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On a class of symplectic 4-orbifolds with vanishing canonical class

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Abstract. A study of certain symplectic 4-orbifolds with vanishing canonical class is initiated. We show that for any such symplectic 4-orbifold X, there is a canonically constructed symplectic 4-orbifold Y, together with a cyclic orbifold covering $Y \to X$, such that Y has at most isolated Du Val singularities and a trivial orbifold canonical line bundle. The minimal resolution of Y, to be denoted by $\tilde{Y}$, is a symplectic Calabi-Yau 4-manifold endowed with a natural symplectic finite cyclic action, extending the deck transformations of the orbifold covering $Y \to X$. Furthermore, we show that when $b_1(X) > 0$, $\tilde{Y}$ is a T^2-bundle over T^2 with symplectic fibers, and when $b_1(X) = 0$, $\tilde{Y}$ is either an integral homology $K3$ surface or a rational homology T^4; in the latter case, the singular set of X is completely classified. To further investigate the topology of X, we introduce a general successive symplectic blowing-down procedure, which may be of independent interest. Under suitable assumptions, the procedure allows us to successively blow down a given symplectic rational 4-manifold to $\mathbb{CP}^2$, during which process we can canonically transform a given configuration of symplectic surfaces to a "symplectic arrangement" of pseudoholomorphic curves in $\mathbb{CP}^2$. The procedure is reversible; by a sequence of successive blowing-ups in the reversing order, one can recover the original configuration of symplectic surfaces up to a smooth isotopy.

1. Introduction and the main results

In this paper, we consider a class of symplectic 4-orbifolds which have vanishing canonical class. Our consideration has its origin in the study of symplectic Calabi-Yau 4-manifolds endowed with certain symplectic finite group actions (cf. [6]); in particular, the quotient orbifolds arising in [6] belong to this class of 4-orbifolds. (By definition, a symplectic 4-manifold is called *Calabi-Yau* if it has trivial canonical line bundle.) We regard these symplectic 4-orbifolds as certain intermediate objects, between the symplectic rational or ruled 4-manifolds and the symplectic Calabi-Yau 4-manifolds. On the one hand, we believe classifying such 4-orbifolds is a more attainable objective, and on the other hand, we hope that these 4-orbifolds may lead to new progress in the topology of symplectic Calabi-Yau 4-manifolds. Finally, these symplectic 4-orbifolds are an interesting object to study in its own right.

Key words and phrases. Symplectic 4-orbifolds, symplectic resolution, finite group actions, symplectic Calabi-Yau, configurations of symplectic surfaces, rational 4-manifolds, successive symplectic blowing-down, symplectic arrangements, pseudo-holomorphic curves.

The class of symplectic 4-orbifolds to be considered in this paper, which will be denoted by X throughout, are specified by the conditions (i)-(iii) below. We denote the underlying space of X by $|X|$.

(i) The canonical line bundle K_X, as an orbifold complex line bundle, has a well-defined first Chern class $c_1(K_X) \in H^2(|X|;\mathbb{Q})$. We assume $c_1(K_X)=0$.

(ii) We assume the singular set of X consists of a disjoint union of embedded surfaces $\{\Sigma_i\}$ and a set of isolated points $\{q_j\}$, where we denote by $m_i>1$ the order of isotropy along Σ_i, and by G_j the isotropy group at q_j. Note that the symplectic G_j-action on the uniformizing system (i.e., orbifold chart) centered at q_j naturally defines G_j as a subgroup of $U(2)$ (i.e., $G_j \subset U(2)$). With this understood, we let H_j be the normal subgroup of G_j which consists of elements of determinant 1 (i.e, $H_j = G_j \cap SU(2)$), and let m_j be the order of the quotient group G_j/H_j, which is easily seen cyclic. (The singular point q_j is called a Du Val singularity if and only if $m_j=1$.)

(iii) We set $n := \text{lcm}\{m_i, m_j\}$ to be the least common multiple of m_i, m_j, and we assume $n>1$, which means that either there is a 2-dimensional component Σ_i in the singular set, or there is a singular point q_j of non-Du Val type.

With the preceding understood, we recall a construction from [5], Theorem 1.5, that is, for any symplectic 4-orbifold, one can canonically associate it with a symplectic 4-manifold, called the *symplectic resolution*. In the present case, the symplectic resolution of X, to be denoted by $\tilde{X}$, is obtained as follows. First, one de-singularizes the symplectic structure on X along the 2-dimensional singular components $\{\Sigma_i\}$, which results a natural symplectic structure on the underlying space $|X|$, making it a symplectic 4-orbifold with only isolated singular points $\{q_j\}$. Each Σ_i descends to an embedded symplectic surface in $|X|$, which will be denoted by B_i. With this understood, the symplectic resolution $\tilde{X}$ is simply the minimal symplectic resolution of the orbifold $|X|$. We refer the readers to [5] for more details. (Compare also [16].)

For each j, let $\{F_{j,k} | k \in I_j\}$ be the exceptional set in the minimal resolution of q_j, and denote by $D_j := \cup_{k\in I_j} F_{j,k}$ be the configuration of symplectic spheres in $\tilde{X}$. Furthermore, we denote by D the pre-image of the singular set of X in $\tilde{X}$ under the resolution map $\tilde{X} \to X$. Then clearly,

$$D = \cup_i B_i \cup \cup_j D_j.$$

With this understood, we note that the assumption $c_1(K_X)=0$ implies that the canonical class of $\tilde{X}$ is supported in $D \subset \tilde{X}$. Indeed, by Proposition 3.2 of [5], $c_1(K_X)=0$ implies

$$c_1(K_{\tilde{X}}) = -\sum_i \frac{m_i-1}{m_i} B_i + \sum_j \sum_{k\in I_j} a_{j,k} F_{j,k},$$

where $\{a_{j,k}\}$ is a set of rational numbers uniquely determined by the following equations: for each j, we set $c_1(D_j) := \sum_{k\in I_j} a_{j,k} F_{j,k}$, then

$$c_1(D_j)\cdot F_{j,l} + F_{j,l}^2 + 2 = 0, \;\; \forall l \in I_j.$$

We remark that $a_{j,k} \leq 0$, and for each j, $a_{j,k} = 0$ for all $k \in I_j$ if and only if $m_j = 1$, i.e., q_j is a Du Val singularity.

The following fact is fundamental to the considerations in this paper.

Proposition 1.0. *The resolution $\tilde{X}$ is a rational or ruled 4-manifold.*

This is an immediate consequence of the assumption that $n := \text{lcm}\{m_i, m_j\} > 1$; indeed, $n > 1$ implies easily that

$$c_1(K_{\tilde{X}}) = -\sum_i \frac{m_i - 1}{m_i} B_i + \sum_j \sum_{k \in I_j} a_{j,k} F_{j,k} \neq 0,$$

and moreover, if $\tilde{\omega}$ is the symplectic structure on $\tilde{X}$, then $c_1(K_{\tilde{X}}) \cdot [\tilde{\omega}] < 0$, which implies that $\tilde{X}$ is rational or ruled. (Compare also [5], Lemma 4.1, for the case of global quotients.)

Now we state the main results of this paper.

Theorem 1.1. *There exists a symplectic 4-orbifold Y with a cyclic symplectic orbifold covering $\pi : Y \to X$ of degree $n := lcm\{m_i, m_j\}$, which has the following properties.*

1. *The orbifold Y has at most Du Val singularities, which are given by the set $\pi^{-1}(\{q_j | H_j \neq \{1\}\})$.*
2. *The canonical line bundle K_Y is trivial as an orbifold complex line bundle. Moreover, there exists a nowhere vanishing section s of K_Y such that the induced $\mathbb{Z}_n$-action on K_Y by the deck transformations is given by the multiplication of $\exp(2\pi i/n)$, i.e., $s \mapsto \exp(2\pi i/n) \cdot s$, for some generator of $\mathbb{Z}_n$.*
3. *The symplectic $\mathbb{Z}_n$-action on Y by the deck transformations has the following fixed-point set structure: for each i, every component in $\pi^{-1}(\Sigma_i)$ is fixed by an element of order m_i in $\mathbb{Z}_n$, and for each j with $m_j > 1$, every point in $\pi^{-1}(q_j)$ is fixed by an element of order m_j in $\mathbb{Z}_n$. The number of components in $\pi^{-1}(\Sigma_i)$ is n/m_i and the number of points in $\pi^{-1}(q_j)$ is n/m_j, for each i, j.*

The construction of $\pi : Y \to X$ is a standard affair in the algebraic geometry setting (see e.g. [1]). Our construction may be regarded as a topological version of it. Note that even if X arises as a global quotient M/G where M is a symplectic Calabi-Yau 4-manifold, Y is not necessarily the same as M; in fact, $Y \neq M$ as long as $X = M/G$ has an isolated singular point q_j with H_j nontrivial, e.g., a Du Val singularity, as in this case Y is singular. Finally, the quotient $Y/\mathbb{Z}_n$ is naturally a smooth 4-orbifold (cf. [5], Lemma 2.1), and $Y/\mathbb{Z}_n = X$ as orbifolds.

Let $\tilde{Y}$ be the (minimal) symplectic resolution of Y. Then $\tilde{Y}$ is a symplectic Calabi-Yau 4-manifold, and furthermore, the symplectic $\mathbb{Z}_n$-action on Y naturally extends to a symplectic $\mathbb{Z}_n$-action on $\tilde{Y}$ (cf. [5], Theorem 1.5(3)). We note that $\tilde{Y}$ only depends on the partial resolution $\tilde{X}^0$ of X, i.e., the symplectic 4-orbifold obtained by only resolving the Du Val singularities of X. It is easy to see that the quotient orbifold $\tilde{Y}/\mathbb{Z}_n$ equals $\tilde{X}^0$ if and only if for each j with $m_j > 1$, the subgroup H_j is trivial. Finally, note that $\tilde{Y} = Y$ (so that $\tilde{Y}/\mathbb{Z}_n = X$) if and only if H_j is trivial for each j. We shall call Y the *Calabi-Yau cover* of X.

The implication of Theorem 1.1 is two-fold. On the one hand, it gives us a way to construct symplectic Calabi-Yau 4-manifolds; we shall explore this in a future occasion. On the other hand, by exploiting the $\mathbb{Z}_n$-action on $\tilde{Y}$, we may obtain information on the singular set of X. Building on earlier work [6], we are led to

Theorem 1.2. *The Calabi-Yau cover Y and its symplectic resolution $\tilde{Y}$ are classified according to the topology of X as follows:*

(1) *Suppose $b_1(X) > 0$. Then the singular set of X consists of only tori with self-intersection zero. In this case, $Y = \tilde{Y}$, which is a T^2-bundle over T^2 with symplectic fibers.*
(2) *Suppose $b_1(X) = 0$. Then $\tilde{Y}$ is an integral homology $K3$ surface, unless X falls into one of the following two cases: (i) the singular set of X consists of 9 non-Du Val isolated points of isotropy of order 3, or (ii) the singular set of X consists of 5 isolated points of isotropy of order 5 which are all of type $(1, 2)$. In both cases (i) and (ii), $Y = \tilde{Y}$, which is a rational homology T^4.*

We remark that, as a consequence of Theorem 1.2, it remains to further investigate the topology of X for the case where $b_1(X) = 0$, as far as the topology of $\tilde{Y}$ is concerned (if $X = M/G$ is a global quotient and $b_1(X) > 0$, then M must be a T^2-bundle over T^2, cf. [6], Theorem 1.1). The following problems are fundamental.

Problem 1.3. Suppose $b_1(X) = 0$.

(1) Determine the singular set, i.e., the possible topological type, including the orders of isotropy of the 2-dimensional singular components, of X.
(2) Determine whether X admits a complex structure.
(3) Determine whether a possible topological type of the singular set of X can be realized.
(4) Determine the orbifold fundamental group of X (as well as that of the partial resolution $\tilde{X}^0$, i.e., the symplectic 4-orbifold obtained by only resolving the Du Val singularities of X).

Concerning Problem 1.3(1), it remains to consider the case where $\tilde{Y}$ is an integral homology $K3$ surface. It is conceivable that there are only finitely many possible topological types for the singular set of X.

Concerning Problem 1.3(2)-(4), our strategy is to consider the resolution $\tilde{X}$ of X, which is a symplectic rational 4-manifold, and to consider the embedding $D \subset \tilde{X}$, which is a disjoint union of configurations of symplectic surfaces in $\tilde{X}$. The relevant questions concerning X are then reduced to corresponding questions concerning the embedding of $D \subset \tilde{X}$. For instance, Problem 1.3(2) is equivalent to the question as whether the embedding of D in $\tilde{X}$ can be made holomorphic.

In order to study the embedding $D \subset \tilde{X}$, our strategy is to first determine the homology classes of the components of D, i.e., the symplectic surfaces B_i, $F_{j,k}$, with respect to a certain standard basis of $H^2(\tilde{X})$ (called a reduced basis, which depends on the choice of symplectic structure on $\tilde{X}$). The necessary technical machinery was developed in [6]; in particular, it is shown that there are essentially only finitely many possible homological expressions of D in $\tilde{X}$. To go beyond the homological classification of D, we introduce in this paper a general successive blowing down procedure, to be applied to a symplectic rational 4-manifold containing a configuration of symplectic surfaces (we remark that a simple version of this procedure was already employed in [6] in the proof of its main result). In particular, this successive blowing down procedure allows us to reduce $\tilde{X}$ to either $\mathbb{CP}^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, and to construct a canonical descendant of D in $\mathbb{CP}^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$. This descendant of D, to be denoted by $\hat{D}$, is a union of pseudoholomorphic curves with controlled singularities and intersection properties, which depend only on the homological expression of D. The procedure is reversible: by successively blowing up $\hat{D}$, we can recover $D \subset \tilde{X}$ up to a smooth isotopy. In this way, we reduce the questions concerning the embedding $D \subset \tilde{X}$ to relevant questions concerning the embedding of $\hat{D}$ in $\mathbb{CP}^2$ or $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, which we believe are more amenable to the current existing techniques in symplectic topology.

The organization of this paper is as follows. In Section 2, we prove Theorems 1.1 and 1.2. In addition, for the purpose of illustration we also include at the end of the section a few examples of the orbifold X. These examples arise as the quotient orbifold of a holomorphic action on a hyperelliptic surface or a complex torus, and their singular sets do not belong to the case (i) or (ii) in Theorem 1.2. The corresponding symplectic Calabi-Yau 4-manifold $\tilde{Y}$ is a $K3$ surface, equipped with a non-symplectic automorphism of finite order. A common feature of these examples is that the $K3$ surface contains a large number of (-2)-curves. Section 3 contains some general constraints on the singular set of X. There are two, seemingly independent, sources for the constraints. One type of the constraints is obtained by analyzing the symplectic $\mathbb{Z}_n$-action on the Calabi-Yau homology $K3$ surface $\tilde{Y}$, while the other type is derived from the Seiberg-Witten-Taubes theory. In Section 4, we give a detailed account of the successive blowing down procedure. Furthermore, at the end of the section we apply the procedure to a few concrete examples for the purpose of illustration.

2. Proof of the main theorems

Recall that an orbifold complex line bundle $p: L \to X$ is said to be trivial if there is a collection of local trivializations of L such that the associated transition functions are given by identity maps, and moreover, for any local trivialization of L over an uniformizing system (U, G), the G-action on $p^{-1}(L|_U) \cong U \times \mathbb{C}$ is trivial on the $\mathbb{C}$-factor. Note that this latter condition is equivalent to the statement that L descends to an ordinary complex line bundle over the underlying topological space $|X|$. It follows easily that if L is a trivial orbifold complex line bundle, then the underlying total space of L, denoted by $|L|$, is given by the product $|X| \times \mathbb{C}$.

Lemma 2.1. *Set $L := K_X$. Then the n-th tensor power L^n is a trivial orbifold complex line bundle over X. Moreover, n is the minimal positive integer having this property, i.e., if L^m is a trivial orbifold complex line bundle for some $m > 0$, then $n|m$.*

Proof. For each i, let $\nu_i := TX/T\Sigma_i$ be the normal bundle of Σ_i in X, and let U_i be the associated disc bundle of ν_i. Then there is a natural smooth $\mathbb{Z}_{m_i}$-action on U_i, fixing the zero section and free in its complement, such that $U_i/\mathbb{Z}_{m_i}$ is identified with a regular neighborhood of Σ_i in $|X|$. (We may regard $(U_i, \mathbb{Z}_{m_i})$ as an uniformizing system of X along Σ_i.) Note that $H^2(U_i)$ is torsion-free, so that $c_1(K_X) = 0$ in $H^2(|X|;\mathbb{Q})$ implies that K_{U_i} is trivial. With this understood, it is easy to see that there is a trivialization $K_{U_i} \cong U_i \times \mathbb{C}$ such that the induced $\mathbb{Z}_{m_i}$-action on the trivialization is given by the multiplication of $\exp(2\pi i/m_i)$ on the $\mathbb{C}$-factor for some generator of $\mathbb{Z}_{m_i}$. On the other hand, for each j, if we let (U_j, G_j) be an uniformizing system centered at q_j, where U_j is a 4-ball, then $K_{U_j} \cong U_j \times \mathbb{C}$, and the induced G_j-action is given by the multiplication of $\exp(2\pi i/m_j)$ for a generator of G_j/H_j. With this understood, we see immediately that L^n descends to an ordinary complex line bundle over $|X|$. Furthermore, it also follows easily that if L^m is a trivial orbifold complex line bundle for some $m > 0$, then m must be divisible by $n = \mathrm{lcm}\{m_i, m_j\}$.

To show that L^n is the trivial orbifold complex line bundle, it remains to prove that L^n descends to a trivial ordinary complex line bundle over $|X|$. With this understood, we note that $c_1(L^n) = nc_1(L) = 0$ in $H^2(|X|;\mathbb{Q})$, and with L^n as an ordinary complex line bundle over $|X|$, $c_1(L^n)$ admits a lift in $H^2(|X|)$ (still denoted by $c_1(L^n)$ for simplicity), which is torsion. The assertion that L^n descends to a trivial ordinary complex line bundle over $|X|$ follows readily from the claim that $c_1(L^n) = 0$ in $H^2(|X|)$.

We shall prove that $H^2(|X|)$ is torsion-free, so that $c_1(L^n) = 0$ in $H^2(|X|)$ as claimed. To see this, we note that the symplectic resolution $\tilde{X}$ of X is either rational or ruled. Moreover, note that $\pi_1(|X|) = \pi_1(\tilde{X})$, where $\pi_1(\tilde{X}) = 0$ when $\tilde{X}$ is rational, and $\pi_1(\tilde{X}) = \pi_1(\Sigma)$ when $\tilde{X}$ is a ruled surface over a Riemann surface Σ. In any event, $H_1(|X|)$ is torsion-free, so that $H^2(|X|) = Hom(H_2(|X|), \mathbb{Z})$, which implies that $H^2(|X|)$ is torsion-free as well. This finishes the proof of the lemma. ☐

Proof of Theorem 1.1:

Let t denote the tautological section of the pull-back bundle of $p : L \to X$ over the total space L, i.e., for each $x \in L$, $t(x) = x \in (p^*L)_x = L_{p(x)}$. Then consider $\xi := t^n$, the n-th tensor power of t, which is a section of the pull-back bundle of L^n over X to the total space L. Since L^n is trivial (as orbifold complex line bundle), we can fix a trivialization $|L| \times \mathbb{C}$ of the pull-back bundle $p^*L^n \to L$, and denote by 1 the constant section $|L| \times \{1\}$. With this understood, we set $Y := \xi^{-1}(1)$, as a subset of the total space L. The map $\pi : Y \to X$ is simply given by the restriction of $p : L \to X$ to Y. Let λ be the generator of $\mathbb{Z}_n$ which acts on L by fiber-wise complex multiplication by $\exp(2\pi i/n)$. Then it is clear that the tautological section t is equivariant under the $\mathbb{Z}_n$-action, i.e., $t(\lambda \cdot x) = \lambda \cdot t(x)$. With this understood, note that $\xi(\lambda \cdot x) = \xi(x)$, which implies that the set Y is invariant under the action of λ. Furthermore, note that the quotient space of Y under the $\mathbb{Z}_n$-action is identified with X under $\pi : Y \to X$.

With the preceding understood, we shall first show that Y is a smooth orbifold and $\pi : Y \to X$ is a smooth orbifold covering. Equipping Y with the pull-back symplectic structure, $\pi : Y \to X$ becomes a symplectic orbifold covering.

To see that Y is a smooth orbifold, we note that the tautological section t is given by an equivariant section for any local trivialization of the pull-back of L over an uniformizing system, and the argument we give below is obviously equivariant. With this understood, let v be any given direction along the fiber of L^n. Suppose $x \in \xi^{-1}(1) = Y$ be any point. We choose a direction u along the fiber of L such that $ux^{n-1} = \frac{1}{n}v$ holds as tensor product (this is possible because $x \neq 0$ in L). Then it is easy to check that

$$\frac{d}{ds}(t^n(x+su))|_{s=0} = nux^{n-1} = v,$$

which implies that the section ξ intersects the constant section 1 transversely. It follows that Y is a smooth orbifold, which is easily seen of dimension 4.

Next we show that $\pi : Y \to X$ is a smooth orbifold covering. We shall only be inspecting the situation near the singular set of X, as the matter is trivial over the smooth locus. To this end, we first consider the singular components Σ_i. We continue to use the notation from Lemma 2.1, where U_i denotes the disc bundle of the normal bundle of Σ_i in X, with a natural $\mathbb{Z}_{m_i}$-action on U_i such that $U_i/\mathbb{Z}_{m_i}$ gives a regular neighborhood of Σ_i in $|X|$. As we have seen before, K_{U_i} is trivial, so we can fix a trivialization $K_{U_i} \cong U_i \times \mathbb{C}$. Let $\delta_i \in \mathbb{Z}_{m_i}$ be the generator such that the action of δ_i on $U_i \times \mathbb{C}$ is given by the multiplication of $\exp(2\pi i/m_i)$ for the $\mathbb{C}$-factor. With this understood, note that $(U_i \times \mathbb{C}, \mathbb{Z}_{m_i})$ is an uniformizing system for the orbifold L, over which the pull-back bundle $p^*L \to L$ admits a natural trivialization $(U_i \times \mathbb{C}) \times \mathbb{C} \to U_i \times \mathbb{C}$, where δ_i also acts as multiplication by $\exp(2\pi i/m_i)$ on the last $\mathbb{C}$-factor. With this understood, we note that $Y = \xi^{-1}(1)$ is given, in the uniformizing system $(U_i \times \mathbb{C}, \mathbb{Z}_{m_i})$, by the subset

$$V_i := \{(y,z) \in U_i \times \mathbb{C} | y \in U_i, z^n = 1\}.$$

Furthermore, the action of $\lambda \in \mathbb{Z}_n$ is given by $(y,z) \mapsto (y, \exp(2\pi i/n)z)$, and the action of δ_i is given by $(y,z) \mapsto (\delta_i \cdot y, \exp(2\pi i/m_i)z)$. It follows easily that the quotient space of $V_i = \{(y,z) \in U_i \times \mathbb{C} | y \in U_i, z^n = 1\}$ under the $\mathbb{Z}_{m_i}$-action can be identified with

$$\{(y,z) \in U_i \times \mathbb{C} | y \in U_i, z^{n/m_i} = 1\},$$

which is a disjoint union of n/m_i many copies of U_i. This shows easily that over $\pi^{-1}(U_i/\mathbb{Z}_{m_i})$, Y is smooth, $\pi : Y \to X$ is given by the quotient map of the action of $\delta_i^{-1}\lambda^{n/m_i}$, and the number of components in $\pi^{-1}(\Sigma_i)$ is n/m_i. In particular, $\pi : Y \to X$ is a smooth orbifold covering near each Σ_i.

The situation near each q_j is similar. If we let (U_j, G_j) be the uniformizing system near q_j, then in $(U_j \times \mathbb{C}, G_j)$, Y is given by the subset

$$V_j := \{(y,z) \in U_j \times \mathbb{C} | y \in U_j, z^n = 1\}.$$

Moreover, the quotient space by the G_j-action can be identified with

$$\{([y], z) \in (U_j/H_j) \times \mathbb{C} | [y] \in U_j/H_j, z^{n/m_j} = 1\},$$

which is a disjoint union of n/m_j many copies of U_j/H_j. It follows easily that over each component of $\pi^{-1}(U_j/G_j)$, Y is a smooth orbifold, uniformized by (U_j, H_j). Furthermore, $\pi : Y \to X$ is a smooth orbifold covering near each q_j, and the number of points in $\pi^{-1}(q_j)$ is n/m_j. Note that in particular, the argument above proved part (1) and part (3) of Theorem 1.1.

Since we endow Y with the pull-back symplectic structure via $\pi : Y \to X$, one has $K_Y = \pi^* K_X = \pi^* L = (p^* L)|_Y$, as $\pi = p|_Y$. With this understood, the restriction of the tautological section $s := t|_Y$ is a nowhere vanishing section of K_Y. Moreover, the action of $\lambda \in \mathbb{Z}_n$ is given by $s \mapsto \exp(2\pi i/n)s$. This proves part (2).

It remains to show that Y is connected, which is a consequence of $n = \text{lcm}\{m_i, m_j\}$ being minimal. To see this, suppose Y is not connected, and let Y_0 be a connected component of Y. Then there is a factor $m > 1$ of n such that λ^m generates the subgroup of $\mathbb{Z}_n$ which leaves Y_0 invariant. With this understood, note that the action of λ^m is trivial on $K_{Y_0}^{n/m} = K_Y^{n/m}|_{Y_0} = (\pi^* L)^{n/m}|_{Y_0}$. Since $(\pi^* L)^{n/m}|_{Y_0}$ is trivial, it follows that $L^{n/m}$ must be the trivial orbifold complex line bundle over X. But this contradicts Lemma 2.1, hence Y is connected. The proof of Theorem 1.1 is complete.

Next we prove Theorem 1.2. The proof relies heavily on the fixed-point set analysis of symplectic finite cyclic actions on symplectic Calabi-Yau 4-manifolds with $b_1 > 0$ carried out in [6]. Furthermore, some standard facts about the topology of symplectic Calabi-Yau 4-manifolds will also be used in the proof.

We shall first have a recollection of these facts, see [2, 13, 14] for more details. Let M be a symplectic Calabi-Yau 4-manifold. Then M is either an integral homology $K3$ surface or a rational homology T^2-bundle over T^2. The latter case corresponds to M having positive first Betti number, and we should note the following facts about this case, which are used frequently:

$$\chi(M) = Sign(M) = 0, \;\; b_2^+(M) = b_2^-(M) = b_1(M) - 1, \text{ and } 2 \leq b_1(M) \leq 4.$$

Moreover, if $b_1(M) = 4$, then M must be a rational homology T^4.

Proof of Theorem 1.2:

We begin by noting that the two cases $b_1(X) > 0$ and $b_1(X) = 0$ correspond to the resolution $\tilde{X}$ being irrational ruled and rational respectively. On the other hand, since we will consider the symplectic $\mathbb{Z}_n$-action on $\tilde{Y}$, it is useful to note that, as $Y/\mathbb{Z}_n = X$ as orbifolds, the resolution of the quotient orbifold $\tilde{Y}/\mathbb{Z}_n$ is in the same symplectic birational equivalence class with $\tilde{X}$ (cf. [5], Theorem 1.5(3)).

Case (1): $b_1(X) > 0$**.** In this case, the resolution of $\tilde{Y}/\mathbb{Z}_n$ is irrational ruled as we noted above. By Theorem 1.1 of [6], $\tilde{Y}$ is a T^2-bundle over T^2, and moreover, from its proof in [6], we also know that the fibers of the T^2-bundle are symplectic.

To see that $Y=\tilde{Y}$, we note that Y must be smooth. This is because if Y has a singular point, then its minimal resolution in $\tilde{Y}$ gives a configuration of symplectic spheres in $\tilde{Y}$, contradicting the fact that $\pi_2(\tilde{Y})=0$ (as $\tilde{Y}$ is a T^2-bundle over T^2). This proves that $Y=\tilde{Y}$. Note that as a consequence of Y being smooth, X has no singular point q_j with H_j nontrivial (cf. Theorem 1.1(1)). With this understood, the proof of Theorem 1.2(1) is completed by the following lemma.

Lemma 2.2. *There is no isolated singular point q_j of X with $m_j>1$, and any 2-dimensional singular component of X is a torus of self-intersection zero.*

Proof. Suppose to the contrary that there is a q_j with $m_j>1$. We pick a $\tilde{q}_j\in\pi^{-1}(q_j)\subset Y$, which, by Theorem 1.1, is fixed by an element of $\mathbb{Z}_n$ of order m_j. We let H be a subgroup of $\mathbb{Z}_n$ of prime order which fixes $\tilde{q}_j$. Then since H_j is trivial, it is easy to see, from the proof of Theorem 1.1, that the image of $\tilde{q}_j$ in Y/H is a non-Du Val singularity. By Lemma 4.1 of [5], the resolution of Y/H is either rational or ruled. We claim that the resolution of Y/H must be irrational ruled. To see this, note that the resolution of $Y/\mathbb{Z}_n$ and the orbifold $Y/\mathbb{Z}_n$ itself have the same first Betti number. Since the resolution of Y/Z_n (which is $\tilde{Y}/\mathbb{Z}_n$ as $Y=\tilde{Y}$) is irrational ruled, it follows that $b_1(Y/\mathbb{Z}_n)>0$, which implies $b_1(Y/H)\geq b_1(Y/\mathbb{Z}_n)>0$. This shows that the resolution of Y/H is irrational ruled because it has the same first Betti number with Y/H. By Theorem 1.2(2) of [6], the resolution of Y/H being irrational ruled implies that the action of H on Y has only 2-dimensional fixed components, which are tori of self-intersection zero. But this is a contradiction as $\tilde{q}_j$ is an isolated fixed point of H.

By the same argument, any singular component Σ_i of X must be a torus of self-intersection zero. This finishes the proof of the lemma. ☐

Case (2): $b_1(X)=0$. In this case, $\tilde{X}$ is rational, so is the resolution of $\tilde{Y}/\mathbb{Z}_n$.

We first assume $\tilde{Y}$ is a symplectic Calabi-Yau 4-manifold with $b_1>0$. We shall prove that $Y=\tilde{Y}$ and $b_1(\tilde{Y})=4$, and determine the singular set of X.

Lemma 2.3. *There are no singular points in Y. Moreover, $b_1(\tilde{Y})\neq 3$.*

Proof. Note that $2\leq b_1(\tilde{Y})\leq 4$. We shall first consider the case where $b_1(\tilde{Y})=4$. In this case, $\tilde{Y}$ is a rational homology T^4. To see that Y is smooth, we recall the following fact from [19], that is, the cohomology ring $H^*(\tilde{Y};\mathbb{R})$ is isomorphic to the cohomology ring $H^*(T^4;\mathbb{R})$. A consequence of this is that the Hurwitz map $\pi_2(\tilde{Y})\to H_2(\tilde{Y})$ has trivial image. If Y has singularities, then the exceptional set of their resolutions in $\tilde{Y}$ consists of symplectic (-2)-spheres, which is a contradiction. The lemma is proved for the case where $b_1(\tilde{Y})=4$.

Next, consider the case where $b_1(\tilde{Y})=3$. For this case, we recall Lemma 2.6 in [6] which says that under the condition $b_1(\tilde{Y})=3$, if the resolution of $\tilde{Y}/\mathbb{Z}_n$ is rational or ruled, then the $\mathbb{Z}_n$-action must be an involution. On the other hand, later in the proof of Theorem 1.1 of [6], the case where the resolution of $\tilde{Y}/\mathbb{Z}_n$ is rational is actually eliminated, as it was shown in this case that $\tilde{Y}$ must be diffeomorphic to a hyperelliptic surface (in particular, $b_1(\tilde{Y}) = 2$ which is a contradiction). Hence $b_1(\tilde{Y}) \neq 3$.

Finally, assume $b_1(\tilde{Y}) = 2$. In this case, $b_2^-(\tilde{Y}) = 1$, which implies easily that Y can have at most one singular point. Let $\tilde{q}_j \in Y$ be such a singular point and let $q_j = \pi(\tilde{q}_j)$ be the singular point in X. Note that $b_2^-(\tilde{Y}) = 1$ implies that the exceptional set of $\tilde{q}_j$ in the minimal resolution in $\tilde{Y}$ consists of a single symplectic (-2)-sphere, which we denote by E. Furthermore, it is easy to see that $H_j = \mathbb{Z}_2$ and $n/m_j = 1$, as $\pi^{-1}(q_j)$ consists of only one point $\tilde{q}_j$. To derive a contradiction, we let H be any prime order subgroup of $\mathbb{Z}_n$ which acts on Y and $\tilde{Y}$, and let p be the order of H. Note that H fixes the point $\tilde{q}_j \in Y$, so the H-action on $\tilde{Y}$ leave the (-2)-sphere E invariant. Finally, according to the fixed-point set analysis of prime order symplectic actions given in Theorem 1.2 of [6], the order p must be 2 or 3. Furthermore, the action of H on $\tilde{Y}$ can only have tori of self-intersection zero as fixed components.

We claim that H must fix the (-2)-sphere E, which is a contradiction to the classification of the fixed components of H mentioned above. To see this, note that from the proof of Theorem 1.1, it is easy to see that the isotropy subgroup of $\tilde{q}_j \in Y$ of the $\mathbb{Z}_n$-action on Y can be naturally identified with G_j/H_j. Since $\tilde{q}_j$ is fixed by H, we may regard H as a subgroup of G_j/H_j. Let $H' \subset G_j$ be the pre-image of H under $G_j \to G_j/H_j$. Then it follows easily from the fact that $H_j = G_j \cap SU(2)$, $H_j = \mathbb{Z}_2$ and H has order 2 or 3, that H' must be a cyclic group of order 4 or 6. Furthermore, the action of H' (as a subgroup of G_j) on the uniformizing system at q_j has weights $(1,1)$. It follows immediately that the (-2)-sphere E in $\tilde{Y}$ is fixed under the H-action. This finishes the proof of the lemma. □

Since Y is nonsingular by Lemma 2.3, we see immediately that $Y = \tilde{Y}$. We shall next prove $b_1(Y) = 4$ and $n = 3$ or 5. But first, note that Lemma 2.3 implies that for any singular point q_j of X, the group H_j is trivial. Consequently, for any prime order subgroup H of $\mathbb{Z}_n$ acting on Y, the orbifold Y/H does not have any Du Val singularity. On the other hand, since $n = \text{lcm}\{m_i, m_j\}$, the order p of H must be a factor of one of m_i or m_j. It follows from Theorem 1.1(3) that the H-action on Y must fix either a component in $\pi^{-1}(\Sigma_i)$ for some i, or a point in $\pi^{-1}(q_j)$ for some j; in particular, the action of H on Y is not free. By Lemma 4.1 of [5], the resolution of Y/H is either rational or ruled.

Lemma 2.4. *There must be a subgroup H of prime order such that the resolution of Y/H is rational.*

Proof. Suppose to the contrary that for every subgroup of prime order, the resolution of the group action is irrational ruled. Then it follows easily from Theorem 1.2(2) of [6] that the orbifold $X = Y/\mathbb{Z}_n$ has only 2-dimensional singular components.

We pick a subgroup Γ of prime order. Since Γ has no isolated fixed points, the resolution of Y/Γ is simply the underlying space $|Y/\Gamma|$, which is a $\mathbb{S}^2$-bundle over T^2

(cf. [6], Theorem 1.2(2)). Moreover, the fixed-point set of Γ consists of tori of self-intersection zero whose images in $|Y/\Gamma|$ intersect transversely with the fibers of the $\mathbb{S}^2$-bundle (see the proof of Theorem 1.1 in [6]). In fact, more is proved in [6], i.e., for any compatible almost complex structure J on $|Y/\Gamma|$ which is integral near the fixed-point set of Γ, the $\mathbb{S}^2$-bundle on $|Y/\Gamma|$ can be chosen to have J-holomorphic fibers. With this understood, we consider the induced action of $\mathbb{Z}_n/\Gamma$ on $|Y/\Gamma|$, which can be made symplectic (cf. [5]). It is clear that the fixed-point set of Γ is invariant under $\mathbb{Z}_n/\Gamma$, so that we may choose J to be $\mathbb{Z}_n/\Gamma$-invariant.

To derive a contradiction, note that the $\mathbb{Z}_n/\Gamma$-action on $|Y/\Gamma|$ preserves the $\mathbb{S}^2$-bundle structure on $|Y/\Gamma|$, so that there is an induced action of $\mathbb{Z}_n/\Gamma$ on the base of the $\mathbb{S}^2$-bundle, which is T^2. Now since $X = Y/\mathbb{Z}_n$ and $b_1(X) = 0$, it follows easily that the $\mathbb{Z}_n/\Gamma$-action on $|Y/\Gamma|$ must induce a homologically nontrivial action on the base T^2. This implies that there must be an element $g \in \mathbb{Z}_n/\Gamma$ which leaves a $\mathbb{S}^2$-fiber invariant. If g fixes the $\mathbb{S}^2$-fiber, then X would have 2-dimensional singular components that are not disjoint, because the $\mathbb{S}^2$-fiber fixed by g intersects with the fixed-point set of Γ. This is a contradiction, so g must fix two isolated points on the $\mathbb{S}^2$-fiber. It is clear that these two isolated fixed points of g can not be contained in the fixed-point set of Γ, hence give two isolated singular points of X. This is also a contradiction as X has only 2-dimensional singular components. Hence the claim that there must be a subgroup H of prime order such that the resolution of Y/H is rational. This proves the lemma. □

An immediate consequence of Lemma 2.4 is that $b_1(Y) = 4$. This is because if $b_1(Y) \neq 4$, then $b_1(Y) = 2$ by Lemma 2.3. In this case, let H be a subgroup of $\mathbb{Z}_n$ of prime order such that the resolution of Y/H is rational. Then by the classification of fixed-point set structures in Theorem 1.2(3) of [6], Y/H has Du Val singularities. But this is a contradiction to Lemma 2.3, hence $b_1(Y) = 4$.

Lemma 2.5. *There is only one subgroup H of prime order such that the resolution of Y/H is rational. Moreover, the order of H is either* 3 *or* 5*, and n does not have any prime factor which is not equal to the order of H.*

Proof. By Theorem 1.2(3) of [6], if H is a prime order subgroup such that the resolution of Y/H is rational, then the order of H must be either 3 or 5. Moreover, if H has order 3, the fixed-point set Y^H consists of 9 isolated points, and if H has order 5, Y^H consists of 5 isolated points. With this understood, we note that since $\mathbb{Z}_n$ is cyclic, there is a unique subgroup of order p for each prime factor p of n.

Suppose to the contrary that there are prime order subgroups H_1, H_2 of order 3, 5 respectively. Then clearly, Y^{H_1} is invariant under H_2, and Y^{H_2} is invariant under H_1. Examining the action of H_2 on Y^{H_1}, which consists of 9 points, it is easy to see that H_2 must fix exactly 4 points in Y^{H_1}. So $Y^{H_1} \cap Y^{H_2}$ consists of 4 points. On the other hand, examining the action of H_1 on Y^{H_2}, it either fixes 2 points or the entire set. This implies that $Y^{H_1} \cap Y^{H_2}$ either consists of 2 points or 5 points, a contradiction. Hence the claim that there is only one subgroup H of prime order such that the resolution of Y/H is rational, and that the order of H is either 3 or 5.

It remains to show that n does not have any prime factor which is not equal to the order of H. To see this, suppose to the contrary that there is a prime factor p of n, which is not equal to the order of H. Then there is a subgroup Γ whose order equals p. Note that the resolution of Y/Γ must be irrational ruled, hence by Theorem 1.2(2), $p = 2$ or 3. Moreover, the fixed-point set Y^Γ consists of a disjoint union of tori of self-intersection zero, which is disjoint from Y^H. If H has order 3, then Γ must be an involution. Examining the action of Γ on Y^H, which consists of 9 points, there must be a point fixed by Γ, a contradiction. If H has order 5, then $p = 2$ or 3. In any event, Γ must also fix a point in Y^H since Y^H consists of 5 points. This shows that Γ can not exist, and the lemma is proved. □

As a consequence, n must be a power of either 3 or 5. If n is neither 3 nor 5, then there must be an element $g \in \mathbb{Z}_n$ of order 9 or 25. Let θ_1, θ_2 be the angles associated to the action of g on $H^1(Y;\mathbb{R})$ in Lemma 2.7 of [6]. Then by Lemma 2.7 of [6], the Lefschetz number $L(g, Y) = 4(1-\cos\theta_1)(1-\cos\theta_2)$, which is an integer, and $2(\cos\theta_1 + \cos\theta_2) \in \mathbb{Z}$. It is easy to check that with the order of g being 9 or 25, this is not possible. Hence $n = 3$ or 5. Now with $n = 3$ or 5, the singular set of $X = Y/\mathbb{Z}_n$ must be as in (i) or (ii) of Theorem 1.2 by the classification of fixed-point sets in Theorem 1.2(3) of [6].

Conversely, if X is given as in (i) or (ii) of Theorem 1.2, then it is clear that $n = 3$ or 5, and $Y = \tilde{Y}$. We claim that $\chi(Y) = 0$. To see this, we use the Lefschetz fixed point theorem. For example, for the case of $n = 3$, we note that the resolution $\tilde{X} = \mathbb{CP}^2\#12\overline{\mathbb{CP}^2}$ because $c_1(K_{\tilde{X}})^2 = -3$. This implies that $\chi(X) = \chi(\tilde{X}) - 9 = 15 - 9 = 6$. By the Lefschetz fixed point theorem, $3\chi(Y/\mathbb{Z}_3) = \chi(Y) + (3-1)\cdot \#Y^{\mathbb{Z}_3}$. With $X = Y/\mathbb{Z}_3$ and $\#Y^{\mathbb{Z}_3} = 9$, we obtain $\chi(Y) = 0$. The case of $n = 5$ is similar. Hence our claim that $\chi(Y) = 0$. It follows that Y has $b_1 > 0$. By the classification in Theorem 1.2(3) of [6], we have $b_1(Y) = 4$. This finishes the proof of Theorem 1.2.

Example 2.6. We list a few examples of holomorphic G-actions on a hyperelliptic surface or complex torus M such that the quotient orbifold $X = M/G$ does not have the singular set in (i) or (ii) of Theorem 1.2 and $b_1(X) = 0$. Hence by Theorem 1.2, the corresponding symplectic Calabi-Yau 4-manifold $\tilde{Y}$ must be a $K3$ surface, with the symplectic $\mathbb{Z}_n$-action defining an automorphism of the $K3$ surface. Note that the automorphism must be non-symplectic, because if it were symplectic, the resolution of $\tilde{Y}/\mathbb{Z}_n$ must also be a $K3$ surface. However, we know that the resolution of $\tilde{Y}/\mathbb{Z}_n$ is in the same symplectic birational equivalence class with $\tilde{X}$ (cf. [5], Theorem 1.5(3)), which is rational. An interesting feature of these examples is that the $K3$ surface contains a large number of (-2)-curves appearing in various types of configurations in the complement of the fixed-point set of the non-symplectic automorphism, coming from the resolution of the Du Val singularities in X.

(1) Take a holomorphic involution on a hyperelliptic surface which fixes 2 tori and 8 isolated points. The orbifold X has a singular set of 2 embedded tori and 8 isolated points of Du Val type. The $K3$ surface $\tilde{Y}$ admits a non-symplectic involution, which fixes 2 tori, and in the complement of the fixed-point set, there are 16 disjoint (-2)-curves.

(2) Take a holomorphic $\mathbb{Z}_3$-action on a hyperelliptic surface, which has 6 isolated fixed points and either no fixed curve or a single fixed torus (both cases are possible), where exactly 3 of the isolated fixed points are Du Val. In this case, $\tilde{Y}$ comes with a non-symplectic automorphism of order 3, which has 3 isolated fixed points and either no fixed curve or a single fixed torus, such that in the complement of the fixed-point set, there are 9 pairs of (-2)-curves, each intersecting transversely in one point.

(3) Consider a holomorphic $\mathbb{Z}_4$-action on a hyperelliptic surface, which has 4 isolated fixed points where 2 of them are Du Val, and 4 isolated points of isotropy of order 2. The quotient orbifold X has 6 singular points, of which 4 are Du Val. It is easy to see that $n = 2$ in this example, so the $K3$ surface $\tilde{Y}$ comes with a non-symplectic involution. Note that the orbifold Y has 4 Du Val singularities of order 4, and 6 Du Val singularities of order 2, where 2 of the order 2 singularities are fixed by the $\mathbb{Z}_2$ deck transformation. It follows easily that the non-symplectic involution on $\tilde{Y}$ has 2 fixed (-2)-curves, and in the complement there are 4 disjoint (-2)-curves and 4 linear chains of (-2)-curves, each containing 3 curves. (Totally, we see 18 (-2)-curves in $\tilde{Y}$.)

(4) Finally, we consider a holomorphic $\mathbb{Z}_8$-action on a complex torus. It has 2 isolated fixed points, all of type $(1,5)$, 2 isolated points of isotropy of order 4 of type $(1,1)$, and 12 isolated points of isotropy of order 2. The quotient orbifold X has 6 singular points, of which 3 are Du Val singularities. It is easy to see that $n = 4$ in this example. Note that the orbifold Y has 16 Du Val singularities of order 2, whose resolution gives 16 disjoint (-2)-curves in the $K3$ surface $\tilde{Y}$. Let τ be the non-symplectic automorphism of order 4. Then the action of τ on $\tilde{Y}$ is as follows: τ^2 fixes 4 of the 16 disjoint (-2)-curves in $\tilde{Y}$, and furthermore, τ switches 2 of the 4 curves of isotropy of order 2, and leaves each of the remaining 2 curves invariant. In particular, note that τ has 4 fixed points, which are contained in the 2 invariant (-2)-curves.

3. Constraints of the singular set

3.1. Group actions on Calabi-Yau homology $K3$ surfaces

Theorem 1.1 has interesting applications on symplectic finite group actions on symplectic 4-manifolds with torsion canonical class. For an illustration, we shall consider the case of symplectic Calabi-Yau homology $K3$ surfaces; the result will be used later in the section.

A well-known property of holomorphic actions on a $K3$ surface is that the fixed-point set does not contain points of mixed types, i.e., of both Du Val and non-Du Val types. The reason is that the canonical line bundle of a $K3$ surface is holomorphically trivial, meaning that there is a nowhere vanishing holomorphic section. If the induced action on the holomorphic section is trivial, then all fixed points are Du Val, and if the induced action is nontrivial, none of the fixed points are Du Val.

In the following theorem, we generalize this phenomenon to the symplectic category. For simplicity, we assume the group action is of prime order.

Theorem 3.1. *Let M be a symplectic Calabi-Yau 4-manifold with $b_1 = 0$, which is equipped with a symplectic G-action of prime order p. Let $X = M/G$ be the quotient orbifold such that the resolution $\tilde{X}$ is rational. Then M with the symplectic G-action is equivariantly symplectomorphic to the Calabi-Yau cover Y equipped with the symplectic $\mathbb{Z}_p$-action of deck transformations (note that $n = p$ in this case). As a consequence, the canonical line bundle K_M admits a nowhere vanishing section s, such that the induced action of G on K_M is given by multiplication of $\exp(2\pi i/p)$ for some generator $g \in G$. In particular, the fixed-point set M^G does not contain any fixed points of Du Val type.*

Proof. First of all, since G is of prime order p, the singular set of $X = M/G$ consists of 2-dimensional components $\{\Sigma_i\}$ and isolated points $\{q_j\}$, where $m_i = p$ for each i and $G_j = G$ for each j. If for some q_j, $m_j = 1$, then $H_j = G$, and if $m_j > 1$, then H_j is trivial and $m_j = p$. Since $\tilde{X}$ is rational, it follows easily that $n = p$, where $n := \mathrm{lcm}\{m_i, m_j\}$.

We claim that the set $\{q_j | m_j = 1\}$ is empty, and $Y = \tilde{Y}$, which is a symplectic Calabi-Yau 4-manifold with $b_1 = 0$. To see this, we first note that the singularities of Y are given by the pre-image $\pi^{-1}(q_j)$ where q_j is a singular point of X with $m_j = 1$, so the canonical symplectic $\mathbb{Z}_p$-action on Y acts freely on the singular set of Y. With this understood, let x be the number of singular points q_j such that $m_j = 1$. Then

$$\chi(\tilde{Y}/\mathbb{Z}_p) = \chi(M/G) + x(p-1) \text{ and } \chi(\tilde{Y}^{\mathbb{Z}_p}) = \chi(M^G) - x.$$

On the other hand, the Lefschetz fixed point theorem implies that

$$p \cdot \chi(M/G) = \chi(M) + (p-1) \cdot \chi(M^G), \quad p \cdot \chi(\tilde{Y}/\mathbb{Z}_p) = \chi(\tilde{Y}) + (p-1) \cdot \chi(\tilde{Y}^{\mathbb{Z}_p}).$$

It follows easily that $\chi(\tilde{Y}) = \chi(M) + x(p^2 - 1)$. With $\chi(M) = 24$, it follows immediately that $\chi(\tilde{Y}) = 24$ and $x = 0$. Hence our claim.

It remains to show that M is G-equivariantly symplectomorphic to Y with the natural $\mathbb{Z}_p = G$ action. This part relies on a well-known property of M that $\pi_1(M)$ has no subgroups of finite index (cf. [2, 13]). To finish the proof, we let $pr : M \to X = M/G$ be the quotient map. We claim that pr can be lifted to a map $\psi : M \to Y$ under the orbifold covering $\pi : Y \to X$ from Theorem 1.1. To this end, we need to examine the image of $pr_* : \pi_1(M) \to \pi_1^{orb}(X)$, and show that $pr_*(\pi_1(M)) \subset \pi_*(\pi_1(Y))$. For this we observe that there is a surjective homomorphism $\rho : \pi_1^{orb}(X) \to \mathbb{Z}_p$ associated to the orbifold covering $\pi : Y \to X$ such that $\pi_*(\pi_1(Y))$ is identified with the kernel of ρ. With this understood, suppose to the contrary that $pr_*(\pi_1(M))$ is not contained in $\pi_*(\pi_1(Y))$. Then the homomorphism $\rho \circ pr_* : \pi_1(M) \to \mathbb{Z}_p$ must be surjective as p is prime. The kernel of $\rho \circ pr_*$ is a subgroup of $\pi_1(M)$ of a finite index, which is a contradiction. Hence our claim that pr can be lifted to a map $\psi : M \to Y$ under the orbifold covering $\pi : Y \to X$. The map $\psi : M \to Y$ is clearly an equivariant diffeomorphism, inducing the identity map on the orbifold X. Since the symplectic structure on Y is the pull-back of the symplectic structure on X via the orbifold covering $\pi : Y \to X$, it follows that ψ is a symplectomorphism. This completes the proof of Theorem 3.1. □

Now we state a theorem which gives some general constraints on the singular set of X (here X is not necessarily a global quotient M/G).

Theorem 3.2. *Suppose the symplectic Calabi-Yau 4-manifold $\tilde{Y}$ is an integral homology $K3$ surface. Then the number $n := lcm\{m_i, m_j\}$ and the 2-dimensional components $\{\Sigma_i\}$ of the singular set of X obey the following constraints.*

1. *If p is a prime factor of n, then $p \leq 19$.*
2. *There can be at most one component in $\{\Sigma_i\}$ which has genus greater than 1. If there is such a component in $\{\Sigma_i\}$, then the remaining components must be all spheres. Moreover, n must equal the order of isotropy along the component of genus > 1, and if p is a prime factor of n, then $p \leq 5$.*
3. *There can be at most two components in $\{\Sigma_i\}$ which are torus, and if this happens, there are no other components in $\{\Sigma_i\}$, and $n = 2$ must be true. If there is only one torus in $\{\Sigma_i\}$, then n must equal the order of isotropy along the torus, and moreover, if p is a prime factor of n, then $p \leq 11$.*

Proof. We will prove the theorem by examining the prime order subgroup actions of the symplectic $\mathbb{Z}_n$-action on $\tilde{Y}$. To this end, we let M be a symplectic Calabi-Yau 4-manifold with $b_1 = 0$, equipped with a symplectic G-action of prime order p. Note that M has the integral homology of $K3$ surface.

The induced action of G on $H^2(M)$, as an integral $\mathbb{Z}_p$-representation, splits into a direct sum of 3 types of $\mathbb{Z}_p$-representations, i.e, the regular type of rank p, the trivial representation of rank 1, and the representation of cyclotomic type of rank $p-1$. If we let r, t, s be the number of summands of the above 3 types of $\mathbb{Z}_p$-representations in $H^2(M)$, then we have the following identities (cf. [8]):

$$b_2(M) = rp + t + s(p-1), \chi(M^G) = t - s + 2, \text{ and } s = b_1(M^G).$$

Note that the second identity is the Lefschetz fixed point theorem. As for the third one, i.e., $s = b_1(M^G)$, it was proved in [8] under the assumption that M is simply connected. However, since its argument is purely cohomological, the identity continues to hold under the weaker condition $H_1(M) = 0$ (cf. [9]). As an immediate corollary, note that if p is a prime factor of n, then there is an induced $\mathbb{Z}_p$-action on $\tilde{Y}$. If $p > 19$, the $\mathbb{Z}_p$-representation on $H^2(\tilde{Y})$ can not have any summands of regular type or cyclotomic type, i.e., $r = s = 0$, because $b_2(\tilde{Y}) = 22$. In other words, the symplectic $\mathbb{Z}_p$-action on $\tilde{Y}$ is homologically trivial. However, since $c_1(K_{\tilde{Y}}) = 0$, this is not possible (cf. [7]). Hence part (1) of Theorem 3.2 follows.

Next we prove part (2) of Theorem 3.2. Let Σ_i be a singular component of X whose genus is denoted by g_i and let B_i be the descendant in the resolution $\tilde{X}$. Applying the adjunction formula to B_i (note that $c_1(K_{\tilde{X}}) = -\sum_i \frac{m_i-1}{m_i} B_i + \sum_j \sum_{k \in I_j} a_{j,k} F_{j,k}$), it follows easily that $B_i^2 = 2m_i(g_i - 1)$. As a consequence, $g_i > 1$ if and only of $B_i^2 > 0$. Since $b_2^+(\tilde{X}) = 1$, it follows immediately that one can have at most one Σ_i with $g_i > 1$. Moreover, suppose there is another component Σ_k which is a torus, then its descendent B_k has $B_k^2 = 0$. It is easy to see that Σ_i, Σ_k can not both exist, because $(B_i + B_k)^2 > 0$

and B_i and $B_i + B_k$ are linearly independent. Hence if there is a singular component Σ_i of genus $g_i > 1$, then all other singular components must be spheres. To see that $n = m_i$ in this case, we observe that the pre-image $\pi^{-1}(\Sigma_i)$ in Y has n/m_i many components, each is fixed by a subgroup of $\mathbb{Z}_n$ of order m_i. The above argument on X, if applied to the orbifold $\tilde{Y}/\mathbb{Z}_{m_i}$, implies immediately that $n/m_i = 1$ must be true. Finally, if p is a prime factor of n, then there is a $\mathbb{Z}_p$-action on $\tilde{Y}$ fixing $\pi^{-1}(\Sigma_i)$. Now observe that in the identity $b_2(\tilde{Y}) = rp + t + s(p-1)$, $s \geq b_1(\pi^{-1}(\Sigma_i)) = 2g_i \geq 4$, which implies that $p \leq 22/4 + 1 < 7$. Hence part (2) of Theorem 3.2 is proved.

Finally, we consider part (3) of Theorem 3.2.

Lemma 3.3. *Let G be a finite cyclic group of order m, and let M be a symplectic Calabi-Yau 4-manifold with $b_1 = 0$. Suppose a symplectic G-action on M has at least two fixed components of torus. Then $m = 2$ and M^G consists of the two tori.*

Proof. Let $X = M/G$ be the quotient orbifold. We first consider the special case where $m = p$ is prime. To this end, we first note that in $b_2(M) = rp + t + s(p-1)$, $s \geq 4$, so that $p \leq 5$. To further analyze the $\mathbb{Z}_p$-action for these cases, we shall consider the resolution $\tilde{X}$ of X, which is a symplectic rational 4-manifold.

Let $\{\Sigma_i\}$ be the 2-dimensional fixed components and $\{q_j\}$ the isolated fixed points of the $\mathbb{Z}_p$-action. Let B_i be the descendent of Σ_i in $\tilde{X}$, and let $D_j \subset \tilde{X}$ be the exceptional set of the minimal resolution of q_j. With this understood, we shall prove the lemma by looking at the expressions of the B_i's and the components in the D_j's with respect to a certain basis of $H^2(\tilde{X})$, which is called a reduced basis. (See Section 4 for more discussions about reduced bases.)

Let $H, E_1, \cdots, E_N$ be a reduced basis. Then $c_1(K_{\tilde{X}}) = -3H + E_1 + \cdots + E_N$ (cf. Section 4). On the other hand, note that

$$c_1(K_{\tilde{X}}) = -\frac{p-1}{p}\sum_i B_i + \sum_j c_1(D_j).$$

Denoting by B_1, B_2 the two torus components in $\{B_i\}$, we write each of B_1, B_2 in the reduced basis, with an expression of the form $aH - \sum_{k=1}^N b_k E_k$, and we shall call the coefficient a in such expressions the a-coefficient of B_1, B_2. With this understood, by Lemma 4.2 in [6], the a-coefficients of both B_1, B_2 are at least 3. Moreover, if the a-coefficient equals 3, then B_1 or B_2 must take the form $B = 3H - E_{j_1} - E_{j_2} - \cdots - E_{j_9}$. On the other hand, if there is a symplectic sphere S in $\{B_i\}$ or $\{D_j\}$ whose a-coefficient is negative, then S must have the homological expression

$$S = aH + (|a|+1)E_1 - E_{i_1} - \cdots - E_{i_l}$$

for some $a < 0$ (cf. [6], Lemma 3.4). With this understood, note that

$$B \cdot S \leq (|a|+1) + 3a = 2a + 1 < 0,$$

which contradicts the fact that B_1, B_2 are disjoint from S. Hence if there is a symplectic sphere S in $\{B_i\}$ or $\{D_j\}$ whose a-coefficient is negative (such a component must be unique, see [6], Lemma 4.2), the a-coefficients of both B_1, B_2 must be at least 4.

To derive a contradiction for the case where $p = 3$ or 5, we first observe that the contribution of B_1, B_2 to the a-coefficient of $-p \cdot c_1(K_{\tilde{X}})$ is at least $6(p-1)$, which is greater than $3p$ for $p = 3$ or 5. Hence there must be a sphere S in $\{B_i\}$ or $\{D_j\}$ whose a-coefficient is negative. With this understood, the contribution of B_1, B_2 to the a-coefficient of $-p \cdot c_1(K_{\tilde{X}})$ is then at least $8(p-1)$. We will get a contradiction again if the contribution of S to the a-coefficient of $-p \cdot c_1(K_{\tilde{X}})$ is greater than $8-5p$.

Consider first the case of $p = 3$. In this case, if S is a component of $\{B_i\}$, then S is a (-6)-sphere. The a-coefficient of S is no less than -2 (cf. [6], Lemma 3.4), and the contribution to $-p \cdot c_1(K_{\tilde{X}})$ is at least $-2(p-1) = -4 > 8-5p$. If S is a component from $\{D_j\}$, then S is a (-3)-sphere, and its contribution to the a-coefficient of $-p \cdot c_1(K_{\tilde{X}})$ equals $3 \times \frac{1}{3} \times (-1) = -1$. In either case, we arrive at a contradiction. Hence $p = 3$ is ruled out. For $p = 5$, the argument is similar. If S is a component of $\{B_i\}$, then S is a (-10)-sphere. In this case, the contribution of S to the a-coefficient of $-p \cdot c_1(K_{\tilde{X}})$ is at least $-4(p-1) = -16 > 8-5p$. If S is a component from $\{D_j\}$, there are several possibilities. Note that D_j either consists of a single (-5)-sphere, or a pair of (-3)-sphere and (-2)-sphere intersecting transversely at one point. With this understood, note that S cannot be a (-2)-sphere as it has negative a-coefficient (cf. [6], Lemma 3.4). If S is a (-5)-sphere, the contribution of S to the a-coefficient of $-p \cdot c_1(K_{\tilde{X}})$ is at least -6, and if S is a (-3)-sphere, the contribution equals -2. In either case, we arrive at a contradiction. Hence $p = 5$ is also ruled out.

It remains to consider the case of $p = 2$. Note that by Theorem 3.1, there are no isolated fixed points, so $\{D_j\} = \emptyset$. We first assume S exists. Then the contribution of B_1, B_2 to the a-coefficient of $-p \cdot c_1(K_{\tilde{X}})$ is at least $8(p-1) = 8$. On the other hand, S as a component in $\{B_i\}$ must be a (-4)-sphere. Its contribution to the a-coefficient of $-p \cdot c_1(K_{\tilde{X}})$ equals -1. This is a contradiction as the a-coefficient of $-p \cdot c_1(K_{\tilde{X}})$ equals 6 for $p = 2$. Hence S cannot exist, and both B_1, B_2 have a-coefficient equal to 3. Then it follows easily that $B_1 = B_2 = 3H - E_{j_1} - E_{j_2} - \cdots - E_{j_9}$ for some classes E_{j_s}, $s = 1, 2, \cdots, 9$. On the other hand, $c_1(K_{\tilde{X}}) = -3H + E_1 + \cdots + E_N$. By comparing with the equation $c_1(K_{\tilde{X}}) = -\frac{1}{2}\sum_i B_i$, it follows easily that there are no other components in $\{B_i\}$ besides B_1, B_2 (and we must have $N = 9$). This proves the lemma for the special case of prime order actions.

For the general case, it follows easily that $m = 2^k$ for some $k > 0$. With this understood, observe that if a point $q \in M$ is fixed by some nontrivial element of G, then it must be fixed by the subgroup of G of order 2. It follows easily that the singular set of $X = M/G$ consists of only the two tori. If we continue to denote by B_1, B_2 the descendants of the fixed tori in the resolution $\tilde{X}$ of $X = M/G$, then we have

$$c_1(K_{\tilde{X}}) = -\frac{m-1}{m}(B_1 + B_2).$$

Again, the a-coefficients of B_1, B_2 are at least 3 (cf. [6], Lemma 4.2), from which the above equation implies that $3 \geq \frac{m-1}{m}(3+3)$ by comparing the a-coefficients of both sides. It follows immediately that $m = 2$, and the proof of the lemma is complete. □

Back to the proof of Theorem 3.2, suppose Σ_1, Σ_2 are two singular components of X which are torus, with m_1, m_2 being the order of the isotropy groups respectively. If $m_1 \neq m_2$, then one of $n/m_1, n/m_2$ must be greater than 1. Without loss of generality, assume $n/m_1 > 1$. Then there are at least two components in $\pi^{-1}(\Sigma_1) \subset \tilde{Y}$, which are fixed by a subgroup of $\mathbb{Z}_n$ of order m_1. By Lemma 3.3, we must have $m_1 = 2$ and $n/m_1 = 2$. It follows that we must have $n = m_2 = 4$ by the assumption that $m_1 \neq m_2$. But this implies that the $\mathbb{Z}_n$-action fixes $\pi^{-1}(\Sigma_2) \subset \tilde{Y}$, so that the subgroup of order $m_1 = 2$, which already fixes two tori in $\pi^{-1}(\Sigma_1)$, also fixes $\pi^{-1}(\Sigma_2)$. This is clearly a contradiction to Lemma 3.3. Hence $m_1 = m_2$. Then the above argument shows that we must have $n/m_1 = n/m_2 = 1$, and $n = 2$ by Lemma 3.3. Moreover, there are no other components in $\{\Sigma_i\}$ besides Σ_1, Σ_2.

Finally, suppose there is only one component Σ_1 which is a torus, with m_1 being the order of the isotropy group along Σ_1. Then if $n > m_1$, there will be at least two components in $\pi^{-1}(\Sigma_1)$, which is fixed by a $\mathbb{Z}_{m_1}$-action on $\tilde{Y}$. By Lemma 3.3, $m_1 = 2$ and $n/m_1 = 2$, so that $n = 4$. If there is a component in $\{\Sigma_i\}$ with $m_i = n = 4$, then this component is also fixed by the $\mathbb{Z}_{m_1}$-action, which contradicts Lemma 3.3. Hence there must be a singular point q_j such that $m_j = n = 4$. Suppose first that the subgroup H_j at q_j is trivial. Then $\pi^{-1}(q_j)$, consists of one point, is a smooth point in Y, and is being fixed by the $\mathbb{Z}_n$-action on $\tilde{Y}$. In particular, it is a fixed point of the subgroup of order $m_1 = 2$. But this contradicts Lemma 3.3. Suppose H_j is nontrivial. Then $\pi^{-1}(q_j)$ is a singular point of Y. Let D_j be the exceptional set of its minimal resolution in $\tilde{Y}$. Then D_j is invariant under the $\mathbb{Z}_n$-action on $\tilde{Y}$. It is easy to see that the action of the subgroup of order $m_1 = 2$ has a fixed point contained in D_j, which is a contradiction to Lemma 3.3. This proves that n must be equal to the order of the isotropy group along the unique torus component Σ_1. Finally, suppose p is a prime factor of n. Then the action of the subgroup of $\mathbb{Z}_n$ of order p on $\tilde{Y}$ fixes the torus $\pi^{-1}(\Sigma_1)$. Now appealing to the identity $b_2(\tilde{Y}) = rp + t + s(p-1)$, we find that $s(p-1) \leq 22$, where $s \geq b_1(\pi^{-1}(\Sigma_1)) = 2$. It follows easily that $p \leq 11$. This completes the proof of Theorem 3.2. □

3.2. Constraints from Seiberg-Witten-Taubes theory

In this subsection, we derive some constraints on the singular set of the orbifold X using the Seiberg-Witten-Taubes theory, by extending an argument of T.-J. Li in [13] to the orbifold setting. It is important to note that X is assumed to have $b_1 = 0$.

The constraints are given in terms of certain numerical contributions of the singular set to the dimension of the moduli space of Seiberg-Witten equations. To describe them, we consider any orbifold complex line bundle L over X such that $c_1(L) = 0 \in H^2(|X|; \mathbb{Q})$. For each singular point q_j of X, we denote by $\rho_j^L : G_j \to \mathbb{C}^*$ the complex representation of the isotropy group G_j on the fiber of L at q_j, and denote by $\rho_{j,k}(g)$, for $k = 1, 2$, the eigenvalues of $g \in G_j$ associated to the complex representation of G_j on the tangent space $T_{q_j}X$. For each 2-dimensional singular component Σ_i, we denote by $G_i := \mathbb{Z}_{m_i}$ the isotropy group along Σ_i, and let $\rho_i^L : G_i \to \mathbb{C}^*$ be the complex representation of G_i on the fibers of L along Σ_i, and let $\rho_i : G_i \to \mathbb{C}^*$ be the complex representation of G_i on the normal bundle ν_{Σ_i} of Σ_i. With this understood, we set

$$I_i(L) := \frac{1}{m_i} \sum_{g \in G_i \setminus \{e\}} \frac{(1+\rho_i(g^{-1}))(\rho_i^L(g)-1)}{(1-\rho_i(g^{-1}))^2},$$

and

$$I_j(L) := \frac{1}{|G_j|} \sum_{g \in G_j \setminus \{e\}} \frac{2(\rho_j^L(g)-1)}{(1-\rho_{j,1}(g^{-1}))(1-\rho_{j,2}(g^{-1}))}.$$

It is easy to check that $I_i(L) = I_i(K_X \otimes L^{-1})$, $I_j(L) = I_j(K_X \otimes L^{-1})$ for any i, j. Finally, we set

$$d(L) := \sum_i I_i(L)\chi(\Sigma_i) + \sum_j I_j(L).$$

One can easily check that, with $c_1(L) = 0$, and with

$$c_1(\nu_{\Sigma_i})[\Sigma_i] = \Sigma_i^2 = 2g_i - 2 = -c_1(T\Sigma_i)[\Sigma_i] = -\chi(\Sigma_i)$$

by the adjunction formula (here g_i is the genus of Σ_i), $d(L)$ equals the dimension of the moduli space of Seiberg-Witten equations associated to the orbifold complex line bundle L (cf. [3], Appendix A). With this understood, we note that $d(L) = d(K_X \otimes L^{-1})$, and $d(L) = 0$ if L is the trivial complex line bundle or $L = K_X$.

Theorem 3.4. *Suppose $b_1(X) = 0$. Then for any orbifold complex line bundle L such that $c_1(L) = 0 \in H^2(|X|; \mathbb{Q})$, one has*

$$d(L) \leq 0,$$

with equality if and only if L is the trivial complex line bundle or $L = K_X$.

Proof. We begin by noting that $b_2^+(X) = b_2^+(\tilde{X}) = 1$. With $b_1(X) = 0$, the wall-crossing number for the Seiberg-Witten invariant of X equals ± 1. With this understood, we denote by $SW_X(L)$ the Seiberg-Witten invariant of X associated to an orbifold complex line bundle L defined in the Taubes chamber. Then if the dimension $d(L) \geq 0$, one has

$$|SW_X(L) - SW_X(K_X \otimes L^{-1})| = 1.$$

Now observe that if $SW_X(L) \neq 0$ and $c_1(L) = 0$, L must be the trivial orbifold complex line bundle (cf. [21]). Since K_X is torsion of order $n > 1$ (recall that $n = \text{lcm}\{m_i, m_j\}$ is the minimal number such that K_X^n is trivial, cf. Lemma 2.1), it is clear that one of the orbifold complex line bundles, L or $K_X \otimes L^{-1}$, must be a non-trivial torsion bundle, hence has vanishing Seiberg-Witten invariant. It follows easily that one of $SW_X(L)$, $SW_X(K_X \otimes L^{-1})$ must equal ± 1. This implies that either L or $K_X \otimes L^{-1}$ must be the trivial orbifold complex line bundle. Theorem 3.4 follows easily. $\square$

Suppose $n = \text{lcm}\{m_i, m_j\} > 2$. Then $L = K_X^k$ is nontrivial and not equal to K_X for $k = 2, 3, \cdots, n-1$. By Theorem 3.4, $d(K_X^k) < 0$ for any $2 \le k \le n-1$. On the other hand, note that a singular component Σ_i makes zero contribution to $d(L)$ for any L if Σ_i is a torus, and one can check directly that $I_j(K_X^k) = 0$ for any k if q_j is a Du Val singularity (i.e., $m_j = 1$). It is easy to see that we have the following

Corollary 3.5. *Suppose $n = lcm\{m_i, m_j\} > 2$. Then the following are true.*

1. *For each $2 \le k \le n-1$, $d(K_X^k)$ is a negative, even integer.*
2. *Either there is a singular component Σ_i which is not a torus, or there is a singular point q_j which is non-Du Val.*

Remarks: It is possible that for a singular point q_j, the number $I_j(K_X^k) > 0$ for some $2 \le k \le n-1$; this depends on the isotropy type of q_j. For example, if q_j is of isotropy of order 5 of type $(1,1)$, then

$$I_j(K_X^2) = \frac{1}{5} \sum_{1 \neq \lambda \in \mathbb{C}^*, \lambda^5 = 1} \frac{2(\lambda^4 - 1)}{(1-\lambda)(1-\lambda)} = \frac{2}{5}.$$

So the conditions $d(K_X^k) < 0$ give rise to nontrivial constraints on the singular set.

4. A successive blowing-down procedure

In this section, we describe a general successive symplectic blowing-down procedure. First, we shall adopt the following notations: we set $X_N := \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$, which is equipped with a symplectic structure denoted by ω_N. In order to emphasize the dependence of the canonical class on the symplectic structure, we shall denote by K_{ω_N} the canonical line bundle of (X_N, ω_N).

4.1. The statement

The successive symplectic blowing-down procedure, to be applied to (X_N, ω_N) for $N \ge 2$, depends on a choice of the so-called **reduced basis** of (X_N, ω_N). To explain this notion, we let $\mathcal{E}_{X_N}$ be the set of classes in $H^2(X_N)$ which can be represented by a smooth (-1)-sphere, and let $\mathcal{E}_{\omega_N} := \{E \in \mathcal{E}_{X_N} | c_1(K_{\omega_N}) \cdot E = -1\}$. Then each class in $\mathcal{E}_{\omega_N}$ can be represented by a symplectic (-1)-sphere; in particular, $\omega_N(E) > 0$ for any $E \in \mathcal{E}_{\omega_N}$. With this understood, a basis $H, E_1, E_2, \cdots, E_N$ of $H^2(X_N)$ is called a reduced basis of (X_N, ω_N) if the following are true:

- it has a standard intersection form, i.e., $H^2 = 1$, $E_i^2 = -1$ and $H \cdot E_i = 0$ for any i, and $E_i \cdot E_j = 0$ for any $i \neq j$;
- $E_i \in \mathcal{E}_{\omega_N}$ for each i, and moreover, the following area conditions are satisfied for $N \ge 3$: $\omega_N(E_N) = \min_{E \in \mathcal{E}_{\omega_N}} \omega_N(E)$, and for any $2 < i < N$, $\omega_N(E_i) = \min_{E \in \mathcal{E}_i} \omega_N(E)$, where $\mathcal{E}_i := \{E \in \mathcal{E}_{\omega_N} | E \cdot E_j = 0 \;\; \forall j > i\}$ for any $i < N$;
- $c_1(K_{\omega_N}) = -3H + E_1 \cdots + E_N$.

Reduced bases always exist. Moreover, if we assume $\omega(E_1) \geq \omega(E_2)$ without loss of generality, then a reduced basis $H, E_1, E_2, \cdots, E_N$ obeys the following constraints in symplectic area (cf. [15]):

- $\omega_N(H) > 0$, and for any $j > i$, $\omega_N(E_i) \geq \omega_N(E_j)$;
- for any $i \neq j$, $H - E_i - E_j \in \mathcal{E}_{\omega_N}$, so that $\omega_N(H - E_i - E_j) > 0$;
- $\omega_N(H - E_i - E_j - E_k) \geq 0$ for any distinct i, j, k.

We remark that a reduced basis is not necessarily unique, however, the symplectic areas of its classes

$$(\omega_N(H), \omega_N(E_1), \omega_N(E_2), \cdots, \omega_N(E_N))$$

uniquely determine the symplectic structure ω_N up to symplectomorphism, cf. [12].

Definition 4.1. The symplectic structure ω_N is called **odd** if $\omega_N(H - E_1 - 2E_2) \geq 0$, and is called **even** if otherwise.

We remark that since ω_N is determined by $\omega_N(H), \omega_N(E_1), \omega_N(E_2), \cdots, \omega_N(E_N)$ up to symplectomorphism, the above definition does not depend on the choice of the reduced basis.

The following technical result from [12] is crucial to our construction.

Suppose $N \geq 2$. Then for **any** *ω_N-compatible almost complex structure J, any class $E \in \mathcal{E}_{\omega_N}$ which has the minimal symplectic area can be represented by an embedded J-holomorphic sphere. In particular, for $N \geq 3$, the class E_N in a reduced basis $H, E_1, \cdots, E_N$ can be represented by a J-holomorphic (-1)-sphere for any given J.*

With the preceding understood, the following lemma makes it possible for a successive blowing-down procedure.

Lemma 4.2. *Let $H, E_1, \cdots, E_N$ be a reduced basis of (X_N, ω_N), and let C_N be any symplectic (-1)-sphere in (X_N, ω_N) representing the class E_N. Denote by (X_{N-1}, ω_{N-1}) the symplectic blowdown of (X_N, ω_N) along C_N. Then $H, E_1, \cdots, E_{N-1}$ naturally descend to a reduced basis $H', E_1', \cdots, E_{N-1}'$ of (X_{N-1}, ω_{N-1}). When $N \geq 3$, ω_{N-1} is odd if and only if ω_N is odd.*

Proof. It is clear that $H, E_1, \cdots, E_{N-1}$ naturally descend to a basis $H', E_1', \cdots, E_{N-1}'$ of $H^2(X_{N-1})$. We need to show that it is a reduced basis of (X_{N-1}, ω_{N-1}), and moreover, when $N \geq 3$, ω_{N-1} is odd if and only if ω_N is odd.

First of all, we note that $H', E_1', \cdots, E_{N-1}'$ has the standard intersection form, and the symplectic canonical class of (X_{N-1}, ω_{N-1}) is given by

$$c_1(K_{\omega_{N-1}}) = -3H' + E_1' + \cdots + E_{N-1}'.$$

It remains to verify that for each i, $E_i' \in \mathcal{E}_{\omega_{N-1}}$, and moreover, the following area conditions are satisfied: $\omega_{N-1}(E_{N-1}') = \min_{E' \in \mathcal{E}_{\omega_{N-1}}} \omega_{N-1}(E')$, and for any $i < N-1$, $\omega_{N-1}(E_i') = \min_{E' \in \mathcal{E}_i'} \omega_{N-1}(E')$, where $\mathcal{E}_i' := \{E' \in \mathcal{E}_{\omega_{N-1}} | E' \cdot E_j' = 0, \forall j > i\}$.

The key step is to show that the set $\mathcal{E}_{N-1} = \{E \in \mathcal{E}_{\omega_N} | E \cdot E_N = 0\}$ may be identified with the set $\mathcal{E}_{\omega_{N-1}}$ by identifying the elements of $\mathcal{E}_{N-1}$ with their descendants in

$H^2(X_{N-1})$, and moreover, under this identification the symplectic forms $\omega_N = \omega_{N-1}$. To see this, let $E \in \mathcal{E}_{N-1}$ be any class and let E' be its descendant in $H^2(X_{N-1})$. We choose a J_1 such that C_N is J_1-holomorphic. Then pick a generic J_0 and connect J_0 and J_1 through a smooth path J_t. Since J_0 is generic, E can be represented by a J_0-holomorphic (-1)-sphere, denoted by C_E. On the other hand, since E_N has minimal symplectic area, for each t, E_N is represented by a J_t-holomorphic (-1)-sphere C_t, which depends on t smoothly, with C_1 at $t = 1$ being the original (-1)-sphere C_N. Note also that the J_0-holomorphic (-1)-spheres C_E and C_0 are disjoint because $E \cdot E_N = 0$. With this understood, we note that the isotopy from C_0 to $C_1 = C_N$ is covered by an ambient isotopy $\psi_t : X_N \to X_N$, where each ψ_t is a symplectomorphism (cf. Proposition 0.3 in [20]). It follows easily that E is represented by the symplectic (-1)-sphere $\psi(C_E)$, which is disjoint from C_N. This shows that the descendant E', which is represented by the symplectic (-1)-sphere $\psi(C_E)$ in X_{N-1}, lies in the set $\mathcal{E}_{\omega_{N-1}}$. Moreover, $\omega_N(E) = \omega_{N-1}(E')$. Finally, let E' be any class in $\mathcal{E}_{\omega_{N-1}}$. Then E' can be represented by a smooth (-1)-sphere, to be denoted by S', and $E' \cdot c_1(K_{\omega_{N-1}}) = -1$. Now recall that the 4-manifold X_{N-1} is obtained from X_N by removing the (-1)-sphere C_N and then filling in a symplectic 4-ball B. Without loss of generality, we may assume S' is lying outside B, because if otherwise, one can always apply an ambient isotopy to push S' outside of B. With this understood, the smooth sphere S' can be lifted to a smooth sphere S in X_N. Let E be the class of S. Then clearly $E \cdot E_N = 0$ and E' is the descendant of E in $H^2(X_{N-1})$. To see that $E \in \mathcal{E}_{N-1}$, we only need to verify that $E \cdot c_1(K_{\omega_N}) = -1$. But this follows easily from the fact that $c_1(K_{\omega_N}) = c_1(K_{\omega_{N-1}}) + E_N$ and $E' \cdot c_1(K_{\omega_{N-1}}) = -1$. Hence the claim that $\mathcal{E}_{N-1}$ and $\mathcal{E}_{\omega_{N-1}}$ are naturally identified and the symplectic forms ω_N and ω_{N-1} agree.

With the preceding understood, it follows easily that for each $i = 1, 2, \cdots, N-1$, $E_i' \in \mathcal{E}_{\omega_{N-1}}$. Moreover, $\omega_{N-1}(E'_{N-1}) = \min_{E' \in \mathcal{E}_{\omega_{N-1}}} \omega_{N-1}(E')$. We further observe that for each $i < N-1$, the subset $\mathcal{E}_i$ of $\mathcal{E}_{N-1}$ is identified with the subset $\mathcal{E}_i'$ of $\mathcal{E}_{\omega_{N-1}}$ under the identification between $\mathcal{E}_{N-1}$ and $\mathcal{E}_{\omega_{N-1}}$. With ω_N and ω_{N-1} agreeing with each other under the identification, it follows immediately that $H', E_1', \cdots, E_{N-1}'$ is a reduced basis of (X_{N-1}, ω_{N-1}). Moreover, when $N \geq 3$, ω_{N-1} is odd if and only if ω_N is odd. This finishes off the proof. $\square$

For simplicity, we shall continue to use the notations $H, E_1, \cdots, E_{N-1}$ to denote the descendants in the symplectic blowdown (X_{N-1}, ω_{N-1}), instead of the notations $H', E_1', \cdots, E_{N-1}'$ in the lemma.

Now fixing any reduced basis $H, E_1, E_2, \cdots, E_N$, we can successively blow down the classes $E_N, E_{N-1}, \cdots, E_3$, reducing (X_N, ω_N) to (X_2, ω_2). To further blow down (X_2, ω_2), we note that ω_N is odd if and only if E_2 has the minimal area among the classes in $\mathcal{E}_2$, i.e., $\omega_N(E_2) = \min_{E \in \mathcal{E}_2} \omega_N(E)$, as it is easy to see that

$$\mathcal{E}_2 = \{E_1, E_2, H - E_1 - E_2\}.$$

Thus when ω_N is odd, we can further blow down E_2 to reach $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$. If ω_N is even, then $\omega_N(H-E_1-E_2)=\min_{E\in\mathcal{E}_2}\omega_N(E)$. In this case, by blowing down the (-1)-class $H-E_1-E_2$, we reach the final stage $\mathbb{S}^2\times\mathbb{S}^2$.

Since at each stage of the blowing-down procedure the (-1)-class has minimal area, it can be represented by a J-holomorphic (-1)-sphere for any given J. This property allows us to construct, in a canonical way, the descendant of a given set of symplectic surfaces $D=\cup_k F_k$ in (X_N,ω_N) under the successive blowing-down procedure. Without loss of much generality, we shall assume D satisfies the following condition:

(†) Any two symplectic surfaces F_k, F_l in D are either disjoint, or intersect transversely and positively at one point, and no three distinct components of D meet in one point.

Further assumptions on D are required so that the procedure is reversible. In order to explain this, observe that the class of each F_k in D can be written with respect to the reduced basis $H, E_1, E_2, \cdots, E_N$ in the following form:

$$F_k = aH-\sum_{i=1}^N b_iE_i, \text{ where } a, b_i\in\mathbb{Z}.$$

We shall call the numbers a and b_i the *a-coefficient* and *b_i-coefficients* of F_k. (See Section 3 of [6] for some general properties of the a-coefficient and b_i-coefficients.) The expression $F_k = aH-\sum_{i=1}^N b_iE_i$ is called the *homological expression* of F_k (with respect to the reduced basis).

With the preceding understood, the assumptions on D are concerned with the homological expressions of the components F_k whose a-coefficients are zero. More concretely, it is known (cf. [6], Lemma 3.3) that such a component must be a symplectic sphere, and its b_i-coefficients are equal to 1 except for one of them, which equals -1. We shall call the E_i-class with the (-1) b_i-coefficient the *leading class* of the component. With this understood, it is easy to show that for any given component S of D which has zero a-coefficient, there are at most two components F_k in D such that the expression of F_k contains the leading class of S and F_k has a zero a-coefficient (cf. Lemma 4.5). The assumptions we shall impose on D are concerned with the homological expressions of such components F_k for any given such S in D.

To be more precise, let $S\subset D$ be any such symplectic sphere, and we write the homological expression of S as

$$S=E_n-E_{l_1}-E_{l_2}-\cdots-E_{l_\alpha}, \text{ where } n<l_s \text{ for all } s.$$

Then the imposed assumptions on D are stated as follows:

(a) Suppose there are two symplectic spheres $S_1, S_2\subset D$ whose a-coefficients equal zero and whose homological expressions contain the leading class E_n of S. Then for any class E_{l_s} which appears in S, but appears in neither S_1 nor S_2, there is at most one component F_k of D other than S, whose homological expression contains E_{l_s} with $F_k\cdot E_{l_s}=1$.

(b) Suppose there is only one symplectic sphere $S_1 \subset D$ whose a-coefficient equals zero and whose homological expression contains the leading class E_n of S. Then there is at most one class E_{l_s} in S, which does not appear in S_1, but either appears in the expressions of more than one components $F_k \neq S$, or appears in the expression of only one component $F_k \neq S$ but with $F_k \cdot E_{l_s} > 1$.

(We remark that when S is a (-2)-sphere or (-3)-sphere, and S_1, S_2 are disjoint from S, the assumptions (a) and (b) are automatically satisfied.)

With the preceding understood, we now state the theorem concerning the descendant of D under the successive symplectic blowing-down procedure. For simplicity, we shall only discuss the case where the symplectic structure ω_N is odd, which is the most relevant case for us. The case where ω_N is even can be similarly dealt with.

Theorem 4.3. *Let $D = D_N = \cup_k F_k$ be a union of symplectic surfaces in (X_N, ω_N), where $N \geq 2$ and ω_N is odd, such that D_N satisfies the condition (†). Fix any reduced basis $H, E_1, E_2, \cdots, E_N$ of (X_N, ω_N) such that the assumptions (a) and (b) are satisfied for the homological expressions of the components F_k in D_N. We set*

$$\mathcal{E}_0(D_N) = \{E_i | \text{there is no } F_k \subset D_N \text{ with zero } a\text{-coefficient such that } E_i \cdot F_k > 0\}.$$

Then there is a well-defined successive symplectic blowing-down procedure associated to the reduced basis, blowing down the classes $E_N, E_{N-1}, \cdots, E_2$ successively, such that (X_N, ω_N) is reduced to (X_1, ω_1) (note that $X_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$), and D_N is transformed to its descendant D_1 in (X_1, ω_1), which is a union of J_1-holomorphic curves with respect to some ω_1-compatible almost complex structure J_1 on (X_1, ω_1), where the singularities and the intersection pattern of the components of D_1 are canonically determined by the homological expressions of the components F_k of D_N. Moreover, under any of the conditions (c), (d), (e) listed below, one can further blow down the class E_1 to reach $\mathbb{CP}^2$ in the final stage of the successive blowing-down, with the descendant D_0 of D_1 in $\mathbb{CP}^2$ having the same properties of D_1:

(c) *The classes E_1, E_2 have the same area, i.e., $\omega_N(E_1) = \omega_N(E_2)$.*
(d) *The class E_1 is the leading class of a symplectic sphere $S \subset D_N$.*
(e) *There is a component $F_k = aH - bE_1 - \sum_{i>1} b_i E_i$ of D_N such that $2b < a$.*

More specifically, let $\mathcal{E}(D_N) := \mathcal{E}_0(D_N) \setminus \{E_1\}$ if the final stage of the successive blowing-down is (X_1, ω_1) and let $\mathcal{E}(D_N) := \mathcal{E}_0(D_N)$ if the final stage is $\mathbb{CP}^2$. Then the new intersection points in D_1 or D_0 are labelled by the elements of $\mathcal{E}(D_N)$. For each new intersection point $\hat{E}_i$ labelled by $E_i \in \mathcal{E}(D_N)$, there is a small 4-ball $B(\hat{E}_i)$ centered at $\hat{E}_i$, with standard symplectic structure and complex structure, such that $D_1 \cap B(\hat{E}_i)$ or $D_0 \cap B(\hat{E}_i)$ consists of a union of holomorphic discs intersecting at $\hat{E}_i$, which are either embedded or singular at $\hat{E}_i$ with a singularity modeled by equations of the form $z_1^n = a z_2^m$ in some compatible complex coordinates (z_1, z_2) (i.e., the link of the singularity is always a torus knot). The orders of tangency of the intersections at $\hat{E}_i$ as well as the singularity types in $B(\hat{E}_i)$ are completely and canonically determined by the pattern of appearance of the class E_i and the classes not contained in $\mathcal{E}(D_N)$ in the homological expressions *of the components F_k in D_N. Finally, a component of D_N descends to a component in D_1 or D_0 if and only if it has nonzero a-coefficient (a component with zero a-coefficient disappears).*

Remarks: (1) We shall call D_0 or D_1 a **symplectic arrangement** of pseudoholomorphic curves. (We borrow the terminology from [18], where in the case when D_0 is a union of degree 1 pseudoholomorphic spheres in $\mathbb{CP}^2$, it is called a symplectic line arrangement.)

(2) Two situations of the new intersection points are worth mentioning, as they occur more generically: let $E_n \in \mathcal{E}(D_N)$ be any element.

(i) If E_n is not the leading class of any symplectic sphere in D_N, then the descendants of the components of D_N containing E_n will intersect the 4-ball $B(\hat{E}_n)$ in a union of holomorphic discs, which are all embedded and intersecting at $\hat{E}_n$ transversely.

(ii) If E_n is the leading class of a symplectic sphere $S \subset D_N$, where

$$S = E_n - E_{l_1} - E_{l_2} - \cdots - E_{l_\alpha},$$

such that the classes E_{l_s} in S are not the leading class of any symplectic spheres in D_N, then the holomorphic discs in $B(\hat{E}_n)$ are all embedded, and moreover, each E_{l_s} determines a complex line (through the origin $\hat{E}_n$) in $B(\hat{E}_n)$, such that the descendants of the components of D_N containing E_{l_s} will intersect the 4-ball $B(\hat{E}_n)$ in a union of holomorphic discs which are all tangent to the complex line determined by E_{l_s}, with a tangency of order 2.

(3) The successive blowing-down procedure is purely a symplectic operation; there are no holomorphic analogs. Note that the descendant D_0 or D_1 depends on the choice of the reduced basis, which in general is not necessarily uniquely determined by the symplectic structure ω_N. On the other hand, there is also flexibility in choosing the symplectic structure ω_N (cf. [6], Lemma 4.1). Hence it is not clear if there is a descendant D_0 or D_1 that is determined by D_N itself.

(4) The successive blowing-down procedure is reversible; by reversing the procedure (with either symplectic blowing-up or holomorphic blowing-up), one can recover $D_N \subset X_N$ up to a smooth isotopy. Note that in each step of the reversing successive blowing-up operation, one either takes the total transform or the proper transform (cf. [17]), depending on whether in the corresponding blowing-down step, the (-1)-sphere being blown down is part of the descendant of D_N or not.

4.2. The construction

Suppose we are given with a union of symplectic surfaces $D = D_N = \cup_k F_k$ in (X_N, ω_N) satisfying the condition (†). We shall first describe how to blow down (X_N, ω_N) along the class E_N and how to define the descendants of the components F_k of D_N in (X_{N-1}, ω_{N-1}). First of all, we slightly perturb the symplectic surfaces F_k if necessary, so that the intersection of F_k is ω_N-orthogonal (cf. [10]). Furthermore, we choose an ω_N-compatible almost complex structure J_N which is integrable near each intersection point of the symplectic surfaces F_k such that D_N is J_N-holomorphic. With this understood, since $N \geq 2$ and ω_N is odd, we may represent the class E_N by an embedded J_N-holomorphic sphere C_N.

4.2.1. Perturbing the (-1)-spheres to a general position

An important feature of the successive blowing-down procedure is that, before we blow down the (-1)-sphere C_N, we shall first put it in a general position, as long as C_N is not part of D_N. We carry out this step as follows.

The intersection of C_N with each F_k is isolated, though not necessarily transverse, and furthermore, C_N may contain the intersection points of the components F_k in D_N. The local models for the intersection of C_N with D_N are as follows. If $p \in C_N \cap D_N$ is the intersection of C_N with a single component F_k, then locally near p, C_N and F_k are given respectively by $z_2 = 0$ and $z_2 = z_1^m +$ higher order terms. If $p \in C_N \cap D_N$ is the intersection of C_N with more than one components of D_N, then near p there is a standard holomorphic coordinate system such that the relevant components of D_N are given by complex lines through the origin, and C_N is given by an embedded holomorphic disc through the origin. With this understood, it is easy to see that one can always slightly perturb C_N to a symplectic (-1)-sphere, still denoted by C_N for simplicity, such that C_N obeys the following **general position condition:**

C_N intersects each F_k transversely and positively, and C_N does not contain any intersection points of the components of D_N. Furthermore, the intersection of C_N with each F_k is ω_N-orthogonal (after a small perturbation if necessary, cf. [10]*). (We should point out that when C_N is part of D_N, there is no need to perturb C_N.)*

By the Weinstein neighborhood theorem, a neighborhood U of C_N is symplectically modeled by a standard symplectic structure on a disc bundle associated to the Hopf fibration, where C_N is identified with the zero-section. With this understood, for each F_k which intersects C_N, we slightly perturb F_k near the intersection points so that F_k coincides with a fiber disc inside U. Now symplectically blowing down (X_N, ω_N) along C_N amounts to cutting X_N open along C_N and then inserting a standard symplectic 4-ball of a certain radius back in (the radius of the 4-ball is determined by the area of C_N). We denote the resulting symplectic 4-manifold by (X_{N-1}, ω_{N-1}). Then the descendant of F_k in X_{N-1} is defined to be the symplectic surface, to be denoted by $\tilde{F}_k$, which is obtained by adding a complex linear disc to $F_k \setminus C_N$ inside the standard symplectic 4-ball for each of the intersection points of F_k with C_N. If F_l is another symplectic surface intersecting C_N, then the descendant $\tilde{F}_l$ of F_l in X_{N-1} will intersect with $\tilde{F}_k$ at the origin of the standard symplectic 4-ball, which is the only new intersection point introduced to F_k, F_l under the blowing down operation along C_N. We denote the origin of the standard symplectic 4-ball by $\hat{E}_N \in X_{N-1}$. Note that under this construction, $\tilde{F}_k$ is immersed in general, where the (transverse) self-intersection at $\hat{E}_N$ is introduced if F_k intersects C_N at more than one point. Finally, we denote by $B(\hat{E}_N)$ a small 4-ball centered at $\hat{E}_N$ such that $B(\hat{E}_N) \cap (\cup_k \tilde{F}_k)$ consists of a union of (linear) complex discs through the origin.

Note that for each k, the number of complex discs in $B(\hat{E}_N) \cap \tilde{F}_k$ equals the intersection number $E_N \cdot F_k$.

To continue with the successive blowing-down procedure, we consider the union of the generally immersed symplectic surfaces $D_{N-1} := \cup_k \tilde{F}_k$ in (X_{N-1}, ω_{N-1}). For simplicity, we shall continue to denote the descendant $\tilde{F}_k$ by the original notation F_k. However, one should note that the initial condition (†) concerning the intersections of the components F_k of D_N is replaced by the following condition:

(‡) There exists an ω_{N-1}-compatible almost complex structure J_{N-1} such that each component F_k in D_{N-1} is J_{N-1}-holomorphic, self-intersecting and intersecting with each other transversely. Moreover, J_{N-1} is integrable near the intersection points.

We shall continue this process if $N-1 \geq 2$. Now suppose we are at the stage of (X_n, ω_n) for some $n < N$, with the descendant of D_N in X_n denoted by D_n, which is J_n-holomorphic with respect to some ω_n-compatible almost complex structure J_n. Suppose $n \geq 2$ and we are trying to blow down the class E_n in the reduced basis of (X_n, ω_n), and to define the descendant of D_n under the blowing-down operation. To this end, we represent the class E_n by a J_n-holomorphic sphere C_n. If C_n is not part of D_n, then as we argued in the case of C_N, one can slightly perturb C_n to a symplectic (-1)-sphere, still denoted by C_n, such that C_n obeys the general position condition. With this understood, we simply blow down (X_n, ω_n) along C_n in the same way as we blow down (X_N, ω_N) along C_N, and move on to the next stage (X_{n-1}, ω_{n-1}).

However, if C_n is part of D_n, then we can no longer perturb C_n before blowing it down, in order to make the successive blowing-down procedure reversible. In the easy situation where C_n is one of the original symplectic surfaces in D_N, we can simply blow it down without perturbing it. In general, C_n is the descendant of a symplectic sphere $S \subset D_N$ to X_n, where the a-coefficient of S is zero and the class E_n appears in S as the leading class, i.e., S has the homological expression

$$S = E_n - E_{l_1} - \cdots - E_{l_\alpha}, \text{ where } n < l_s \text{ for all } s.$$

In this case, more care needs to be given in defining the descendant D_{n-1} of D_n in the next stage (X_{n-1}, ω_{n-1}).

4.2.2. Tangency of higher orders and singularities

When C_n is part of D_n, intersection of higher order tangency as well as singularities may occur in D_{n-1}. In order to construct D_{n-1}, we need the following technical lemma.

Lemma 4.4. *Let (M, ω) be a symplectic 4-manifold and C be a symplectic (-1)-sphere in (M, ω). Let (M', ω') be the symplectic blow-down of (M, ω) along C, obtained by removing C and gluing back a standard symplectic 4-ball (with an appropriate size depending on the area of C). Note that the set of points on C corresponds naturally to the set of complex lines through the origin in the standard symplectic 4-ball in (M', ω'). With this understood, the following statements hold.*

(1) *Let $S_0, S_1, \cdots, S_k$ be symplectic surfaces in (M,ω), which intersect C at a point p. Moreover, suppose there is a complex coordinate system (w_1, w_2) centered at p in which the symplectic structure ω is standard, such that C is defined by $w_2 = 0$, S_0 is defined by $w_1 = 0$, and each S_i, $i > 0$, is defined by the complex line $w_2 = a_i w_1$ for some distinct complex numbers $a_i \neq 0$. Then the descendant S_i' of S_i in the blow-down (M', ω') can be defined as follows: let (z_1, z_2) be the complex coordinates of the standard symplectic 4-ball in (M', ω'), such that the complex line corresponding to the intersection point $p \in C$ is given by $z_1 = 0$, then S_0' is obtained by gluing a complex disc to $S_0 \setminus C$ contained in $z_1 = 0$, and for each $i > 0$, S_i' is obtained by gluing a holomorphic disc to $S_i \setminus C$ defined by the equation $z_1 = b_i z_2^2$ for some distinct complex numbers $b_i \neq 0$.*

(2) *Let S be a symplectic surface intersecting C at p, such that there is a Darboux complex coordinate system (w_1, w_2) centered at p, in which C and S are given by $w_2 = 0$ and $w_2^n = a w_1^m$ for some relative prime integers $m, n > 0$ and a complex number $a \neq 0$. Then the descendant S' of S in the blow-down (M', ω') can be defined as follows: let (z_1, z_2) be the complex coordinates of the standard symplectic 4-ball in (M', ω'), such that the complex line corresponding to the intersection point $p \in C$ is given by $z_1 = 0$, then S' is obtained by gluing a holomorphic disc to $S \setminus C$ defined by the equation $z_1^m = b z_2^{m+n}$ for some complex number $b \neq 0$, which is explicitly determined by a, m and n.*

Proof. Let the symplectic area of C be $\omega(C) = \pi\delta_0^2$ for some $\delta_0 > 0$. Then by the Weinstein neighborhood theorem, a neighborhood of C in (M, ω) has a standard model which we describe below.

Let (z_1, z_2) be the coordinates of $\mathbb{C}^2$ such that the standard symplectic structure ω_0 is given by $\omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$. Let $B^4(\delta) = \{(z_1, z_2) | |z_1|^2 + |z_2|^2 < \delta^2\}$ denote the open ball of radius $\delta > 0$ in $\mathbb{C}^2$, and for any $\delta_1 > \delta_0$, let $W(\delta_1)$ be the symplectic 4-manifold which is obtained by collapsing the fibers of the Hopf fibration on the boundary of $B^4(\delta_1) \setminus B^4(\delta_0)$. Then a neighborhood of C in (M, ω) is symplectomorphic to $W(\delta_1)$ for some δ_1 where $\delta_1 - \delta_0$ is sufficiently small. With this understood, the symplectic blow-down (M', ω') is obtained by cutting (M, ω) open along C and gluing in the standard symplectic 4-ball $B^4(\delta_0)$ after fixing an identification of a neighborhood of C with $W(\delta_1)$. In the present situation, in order to extend the symplectic surfaces $S_i \setminus C$ or $S \setminus C$ across the 4-ball $B^4(\delta_0)$, we need to choose the identification of a neighborhood of C with $W(\delta_1)$ more carefully.

To this end, we consider the following reparametrization of a neighborhood of the circle $\{z_1 = 0\} \cap \mathbb{S}^3(\delta_0)$ in $\mathbb{C}^2$, where $\mathbb{S}^3(\delta_0)$ is the sphere of radius δ_0, by the map

$$(z_1, z_2) = (\frac{r\delta}{\sqrt{1+r^2}} e^{i(\theta+\phi)}, \frac{\delta}{\sqrt{1+r^2}} e^{i\phi}),$$

for $0 \leq r < r_0$, $\theta, \phi \in \mathbb{R}/2\pi\mathbb{Z}$, and δ lying in a small interval containing δ_0. We note that (r, θ, ϕ) gives a trivialization of the Hopf fibration near $z_1 = 0$ in $\mathbb{S}^3(\delta_0)$, with (r, θ) for the base and ϕ for the fiber. In the new coordinates $(r, \theta, \delta, \phi)$, the standard symplectic structure on $\mathbb{C}^2$ takes the form

$$\omega_0 = \frac{r^2\delta}{1+r^2} d\delta\wedge d\theta + \frac{\delta^2 r}{(1+r^2)^2} dr\wedge d\theta + \delta d\delta\wedge d\phi.$$

Replacing δ^2 by $\delta^2+\delta_0^2$ and assuming $0\leq\delta<\sqrt{\delta_1^2-\delta_0^2}$, we obtain a description of the symplectic structure on $W(\delta_1)$ in a neighborhood of the image of $\{z_1=0\}\cap\mathbb{S}^3(\delta_0)$ in $W(\delta_1)$ (where the image of $\{z_1=0\}\cap\mathbb{S}^3(\delta_0)$ has coordinates $\lambda=\delta=0$):

$$\omega_0 = \lambda d\lambda\wedge d\theta + \delta d\delta\wedge d\phi, \text{ where } \lambda = \frac{r\sqrt{\delta^2+\delta_0^2}}{\sqrt{1+r^2}}.$$

With this understood, the map $(w_1,w_2)=(\lambda e^{i\theta},\delta e^{i\phi})$ is a symplectomorphism which identifies a neighborhood of the image of $\{z_1=0\}\cap\mathbb{S}^3(\delta_0)$ in $W(\delta_1)$ with a neighborhood of $p\in C$ in (M,ω). Then by the relative version of the Weinstein neighborhood theorem, we may extend this symplectomorphism to a symplectomorphism which identifies $W(\delta_1)$ with a neighborhood of C in (M,ω).

With the preceding understood, we now consider case (1) of the lemma. First, note that the symplectic surface S_0 is given by $w_1=0$ near the point p. Hence under the symplectomorphism $(w_1,w_2)=(\lambda e^{i\theta},\delta e^{i\phi})$ where $\lambda=\frac{r\sqrt{\delta^2+\delta_0^2}}{\sqrt{1+r^2}}$, the part of S_0 near p as a symplectic surface in $W(\delta_1)$ is given by the equation $r=0$ in the coordinate system (r,θ,δ,ϕ), which implies that, as a symplectic surface in $\mathbb{C}^2$, it is given by the equation $z_1=0$. It follows immediately that one can extend $S_0\setminus C$ across the standard symplectic 4-ball in (M',ω') by gluing in a complex disc contained in the complex line $z_1=0$. This is the descendant S_0' of S_0 in (M',ω').

For each $i>0$, S_i is given by the complex line $w_2=a_iw_1$ near the point p. Writing $a_i=\rho_ie^{i\kappa_i}$, we parametrize S_i near p by the equations $w_1=te^{is}$ and $w_2=t\rho_ie^{i(s+\kappa_i)}$. Under the symplectomorphism $(w_1,w_2)=(\lambda e^{i\theta},\delta e^{i\phi})$ where $\lambda=\frac{r\sqrt{\delta^2+\delta_0^2}}{\sqrt{1+r^2}}$, it is parametrized in the (r,θ,δ,ϕ) coordinate system by the following equations:

$$r=\frac{t}{\sqrt{\delta_0^2+(\rho_i^2-1)t^2}},\ \theta=s,\ \delta=t\rho_i,\ \phi=s+\kappa_i.$$

Now reviewing the part of S_i near p as a subset in $\mathbb{C}^2$, it is parametrized in the coordinates (z_1,z_2) by the following equations (recall we have replaced δ^2 by $\delta_0^2+\delta^2$):

$$z_1=\frac{r\sqrt{\delta_0^2+\delta^2}}{\sqrt{1+r^2}}e^{i(\theta+\phi)}=te^{i(2s+\kappa_i)},\ \ z_2=\frac{\sqrt{\delta_0^2+\delta^2}}{\sqrt{1+r^2}}e^{i\phi}=\sqrt{\delta_0^2+(\rho_i^2-1)t^2}\cdot e^{i(s+\kappa_i)}.$$

With this understood, we observe that z_1,z_2 satisfy the equation $z_1=b_iz_2^2$, where

$$b_i=\frac{te^{-i\kappa_i}}{\delta_0^2+(\rho_i^2-1)t^2},$$

for any $t>0$ which is sufficiently small. It is clear that $b_i\neq 0$ for each $i>0$, and that $\{a_i\}$ being distinct implies that $\{b_i\}$ are also distinct (for each fixed t). Now we fix a value $t_0 > 0$ which is sufficiently small, and remove the part $\{t \le t_0\}$ from S_i and glue onto it the holomorphic disc defined by the equation $z_1 = b_i z_2^2$, where

$$b_i = \frac{t_0 e^{-i\kappa_i}}{\delta_0^2 + (\rho_i^2 - 1)t_0^2}.$$

For t_0 small, one can smooth off the corners near the gluing region to obtain a symplectic surface in (M', ω'), which is defined to be the descendant S_i' of S_i in the symplectic blow-down. This finishes the proof for case (1).

The argument for case (2) is similar. The surface S near p is given by the equation $w_2^n = a w_1^m$. Writing $a = \rho e^{i\kappa}$, we parametrize S near p by the equations

$$w_1 = t^n e^{ins} \text{ and } w_2 = t^m \rho^{\frac{1}{n}} e^{i(ms + \frac{\kappa}{n})}.$$

Under the symplectomorphism $(w_1, w_2) = (\lambda e^{i\theta}, \delta e^{i\phi})$ where $\lambda = \frac{r\sqrt{\delta^2+\delta_0^2}}{\sqrt{1+r^2}}$, it is parametrized in the $(r, \theta, \delta, \phi)$ coordinate system by the following equations:

$$r = \frac{t^n}{\sqrt{\delta_0^2 + \rho^{2/n} t^{2m} - t^{2n}}},\ \theta = ns,\ \delta = \rho^{1/n} t^m,\ \phi = ms + \kappa/n.$$

In the coordinates (z_1, z_2) on $\mathbb{C}^2$, the part of S near p is parametrized by the following equations:

$$z_1 = t^n e^{i\kappa/n} \cdot e^{i(m+n)s},\ \ z_2 = \sqrt{\delta_0^2 + \rho^{2/n} t^{2m} - t^{2n}} \cdot e^{i\kappa/n} e^{ims}.$$

It follows easily that z_1, z_2 satisfy the equation $z_1^m = b z_2^{m+n}$, where

$$b = \frac{t^{mn} e^{-i\kappa}}{(\delta_0^2 + \rho^{2/n} t^{2m} - t^{2n})^{\frac{m+n}{2}}}$$

for any $t > 0$ which is sufficiently small. Clearly, $b \neq 0$. As in case (1), we fix a value $t_0 > 0$ sufficiently small, remove the part $\{t \le t_0\}$ from the surface S and glue onto it the holomorphic disc (singular in this case) defined by the equation $z_1^m = b z_2^{m+n}$, where in b the variable t is evaluated at t_0. The resulting surface (after smoothing off the corners) is the descendant S' of S in the symplectic blow-down (M', ω'). This finishes the proof for case (2), and the proof of the lemma is complete. □

With Lemma 4.4 at hand, we shall define the descendant D_{n-1} of D_n in the next stage (X_{n-1}, ω_{n-1}) as follows. First, since $n < l_s$ for each s, the classes E_{l_s} all have been blown down in the earlier stages. We assume that for each s, the class E_{l_s} does not appear in any of the components of D_N as the leading class (i.e., this is the first time we cannot perturb the (-1)-sphere to a general position). With this understood, for each s, there is a point $\hat{E}_{l_s}$ and a small, standard symplectic 4-ball $B(\hat{E}_{l_s}) \subset X_n$ centered at $\hat{E}_{l_s}$, such that $\hat{E}_{l_s} \in C_n$ for each s, and the intersection $B(\hat{E}_{l_s}) \cap C_n$ is a disc lying in a complex line (called a complex linear disc).

Case (1): Suppose the class E_n does not appear in any of the components of D_N which has zero a-coefficient. In this case, we can simply blow down (X_n, ω_n) along C_n to the next stage (X_{n-1}, ω_{n-1}), which means that we will cut X_n open along C_n and then insert a standard symplectic 4-ball of appropriate size. For any component F_k in D_n which intersects with C_n, there are two possibilities. If an intersection point of F_k with C_n is inherited from the original intersection in D_N, then by the condition (†), there is no other component F_l passing through this intersection point. For such an intersection point on C_n, we shall simply glue a disc to $F_k \setminus C_n$ which is lying on a complex line in the standard symplectic 4-ball. Any other intersection point of F_k with C_n should occur at one of the points $\hat{E}_{l_s}$. For any such intersection points, we shall define the descendant of F_k in X_{n-1} by extending the surface $F_k \setminus C_n$ across the standard symplectic 4-ball according to Lemma 4.4(1). With this understood, we denote the center of the standard symplectic 4-ball by $\hat{E}_n$. Then it is easy to see that there is a small 4-ball $B(\hat{E}_n)$ centered at $\hat{E}_n$, such that each original intersection point on C_n from D_N determines a linear complex disc in $B(\hat{E}_n)$ as part of the descendant D_{n-1}, and each point $\hat{E}_{l_s} \in C_n$ determines a complex line in $B(\hat{E}_n)$ with the property that each linear complex disc in $B(\hat{E}_{l_s}) \cap D_n$ which is not part of C_n determines a holomorphic disc in $B(\hat{E}_n)$ as part of the descendant D_{n-1}, which has tangency of order 2 with the complex line determined by the point $\hat{E}_{l_s}$. Finally, we remark that after shrinking the size, the 4-ball $B(\hat{E}_n)$, particularly the point $\hat{E}_n$, will survive to the last stage of the successive blowing-down. Note that $E_n \in \mathcal{E}_0(D_N)$, but $\forall s$, E_{l_s} does not belong to $\mathcal{E}_0(D_N)$.

Case (2): If the class E_n appears in the expression of a symplectic sphere in D_N whose a-coefficient is zero (note that in this case, E_n is not an element of $\mathcal{E}_0(D_N)$), then more care is needed in defining the descendant D_{n-1}. And here is the reason: suppose E_n is contained in S_1 whose a-coefficient is zero, and let E_m be the leading class in S_1. Then $m < n$, and in a later stage of (X_m, ω_m) when we blow down the class E_m, we will be again in a situation where we cannot perturb the (-1)-sphere C_m to a general position (because C_m is the descendant of S_1 in D_m, so is part of D_m). In particular, we will have to apply Lemma 4.4 when blowing down the class E_m. With this understood, observe that in Lemma 4.4, near the point $p \in C$, the symplectic surfaces under consideration have to be in certain standard forms with respect to a complex coordinate system (w_1, w_2) with standard symplectic structure, and in particular, the (-1)-sphere C has to be given by a complex coordinate line $w_2 = 0$. This requires that, when we blow down the (-1)-sphere C_n, we need to arrange so that in the small 4-ball $B(\hat{E}_n) \subset X_{n-1}$, the holomorphic discs $B(\hat{E}_n) \cap D_{n-1}$ can be placed in the model required in Lemma 4.4.

With this understood, we first make the following observation.

Lemma 4.5. *There are at most two components F_k in D_N such that* (1) *the a-coefficient of F_k is zero,* (2) *the homological expression of F_k contains the class E_n. Moreover, such a component F_k can contain at most one of the classes E_{l_s} in its homological expression, and the classes E_{l_s} contained in two distinct such components F_k must be distinct.* (*Recall $S = E_n - E_{l_1} - \cdots - E_{l_\alpha}$ is the symplectic sphere in D_N that is under consideration.*)

Proof. Suppose S_1 is such a component in D_N, i.e, the a-coefficient of S_1 is zero and the homological expression of S_1 contains the class E_n. Let E_{j_1} be the leading class in S_1. Then the fact that E_n is contained in S_1 implies that $j_1 < n$ must be true. On the other hand, $S \cdot S_1 \geq 0$ implies that $S \cdot S_1$, in fact, equals either 0 or 1. In the former case, S_1 contains exactly one of the classes E_{l_s}, and in the latter case, S_1 contains none of the classes E_{l_s}.

Suppose S_2 is another such component in D_N, with E_{j_2} being the leading class in S_2. Without loss of generality, we assume $j_2 < j_1$. Then since S_1, S_2 both contain the class E_n, it follows easily from $S_1 \cdot S_2 \geq 0$ that E_{j_1} must appear in the expression of S_2, the intersection $S_1 \cdot S_2 = 0$, and the classes E_{l_s} which are contained in S_1, S_2 must be distinct. With this understood, suppose to the contrary that there are more than two such components, and let S_3 be a third such component. Then the same argument as in the case of S_2 implies that the expression of S_3 must contain both E_{j_1} and E_n. But then this would imply $S_2 \cdot S_3 < 0$, which is a contradiction. The lemma follows easily from these considerations. $\square$

We shall consider separately according to the number of the symplectic spheres described in Lemma 4.5.

Case (a): Suppose there are two symplectic spheres $S_1, S_2 \subset D_N$ with zero a-coefficient whose homological expressions contain the class E_n in S. We shall need to make some very specific identification of a neighborhood of C_n in (X_n, ω_n) with the standard model, which is described below. Assume $\omega_n(C_n) = \pi\delta_0$.

Fix a coordinate system (z_1, z_2) of $\mathbb{C}^2$ such that the standard symplectic structure ω_0 on $\mathbb{C}^2$ is given by $\omega_0 = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$. Let $B^4(\delta) = \{(z_1, z_2) || z_1|^2 + |z_2|^2 < \delta^2\}$ denote the open ball of radius $\delta > 0$ in $\mathbb{C}^2$, and for any $\delta_1 > \delta_0$, let $W(\delta_1)$ be the symplectic 4-manifold which is obtained by collapsing the fibers of the Hopf fibration on the boundary of $B^4(\delta_1) \setminus B^4(\delta_0)$. Then by the Weinstein neighborhood theorem, a neighborhood of C_n in (X_n, ω_n) is symplectomorphic to $W(\delta_1)$ for some δ_1 where $\delta_1 - \delta_0$ is sufficiently small. With this understood, the symplectic blow-down (X_{n-1}, ω_{n-1}) is obtained by cutting (X_n, ω_n) open along C_n and gluing in the standard symplectic 4-ball $B^4(\delta_0)$ after fixing an identification of a neighborhood of C_n with $W(\delta_1)$.

With the preceding understood, let p_1, p_2 be the intersection points of the descendants of S_1, S_2 in D_n with C_n. Then by a relative version of the Weinstein neighborhood theorem, we can choose an identification of a neighborhood of C_n with $W(\delta_1)$ such that p_1 and p_2 are identified with the images of the Hopf fibers at $z_1 = 0$ and $z_2 = 0$ respectively. With this understood, when we apply Lemma 4.4 to the points p_1, p_2, we can furthermore arrange the descendants of S_1, S_2 in D_n to be the symplectic surface S_0 in Lemma 4.4, so that after applying Lemma 4.4, the descendants of S_1, S_2 in $D_{n-1} \cap B^4(\delta_0)$ are given by the complex lines $z_1 = 0$ and $z_2 = 0$ respectively. Moreover, any other component of D_n which intersects C_n at either p_1 or p_2 will have its descendant in D_{n-1} given by a holomorphic disc in $B^4(\delta_0)$ of the form $z_1 = bz_2^2$ or $z_2 = bz_1^2$ respectively (more generally, of the form $z_1^m = bz_2^{m+n}$ if before blowing down it is given by $w_2^n = aw_1^m$, etc.). It remains to deal with the intersection points $\hat{E}_{l_s} \in C_n$ which are not p_1, p_2. By the assumption (a) in Theorem 4.3, for any such an $\hat{E}_{l_s}$, there is only one component in D_n which intersects C_n at $\hat{E}_{l_s}$, with intersection number $+1$. (Equivalently, there is only one holomorphic disc in the small 4-ball $B(\hat{E}_{l_s})$ which does not lie in C_n.) By a small perturbation, we can arrange this component to coincide with the fiber at $\hat{E}_{l_s} \in C_n$ in $W(\delta_1)$, so that it can be extended across the 4-ball $B^4(\delta_0)$ by a linear complex disc (given by equation $z_2 = az_1$) when we blow down C_n. In summary, the holomorphic discs $B^4(\delta_0) \cap D_{n-1}$ can be placed in a model that is required in Lemma 4.4 before the blowing down, so that in a later stage, when we blow down the (-1)-sphere which is the descendant of S_1 or S_2, Lemma 4.4 can be applied in the process.

Case (b): Suppose there is only one symplectic sphere $S_1 \subset D_N$ with zero a-coefficient whose homological expression contains the class E_n in S. Let p_1 be the intersection point of C_n with the descendant of S_1 in D_n. Then by the assumption (b) in Theorem 4.3, there is at most one intersection point $\hat{E}_{l_s} \neq p_1$ such that the small 4-ball $B(\hat{E}_{l_s})$ contains more than one holomorphic discs which do not lie in C_n. With this understood, we shall choose an identification of a neighborhood of C_n in (X_n, ω_n) with $W(\delta_1)$ such that p_1 and the intersection point $\hat{E}_{l_s}$ are identified with the images of the Hopf fibers at $z_1 = 0$ and $z_2 = 0$ respectively. Then by the same argument as in Case (a), we can arrange such that the holomorphic discs $B^4(\delta_0) \cap D_{n-1}$ can be placed in an appropriate model, so that when we blow down the (-1)-sphere which is the descendant of S_1 in a later stage, Lemma 4.4 can be applied in the process.

With the preceding understood, it follows easily that under assumptions (a) and (b), one can continue the process and successively blow down the classes $E_N, E_{N-1}, \cdots, E_2$ to reach to the stage (X_1, ω_1) (where $X_1 = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$), obtaining a canonically constructed descendant D_1 of D_N in (X_1, ω_1). We remark that there is an ω_1-compatible almost complex structure J_1, such that D_1 is J_1-holomorphic.

It remains to show that if any of the conditions (c), (d), (e) is satisfied, then one can further blow down the class E_1 to reach $\mathbb{CP}^2$ in the final stage. First, assume (c) is true. In this case, since $\omega_N(E_1) = \omega_N(E_2)$, the class E_1 also has the minimal area in (X_2, ω_2), so that we can represent both E_1, E_2 by a J_2-holomorphic sphere. It follows that we can blow down both (-1)-classes at the same time.

Next, suppose condition (d) is satisfied. In this case, there is a symplectic sphere S in D_N such that E_1 appears in the expression of S as the leading class. We observe that the descendant of S in D_1 is a symplectic (-1)-sphere representing the class E_1. We simply blow down (X_1, ω_1) along this (-1)-sphere to reach the final stage $\mathbb{CP}^2$.

Finally, suppose condition (e) is satisfied. In this case, we appeal to Lemma 2.3 of [4], which says that either E_1 is represented by a J_1-holomorphic sphere, or there is a J_1-holomorphic sphere C such that $E_1 = m(H - E_1) + C$ for some $m \geq 1$. In the former case, we can blow down the class E_1. In the latter case, we reach a contradiction as follows. By condition (e), there is a component F_k of D_N whose a-coefficient, a, and the

b_i-coefficient for E_1, b, obeys $2b < a$. Let $\hat{F}_k$ denote the descendant of F_k in D_1, which is J_1-holomorphic and has class $aH - bE_1$. Then we have

$$0 \leq C \cdot \hat{F}_k = (m+1)b - ma,$$

contradicting the assumption $2b < a$ and the fact $m \geq 1$. The proof of Theorem 4.3 is complete.

4.3. Examples

For the purpose of illustration, we shall apply the successive symplectic blowing down procedure to some concrete examples, where X_N is the resolution $\tilde{X}$ of the symplectic 4-orbifold X and $D_N = D$, the pre-image of the singular set of X under the map $\tilde{X} \to X$.

Example 4.6. (1) Consider the case where X has a singular set described in (i) of Theorem 1.2(2), i.e., the singular set consists of 9 isolated non-Du Val singularities of isotropy of order 3. In this case, the symplectic configuration D is a disjoint union of 9 symplectic (-3)-spheres, to be denoted by $F_1, F_2, \cdots, F_9$. Note that the canonical class of the resolution $\tilde{X}$ is given by

$$c_1(K_{\tilde{X}}) = -\frac{1}{3}(F_1 + F_2 + \cdots + F_9).$$

It follows immediately that $\tilde{X} = \mathbb{CP}^2 \# 12\overline{\mathbb{CP}^2}$.

The following is a set of possible homological expressions for $F_1, F_2, \cdots, F_9$:

- $H - E_i - E_r - E_s - E_t$, $H - E_i - E_u - E_v - E_w$, $H - E_i - E_x - E_y - E_z$,
- $H - E_j - E_r - E_u - E_x$, $H - E_j - E_s - E_v - E_y$, $H - E_j - E_t - E_w - E_z$,
- $H - E_k - E_r - E_v - E_z$, $H - E_k - E_s - E_w - E_x$, $H - E_k - E_t - E_u - E_y$.

Each class can be represented by a symplectic (-3)-sphere, each pair of distinct classes has zero intersection number, and the sum of the 9 classes equals $-3c_1(K_{\tilde{X}})$. Furthermore, one can arrange so that the symplectic structure on $\tilde{X}$ is odd, e.g., when $F_1, F_2, \cdots, F_9$ have the same area (cf. [6], Lemma 4.1).

It is easy to see that the assumptions (a) and (b) are satisfied, and also, the condition (e) is satisfied. Furthermore, the set $\mathcal{E}_0(D)$ consists of all the 12 E_i-classes. Thus by the successive blowing-down procedure, we obtain a symplectic arrangement $\hat{D}$ in $\mathbb{CP}^2$, which is a union of 9 symplectic lines (i.e., a symplectic sphere of degree 1) intersecting at 12 points. Note that each line contains 4 intersection points, each intersection point is contained in 3 lines, so $\hat{D}$ has an incidence relation which is the same as that of the dual configuration of the famous Hesse configuration (cf. [11]). In particular, $\hat{D}$ can be realized by an arrangement of complex lines.

(2) Consider the case where X has a singular set as in (ii) of Theorem 1.2(2). In this case, D is a disjoint union of 5 pairs of a symplectic (-3)-sphere and a symplectic (-2)-sphere, denoted by $F_{1,k}, F_{2,k}$ for $k = 1, 2, \cdots, 5$, where each pair of symplectic spheres

$F_{1,k}$, $F_{2,k}$ intersect transversely and positively in one point. Moreover,

$$c_1(K_{\tilde{X}}) = -\frac{1}{5}\sum_{k=1}^{5}(2F_{1,k} + F_{2,k}).$$

It follows easily that $\tilde{X} = \mathbb{CP}^2 \# 11\overline{\mathbb{CP}^2}$.

The following is a set of possible homological expressions for $F_{1,k}, F_{2,k}$, $1 \leq k \leq 5$:

- $F_{1,1} = H - E_{i_1} - E_{i_2} - E_{i_3} - E_{i_4}$, $F_{2,1} = H - E_r - E_{i_5} - E_{i_{10}}$,
- $F_{1,2} = H - E_{i_1} - E_{i_5} - E_{i_6} - E_{i_7}$, $F_{2,2} = H - E_r - E_{i_3} - E_{i_9}$,
- $F_{1,3} = H - E_{i_2} - E_{i_5} - E_{i_8} - E_{i_9}$, $F_{2,3} = H - E_r - E_{i_4} - E_{i_6}$,
- $F_{1,4} = H - E_{i_3} - E_{i_6} - E_{i_8} - E_{i_{10}}$, $F_{2,4} = H - E_r - E_{i_2} - E_{i_7}$,
- $F_{1,5} = H - E_{i_4} - E_{i_7} - E_{i_9} - E_{i_{10}}$, $F_{2,5} = H - E_r - E_{i_1} - E_{i_8}$,

where the symplectic structure on $\tilde{X}$ can be arranged so that it is odd, e.g., by requiring that the symplectic spheres $F_{1,k}, F_{2,k}$, where $k = 1, 2, \cdots, 5$, have the same area (cf. [6], Lemma 4.1). Again, the assumptions (a), (b) and the condition (e) are satisfied, so we can blow down $\tilde{X}$ and transform D to a symplectic arrangement $\hat{D} \subset \mathbb{CP}^2$. In this case, $\hat{D}$ is also a symplectic line arrangement, consisting of 10 lines which intersect at 16 points. There are 5 original intersection points, i.e., those inherited from D, and 11 new intersection points corresponding to the 11 E_i-classes. The original intersection points are double points, and among the 11 new intersection points, 10 are triple points and one point is contained in 5 lines. We note that this incidence relation is realized by the real line arrangement $A_1(2m)$ for $m = 5$. (Recall that $A_1(2m)$, for $m \geq 3$, is the arrangement of $2m$ lines in $\mathbb{RP}^2$, of which m are the lines determined by the edges of a regular m-gon in $\mathbb{R}^2$, while the other m are the lines of symmetry of that m-gon, cf. [11].) In particular, $\hat{D}$ can be realized by the complexification of a real line arrangement.

Example 4.7. Here we consider the orbifold X in Example 4.6(1) again, but with the following possible set of homological expressions for $F_1, F_2, \cdots, F_9$:

- $F_1 = E_u - E_j - E_w$, $F_2 = E_y - E_k - E_z$,
- $F_3 = H - E_i - E_r - E_s - E_t$, $F_4 = H - E_i - E_u - E_v - E_w$,
- $F_5 = H - E_i - E_x - E_y - E_z$, $F_6 = H - E_j - E_r - E_u - E_x$,
- $F_7 = H - E_k - E_r - E_v - E_y$, $F_8 = 2H - E_s - E_t - E_u - E_y - E_j - E_v - E_z$,
- $F_9 = 2H - E_s - E_t - E_u - E_y - E_k - E_x - E_w$.

Again, the assumptions (a), (b) and the condition (e) are satisfied. In this case, the symplectic arrangement $\hat{D}$ is a union of 5 symplectic lines and 2 symplectic spheres of degree 2, consisting of the descendants of F_k for $3 \leq k \leq 9$. As for the intersection points, note that $\mathcal{E}_0(D) = \{E_s, E_t, E_x, E_v, E_r, E_i, E_u, E_y\}$, so there are totally 8 intersection points labelled by these classes. Moreover, each of the 6 intersection points $\hat{E}_s$, $\hat{E}_t$, $\hat{E}_x$, $\hat{E}_v$, $\hat{E}_r$, $\hat{E}_i$ is a triple point; it is contained in 3 components in $\hat{D}$ intersecting at it transversely. As for $\hat{E}_u$ and $\hat{E}_y$, let's denote by $\hat{F}_k$ the descendant of F_k in $\hat{D}$, for $3 \leq k \leq 9$. Then the class E_j (resp. E_w) determines a complex line in the 4-ball $B(\hat{E}_u)$, such that $\hat{F}_6$ and $\hat{F}_8$ (resp. $\hat{F}_4$ and $\hat{F}_9$) are tangent to it at $\hat{E}_u$, with the intersection of $\hat{F}_6$

and $\hat{F}_8$ (resp. $\hat{F}_4$ and $\hat{F}_9$) at $\hat{E}_u$ being of tangency of order 2 (cf. Remark (2)(ii) following Theorem 4.3). Similar discussions apply to $\hat{E}_y$, and the classes E_k, E_z.

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