frac{\kappa}{4} |\chi_i \phi|^2 + \int_X \frac{\chi_i^2 |\phi|^4}{4} - \|u\|_{L^\infty} \|\chi_i \phi\|_{L^2(K_0)}^2 \\ &\geq \|\nabla(\chi_i \phi)\|_{L^2(X)}^2 + \int_{X \setminus K_0} \frac{\kappa}{4} |\chi_i \phi|^2 + \int_X \frac{\chi_i^2 |\phi|^4}{4} + (\kappa_0 - \|u\|_{L^\infty}) \|\chi_i \phi\|_{L^2(K_0)}^2 \end{aligned}$$

where $\kappa_0 := \inf_{x \in K_0} \kappa(x) > 0$. By the assumption, one may assume

$$\inf_{K_0} \kappa \geq \|u\|_{L^\infty}.$$

Hence, this implies the equality $\phi \equiv 0$, since the left-hand side converges to zero as $i \rightarrow \infty$, and the limit-inf of the right-hand side is at least $\frac{1}{4} \|\phi\|_{L^4(X)}^4$. \square

4.3. Proof of Theorem 1.3

This subsection is devoted to giving a proof of the remainder of Theorem 1.3. First, let us state a general result on differential forms on manifolds with boundary. Let X_0 be a compact smooth manifold with boundary. Let us equip with a Riemannian metric on X_0 , and let $L_l^2(X_0; \Lambda^k)$ be the Sobolev l -space. Let $Y_0 \subset X_0$ be an embedding of a compact submanifold with boundary that satisfies $\partial Y_0 \cap \partial X_0 = \phi$. The following result is standard.

Lemma 4.4. *Suppose the natural map $\pi_1(Y_0) \rightarrow \pi_1(X_0)$ is zero. Let $\eta \in L_1^2(X_0; \Lambda^1)$. Then there is an exact form $d\mu' \in L_1^2(Y_0; \Lambda^1)$ such that*

$$\omega := \eta - d\mu' \in L_1^2(Y_0; \Lambda^1)$$

satisfies the lower bound

$$\|d\omega\|_{L^2(X_0)} \geq c \|\omega\|_{L^2(Y_0)}$$

with $d^(\omega) = 0$.*

This is based on Hodge theory on manifolds with boundary [S, W]. See also the Appendix in [K3]. Let us give a proof of the remainder of Theorem 1.3.

Step 1: Let M be a $K3$ surface and denote $X' := 3(S^2 \times S^2) \setminus \text{pt.}$

Lemma 4.5. *Let M be as above. Then there exists an open subset $X \subset M$ such that X is homeomorphic to X' , but is not diffeomorphic to the latter manifold with respect to the induced smooth structure by the embedding $X \subset M$.*

Proof. Actually there is a topological decomposition $M \cong 2| - E_8 | \# 3(S^2 \times S^2)$, and X is obtained as an open subset of the complement of $2| - E_8 |$ term. See [FU], [DK]. \square

Required properties have been given for X' in the introduction. We now focus on X .

The following is known (see [M]).

Lemma 4.6. *The Seiberg-Witten invariant is non zero over M with respect to the spin structure.*

We shall deduce a contradiction, assuming that the above X admits a complete Riemannian metric which satisfies the conditions $(*)$ in Theorem 1.3.

Step 2: Let (X, g) be a quasi-cylindrical-end Riemannian 4-manifold whose scalar curvature is positive, and let us take any non zero L^2 harmonic self-dual 2-form u on (X, g) , which is exact at infinity. Let $v^+ \in \Omega_c^+(K_0)$ be the self-dual 2-form in Corollary 2.11.

Take a family of metrics h_i on M as in Lemma 4.1. The (perturbed) SW invariant is invariant for any choice of generic Riemannian metric and perturbation. Hence, there

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is a solution to any metric h_i and perturbation by Lemma 4.6. Let $(A_i = \nabla + ia_i, \phi_i)$ be a solution to the perturbed SW equation by v^+ with respect to (M, h_i) . It obeys the equation

$$id^+ a_i - \sigma(\phi_i) = \sqrt{-1}v^+.$$

Step 3: It follows from Proposition 3.2 (1) and Lemma 4.1 that there is a constant C such that the uniform bounds

$$\|\phi_i\|_{L^4(K_i)}, \quad \|da_i\|_{L^2(K_i)} \leq C$$

hold.

Let us fix i_0 . It follows from Lemma 4.4 with Remark 3.3 that after gauge transform, the estimates

$$\|a_i\|_{L^2(K_{i_0})} \leq C_{i_0} \|da_i\|_{L^2(K_{i_0+1})} \leq C'_{i_0}$$

hold for some constants C_{i_0} and C'_{i_0} , and $i \geq i_0 + 1$. Moreover one may assume the gauge-fixing

$$d^*(a_i) = 0.$$

Hence we obtain the L^2_1 bound

$$\|a_i\|_{L^2_1(K_{i_0})} \leq C''_{i_0}$$

by the elliptic estimate.

Step 4: Since (A_i, ϕ_i) is a solution to the perturbed SW equation, the equality

$$0 = D_{A_i}(\phi_i) = D(\phi_i) + a_i \cdot \phi_i$$

holds. Thus, we obtain the estimates

$$\begin{aligned} \|D(\phi_i)\|_{L^2(K_{i_0})} &\leq \|a_i \cdot \phi_i\|_{L^2(K_{i_0})} \leq \|a_i\|_{L^4(K_{i_0})} \|\phi_i\|_{L^4(K_{i_0})} \\ &\leq C_{i_0} \|a_i\|_{L^2_1(K_{i_0})} \|\phi_i\|_{L^4(K_{i_0})} \leq C'_{i_0} \end{aligned}$$

using the Sobolev embedding $L^2_{1,loc} \hookrightarrow L^4_{loc}$. Again by the elliptic estimate, we obtain the uniform bound

$$\|\phi_i\|_{L^2_1(K_{i_0})} \leq C_{i_0}.$$

Step 5: It is well known that the perturbed SW solution admits an L^∞ bound

$$\|\phi_i\|_{L^\infty(M)} \leq \sup_{m \in M} \max(0, -\kappa_i(m) + \|v^+\|_{L^\infty}) \leq C$$

(see [M] page 77, proof of Corollary 5.2.2). Since $F_{A_i}^+ = \phi_i \otimes \phi_i^* - \frac{1}{2}|\phi_i|^2 \text{id} + \sqrt{-1}v^+$ holds, the equality

$$\nabla F_{A_i}^+ = \nabla(\phi_i) \otimes \phi_i^* + \phi_i \otimes \nabla(\phi_i^*) - \langle \nabla(\phi_i), \phi_i \rangle \text{id} + \sqrt{-1}\nabla v^+.$$

holds. Hence we have the estimates

$$\|\nabla F_{A_i}^+\|_{L^2(K_{i_0})} \leq C \|\phi_i\|_{L^\infty(M)} \|\nabla(\phi_i)\|_{L^2(K_{i_0})} + \|\nabla v^+\|_{L^2(K_0)} \leq C'_{i_0}.$$

Then it follows from Step 3 with the elliptic estimate that the bound

$$\|a_i\|_{L^2_2(K_{i_0})} \leq C_{i_0}$$

holds, since $F_{A_i}^+ = \sqrt{-1}d^+a_i$ and $d^*a_i = 0$ holds by Step 3. In summary, we have the estimates as below

$$\begin{cases} \|a_i\|_{L^2_2(K_{i_0})} \leq C_{i_0}, & \|da_i\|_{L^2(K_i)} \leq C, \\ \|\phi_i\|_{L^2_1(K_{i_0})}^2 \leq C_{i_0}, & \|\phi_i\|_{L^4(K_i)} \leq C. \end{cases}$$

Step 6: By Steps 3 and 4 with local compactness of the Sobolev embedding, we can choose a subsequence of spinors so that they converge to $\phi \in L^4((X, g); S^+)$ on each compact subset. Moreover, the subsequence is locally in L^2_1 .

By Steps 3 and 5 with local compactness of the Sobolev embedding, we can choose a subsequence of 1-forms so that they converge to $a \in (L^2_1)_{loc}((X, g); \Lambda^1)$ on each compact subset. Moreover da is in $L^2((X, g); \Lambda^2)$.

Since $(d + a, \phi)$ is a solution to the perturbed SW equation by v^+ with respect to (X, g) , we conclude $\phi \equiv 0$ by Lemma 4.3.

Hence, a subsequence $\{d^+a_i\}_i$ will converge to v^+ in L^2 on each compact subset. However, this contradicts Corollary 2.11, completing the proof of Theorem 1.3.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, KYOTO UNIVERSITY, KYOTO 606-8502 JAPAN
Email address: `tkato@math.kyoto-u.ac.jp`