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Homotopy 4-spheres associated to an infinite order loose cork

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Abstract. We prove that the homotopy spheres $\Sigma_n = -W \smile_{f^n} W$, formed by doubling the infinite order loose-cork (W, f), by the iterates of the cork automorphism $f : \partial W \to \partial W$, is S^4. To do this we first show that Σ_n are obtained by Gluck twistings of S^4. Then, from this we show how to cancel 3-handles of Σ_n and identify it by S^4.

1. Introduction

Let (W, f) be the infinite order loose-cork of [A1], shown in Figure 1. As indicated in [A1], this W can be identified with the one described in [G1]. Recall that the diffeomorphism $f : \partial W \to \partial W$ here is given by the δ-move along the curve δ of Figure 2 as defined in [A1]. For simplicity we will refer the iterates f^n of f as δ^n or δ -move.

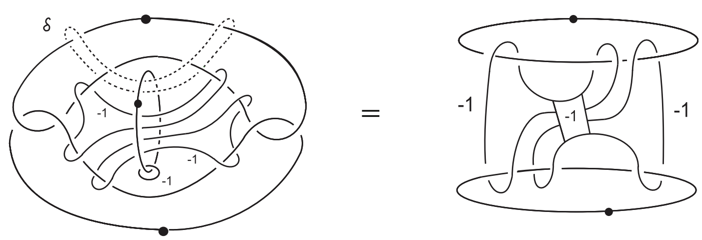

Figure 1. W

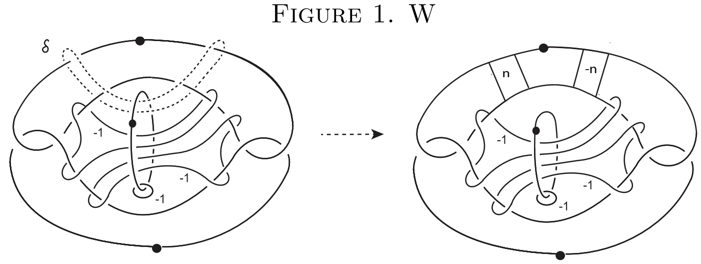

Figure 2. δ^n - move

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Even though the δ-move diffeomorphism $f : \partial W \to \partial W$ by itself looks like an innocuous operation, when W appears as a codimension zero submanifold $W \subset M^4$, the operation of cutting W from M and regluing with the composition map $f^n = f \circ f \circ ... \circ f$ could result infinitely many different diffeomorphism types of M ([A1]). For example, reader can easily observe that if we attach a handle to W along γ, then the δ-move operation applied to W alters the position of γ as shown in Figure 26. Consider the homotopy 4-spheres obtained by doubling of the contractible manifold W by the iterates $f^n = f \circ f \circ ... \circ f$:

$$\Sigma_n = -W \smile_{f^n} W \tag{1}$$

The obvious questions is whether Σ_n are diffeomorphic to S^4, or if this family contains an exotic copy of S^4. The answer is given by:

Theorem 1.1. *Each Σ_n is diffeomorphic to S^4.*

For brevity call $\Sigma_n^o = \Sigma_n - B^4$. Proof will proceed as follows: We will first show each Σ_n is obtained by Gluck twisting S^4 along some knotted $S^2 \subset S^4$, then find 3-handle free handle pictures of Σ_n. We then cancel 3-handles of Σ_n by the trick which was used in the solution of the Cappell-Shaneson homotopy sphere problem [A2]. From this we will see that Σ_n^o is obtained from the ribbon complement $Q = B^4 - N(D^2)$ by attaching a 2-handle along a knot $\gamma_n \subset \partial Q$ where $D \subset B^4$ is the standard ribbon bounded by $\partial D = K \# K$ where K is the figure-8 knot, and $N(D)$ is the tubular neighborhood of D.

$$\Sigma_n^o = Q \smile h^2_{\gamma_n} \tag{2}$$

Knots γ_n are related each other by $f(\gamma_{n-1}) = \gamma_n$, where f is a δ-move diffeomorphism $f : \partial Q \to \partial Q$. Next we show $\Sigma_0^o = B^4$, and Σ_n^0 is obtained by gluing Σ_{n-1}^0 to $\partial \Sigma_{n-1}^0 \times [0,1]$ along $\partial \Sigma_{n-1} \times 0$, by a δ-move diffeomorphism $g_{n-1} : \partial \Sigma_{n-1}^0 \to \partial \Sigma_{n-1}^0$, associated to f.

$$\Sigma_n^0 = \Sigma_{n-1}^0 \smile_{g_{n-1}} \partial \Sigma_{n-1}^0 \times [0,1]$$

Then this fact coupled with the fact that any δ-move diffeomorphism $g : S^3 \to S^3$ is isotopic to identity, finishes the proof by induction.

2. Construction

Our first goal is to determine how the δ-move diffeomorphism f moves curves on the boundary ∂W (see also [A3]). This is important because by using this we will construct the handlebody picture of the manifolds Σ_n by drawing the attaching circles of the dual handles of the upside down $-W$. This is a nontrivial task because δ-move is performed by first introducing and then canceling 2/3-handle pairs. So the attaching S^2 of the 3-handle might puncture the dual 2-handle curves on $-\partial W$ forcing us to push them into the interior of W. To go around this problem, we will describe the δ-move diffeomorphism in an alternative way, as a carving and uncarving operations, that is 1- and 2- handle exchanges in the interior (this is also referred as "dot and zero exchanges" in short). This technique was exploited in [A3].

In Figure 3 we first replace dot with zero (turning 1-handle to 2-handle), then perform the 2-handle slide (indicated by the arrows), resulting the handlebody on the right. The reverse operation (i.e. going from right to left of the figure) can be obtained by first doing the 2-handle slide, indicated by the dotted arrows, and then by replacing "zero with dot". Here we also traced the dual circles to the 2- handles during this operation (small red circles), where attaching 2-handles to these circles gives the double of W, which we denoted by Σ_0. To construct the handlebody of Σ_n, we need to modify $W \subset \Sigma_n$ along its boundary by a δ-move.

Figure 4 indicates how the δ^n-move $f : \partial W \to \partial W$ affects the dual handles (red) circles (figure drawn for $n = 2$). Going back to the original Figure 1 via the reverse δ-move (as indicated in Figure 3) shows that the effect of the δ-move on dual circles, is as in Figure 5.

Now comes a crucial point: A reader gazing at the first picture of Figure 4 might conclude that δ-move does not move the dual circles because n and $-n$ twists cancel each other. Here are two explanations: First of all, here we are dealing with circles-with-dots not framed circles, transferring twist across them has the affect of changing the carvings (i.e. changing the interiors). Secondly, the original δ move takes place on ∂W, not on the homotopy ball $\Sigma_n^0 = W \smile$ [dual 2 and 3$-$handles], that is δ may not be an on unknot on $\partial\Sigma_n^0$. Surprisingly, we can obtain Figure 5 by performing δ-move to Σ_n^0 (the first picture of Figure 3, with dual handles) by using the curve d of Figure 6. This d is in fact an unknot on $\partial(\Sigma_n^0)$ which can be checked by the boundary correspondence of Figure 3. Also d happens to be an unknot on ∂W, so we could use d for the place of δ to serve for the dual purpose.

To sum up, the first picture of Figure 7 represents a handlebody of Σ_n. Now it is easy to check that the middle dotted curve in the second picture of Figure 7 is an unknot (to see this, do the reverse δ-move go back to the first picture of Figure 3, and then observe that in the presence of the dual 2-handles, the dotted circle becomes an unknot there). From this we see that Σ_n is obtained from $S^4 = \Sigma_0$ by Gluck twisting (this requires a simple check here, namely remove the dotted circle, and the -1 twist on the curves it links from the middle of Figure 7, then see that you get S^4). Now by using this unknot, we can attach a 2/3-handle pair (the new 2-handle is the 0-framed dotted curve in the figure). Next we employ a trick , which was used solving the "Cappell-Shaneson homotopy sphere problem" (Figure 14.11 of [A2]): After the obvious handle slide over the middle 0-framed 2-handle in Figure 7, we obtain the pictures of Figure 8, where we can see two cancelling 1/2 -handle pairs! The two 0-framed middle 2-handles cancel the two 1-handles (represented with large dotted circles)! So this picture can be thought of a handlebody without 1-handles, and hence turning it upside down we will get a handlebofy without 3-handles! Having noted this, we can turn this handlebody upside down (as the process described in [A2]). That is, we ignore the cancelled 1/2 handle pairs, and carry the duals of the remaing 2-handles to the boundary of $\#3(S^1 \times B^3)$ by a diffeomorphism (duals to 2-handles are indicated by the dashed little circles in Figure 8).

Now our task is to find a diffeomorphism from the boundary of the pictures of Figure 8, to $\#3(S^1 \times B^3)$ and carry the dual 2-handles. By applying the Figure 3 boundary identification, we see that the boundary of the last picture of Figure 8 can be identified with the boundary of the first picture of Figure 9, then the obvious isotopy gives the second picture of Figure 9. Note that we do not draw 3- and 4-handles here, the handldebodies of Σ_n and Σ_n^0 will be drawn the same.

Again by applying the reverse boundary identification of Figure 3 to Figure 9 we get Figure 10, which is a handlebody picture of Σ_n, without 3-handles! Finally the indicated simple handler slide gives Figure 11 (the picture is drawn for $n = 2$). To indicate how the pattern changes as we increse $n \to n+1$, in Figure 12, we drew Σ_n for $n = 1$.

Now let us check the identification 2 of Section 1. We will demonstrate a proof for Σ_1 (from this the reader can see the proof for the general case). For this we first isotope Figure 12 to Figure 13, then do the handle slides and cancellations of the figures Figures 13 $\rightsquigarrow .. \rightsquigarrow$ 18, as indicated in the pictures. During these operations we trace the ribbon which the unknot T of Figure 13 bounds. In this figure this ribbon is the trivial ribbon bounding the unknot, where its ribbon move indicated with an unknotted arc in Figure 15. But during the handle slide Figure 15 $\to$ Figure 16 this trivial ribbon turns into the nontrivial ribbon D, mentioned above. By performing the ribbon move in Figure 16, along the indicated dotted arc, we get Figure 17 (the dotted blue line of his figure is the dual of dotted red line of Figure 16). Then the 1/2 handle cancellation by using the -1 framed 2-handle, gives Figure 18 which is Σ_1. To see the general pattern, we can apply the same steps to Figure 11 rather than Figure 12, then we see that we get Figure 19 picture of Σ_2. Now the handlebody patterns of Σ_n is as required in 2.

3. Rolling versus carving

Notice that the loop $c \subset \partial M_n = S^3$ which links the ribbon in Figure 18 (and in Figure 19) is the unknot in S^3. This is because doing -1 surgery to c (which corresponds to putting 0-framing on c on the figure) gives S^3. Hence by Property P the loop c must be the unknot. Now we can attach a cancelling 2/3 handle pair to Σ_n along c (this corresponds to adding $+1$ framed 2-handle to c). This gives an alternative description of Σ_n which contains a copy of W:

$$\Sigma_n^0 = W \smile_{f^n(\gamma)} h_\gamma^2 \tag{3}$$

This is because W is in the form $W = Q \smile h_c^2$, where $Q = B^4 - N(D)$ and h_c is a $+1$-framed 2 handle attached along c ([A1] Remark 1, and [G1])), i.e. Σ_n^0 is obtained by attaching 2-handle to W along the n-th iterate of a loop $\gamma \subset \partial Q$ by some diffeomorphism

$$f : \partial Q \to \partial Q$$

Remark 1. The handlebody picture of Σ_n (Figures 18 and 19) shows that, by changing the carving, which $K\#K$ bounds in B^4 by a diffeomorphism will move the position of the 2-handle γ to the 2-handle of Figure 21, which can easily be identified with B^4. This diffeomorphism is obtained by first moving the knot $K\#K$ by an isotopy $g_t : S^3 \to S^3$ back to itself as indicated in Figure 20 along the dotted arrow (i.e. *rolling* one of the factors of the connected sum over $K\#K$ back to itself), then letting $g_1(K\#K)$ bound the standard slice disk D in B^4, which $K\#K$ bounds. Call this new ribbon disk D'. Recall that in [A4] relatively exotic but diffeomorphic ribbon complements in B^4 were constructed. Here the ribbon complements D and D' have the similar property (otherwise W would not be a loose cork).

Remark 2. Reader should compare this to the infinitely many absolutely exotic manifolds of [A3], which also decompose as 3. To study Σ_n we have two options: (1) Either attach the rolled 2-handle to the standard ribbon complement $B^4 - D$ as in Figure 19, or (2) Attach the standard 2 handle of Figure 21 to the nonstandard ribbon complement $B^4 - D'$, carved by rolling.

Note that the "Dehn twist diffeomorphism $\partial W \to \partial W$ along an imbedded torus", discussed in [G1] and [RR], corresponds to δ-move diffeomorphism along some $\delta \subset \partial W$. Patient reader can check this by tracing the steps outlined by Remark 1 of [A1], one gets the identification of Figure 24. Then by doubling and connect summing the circle δ_1 and the arc δ_2 one can recover the position of δ on the right picture of W in Figure 24 (cf.[A2]). This shows that the δ move of W corresponds to Dehn twisting boundary of W along the imbedded torus of Figure 25. Also note that since any imbedded torus $T \subset S^3$ bounds a solid torus any Dehn twist diffeomorphism $S^3 \to S^3$ is isotopic to identity. Hence δ -move diffeomorphisms of S^3 are isotopic to identity.

We are now ready for the proof of Theorem 1.1: Recall that we have the identification of the ribbon complements of Figure 11 (without the curve γ), which is Figure 19 (without the curve γ), with Figure 1 (without the small -1 linking circle). Now a patient reader can easily check that under these identifications the curve γ of Figure 19 (which we also denote by γ_n) corresponds to the position of the curve curve γ of Figure 26, after δ^n -move diffeomorphism (as described by Figure 2). It is amusing that δ of Figure 26 remains unknot even after we attached a 2-handle γ to W, so in particular δ-move commutes with attaching the 2-handle handle γ (this can be checked by using the Figure 3 identification). Now remarks of the last paragraph of Section 1 finishes the proof. □

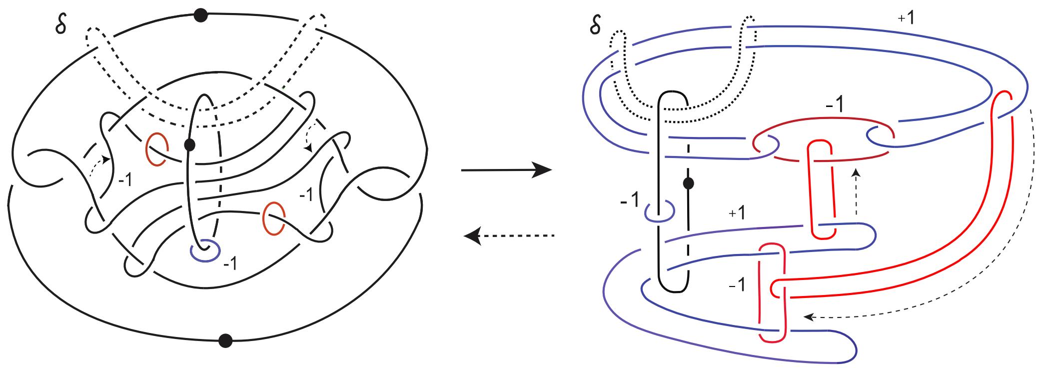

FIGURE 3. Changing the carvings

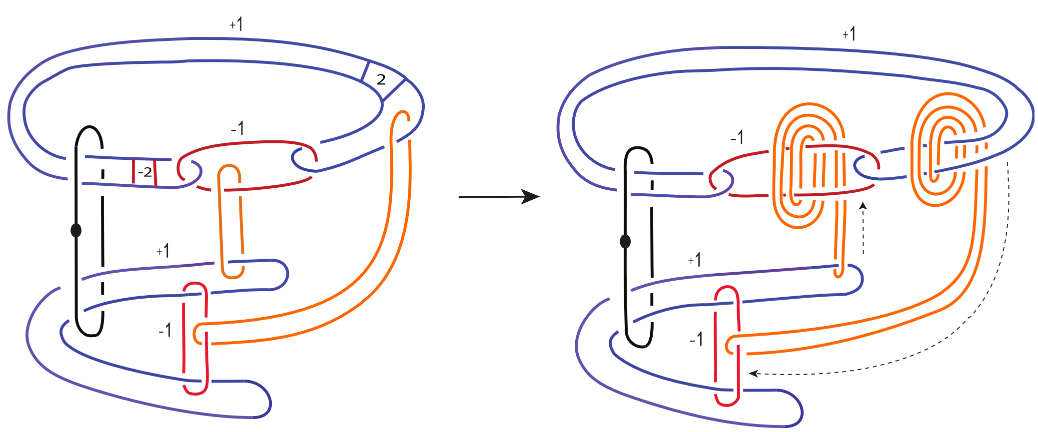

FIGURE 4. Affect of δ-move on the boundary, n=2

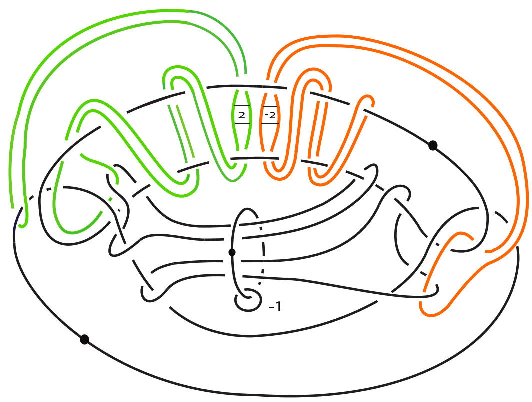

FIGURE 5. Σ_2

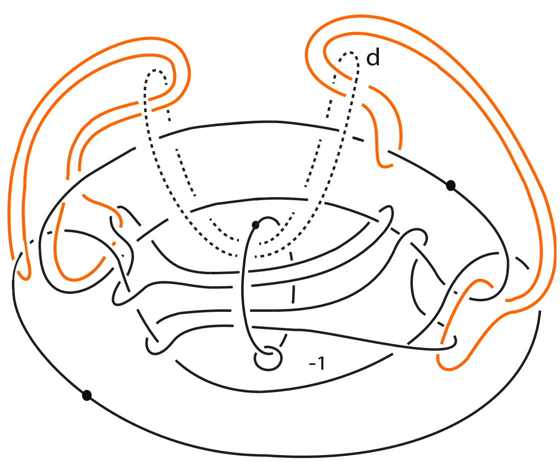

FIGURE 6. $d \subset \Sigma_0 = B^4$

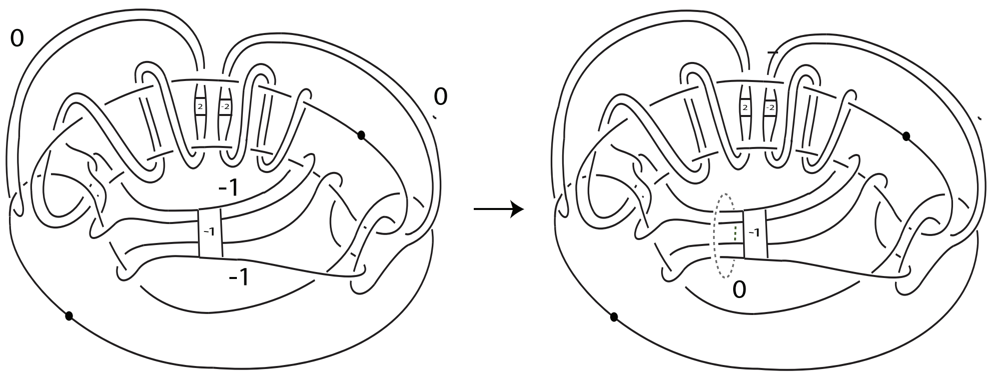

Figure 7. Σ_n

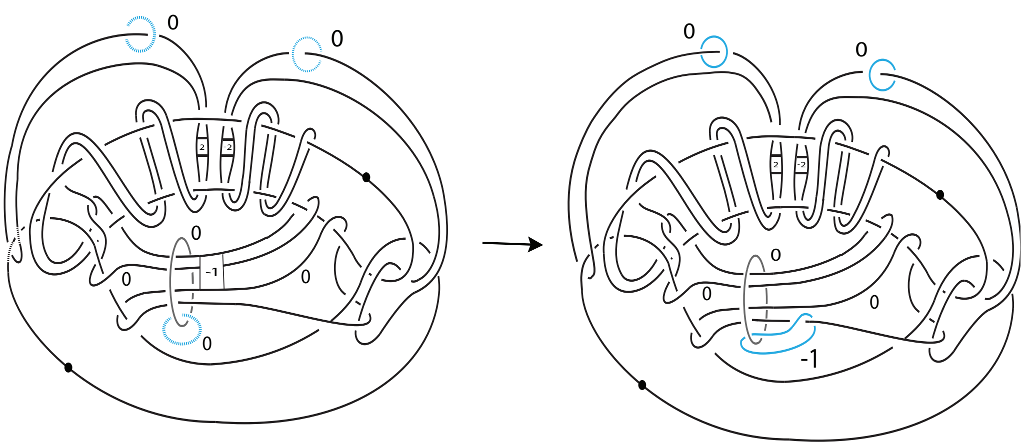

Figure 8. Turning Σ_n upside down

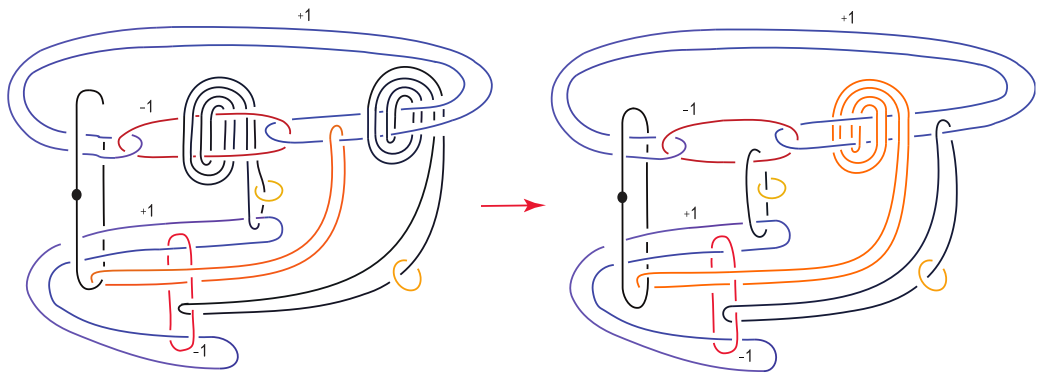

FIGURE 9. Σ_n, for n=2

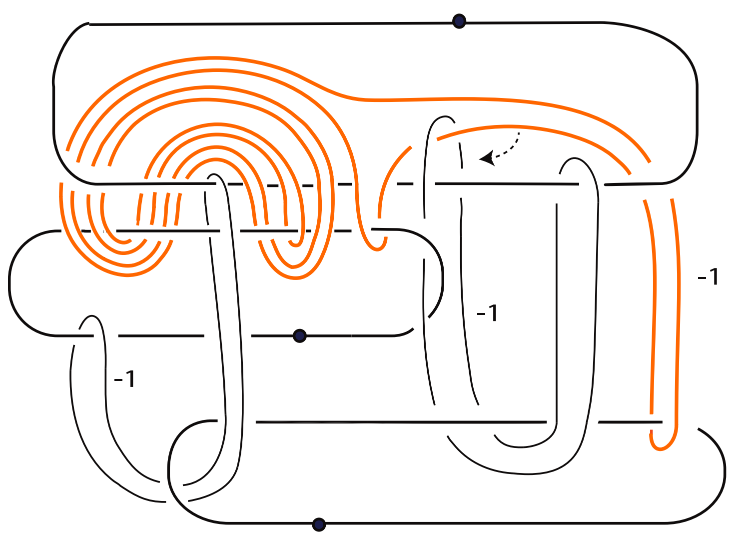

FIGURE 10. Σ_n, for $n = 2$

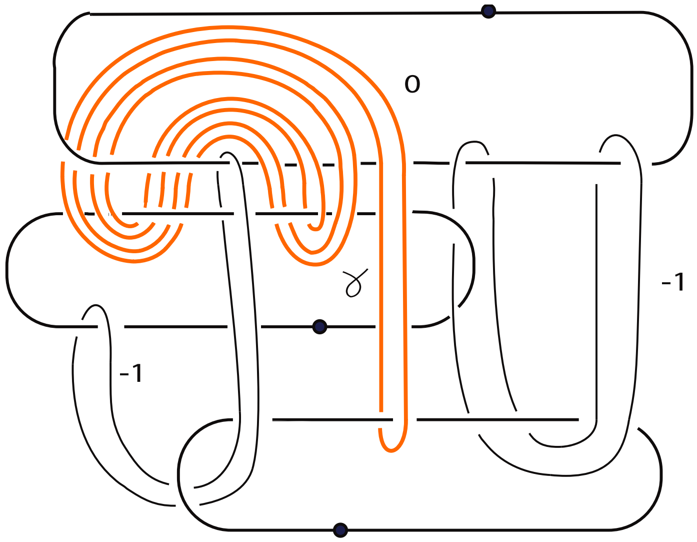

FIGURE 11. Σ_n, for $n = 2$

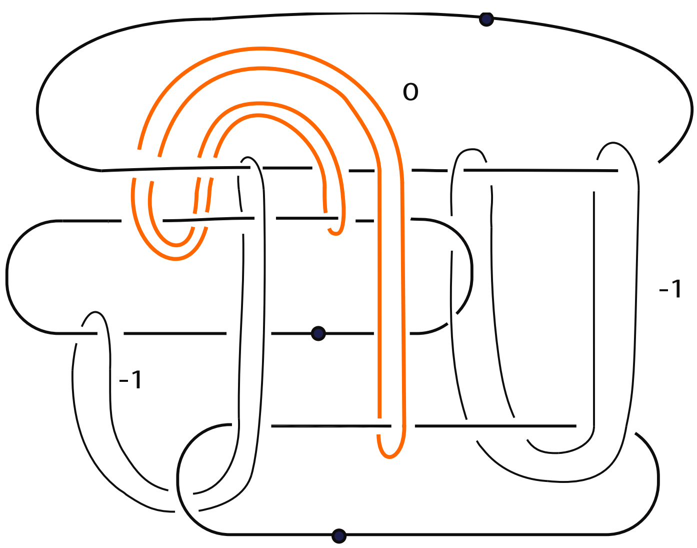

FIGURE 12. Σ_1

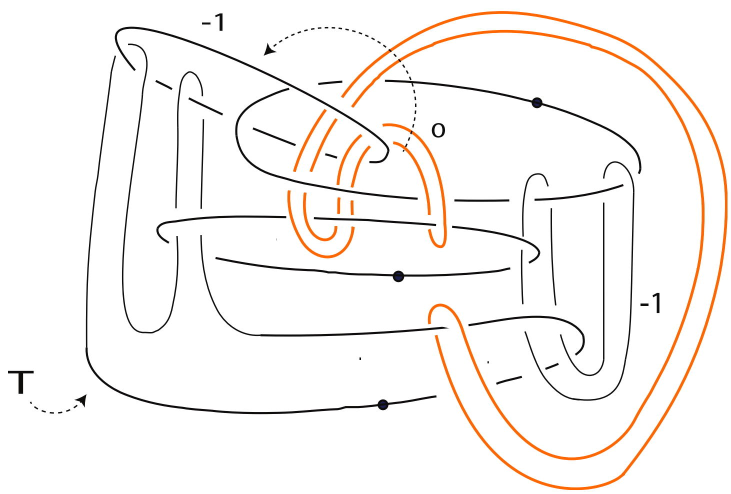

FIGURE 13. Σ_1

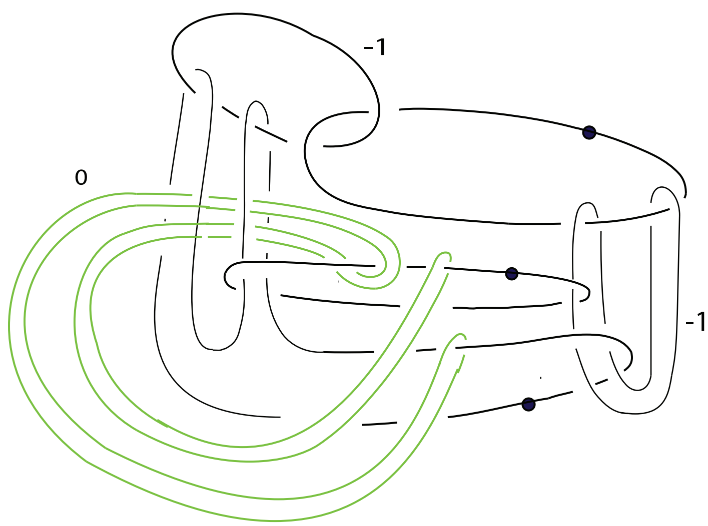

FIGURE 14. Σ_1

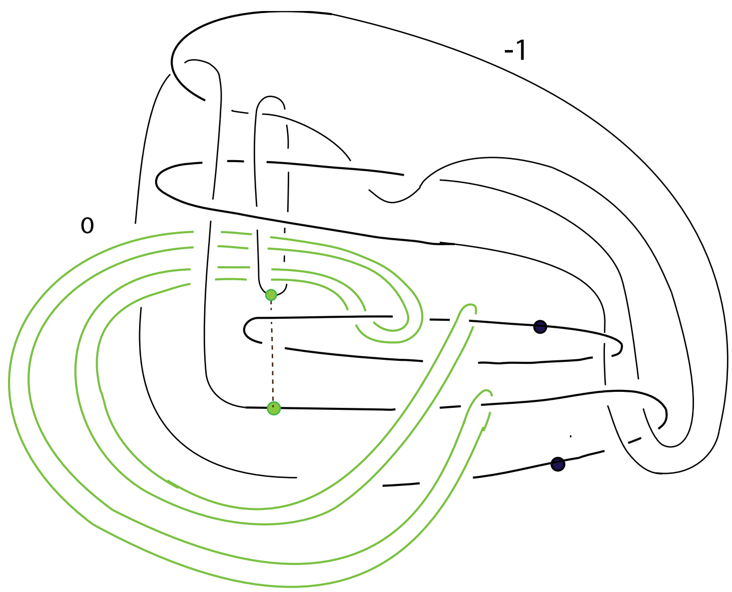

FIGURE 15. Σ_1

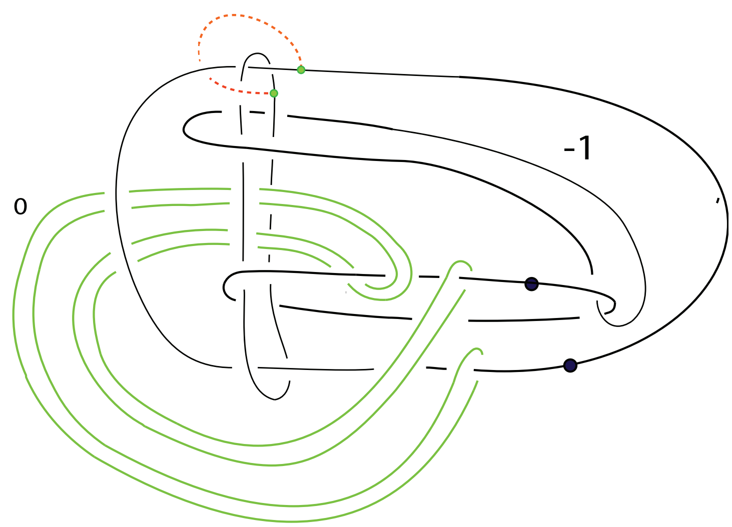

FIGURE 16. Σ_1

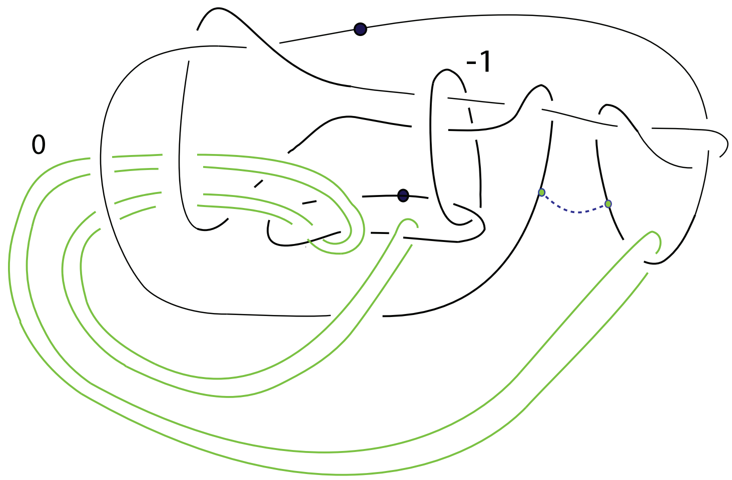

FIGURE 17. Σ_1

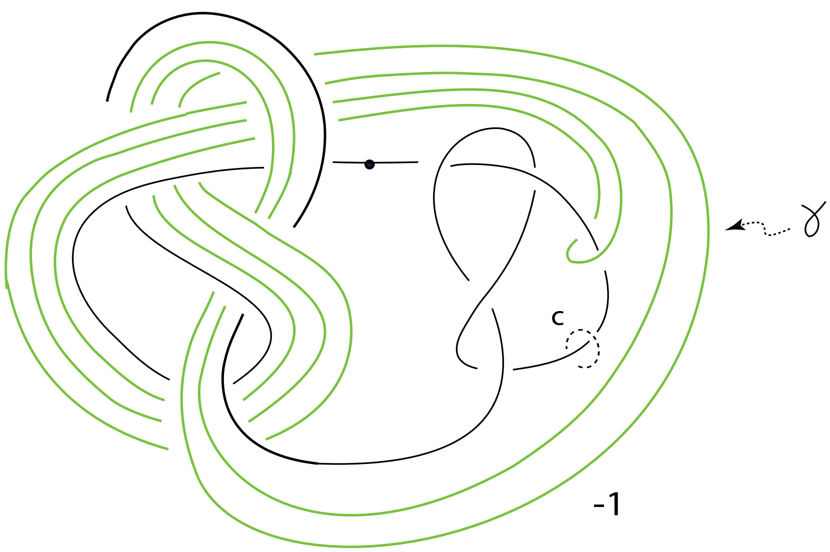

FIGURE 18. Σ_1

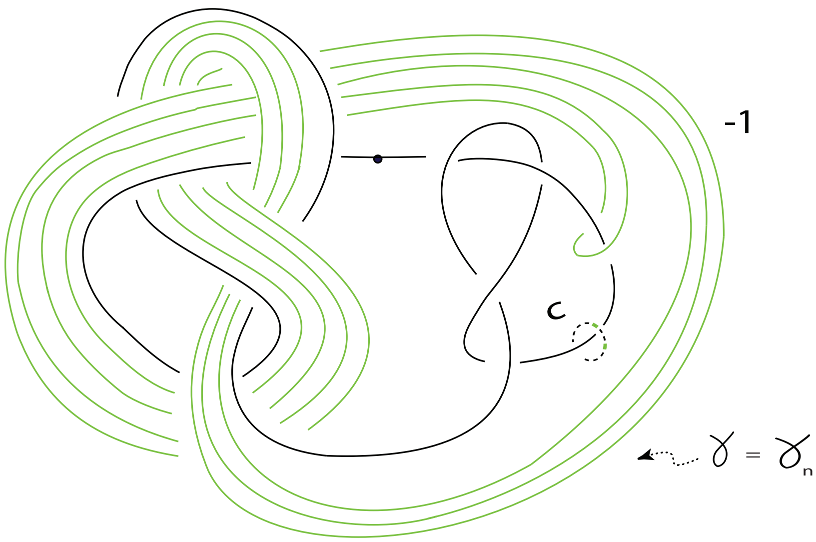

FIGURE 19. Σ_n, $n = 2$

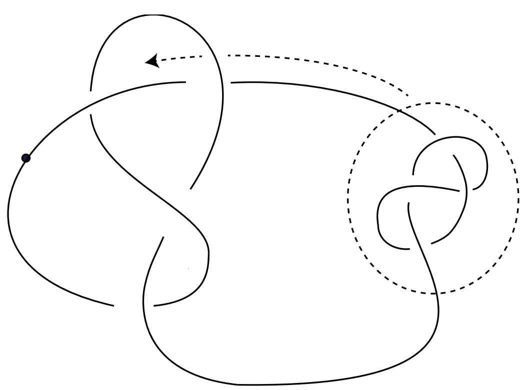

FIGURE 20. Rolling f^n

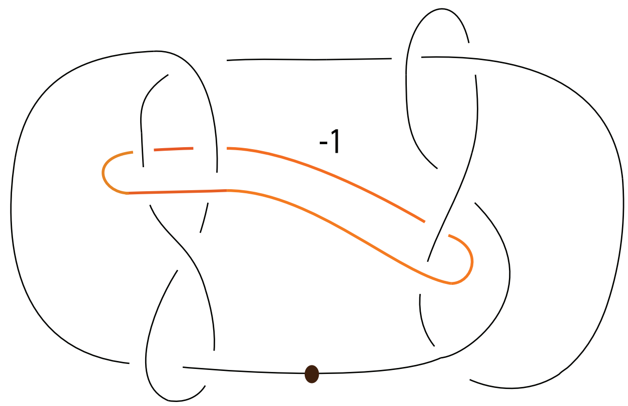

FIGURE 21. $\Sigma_0 = B^4$

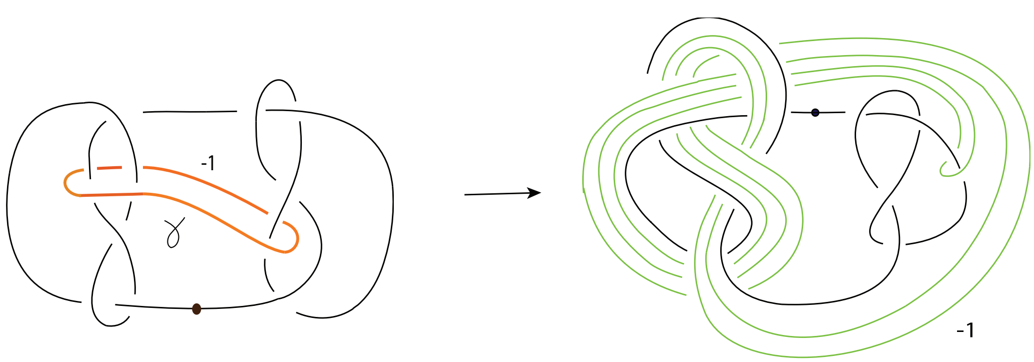

FIGURE 22. Rolling 2-handle γ by f^n

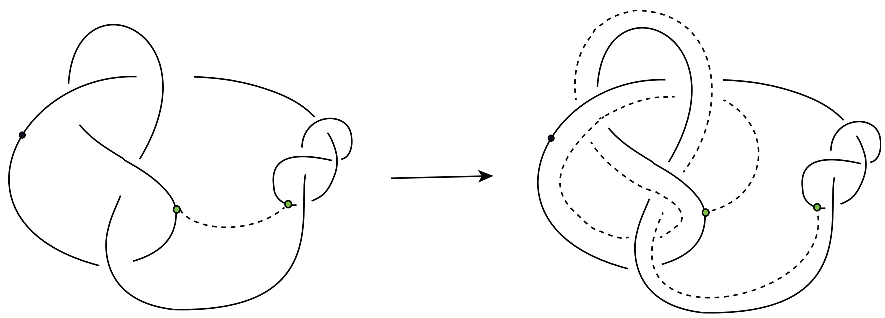

Figure 23. Carving ribbon 1-handle by f^n

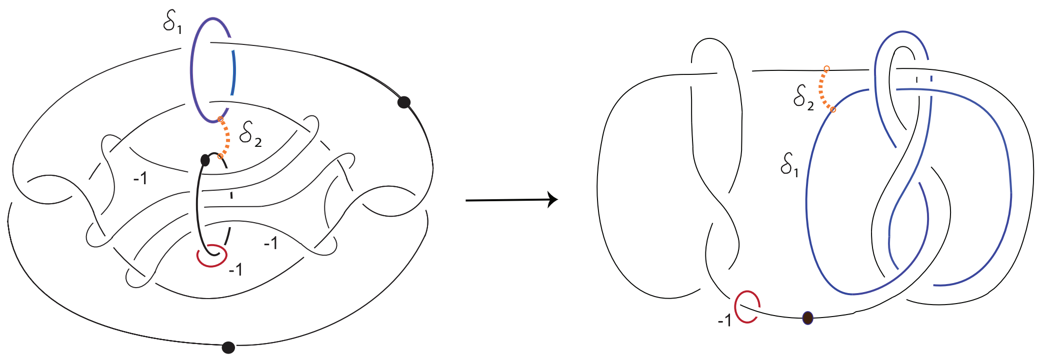

Figure 24. δ-move $\rightsquigarrow$ Dehn surgery

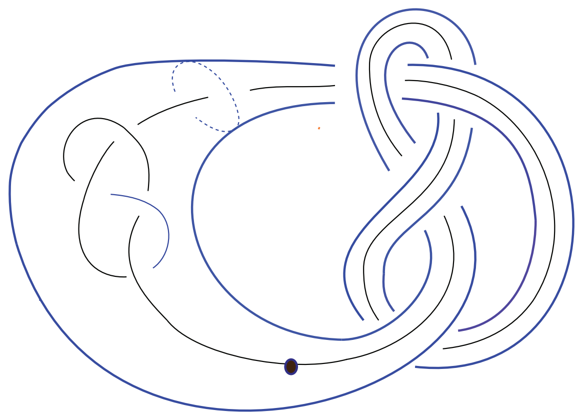

Figure 25. Dehn surgered torus

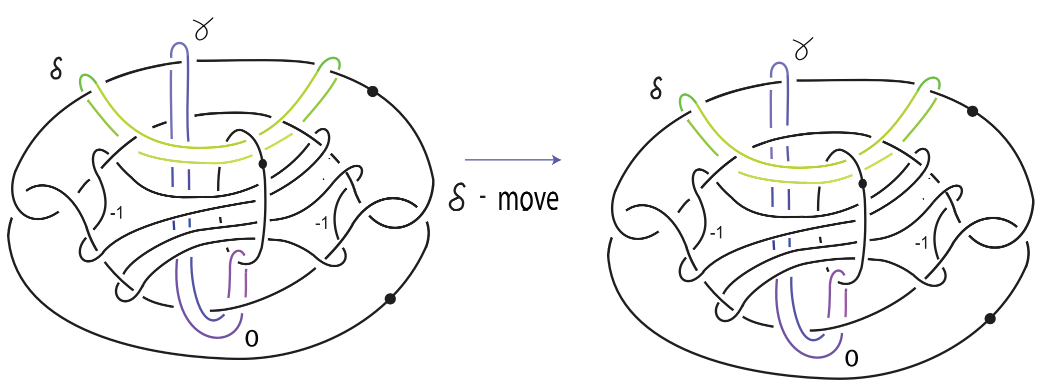

Figure 26. δ-move

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