

Finite group actions on symplectic Calabi-Yau 4-manifolds with $b_1 > 0$

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ABSTRACT. This is the first of a series of papers devoted to the topology of symplectic Calabi-Yau 4-manifolds endowed with certain symplectic finite group actions. We completely determine the fixed-point set structure of a finite cyclic action on a symplectic Calabi-Yau 4-manifold with $b_1 > 0$. As an outcome of this fixed-point set analysis, the 4-manifold is shown to be a T^2 -bundle over T^2 in some circumstances, e.g., in the case where the group action is an involution which fixes a 2-dimensional surface in the 4-manifold. Our project on symplectic Calabi-Yau 4-manifolds is based on an analysis of the existence and classification of disjoint embeddings of certain configurations of symplectic surfaces in a rational 4-manifold. This paper lays the ground work for such an analysis at the homological level. Some other result which is of independent interest, concerning the maximal number of disjointly embedded symplectic (-2) -spheres in a rational 4-manifold, is also obtained.

1. Introduction and the main results

In this paper, we study symplectic finite group actions on symplectic Calabi-Yau 4-manifolds with $b_1 > 0$. (Recall that a symplectic 4-manifold M is called *Calabi-Yau* if K_M is trivial.) Our starting point is the recent construction in [7] (see also [32]), where to each symplectic 4-manifold M equipped with a finite symplectic G -action, we associate a symplectic 4-manifold, denoted by M_G , and an embedding $D \rightarrow M_G$ of a disjoint union of configurations of symplectic surfaces. Roughly speaking, the 4-manifold M_G is constructed by first de-singularizing the symplectic structure of the quotient orbifold M/G along the 2-dimensional singular strata, making the underlying space $|M/G|$ into a symplectic 4-orbifold with only isolated singularities. Then M_G is taken to be the minimal symplectic resolution of the symplectic 4-orbifold $|M/G|$, and D is simply the pre-image of the singular set of the original orbifold M/G in M_G . See [7] for more details. The idea is to recover the G -action on M , in particular the 4-manifold M , by analyzing the embedding $D \rightarrow M_G$. With this understood, it was shown (cf. [7], Theorem 1.9) that if M is Calabi-Yau, then M_G is either of torsion canonical class, or is a rational 4-manifold, or an irrational ruled 4-manifold over T^2 . Moreover, M_G is of torsion canonical class if

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and only if the quotient orbifold M/G has at most isolated Du Val singularities (cf. [7], Lemma 4.1). Our basic observation is that, when M_G is rational or ruled, it is possible to effectively recover the original 4-manifold M by analyzing the embedding $D \rightarrow M_G$. Moreover, as it turns out, one can also derive new constraints on the fixed-point set structure of the G -action from non-existence results for the embedding $D \rightarrow M_G$.

As an initial step toward understanding the topology of symplectic Calabi-Yau 4-manifolds endowed with a symplectic finite group action, we consider first the case where the 4-manifold M has $b_1 > 0$, and determine the fixed-point set structure of a finite cyclic action on M . As a result of our analysis, we obtain the following

Theorem 1.1. *Suppose M is a symplectic Calabi-Yau 4-manifold with $b_1 > 0$ which is endowed with a finite symplectic G -action. If the resolution M_G is irrational ruled, or M_G is rational and $G = \mathbb{Z}_2$, then M must be diffeomorphic to a T^2 -bundle over T^2 with homologically essential fibers.*

We remark that in Theorem 1.1, M is in fact diffeomorphic to a hyperelliptic surface in the case of $G = \mathbb{Z}_2$ and M_G is rational. On the other hand, we note that in the case of $G = \mathbb{Z}_2$, M_G is rational or ruled if and only if the fixed-point set M^G contains a 2-dimensional component. We state this special case in the following

Corollary: *Let M be a symplectic Calabi-Yau 4-manifold with $b_1 > 0$, which is equipped with a symplectic involution whose fixed-point set contains a 2-dimensional component. Then M must be diffeomorphic to a T^2 -bundle over T^2 with homologically essential fibers.*

R. Inanc Baykur [2] informed us that he has examples of symplectic Calabi-Yau 4-manifolds with $b_1 = 2$ and 4, which are constructed using symplectic Lefschetz pencils, and which come with a natural symplectic involution whose fixed-point set contains a 2-dimensional component. Our theorem shows that these symplectic Calabi-Yau 4-manifolds all have the standard smooth structure.

To put Theorem 1.1 in a perspective, recall that symplectic 4-manifolds can be classified into four classes according to their symplectic Kodaira dimension κ^s , which is a smooth invariant and takes values in $\{-\infty, 0, 1, 2\}$. (The classification is analogous to the classification in complex surface theory, but the relevant definitions are given in completely different ways. For Kähler surfaces, the two classifications coincide. See [27].) Furthermore, as a culmination of the seminal works of Gromov, McDuff, and Taubes [23, 33, 42], the case of $\kappa^s = -\infty$ is completely determined: these symplectic 4-manifolds are precisely the rational or ruled surfaces.

Much effort has also been devoted to the next case, i.e., $\kappa^s = 0$. First, based on Taubes' theory [42], T.-J. Li (cf. [27]) showed that a minimal symplectic 4-manifold M with $\kappa^s = 0$ is either Calabi-Yau (i.e., K_M is trivial), or a double cover of M is Calabi-Yau. Note that a symplectic Calabi-Yau 4-manifold is spin. Using the Bauer-Furuta theory of spin 4-manifolds, together with Taubes' theorem [42] and the classical Rochlin Theorem, the following homological constraints were obtained, see [1, 27, 28, 34]:

- A symplectic Calabi-Yau 4-manifold M either has the integral homology and intersection form of a $K3$ surface, or has the rational homology and intersection form of a T^2 -bundle over T^2 ; in particular, $0 \leq b_1(M) \leq 4$, and if $b_1(M) > 0$, M has zero Euler number and signature. (If M is non-Calabi-Yau but a double cover of M is Calabi-Yau, then M is an integral homology Enriques surface.)
- In addition, for the case of $b_1(M) = 4$, the cohomology ring $H^*(M; \mathbb{R})$ is isomorphic to $H^*(T^4; \mathbb{R})$ (cf. [39]).

The above homological constraints are in sharp contrast to the flexibility known in higher dimensional symplectic Calabi-Yau manifolds, see e.g., [14]. Using a covering trick, one can also obtain interesting constraints on the fundamental group (as well as homotopy type in the case of $b_1 > 0$) of a symplectic Calabi-Yau 4-manifold (cf. [16]), e.g., in the case of $b_1 = 0$, the fundamental group has no subgroup of finite index.

As for examples, besides $K3$ surfaces, all orientable T^2 -bundles over T^2 are symplectic Calabi-Yau 4-manifolds (cf. [21, 27]). (A topological classification of T^2 -bundles over T^2 is given in [40].) We remark that not all T^2 -bundles over T^2 admit a complex structure, and not all T^2 -bundles over T^2 have homologically essential fibers (cf. [21]). If a complex surface is a symplectic Calabi-Yau 4-manifold, then it is either a $K3$ surface, a complex torus, a primary Kodaira surface, or a hyperelliptic surface. With this understood, the following has been an open question (cf. [12, 27]):

Does there exist a symplectic Calabi-Yau 4-manifold other than the known examples, i.e., a T^2 -bundle over T^2 or a $K3$ surface?

We remark that the basic smooth invariants in 4-manifold theory (e.g., the Seiberg-Witten invariants) are ineffective in distinguishing homeomorphic symplectic Calabi-Yau 4-manifolds. As a result, one hopes to construct new examples which have different topological invariants such as the fundamental group. On the other hand, concerning characterizing the diffeomorphism types of symplectic Calabi-Yau 4-manifolds, Theorem 1.1 is the first result of such kind (under a finite symmetry condition). Finally, for connections of this question with hypersymplectic structures and Donaldson's conjecture, we refer the readers to the recent article [15].

With the preceding understood, the idea of our project is to specialize in symplectic Calabi-Yau 4-manifolds M which admits a G -action such that M_G is rational or ruled, and through $D \rightarrow M_G$, to gain insight about the topology of M . Note that with Theorem 1.1, the case where M_G is irrational ruled is settled.

Now we state the results on the fixed-point set structure of a finite cyclic action on symplectic Calabi-Yau 4-manifolds with $b_1 > 0$. We shall separate the prime order and non-prime order cases.

Theorem 1.2. *Let G be a cyclic group of prime order, and let M be a symplectic Calabi-Yau 4-manifold with $b_1 > 0$, equipped with a non-free symplectic G -action. Then the fixed-point set structure of the G -action and the symplectic resolution M_G must belong to one of the following cases:*

- (1) Suppose M_G has torsion canonical class. Then either $G = \mathbb{Z}_2$ or $G = \mathbb{Z}_3$. In the former case, G either has 8 isolated fixed points, with $b_1(M) < 4$ and M_G being an integral homology Enriques surface, or has 16 isolated fixed points, with $b_1(M) = 4$ and M_G being an integral homology K3 surface. In the latter case where $G = \mathbb{Z}_3$, the fixed point set consists of 9 isolated points of type (1, 2), with $b_1(M) = 4$ and M_G being an integral homology K3 surface.
- (2) Suppose M_G is irrational ruled. Then $G = \mathbb{Z}_2$ or \mathbb{Z}_3 , the fixed point set consists of only tori with self-intersection zero, and M_G is a \mathbb{S}^2 -bundle over T^2 .
- (3) Suppose M_G is rational. Then $G = \mathbb{Z}_2, \mathbb{Z}_3$ or \mathbb{Z}_5 . The fixed-point set structure and M_G are listed below:
 - (i) If $G = \mathbb{Z}_2$, the fixed point set consists of one or two torus of self-intersection zero and 8 isolated points, and $M_G = \mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$, $b_1(M) = 2$.
 - (ii) If $G = \mathbb{Z}_3$, there are three possibilities, where $b_1(M) = 2$ in (a), (b), and $b_1(M) = 4$ in (c):
 - (a) the fixed point set consists of 6 isolated points, where exactly 3 of the fixed points are of type (1, 1), and $M_G = \mathbb{C}\mathbb{P}^2 \# 10\overline{\mathbb{C}\mathbb{P}^2}$;
 - (b) the fixed point set consists of one torus with self-intersection zero and 6 isolated points, where exactly 3 of the fixed points are of type (1, 1), and $M_G = \mathbb{C}\mathbb{P}^2 \# 10\overline{\mathbb{C}\mathbb{P}^2}$;
 - (c) the fixed point set consists of 9 isolated points of type (1, 1), and $M_G = \mathbb{C}\mathbb{P}^2 \# 12\overline{\mathbb{C}\mathbb{P}^2}$.
 - (iii) If $G = \mathbb{Z}_5$, the fixed point set consists of 5 isolated points of type (1, 2), and $M_G = \mathbb{C}\mathbb{P}^2 \# 11\overline{\mathbb{C}\mathbb{P}^2}$, $b_1(M) = 4$.

Theorem 1.3. *Let G be a cyclic group of non-prime order, and let M be a symplectic Calabi-Yau 4-manifold with $b_1 > 0$, equipped with a symplectic G -action such that no subgroups of G act freely on M . Suppose M_G is rational or ruled, but for any prime order subgroup H , M_H has torsion canonical class. Then $G = \mathbb{Z}_4$ or \mathbb{Z}_8 . Moreover,*

- (i) If $G = \mathbb{Z}_4$, there are two possibilities:
 - (a) the G -action has 4 isolated fixed points, where exactly 2 of the fixed points are of type (1, 1), and 4 isolated points of isotropy of order 2, with $M_G = \mathbb{C}\mathbb{P}^2 \# 11\overline{\mathbb{C}\mathbb{P}^2}$; in this case, $b_1(M) = 2$,
 - (b) the G -action has 4 isolated fixed points, all of type (1, 1), and 12 isolated points of isotropy of order 2, with $M_G = \mathbb{C}\mathbb{P}^2 \# 13\overline{\mathbb{C}\mathbb{P}^2}$; in this case, $b_1(M) = 4$.
- (ii) If $G = \mathbb{Z}_8$, there are two possibilities, where in both cases, $b_1(M) = 4$:
 - (a) the G -action has 2 isolated fixed points, all of type (1, 3), and 2 isolated points of isotropy of order 4 of type (1, 3), and 12 isolated points of isotropy of order 2, with $M_G = \mathbb{C}\mathbb{P}^2 \# 11\overline{\mathbb{C}\mathbb{P}^2}$;
 - (b) the G -action has 2 isolated fixed points, all of type (1, 5), and 2 isolated points of isotropy of order 4 of type (1, 1), and 12 isolated points of isotropy of order 2, with $M_G = \mathbb{C}\mathbb{P}^2 \# 11\overline{\mathbb{C}\mathbb{P}^2}$.

Remarks: (1) Let n be the order of G (prime or non-prime). An isolated fixed point q is said to be of type $(1, b)$ (where $0 < b < n$) if there is a generator g of G such that the induced action of g on $T_q M$ has eigenvalues $\exp(2\pi i/n)$ and $\exp(2\pi i b/n)$ (with respect to a complex structure on $T_q M$ compatible with the symplectic structure on M). More generally, an isolated point q is of isotropy order m of type $(1, b)$ (where $0 < b < m$) if there is a generator g of the isotropy subgroup G_q at q , such that $m = |G_q|$ and the induced action of g on $T_q M$ has eigenvalues $\exp(2\pi i/m)$ and $\exp(2\pi i b/m)$. Note that if $G = \mathbb{Z}_2$ or the isotropy order $m = 2$, q is always of type $(1, 1)$.

(2) In light of Theorem 1.1, it remains to determine M when M_G is rational and $G \neq \mathbb{Z}_2$. Examining the cases in Theorems 1.2 and 1.3 where M_G is rational, we see that either $b_1(M) = 2$ or $b_1(M) = 4$. In particular, if M admits a complex structure, then M must be either a hyperelliptic surface or a complex torus.

(3) We point out that for all the cases where M_G is rational, the fixed-point set structures can be realized by holomorphic actions (either on a hyperelliptic surface or a complex torus). Explicit examples realizing the fixed-point set structures listed in Theorem 1.2(3)(iii) and Theorem 1.3(ii) (where $G = \mathbb{Z}_5$ or \mathbb{Z}_8) can be found in Fujiki [18], Table 6 (examples for the remaining cases can be easily constructed by hand).

For a large part, the proofs of Theorems 1.2 and 1.3 employ the standard techniques in group actions, i.e., the Lefschetz fixed point theorem and the G -signature theorem, coupled with the standard results in symplectic topology of rational and ruled surfaces and the topological constraints of minimal symplectic 4-manifolds with $\kappa^s = 0$ through the use of M_G . Some of the cases also require the use of G -index theorem for Dirac operators and Seiberg-Witten theory. These traditional methods are quite efficient in determining the fixed-point set structure for the isolated fixed points, however, for the 2-dimensional fixed components, these methods have their natural limitations. The reason is that the 2-dimensional fixed components (particularly the tori of self-intersection zero) often do not make any contribution in the various G -index theorem calculations, hence cannot be detected by these methods. (See [10], Section 3, for a summary of these traditional methods.)

With this understood, in order to obtain further constraints on the 2-dimensional fixed components (as well as for a proof of Theorem 1.1), we shall analyze the embedding of D in M_G . It turns out that the main difficulty occurs when M_G is rational.

To explain this aspect of the story, we let (X, ω) be a symplectic rational 4-manifold, where $X = \mathbb{C}\mathbb{P}^2 \# N \overline{\mathbb{C}\mathbb{P}^2}$. We fix a reduced basis H, E_1, E_2, \dots, E_N of (X, ω) (a more detailed discussion on reduced bases will be given in Section 3). Then for any symplectic surface in X , its homology class A can be expressed in terms of the reduced basis H, E_1, E_2, \dots, E_N :

$$A = aH - \sum_{i=1}^N b_i E_i, \text{ where } a \in \mathbb{Z}, b_i \in \mathbb{Z}.$$

The numbers a, b_i are called the a -coefficient and b_i -coefficients of A . By the adjunction formula, the numbers a and b_i are bound by a set of equations involving the self-intersection number A^2 and the genus of the surface. It follows easily from these equations that for each fixed value of the a -coefficient, there are only finitely many possible values for the b_i -coefficients. However, for each given symplectic surface, there is no a priori upper bound for the a -coefficient of its class A , although one can show that there is a lower bound for the a -coefficient (cf. Lemmas 3.3 and 3.4).

Now suppose D is a disjoint union of configurations of symplectic surfaces embedded in X , where its components are denoted by F_k . The first step in approaching the problem of existence and classification of $D \rightarrow X$ is to look at the classes of the components F_k in a given reduced basis. This process often involves a case-by-case examination, hence it is important that for each component F_k , there are only finitely many possible homological expressions. Such a finiteness can be achieved by bounding the values of the a -coefficient of each F_k , as the self-intersection number F_k^2 and the genus of F_k are all pre-determined by $D \rightarrow X$.

In the present situation, $c_1(K_X)$ is supported in D . More precisely,

$$c_1(K_X) = \sum_k c_k F_k, \text{ where } c_k \in \mathbb{Q} \text{ and } c_k \leq 0.$$

As $c_1(K_X) = -3H + \sum_{i=1}^N E_i$, the a -coefficient of $c_1(K_X)$ equals -3 . It follows easily that for those components F_k with $c_k \neq 0$, the a -coefficient is bounded from above. However, if F_k is a (-2) -sphere, which is either disjoint from the other components, or appears in a configuration of only (-2) -spheres, then $c_k = 0$ and there is no bearing on the a -coefficient of F_k from $c_1(K_X)$.

It turns out that we can remedy this issue by imposing an auxiliary area condition. More concretely, let A be the class of a symplectic $(-\alpha)$ -sphere where $\alpha = 2$ or 3 . If the area condition $\omega(A) < -c_1(K_X) \cdot [\omega]$ is satisfied, then A must take the following expression in a given reduced basis:

$$A = aH - (a-1)E_{j_1} - E_{j_2} - \cdots - E_{j_{2a+\alpha}}.$$

In particular, the a -coefficient of A has an upper bound in terms of N :

$$a \leq \frac{1}{2}(N - \alpha)$$

(See Lemma 3.6.) On the other hand, for the problem of existence and topological classification of embeddings of D in X , one can always freely impose such an area condition by working with a different symplectic structure (cf. Lemma 4.1). Thus in principle, at least for the problem we have at hand, we have developed the necessary tools in this paper to classify the possible embeddings $D \rightarrow M_G$ at the homological level. In particular, by choosing an appropriate symplectic structure ω on M_G , there are only finitely many possible homological expressions for the components of D with respect to any given reduced basis of (M_G, ω) . In forthcoming papers, we shall further develop techniques in order to

understand the possible embeddings $D \rightarrow M_G$ beyond the homological level. (See [8] for more discussions.)

In the course of the proof of Theorem 1.1, we also discover the following result which is of independent interest.

Theorem 1.4. *Let $X = \mathbb{C}\mathbb{P}^2 \# N \overline{\mathbb{C}\mathbb{P}^2}$ where $N = 7, 8$ or 9 . There exist no N disjointly embedded symplectic (-2) -spheres in X .*

We remark that by a theorem of Ruberman [37], there exist N disjointly embedded smooth (-2) -spheres in $X = \mathbb{C}\mathbb{P}^2 \# N \overline{\mathbb{C}\mathbb{P}^2}$ for any $N \geq 2$. On the other hand, for $N = 7$ and 8 , there exist N homology classes $F_1, F_2, \dots, F_N \in H_2(X)$, where $F_i \cdot F_j = 0$ for any $i \neq j$, and each individual F_i can be represented by a symplectic (-2) -sphere (cf. Lemma 5.1). The above theorem says that these homology classes can not be represented simultaneously by disjoint symplectic (-2) -spheres. For $N = 9$, the corresponding homology classes do not exist (cf. Lemma 5.1).

The proof of Theorem 1.4 relies on a recent theorem of Ruberman and Starkston, which asserts that the combinatorial line arrangement coming from the Fano plane has no topological \mathbb{C} -realization (cf. [38]). Our result and method raises naturally the following interesting

Question: *For each $N \geq 2$, what is the maximal number of disjointly embedded symplectic (-2) -spheres in the rational 4-manifold $\mathbb{C}\mathbb{P}^2 \# N \overline{\mathbb{C}\mathbb{P}^2}$?*

We point out that for any $N \geq 3$ and odd, there always exist $N - 1$ disjointly embedded symplectic (-2) -spheres in $\mathbb{C}\mathbb{P}^2 \# N \overline{\mathbb{C}\mathbb{P}^2}$. So for $N = 7$ and 9 , the maximal number is 6 and 8 respectively.

As for the proof of Theorem 1.1, the case where $G = \mathbb{Z}_2$ and M_G is rational is the most delicate one. Here the key technical result, stated as Lemma 5.1, is a classification of all possible homological expressions (in a reduced basis) of the classes of any given set of 8 disjointly embedded symplectic (-2) -spheres in the rational elliptic surface $\mathbb{C}\mathbb{P}^2 \# 9 \overline{\mathbb{C}\mathbb{P}^2}$, where the symplectic structure on $\mathbb{C}\mathbb{P}^2 \# 9 \overline{\mathbb{C}\mathbb{P}^2}$ is chosen to obey a certain set of delicate area constraints on the (-2) -spheres (such a symplectic structure always exists by Lemma 4.1). The proof of Theorem 1.4 also relies on this technical result.

The organization of the paper is as follows. In Section 2, we give an examination of the fixed-point set structure using the traditional methods in group actions, coupled with some standard results and techniques in symplectic 4-manifolds and Seiberg-Witten theory. Section 3 is occupied by a study of symplectic surfaces in rational 4-manifolds. We begin by deriving some basic constraints on the a, b_i -coefficients of a class A which is represented by a connected, embedded symplectic surface. The later part of the section focuses on the classes of symplectic spheres; in particular, it contains Lemma 3.6, which gives an upper bound on the a -coefficient of a symplectic (-2) -sphere or (-3) -sphere under an area condition. In Section 4, we begin by proving a lemma (i.e., Lemma 4.1) which allows us to freely impose certain auxiliary area conditions. This lemma, especially when combined with Lemma 3.6, proves to be very critical in our analysis of the embedding

$D \rightarrow M_G$. We then prove several non-existence results concerning certain symplectic configurations in rational 4-manifolds. These results are used to further remove some ambiguities concerning 2-dimensional fixed components in Section 2. In Section 5, we give proofs of the main theorems.

2. The fixed-point set: a preliminary examination

We give a preliminary analysis of the fixed-point set structure, using mainly the traditional methods. For the reader's convenience, we shall begin with a brief review of the various G -index theorems that will be frequently used in this section. To this end, let M be a symplectic 4-manifold equipped with a symplectic G -action, where G is cyclic of order n . For any nontrivial element $g \in G$, the fixed-point set $Fix(g)$ of g consists of a disjoint union of symplectic surfaces $\{Y_i\}$ and isolated points $\{q_j\}$. In general, $Fix(g)$ depends on g , but when G is of prime order, $Fix(g)$ coincides with the fixed-point set M^G of the G -action. The local action of g near $Fix(g)$ is determined by a set of weights $\{c_i\}, \{(a_j, b_j)\}$ (where $0 < c_i, a_j, b_j < |g|$) as follows. Along each fixed symplectic surface Y_i , the symplectic structure on M determines a complex structure on the normal bundle of Y_i . With this understood, the action of g on the normal bundle of Y_i is given by multiplication of $\exp(2\pi i c_i / |g|)$. Likewise, at each isolated fixed point q_j , the action of g on the tangent space at q_j has eigenvalues $\exp(2\pi i a_j / |g|), \exp(2\pi i b_j / |g|)$ with respect to a complex structure compatible to the symplectic structure on M . We recall that an isolated point $q \in M$ is of isotropy of order m of type $(1, b)$ if $q \in Fix(g)$ for some element $g \in G$ of order $|g| = m$ with weights $(1, b)$. (Note that when $m = n$, the order of G , $q \in M^G$ is an isolated fixed point of G .) We remark that $q \in M$ corresponds to an isolated Du Val singularity in M/G precisely when $b = m - 1$.

The fixed-point set $Fix(g)$ and the associated weights $\{c_i\}, \{(a_j, b_j)\}$ play a prominent role in the various G -index theorems, which we review next. See [10] and the references therein for more details. We begin with the Lefschetz fixed point theorem and the G -signature theorem. Recall that the Lefschetz number $L(g, M)$ is defined, for any $g \in G$, as

$$L(g, M) = \sum_{k=0}^4 (-1)^k \text{tr}(g|_{H^k(M; \mathbb{R})}),$$

where $\text{tr}(g|_{H^k(M; \mathbb{R})})$ stands for the trace of the induced action of g on $H^k(M; \mathbb{R})$. Likewise, the number $Sign(g, M)$ is defined as

$$Sign(g, M) = \text{tr}(g|_{H^{2,+}(M; \mathbb{R})}) - \text{tr}(g|_{H^{2,-}(M; \mathbb{R})}),$$

where for the action of g on $H^{2,+}(M; \mathbb{R})$ and $H^{2,-}(M; \mathbb{R})$, we fix a G -invariant Riemannian metric on M and look at the action of G on the space of self-dual and anti-self-dual harmonic forms respectively. With this understood, the Lefschetz fixed point theorem states that

$$L(g, M) = \chi(Fix(g)) = \sum_i \chi(Y_i) + \#\{q_j\},$$

and the G -signature theorem states that

$$Sign(g, M) = - \sum_j \cot\left(\frac{a_j \pi}{|g|}\right) \cdot \cot\left(\frac{b_j \pi}{|g|}\right) + \sum_i \csc^2\left(\frac{c_i \pi}{|g|}\right) \cdot Y_i^2.$$

We remark that when G is of prime order $n = p$, one can sum over all the nontrivial elements $g \in G$ and obtain the following (weak) version of the Lefschetz fixed point theorem

$$p \cdot \chi(M/G) = \chi(M) + (p-1) \cdot \chi(M^G),$$

and the G -signature theorem

$$p \cdot Sign(M/G) = Sign(M) + \sum_j def_{q_j} + \sum_i def_{Y_i},$$

where $def_{q_j} = \sum_{1 \neq \lambda \in \mathbb{C}, \lambda^p = 1} \frac{(1+\lambda^{a_j})(1+\lambda^{b_j})}{(1-\lambda^{a_j})(1-\lambda^{b_j})}$, $def_{Y_i} = \frac{p^2-1}{3} \cdot Y_i^2$, which are called the signature defects.

Next we review the G -index theorem for Dirac operators, where we further assume that M is a spin 4-manifold and G is of an odd prime order $n = p$. In this case, it was shown in [10] that the G -action on M must be spin; in particular, the orbifold M/G is spin. With this understood, we fix a G -invariant Riemannian metric on M and let \mathbb{D} be the corresponding Dirac operator. Then $\text{Ker} \mathbb{D}$ and $\text{Coker} \mathbb{D}$ are complex G -representations. For any nontrivial element $g \in G$, we write

$$\text{Ker} \mathbb{D} = \bigoplus_{k=0}^{p-1} V_k^+, \quad \text{Coker} \mathbb{D} = \bigoplus_{k=0}^{p-1} V_k^-,$$

where V_k^+ , V_k^- are the eigenspaces of g with eigenvalue $\mu_p^k := \exp \frac{2k\pi i}{p}$. Then the Spin number $Spin(g, M)$ is defined as

$$Spin(g, M) = \sum_{k=0}^{p-1} d_k \mu_p^k, \quad \text{where } d_k \equiv \dim_{\mathbb{C}} V_k^+ - \dim_{\mathbb{C}} V_k^-.$$

Since both $\text{Ker} \mathbb{D}$ and $\text{Coker} \mathbb{D}$ are quaternion vector spaces, and the quaternions i and j are anti-commutative, it follows that V_0^\pm are quaternion vector spaces, and that multiplication by j maps V_k^\pm isomorphically to V_{p-k}^\pm for $k > 0$. This implies that d_0 is even and $d_k = d_{p-k}$ for $k > 0$. Finally, we note that d_0 equals the index of the Dirac operator on the spin orbifold M/G .

The following formula for $Spin(g, M)$ is given in Lemma 3.8 of [10], assuming that the weights of the action of g near $Fix(g) = \{Y_i\} \cup \{q_j\}$ are $\{c_i\}$, $\{(a_j, b_j)\}$:

$$Spin(g, M) = - \sum_j (-1)^{k(g, q_j)} \frac{1}{4} \csc\left(\frac{a_j \pi}{p}\right) \csc\left(\frac{b_j \pi}{p}\right) + \sum_i (-1)^{k(g, Y_i)} \frac{Y_i^2}{4} \csc\left(\frac{c_i \pi}{p}\right) \cot\left(\frac{c_i \pi}{p}\right),$$

where $k(g, q_j)$ is given by the equation $k(g, q_j) \cdot p = 2r_j + a_j + b_j$ for $0 \leq r_j < p$, and $k(g, Y_i)$ is given by the equation $k(g, Y_i) \cdot p = 2r_i + c_i$ for $0 < r_i < p$. This concludes the review of the G -index theorems to be used in this section.

With the preceding understood, for the rest of this section, we assume that M is Calabi-Yau with $b_1 > 0$; in particular, M is spin. Furthermore, we assume that no subgroups of G act freely on M . We shall denote by g_i the genus of the symplectic surface Y_i . Then the adjunction formula, together with the fact that $c_1(K_M) = 0$, implies that $Y_i^2 = 2g_i - 2$ for each i .

The following homological constraints on M will be frequently used: $2 \leq b_1(M) \leq 4$, and $\chi(M) = \text{Sign}(M) = 0$, which implies

$$b_2^+(M) = b_2^-(M) = b_1(M) - 1.$$

Finally, recall from [7], Theorem 1.9 and Lemma 4.1, that M_G is of torsion canonical class if M/G has at most isolated Du Val singularities; otherwise, M_G is either a rational surface, which occurs precisely when $b_1(M/G) = b_1(M_G) = 0$, or M_G is a ruled surface over T^2 and $b_1(M/G) = b_1(M_G) = 2$. We note that the above homological constraints on M apply to M_G as well when M_G is of torsion canonical class and $b_1(M_G) = b_1(M/G) > 0$. When M_G is of torsion canonical class and $b_1(M_G) = 0$, we note that either $\chi(M_G) = 12$ (where M_G is a homology Enriques surface) or $\chi(M_G) = 24$ (where M_G is a homology $K3$ surface).

Now we begin with our analysis on the fixed-point set structures. First, we observe the following lemma.

Lemma 2.1. *Suppose $b_1(M) = 2$ or 3 , and G is of prime order p such that M_G has torsion canonical class. Then $p = 2$ and M^G consists of 8 isolated points. Furthermore, $b_1(M/G) = 0$ and $b_2^+(M/G) = 1$.*

Proof. Since M_G has torsion canonical class, M/G has only isolated Du Val singularities (cf. [7], Lemma 4.1). By the Lefschetz fixed point theorem,

$$p \cdot \chi(M/G) = \chi(M) + (p - 1) \cdot \#M^G.$$

With $\chi(M) = 0$, and observing that the resolution of each singular point of M/G is a chain of $p - 1$ spheres, we obtain the following expression

$$\chi(M_G) = \chi(M/G) + (p - 1) \cdot \#M^G = (p - 1)\left(\frac{1}{p} + 1\right) \cdot \#M^G.$$

On the other hand, note that $\chi(M_G) = 0, 12$, or 24 . It is clear that $\chi(M_G) > 0$, as $M^G \neq \emptyset$, so that $\chi(M_G) = 12$ or 24 . We also note that $b_1(M_G) = 0$ in these two cases. Moreover, since $b_1(M) = 2$ or 3 , we have $b_2^+(M/G) \leq b_2^+(M) = b_1(M) - 1 \leq 2$, so that $\chi(M_G) = 12$ must be true. The equation $(p - 1)\left(\frac{1}{p} + 1\right) \cdot \#M^G = 12$ has only one solution: $p = 2$ and $\#M^G = 8$. Finally, note that $b_1(M/G) = b_1(M_G) = 0$, and $b_2^+(M/G) = b_2^+(M_G) = 1$. This finishes off the proof. \square

2.1. The case where $b_1 = 2$

We first assume G is of prime order p . Let $g \in G$ be a generator of G . Let $\{q_j\}$ be the set of isolated fixed points and set $z := \#\{q_j\}$, and let $\{Y_i\}$ be the set of 2-dimensional fixed components, with g_i the genus of Y_i .

We begin with the case where M_G is irrational ruled. Note that this happens exactly when $b_1(M/G) = 2 = b_1(M)$, which means that the action of G on $H^1(M; \mathbb{R})$ is trivial.

Lemma 2.2. *Suppose G is of prime order and M_G is irrational ruled. Then the fixed-point set M^G consists of a disjoint union of tori of self-intersection zero.*

Proof. We begin by observing $b_2^-(M) = b_1(M) - 1 = 1$, so that either $b_2^-(M/G) = 0$ or $b_2^-(M/G) = 1$. We claim $b_2^-(M/G) = 1$. To see this, suppose to the contrary that $b_2^-(M/G) = 0$. Then $G = \mathbb{Z}_2$ must be true. With this understood, with $b_2^+(M) = b_1(M) - 1 = 1$ and $b_2^+(M/G) = b_2^+(M_G) = 1$, the Lefschetz fixed point theorem gives

$$\sum_i (2 - 2g_i) + z = L(g, M) = 2 - 2 \times 2 + 1 - 1 = -2,$$

and the G -signature theorem gives

$$\sum_i Y_i^2 = \text{Sign}(g, M) = 1 - (-1) = 2.$$

With $Y_i^2 = 2g_i - 2$ for each i , it follows easily that $z = 0$, i.e., there are no isolated fixed points. As a consequence, we note that the underlying space of M/G is smooth, and it is simply the resolution M_G , which is an irrational ruled 4-manifold by the assumption. But this implies that $b_2^-(M/G) = b_2^-(M_G) \geq 1$, contradicting the assumption $b_2^-(M/G) = 0$. Hence we must have $b_2^-(M/G) = 1$. With $b_2^-(M/G) = 1$, it follows easily that $L(g, M) = 0$ and $\text{Sign}(g, M) = 0$.

The equation $L(g, M) = 0$ implies $z = \sum_i (2g_i - 2) = \sum_i Y_i^2$. Suppose to the contrary that $z > 0$. Then there must be a component Y_i such that $Y_i^2 > 0$. Since $b_2^+(M/G) = b_2^+(M_G) = 1$ as M_G is irrational ruled, it follows easily that there can be only one such component. As a consequence, by replacing g with a suitable power, we may assume that the weight of the action of g along the component Y_i with $Y_i^2 > 0$ equals 1 (i.e., $c_i = 1$ if $Y_i^2 > 0$). It follows from the G -signature theorem that

$$\begin{aligned} \text{Sign}(g, M) &= - \sum_j \cot\left(\frac{a_j \pi}{p}\right) \cdot \cot\left(\frac{b_j \pi}{p}\right) + \sum_i \csc^2\left(\frac{c_i \pi}{p}\right) Y_i^2 \\ &> \sum_i \left(\csc^2\left(\frac{c_i \pi}{p}\right) - \csc^2\left(\frac{\pi}{p}\right)\right) \cdot Y_i^2, \end{aligned}$$

where we use the fact that $z = \sum_i Y_i^2$ and $\cot\left(\frac{a_j \pi}{p}\right) \cdot \cot\left(\frac{b_j \pi}{p}\right) < \csc^2\left(\frac{\pi}{p}\right)$ for each j . Since $c_i = 1$ when $Y_i^2 > 0$, it follows easily that $\sum_i \left(\csc^2\left(\frac{c_i \pi}{p}\right) - \csc^2\left(\frac{\pi}{p}\right)\right) \cdot Y_i^2 \geq 0$. This leads to a contradiction that $\text{Sign}(g, M) > 0$, hence $z = 0$ must be true.

With $z = 0$, M_G is simply the underlying manifold $|M/G|$, which must be a \mathbb{S}^2 -bundle over T^2 as $b_2^-(M/G) = 1$. It remains to show that each Y_i is a torus. This follows easily by observing that $\sum_i (2g_i - 2) = z = 0$, and that $g_i > 0$ for each i . The latter is true because if Y_i is a sphere, then $Y_i^2 = -2$, so that Y_i descends to a $(-2p)$ -sphere in M_G . But M_G is a \mathbb{S}^2 -bundle over T^2 , it does not contain any $(-2p)$ -sphere. This finishes off the proof. \square

Next we consider the case where M_G is rational; note that this happens exactly when $b_1(M/G) = 0$. With $b_1(M) = 2$, the action of G on $H^1(M; \mathbb{Z})/\text{Tor}$ is given by elements of $SL(2, \mathbb{Z})$. It follows easily that G is either \mathbb{Z}_2 or \mathbb{Z}_3 .

Lemma 2.3. *Suppose M_G is rational and $G = \mathbb{Z}_2$. Then G has 8 isolated fixed points. Furthermore, $\{Y_i\} \neq \emptyset$ and $\sum_i Y_i^2 = 2(1 - b_2^-(M/G))$.*

Proof. For $G = \mathbb{Z}_2$, we first observe that the G -Signature theorem gives

$$1 - \text{tr}(g|_{H^2, -}) = \text{Sign}(g, M) = \sum_i Y_i^2 = \sum_i (2g_i - 2).$$

On the other hand, the Lefschetz fixed point theorem implies that

$$z + \sum_i (2 - 2g_i) = L(g, M) = 2 - 4 \times (-1) + 1 + \text{tr}(g|_{H^2, -}) = 8 - \sum_i (2g_i - 2).$$

It follows that $z = 8$. Finally, $\sum_i Y_i^2 = 1 - \text{tr}(g|_{H^2, -}) = 2(1 - b_2^-(M/G))$ because $b_2^-(M) = 1$. Note that $\{Y_i\} \neq \emptyset$ because M_G is rational. This finishes the proof. \square

Lemma 2.4. *Suppose M_G is rational and $G = \mathbb{Z}_3$. Then G has 6 isolated fixed points, exactly three of which are of type $(1, 1)$. Furthermore, $\sum_i Y_i^2 = 0$, and at most one of the components in $\{Y_i\}$ is a sphere.*

Proof. First of all, observe that $b_2^-(M/G) = 1$ as $G = \mathbb{Z}_3$, and consequently,

$$L(g, M) = 2 - 4 \times \left(-\frac{1}{2}\right) + 1 + 1 = 6, \quad \text{Sign}(g, M) = 1 - 1 = 0.$$

If we let x, y be the number of isolated fixed points of G which are of type $(1, 1)$ and type $(1, 2)$ respectively, then the Lefschetz fixed point theorem and the G -Signature theorem imply, respectively, that

$$x + y + \sum_i (2 - 2g_i) = 6 \quad \text{and} \quad -\frac{1}{3}x + \frac{1}{3}y + \frac{4}{3} \cdot \sum_i Y_i^2 = 0.$$

With $Y_i^2 = 2g_i - 2$, we eliminate the variable x and obtain $2y + 3 \sum_i Y_i^2 = 6$. On the other hand, observe that $b_2^-(M/G) = 1$ implies that there is at most one component Y_i such that $Y_i^2 < 0$ (note that these are precisely the spherical components in $\{Y_i\}$). Consequently, it is easily seen that $\sum_i Y_i^2 \geq -2$, and with this, it follows easily that $y = 0$ or 3 are the only possibilities, where $x = 8$ or 3 and $\sum_i Y_i^2 = 2$ or 0 respectively.

It remains to eliminate the possibility that $x = 8, y = 0$ and $\sum_i Y_i^2 = 2$. To this end, we observe that the G -action is spin because the order of G is an odd prime (cf. [10]). Moreover, the index of the Dirac operator on the spin orbifold M/G must be zero because $b_2^+(M/G) = b_2^-(M/G) = 1$ (cf. Fukumoto-Furuta [19], Corollary 1). We shall prove the index is nonzero, thus eliminating the case $x = 8, y = 0$ and $\sum_i Y_i^2 = 2$.

First, let \mathbb{D} be the Dirac operator on M . Then as $G = \mathbb{Z}_3$, it follows easily from $\text{Index } \mathbb{D} = -\frac{1}{8} \text{Sign}(M) = 0$ that $\text{Spin}(g, M) = \frac{3}{2}d_0$, where d_0 equals the index of the Dirac operator on M/G .

Next we compute $Spin(g, M)$ using the G -index theorem for Dirac operators (cf. [10], Lemma 3.8). In order to apply the formula for $Spin(g, M)$, we note that for a type (1, 1) isolated fixed point q_j , the number $k(g, q_j) = 2$, and for a type (1, 2) isolated fixed point q_j , $k(g, q_j) = 1$. On the other hand, it is easy to check that $k(g, Y_i) = c_i$ for any Y_i . With these understood, it follows easily that the Spin number

$$Spin(g, M) = -\frac{1}{3}x + \frac{1}{3}y + \sum_i \left(-\frac{1}{6}Y_i^2\right) = -\frac{1}{3} \times 8 + \frac{1}{3} \times 0 - \frac{1}{6} \times 2 = -3.$$

Consequently, $d_0 = \frac{2}{3}Spin(g, M) = -2$, which is nonzero. This finishes the proof. \square

Finally, we consider the case where G is of non-prime order n . We assume that M_G is rational or ruled, and that for any subgroup H of prime order, M_H has torsion canonical class.

First, by Lemma 2.1, the order n must be a power of 2; more precisely, $n = 2^k > 2$. Furthermore, $b_1(M/G) = 0$, so that M_G must be rational. Finally, note that the action of G on $H^1(M; \mathbb{Z})/\text{Tor}$ is given by elements of $SL(2, \mathbb{Z})$. It follows easily that $n = 4$.

With the preceding understood, we fix a generator g of G , and let H be the subgroup of order 2 generated by $h := g^2$. Then by our assumption, M_H has torsion canonical class. By Lemma 2.1, M^H consists of 8 isolated fixed points. Since M^G is contained in M^H , the action of G has no 2-dimensional fixed components.

To proceed further, note that there are two possibilities: $b_2^-(M/G) = 0$ or 1. Consider first the case where $b_2^-(M/G) = 0$. In this case, $L(g, M) = 2 - 4 \times 0 + 1 - 1 = 2$, so the G -action has 2 isolated fixed points. Examining the induced action of G on M^H , the remaining 6 fixed points of H are of isotropy of order 2, and consequently, the orbifold M/G has 5 singular points – two of order 4 and three of order 2. Let x, y be the number of fixed points of G of type (1, 1) and type (1, 3) respectively. Note that the resolution of a type (1, 1) fixed point in M_G is a (-4) -sphere and the resolution of a type (1, 3) fixed point is a linear chain of three (-2) -spheres. A point of isotropy of order 2 gives rise to a (-2) -sphere in M_G . As a result, we have

$$b_2^-(M_G) = b_2^-(M/G) + x + 3y + 3 = x + 3y + 3.$$

On the other hand, $c_1(K_{M_G}) = \sum_i -\frac{1}{2}E_i$, where E_i are the (-4) -spheres in M_G coming from the resolution of type (1, 1) fixed points of G (cf. [7], Proposition 3.2). Thus $c_1(K_{M_G})^2 = \sum_i \frac{1}{4}E_i^2 = -x$. Since M_G is rational, we have $c_1(K_{M_G})^2 = 9 - b_2^-(M_G)$, which is $-x = 9 - (x + 3y + 3)$. It follows that $y = 2$, and $x = 2 - y = 0$. But this is a contradiction as it implies that $c_1(K_{M_G}) = 0$. Hence the case where $b_2^-(M/G) = 0$ is eliminated.

For the case where $b_2^-(M/G) = 1$, it is easy to see that $L(g, M) = 4$, so the G -action has 4 isolated fixed points. A similar calculation results

$$b_2^-(M_G) = x + 3y + 3 \text{ and } c_1(K_{M_G})^2 = -x.$$

It follows easily that $x = y = 2$. We summarize our discussions in the following

Lemma 2.5. *Suppose M_G is rational or ruled, but for any subgroup H of prime order, M_H has torsion canonical class. Then M_G must be rational, and G is of order 4. Furthermore, the fixed-point set M^G consists of 4 isolated points, exactly two of which are of type $(1, 1)$, and there are 4 isolated points of isotropy of order 2 in M .*

2.2. The case where $b_1 = 3$

Assume M_G is rational or ruled. We first observe that G must be \mathbb{Z}_2 , which, with $b_2^+(M) = b_1(M) - 1 = 2$ and $b_2^+(M/G) = 1$, follows easily by Lemma 2.1.

Lemma 2.6. *Suppose M_G is rational or ruled. Then G must be of order 2. Moreover,*

- (i) *if M_G is irrational ruled, then the fixed-point set M^G consists of a disjoint union of tori of self-intersection zero;*
- (ii) *if M_G is rational, then the fixed-point set M^G contains 8 isolated points, and the 2-dimensional fixed components $\{Y_i\} \neq \emptyset$ and $\sum_i Y_i^2 = 2(1 - b_2^-(M/G))$.*

Proof. Since M_G is rational or ruled and $G = \mathbb{Z}_2$, $\{Y_i\} \neq \emptyset$. We denote by z the number of isolated fixed points and let $1 \neq g \in G$. Then by the G -Signature theorem,

$$\sum_i Y_i^2 = \text{Sign}(g, M) = (1 - 1) - \text{tr}(g|_{H^2, -}) = -\text{tr}(g|_{H^2, -}).$$

First, consider case (i) where M_G is irrational ruled. In this case, $b_1(M/G) = 2$, so the Lefschetz fixed point theorem implies that

$$z + \sum_i (2 - 2g_i) = L(g, M) = 2 - 2 \times (1 + 1 - 1) + (1 - 1) + \text{tr}(g|_{H^2, -}) = \text{tr}(g|_{H^2, -}).$$

With $Y_i^2 = 2g_i - 2$ for each i , it follows immediately that $z = 0$. As a consequence, M_G is simply the underlying manifold of M/G . This immediately ruled out the possibility that $b_2^-(M/G) = 0$, because as an irrational ruled 4-manifold, M_G has non-zero b_2^- .

Next, assume $b_2^-(M/G) = 1$. In this case, by the same argument as in Lemma 2.2, each Y_i is a torus of self-intersection zero and M_G is a \mathbb{S}^2 -bundle over T^2 .

Finally, we rule out the possibility that $b_2^-(M/G) = 2$. In this case, $\sum_i Y_i^2 = -\text{tr}(g|_{H^2, -}) = -(1 + 1) = -2$, so that there must be a Y_i which is a (-2) -sphere. On the other hand, $b_2^-(M/G) = 2$ implies that M_G is a \mathbb{S}^2 -bundle over T^2 blown up at one point. The descendent of Y_i is a symplectic (-4) -sphere in M_G , to be denoted by C . To derive a contradiction, let F and E be the fiber class and the exceptional (-1) -class of M_G respectively. Note that $c_1(K_{M_G}) \cdot F = -2$ and $c_1(K_{M_G}) \cdot E = -1$. With this understood, since $\pi_2(M_G)$ is generated by F and E , we write $C = aF + bE$. Then $-4 = C^2 = -b^2$ and $2 = c_1(K_{M_G}) \cdot C = -2a - b$, giving either $C = -2F + 2E$ or $C = -2E$. Note that in both cases, C has a negative symplectic area. Hence the possibility $b_2^-(M/G) = 2$ is ruled out.

For case (ii) where M_G is rational, $b_1(M/G) = 0$. In this case, the Lefschetz number $L(g, M) = 2 - 2 \times (-1 - 1 - 1) + (1 - 1) + \text{tr}(g|_{H^2, -}) = 8 + \text{tr}(g|_{H^2, -})$, which implies $z = 8$. The assertion $\sum_i Y_i^2 = 2(1 - b_2^-(M/G))$ follows easily from the fact that $\text{tr}(g|_{H^2, -}) = 2(b_2^-(M/G) - 1)$. This finishes the proof. \square

2.3. The case where $b_1 = 4$

The fact that the cohomology ring $H^*(M; \mathbb{R})$ is isomorphic to that of T^4 (cf. [39]) plays a crucial role in the analysis of the fixed-point set structure in this case. In particular, this fact has the following two corollaries: (1) it allows us to express the action of G on the entire cohomology $H^*(M; \mathbb{R})$ in terms of its action on $H^1(M; \mathbb{R})$, and (2) since the Hurwitz map $\pi_2(M) \rightarrow H_2(M)$ has trivial image, the fixed-point set M^G does not have any spherical components. With the help of the adjunction formula, this is equivalent to the statement that all the 2-dimensional fixed components have nonnegative self-intersection.

For the first point above, to be more concrete, let $g \in G$ be any nontrivial element. Since the action of g on M is orientation-preserving, the representation of g on $H^1(M; \mathbb{R})$ splits into a sum of two complex 1-dimensional representations. This said, there is a basis $\{\alpha_i\}$, $i = 1, 2, 3, 4$, of $H^1(M; \mathbb{R})$ such that $\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4 \in H^4(M; \mathbb{R})$ is positive according to the natural orientation of M . Furthermore, we assume that the span of α_1, α_2 and the span of α_3, α_4 are invariant under the action of g , and with respect to the orientation given by the above order, the action of g is given by a rotation of angle θ_1, θ_2 respectively.

Lemma 2.7. *With g, θ_1, θ_2 as given above, the following hold true:*

- (1) $2(\cos \theta_1 + \cos \theta_2), 4 \cos \theta_1 \cos \theta_2 \in \mathbb{Z}$.
- (2) *The Lefschetz number $L(g, M) = 4(1 - \cos \theta_1)(1 - \cos \theta_2)$.*
- (3) *The representation of g on $H^{2,+}(M; \mathbb{R})$ (resp. $H^{2,-}(M; \mathbb{R})$) splits into a trivial 1-dimensional representation and a 2-dimensional one on which g acts as a rotation of angle $\theta_1 + \theta_2$ (resp. $\theta_1 - \theta_2$). Consequently,*

$$\text{Sign}(g, M) = 2(\cos(\theta_1 + \theta_2) - \cos(\theta_1 - \theta_2)) = -4 \sin \theta_1 \sin \theta_2.$$

Proof. Let $\gamma_1 := \alpha_1 \cup \alpha_3$, $\gamma_2 := \alpha_1 \cup \alpha_4$, $\gamma_3 := \alpha_2 \cup \alpha_3$, and $\gamma_4 := \alpha_2 \cup \alpha_4$. Then a straightforward calculation gives

$$g \cdot (\alpha_1 \cup \alpha_2) = \alpha_1 \cup \alpha_2, \quad g \cdot (\alpha_3 \cup \alpha_4) = \alpha_3 \cup \alpha_4,$$

$$g \cdot \gamma_1 = \cos \theta_1 \cos \theta_2 \gamma_1 + \cos \theta_1 \sin \theta_2 \gamma_2 + \sin \theta_1 \cos \theta_2 \gamma_3 + \sin \theta_1 \sin \theta_2 \gamma_4,$$

$$g \cdot \gamma_2 = -\cos \theta_1 \sin \theta_2 \gamma_1 + \cos \theta_1 \cos \theta_2 \gamma_2 - \sin \theta_1 \sin \theta_2 \gamma_3 + \sin \theta_1 \cos \theta_2 \gamma_4,$$

$$g \cdot \gamma_3 = -\sin \theta_1 \cos \theta_2 \gamma_1 - \sin \theta_1 \sin \theta_2 \gamma_2 + \cos \theta_1 \cos \theta_2 \gamma_3 + \cos \theta_1 \sin \theta_2 \gamma_4,$$

and

$$g \cdot \gamma_4 = \sin \theta_1 \sin \theta_2 \gamma_1 - \sin \theta_1 \cos \theta_2 \gamma_2 - \cos \theta_1 \sin \theta_2 \gamma_3 + \cos \theta_1 \cos \theta_2 \gamma_4.$$

The action on $H^3(M; \mathbb{R})$ can be similarly determined. From these calculations we deduce easily that

$$L(g, M) = 2 - 4(\cos \theta_1 + \cos \theta_2) + (2 + 4 \cos \theta_1 \cos \theta_2) = 4(1 - \cos \theta_1)(1 - \cos \theta_2).$$

In order to understand the action of g on $H^{2,+}(M; \mathbb{R})$ and $H^{2,-}(M; \mathbb{R})$, and to compute $\text{Sign}(g, M)$, we note that $H^{2,+}(M; \mathbb{R})$ is spanned by β_i , $i = 1, 2, 3$, where

$$\beta_1 = \alpha_1 \cup \alpha_2 + \alpha_3 \cup \alpha_4, \quad \beta_2 = \alpha_1 \cup \alpha_3 - \alpha_2 \cup \alpha_4, \quad \beta_3 = \alpha_1 \cup \alpha_4 + \alpha_2 \cup \alpha_3.$$

Likewise, $H^{2,-}(M; \mathbb{R})$ is spanned by β'_i , $i = 1, 2, 3$, where

$$\beta'_1 = \alpha_1 \cup \alpha_2 - \alpha_3 \cup \alpha_4, \quad \beta'_2 = \alpha_1 \cup \alpha_3 + \alpha_2 \cup \alpha_4, \quad \beta'_3 = \alpha_1 \cup \alpha_4 - \alpha_2 \cup \alpha_3.$$

With this understood, the action of g on $H^{2,+}(M; \mathbb{R})$ and $H^{2,-}(M; \mathbb{R})$ is as follows: both β_1 and β'_1 are fixed by g , and g acts on the span of β_2, β_3 and the span of β'_2, β'_3 as a rotation of angle $\theta_1 + \theta_2, \theta_1 - \theta_2$ respectively. It follows in particular that $Sign(g, M) := tr(g|_{H^{2,+}}) - tr(g|_{H^{2,-}})$ is given by

$$Sign(g, M) = 2(\cos(\theta_1 + \theta_2) - \cos(\theta_1 - \theta_2)) = -4 \sin \theta_1 \sin \theta_2.$$

Finally, note that $tr(g|_{H^1(M; \mathbb{R})}) = 2(\cos \theta_1 + \cos \theta_2)$, hence $2(\cos \theta_1 + \cos \theta_2) \in \mathbb{Z}$. With this, $L(g, M) = 4(1 - \cos \theta_1)(1 - \cos \theta_2) \in \mathbb{Z}$ implies that $4 \cos \theta_1 \cos \theta_2 \in \mathbb{Z}$ as well. This completes the proof of the lemma. \square

With Lemma 2.7 at hand, we shall first examine the fixed-point set structure when G is of prime order.

Lemma 2.8. *Suppose G is of prime order $p > 1$. Then the following hold true.*

- (1) *Either $b_2^+(M/G) = 1$ or $b_2^+(M/G) = 3$. Moreover, M_G has torsion canonical class if and only if $b_2^+(M/G) = 3$ and $b_1(M/G) = 0$.*
- (2) *If M_G has torsion canonical class, then $p = 2$ or $p = 3$, where in the former case, the fixed-point set M^G consists of 16 isolated points, and in the latter case, M^G consists of 9 isolated points of type $(1, 2)$.*
- (3) *If M_G is irrational ruled, then M^G consists of a disjoint union of tori of self-intersection zero.*
- (4) *If M_G is rational, then $p \neq 2$ and $p \leq 5$.*

Proof. For (1), note that by Lemma 2.7, $b_2^+(M/G) = 3$ if and only if $\theta_1 + \theta_2 = 2\pi$ for a generator g of G . If $\theta_1 + \theta_2 \neq 2\pi$, then $b_2^+(M/G) = 1$. Hence either $b_2^+(M/G) = 1$ or $b_2^+(M/G) = 3$ as claimed. It remains to show that if M_G has torsion canonical class, then $b_2^+(M/G) \neq 1$ but $b_1(M/G) = 0$. To see this, suppose M_G has torsion canonical class. Then the same argument as in Lemma 2.1 shows that $\chi(M_G) = 12$ or 24 , and $b_1(M/G) = 0$. If $b_2^+(M/G) = 1$, then $\chi(M_G) = 12$, and as in Lemma 2.1, $p = 2$ must be true. With $p = 2$ and $b_1(M/G) = 0$, the angles θ_1, θ_2 in Lemma 2.7 must be both equal to π . But this implies that $b_2^+(M/G) = 3$, contradicting the assumption of $b_2^+(M/G) = 1$. Hence part (1) is proved.

Part (2) follows readily from the same argument as in Lemma 2.1. Note that when $\chi(M_G) = 24$, $p = 2, 3$ or 5 . The case of $p = 5$ can be further eliminated by the (weak version) G -signature theorem.

For part (3), if M_G is irrational ruled, then $b_1(M/G) = 2$. This means that in Lemma 2.7, one of the angles θ_1, θ_2 must be 0. As a corollary, $L(g, M) = Sign(g, M) = 0$ for any nontrivial element $g \in G$, and $b_2^-(M/G) = 1$. With this understood, part (3) follows by the same argument as in Lemma 2.2.

Finally, for part (4) we assume M_G is rational. Then $b_2^+(M/G) = 1$ and $b_1(M/G) = 0$, so that by Lemma 2.7, $p \neq 2$. On the other hand, assume $p \geq 5$. We fix a generator

$g \in G$ such that in Lemma 2.7, the angles $\theta_1 = \frac{2\pi}{p}$ and $\theta_2 = \frac{2q\pi}{p}$ for some $0 < q < p - 1$ (note that $q \neq p - 1$ as $b_2^+(M/G) = 1$). Then it follows easily from $p \geq 5$ that $L(g, M) = 4(1 - \cos \theta_1)(1 - \cos \theta_2)$ satisfies the bound $L(g, M) \leq 7$. With this understood, we appeal to the following version of Lefschetz fixed point theorem

$$p \cdot \chi(M/G) = \chi(M) + (p - 1) \cdot L(g, M),$$

where $\chi(M) = 0$ and $L(g, M) \in \mathbb{Z}$. It follows easily that $L(g, M)$ is divisible by p , and with $p \geq 5$ and $L(g, M) \leq 7$, we have $L(g, M) = p$. A further examination easily removes the possibility that $p = 7$. Hence $p \leq 5$. This finishes the proof of the lemma. \square

In the next two lemmas, we shall determine the fixed-point set structure where M_G is rational and $G = \mathbb{Z}_3$ or \mathbb{Z}_5 . Let $g \in G$ be a generator.

Lemma 2.9. *Assume M_G is rational and $G = \mathbb{Z}_3$. Then the fixed-point set M^G consists of 9 isolated points of type (1, 1), plus possible 2-dimensional components $\{Y_i\}$ which are tori of self-intersection zero.*

Proof. We observe that since M_G is rational, $b_1(M/G) = 0$, which implies that the angles θ_1, θ_2 in Lemma 2.7 are both nonzero. Furthermore, $b_2^+(M/G) = 1$ and $G = \mathbb{Z}_3$, which implies $\theta_1 = \theta_2$. It follows easily that $L(g, M) = 9$ and $Sign(g, M) = -3$.

With this understood, let x, y be the number of isolated fixed points of type (1, 1) and type (1, 2) respectively. Then the Lefschetz fixed point theorem and the G -signature theorem imply that

$$x + y - \sum_i Y_i^2 = 9 \text{ and } -\frac{1}{3}x + \frac{1}{3}y + \frac{4}{3} \sum_i Y_i^2 = -3.$$

Combining the two equations, we get $x + \frac{5}{3}y = 9$. It is easy to see that the solutions are $x = 9, y = 0$ or $x = 4, y = 3$. In the former case, $\sum_i Y_i^2 = 0$, while in the latter case, $\sum_i Y_i^2 = -2$. The latter case is not possible since $Y_i^2 \geq 0$ for all i . For the same reason, we must have $Y_i^2 = 0$ for all i in the former case. By the adjunction formula, each Y_i is a torus. This finishes the proof. \square

Lemma 2.10. *Assume M_G is rational and $G = \mathbb{Z}_5$. Then the fixed-point set M^G consists of 5 isolated points of type (1, 2), plus possible 2-dimensional components $\{Y_i\}$ which are tori of self-intersection zero.*

Proof. We shall first apply the Lefschetz fixed point theorem and the weak version of the G -signature theorem. To this end, recall from the proof of Lemma 2.8(4), that $L(g, M) = 5$ and $\chi(M/G) = 4$. The latter easily implies that $Sign(M/G) = 0$. On the other hand, note that the signature defect for an isolated fixed point of type (1, 1), (1, 2) (the same as (1, 3)) and (1, 4) is $-4, 0, 4$ respectively (cf. [9]). Thus if we let x, y, z be the number of fixed points of type (1, 1), (1, 4) and (1, 2) respectively, then

$$x + y + z - \sum_i Y_i^2 = 5 \text{ and } -4x + 4y + \sum_i \frac{5^2 - 1}{3} Y_i^2 = 0.$$

Combining the two equations, we have $x + 3y + 2z = 10$. Note that $x + y + z$ must be odd, because $\sum_i Y_i^2 = \sum_i (2g_i - 2)$ is even. It follows that z must be odd. The solutions of x, y, z and $\sum_i Y_i^2$ are listed below:

- (1) $x = 8, y = 0, z = 1$, and $\sum_i Y_i^2 = 4$,
- (2) $x = 5, y = 1, z = 1$, and $\sum_i Y_i^2 = 2$,
- (3) $x = 2, y = 2, z = 1$, and $\sum_i Y_i^2 = 0$,
- (4) $x = 4, y = 0, z = 3$, and $\sum_i Y_i^2 = 2$,
- (5) $x = 1, y = 1, z = 3$, and $\sum_i Y_i^2 = 0$,
- (6) $x = 0, y = 0, z = 5$. and $\sum_i Y_i^2 = 0$.

Next we shall first eliminate cases (1),(2), and (4) where $\sum_i Y_i^2 \neq 0$ by computing with the G -index theorem for Dirac operators, using the formula for the Spin number $Spin(g, M)$ in Lemma 3.8 of [10]. To this end, we divide the isolated fixed points $\{q_j\}$ of each type and the fixed components $\{Y_i\}$ into two groups, I and II, according to the following rule: for type (1, 1), group I consists of fixed points q_j with $(a_j, b_j) = (1, 1)$ or $(4, 4)$ (and the rest are group II), for type (1, 4), a fixed point q_j belongs to group I if $(a_j, b_j) = (1, 4)$, and to group II if $(a_j, b_j) = (2, 3)$, and for type (1, 2), group I consists of fixed points q_j with $(a_j, b_j) = (1, 2)$ or $(3, 4)$, and group II consists of fixed points q_j with $(a_j, b_j) = (2, 4)$ or $(1, 3)$, and finally, for a fixed component Y_i , it belongs to group I if and only if $c_i = 1$ or 4. With this understood, the contribution to the Spin number $Spin(g, M)$ from an isolated fixed point q_j takes values as follows:

- $-\frac{1}{4} \csc^2 \frac{\pi}{5}$ if q_j is in group I and of type (1, 1),
- $-\frac{1}{4} \csc^2 \frac{2\pi}{5}$ if q_j is in group II and of type (1, 1),
- $\frac{1}{4} \csc^2 \frac{\pi}{5}$ if q_j is in group I and of type (1, 4),
- $\frac{1}{4} \csc^2 \frac{2\pi}{5}$ if q_j is in group II and of type (1, 4),
- $\frac{1}{4} \csc \frac{\pi}{5} \csc \frac{2\pi}{5}$ if q_j is in group I and of type (1, 2),
- $-\frac{1}{4} \csc \frac{\pi}{5} \csc \frac{2\pi}{5}$ if q_j is in group II and of type (1, 2),

and the contribution from a fixed component Y_i takes values as follows:

- $-\frac{1}{4} Y_i^2 \csc \frac{\pi}{5} \cot \frac{\pi}{5}$ if Y_i is in group I,
- $\frac{1}{4} Y_i^2 \csc \frac{2\pi}{5} \cot \frac{2\pi}{5}$ if Y_i is in group II.

If we denote by x_k, y_k, z_k , for $k = 1, 2$, the number of fixed points q_j belonging to group I, II, of type (1, 1), (1, 4), and (1, 2) respectively, and we denote by w_1, w_2 the sum of Y_i^2 for Y_i belonging to group I, II respectively, then the Spin number

$$Spin(g, M) = \frac{1}{4} \left(\sum_{k=1}^2 (y_k - x_k) \csc^2 \frac{k\pi}{5} + (-1)^k w_k \csc \frac{k\pi}{5} \cot \frac{k\pi}{5} + (z_1 - z_2) \csc \frac{\pi}{5} \csc \frac{2\pi}{5} \right).$$

Now the key observation is that for g^2 , the contributions to the Spin number for group I and group II switch values. It follows easily then, with the identities $\sum_{k=1}^2 \csc^2 \frac{k\pi}{5} = 4$

and $\sum_{k=1}^2 (-1)^k \csc \frac{k\pi}{5} \cot \frac{k\pi}{5} = -2$, that

$$\text{Spin}(g, M) + \text{Spin}(g^2, M) = \sum_{k=1}^2 (y_k - x_k - \frac{1}{2}w_k) = y - x - \frac{1}{2} \sum_i Y_i^2 = -\frac{5}{2} \sum_i Y_i^2.$$

(Note that $2 \sum_i Y_i^2 = x - y$ from the weak version of G -signature theorem.) On the other hand, recall that in the definition of Spin number

$$\text{Spin}(g, M) = d_0 + d_1\mu + d_2\mu^2 + d_3\mu^3 + d_4\mu^4, \text{ where } \mu = \exp(2\pi i/5),$$

one has $d_1 = d_4$, $d_2 = d_3$. As $\text{Spin}(g^2, M) = d_0 + d_1\mu^2 + d_2\mu^4 + d_3\mu + d_4\mu^3$, it follows easily that

$$-\frac{5}{2} \sum_i Y_i^2 = \text{Spin}(g, M) + \text{Spin}(g^2, M) = 2d_0 - d_1 - d_2.$$

Finally, $d_0 + d_1 + d_2 + d_3 + d_4 = \text{Ind } \mathbb{D} = -\text{Sign}(M)/8 = 0$. It follows immediately that $d_0 = -\sum_i Y_i^2$. The integer d_0 is the index of Dirac operator on the spin orbifold M/G , which equals 0 because $b_2^-(M/G) = b_2^+(M/G) = 1$ (see Fukumoto-Furuta [19], Corollary 1). This rules out the cases (1),(2),(4), where $d_0 = -\sum_i Y_i^2 \neq 0$.

The above calculation also shows that in the remaining cases, $d_0 = d_1 + d_2 = 0$. Moreover, note that each Y_i is a torus with $Y_i^2 = 0$. In particular, $w_1 = w_2 = 0$.

To deal with the remaining possibilities, we use the Mod p vanishing theorem of Seiberg-Witten invariants (cf. [35]). We shall first compute with the G -signature theorem (not the weak version). First, recall that $\chi(M/G) = 4$, so that $b_2^-(M/G) = 1$ is true. It follows that in Lemma 2.7, the angles $\theta_1 \neq \pm\theta_2$. Without loss of generality, we assume $\theta_1 = \frac{2\pi}{5}$ and $\theta_2 = \frac{4\pi}{5}$ in Lemma 2.7. With this we have

$$\text{Sign}(g, M) = 2(\cos \frac{6\pi}{5} - \cos \frac{-2\pi}{5}) = -2(\cos \frac{\pi}{5} + \cos \frac{2\pi}{5}).$$

On the other hand, we observe that the same division of fixed points or components into group I or group II works here too. With this understood, noting that $w_1 = w_2 = 0$, it follows easily from the G -signature theorem that

$$\text{Sign}(g, M) = \sum_{k=1}^2 (y_k - x_k) \cot^2 \frac{k\pi}{5} + (z_2 - z_1) \cot \frac{\pi}{5} \cot \frac{2\pi}{5}.$$

Next we observe that $\text{Sign}(g^2, M) = 2(\cos \frac{12\pi}{5} - \cos \frac{-4\pi}{5}) = -\text{Sign}(g, M)$, and moreover, for g^2 the contributions to the Sign number for group I and group II switch values. Taking the difference $\text{Sign}(g, M) - \text{Sign}(g^2, M)$, and using the identities (see Lemma 6.4 in [10])

$$\cot^2 \frac{\pi}{5} - \cot^2 \frac{2\pi}{5} = \csc^2 \frac{\pi}{5} - \csc^2 \frac{2\pi}{5} = 4 \cot \frac{\pi}{5} \cot \frac{2\pi}{5},$$

we obtain

$$\text{Sign}(g, M) = (2(y_1 - y_2 + x_2 - x_1) + (z_2 - z_1)) \cdot \cot \frac{\pi}{5} \cot \frac{2\pi}{5}.$$

Now finally, observing the identity $5 \cot \frac{\pi}{5} \cot \frac{2\pi}{5} = 2(\cos \frac{\pi}{5} + \cos \frac{2\pi}{5}) = -\text{Sign}(g, M)$, we obtain the following constraint

$$2(y_1 - y_2 + x_2 - x_1) + z_2 - z_1 = -5.$$

With these preparations, we examine the remaining cases (3), (5) in more detail. First consider case (3), where $x = y = 2, z = 1$. Observe that $y_1 - y_2 + x_2 - x_1$ is always even. It follows easily that $z_2 - z_1 = -1$ and $y_1 - x_1 = -(y_2 - x_2) = -1$ in this case. For case (5) where $x = y = 1, z = 3$, note that $y_1 - y_2 + x_2 - x_1 = \pm 2$. It follows that $z_2 - z_1 = -1$ and $y_1 - x_1 = -(y_2 - x_2) = -1$ as well.

Next we check this against the formula for the Spin number $\text{Spin}(g, M)$. To this end, we will use the following identities:

$$\csc \frac{\pi}{5} \csc \frac{2\pi}{5} = 4 \cot \frac{\pi}{5} \cot \frac{2\pi}{5}, \quad \csc \frac{\pi}{5} \cot \frac{\pi}{5} + \csc \frac{2\pi}{5} \cot \frac{2\pi}{5} = 6 \cot \frac{\pi}{5} \cot \frac{2\pi}{5},$$

which can be easily verified by direct calculation. Now with this understood, note that on the one hand, the definition of the Spin number gives

$$\text{Spin}(g, M) = \sum_{k=0}^4 d_k \mu^k = 2d_1(\cos \frac{\pi}{5} + \cos \frac{2\pi}{5}) = 5d_1 \cot \frac{\pi}{5} \cot \frac{2\pi}{5},$$

and on the other hand, we have from the formula in Lemma 3.8 of [10] that

$$\text{Spin}(g, M) = \frac{1}{4}(-\csc^2 \frac{\pi}{5} + \csc^2 \frac{2\pi}{5} + \csc \frac{\pi}{5} \csc \frac{2\pi}{5}) = 0.$$

It follows immediately that in cases (3), (5), we have $d_1 = 0$, and as a result, $d_k = 0$ for all $k = 0, 1, \dots, 4$.

With the preceding understood, recall that the condition in the Mod p vanishing theorem of Seiberg-Witten invariants (cf. [35]) is $2d_k < 1 - b_1^G + b_+^G$ for any $k = 0, 1, \dots, 4$, where $b_1^G = b_1(M/G) = 0$ and $b_+^G = b_2^+(M/G) = 1$ (note that since $b_1(M/G) = 0$, the fixed-point set J^G in the Mod p vanishing theorem consists of a single point, i.e., $[0]$, so the integers $\{k_j^l\}$ in the theorem are given by $\{d_k\}$ for any l , and the integer $d(c) = 0$). With $d_k = 0$ for all k , the above condition in the Mod p vanishing theorem is satisfied, so the Seiberg-Witten invariant for the canonical Spin^c structure (which is induced by a spin structure on M) vanishes (mod 5). But by Taubes' theorem [42], the Seiberg-Witten invariant equals 1, which is a contradiction. Hence cases (3), (5) are ruled out. This finishes the proof. \square

It remains to consider the case where G is of non-prime order, M_G is rational or ruled, but for any prime order subgroup H , M_H has torsion canonical class. Let n be the order of G . Then by Lemma 2.8, $n = 2^k 3^l$. We first note that $n \neq 6$. This is because if $n = 6$, then $G = \mathbb{Z}_2 \times \mathbb{Z}_3$, and with the assumption that for any prime order subgroup H , M_H has torsion canonical class, it follows easily that M_G has torsion canonical class as well, which is a contradiction. Consequently, either $k > 1$ or $l > 1$ in $n = 2^k 3^l$. Finally, note that for any nontrivial element $g \in G$, the angles θ_1, θ_2 in Lemma 2.7 are both nonzero. In particular, $b_1(M/G) = 0$, and M_G must be rational.

First of all, we have

Lemma 2.11. *Suppose $G = \mathbb{Z}_4$ and for the order 2 subgroup H , M_H has torsion canonical class. Then there are two possibilities:*

- (i) $b_2^+(M/G) = 3$, and the G -action has 4 isolated fixed points, all of type (1,3), and 12 isolated points of isotropy of order 2.
- (ii) M_G is rational, and the G -action has 4 isolated fixed points, all of type (1,1), and 12 isolated points of isotropy of order 2.

Proof. Fix a generator $g \in G$. It is easy to see that in Lemma 2.7, either $\theta_1 = -\theta_2$ or $\theta_1 = \theta_2$. So either $b_2^+(M/G) = 3$, $b_2^-(M/G) = 1$, or $b_2^+(M/G) = 1$, $b_2^-(M/G) = 3$. In any case, we have $\chi(M/G) = 6$. Finally, observe that $L(g, M) = 4$ in both cases.

On the other hand, by examining the action of G on M^H , which consists of 16 isolated points, and with $L(g, M) = 4$, it follows easily that M/G has 10 isolated singularities. With $\chi(M/G) = 6$, it follows that $\chi(M_G) > 12$, so that if M_G has torsion canonical class, then $b_2^+(M/G) = 3$ must be true. Case (i) follows immediately.

Suppose M_G is rational, and let x, y be the number of fixed points of type (1,1) and (1,3) respectively. Then note that each type (1,1) fixed point contributes a (-4) -sphere in M_G , which in turn contributes -1 to $c_1(K_{M_G})^2$. The other singular points of M/G contribute zero, hence $c_1(K_{M_G})^2 = -x$. On the other hand, note that $\chi(M_G) = \chi(M/G) + x + 3y + 6 = 12 + x + 3y$. As M_G is rational, $c_1(K_{M_G})^2 = 12 - \chi(M_G)$, which implies $y = 0$. Hence $x = 4$, and case (ii) follows. This finishes the proof. \square

Now finally, we have

Lemma 2.12. *Suppose M_G is rational, but for any prime order subgroup H , M_H has torsion canonical class. Then the order n of G must either 4 or 8. Moreover, if $n = 8$, then the G -action falls into one of the following two cases:*

- (i) the G -action has 2 isolated fixed points, all of type (1,3), 2 isolated points of isotropy of order 4 of type (1,3), and 12 isolated points of isotropy of order 2;
- (ii) the G -action has 2 isolated fixed points, all of type (1,5), 2 isolated points of isotropy of order 4 of type (1,1), and 12 isolated points of isotropy of order 2.

Proof. It is easy to check that if G contains an element g of order 9, 12, or 16, then for the angles θ_1, θ_2 of g in Lemma 2.7, the integrability conditions in Lemma 2.7(1) are violated. It follows easily that $k \leq 3$ and $l = 0$ in $n = 2^k 3^l$, i.e., $n = 4$ or 8.

With the preceding understood, suppose $n = 8$. We fix a generator g such that $\theta_1 = \frac{2\pi}{8}$, $\theta_2 = \frac{2\pi q}{8}$ in Lemma 2.7, where q is odd and $0 < q < 8$. We note that $q \neq 1$ or 7, for otherwise, the integrability conditions in Lemma 2.7(1) are violated. On the other hand, let H be the subgroup of order 4 generated by g^2 . Then by Lemma 2.11, there are two cases, (i) and (ii), as listed therein.

Suppose we are in case (i) of Lemma 2.11 where M_H has torsion canonical class. In this case, $b_2^+(M/H) = 3$, which easily implies that $q = 3$ in θ_2 . As a corollary, $L(g, M) = 2$, and $b_2^+(M/G) = b_2^-(M/G) = 1$, so that $\chi(M/G) = 4$. Examining the action of g on M^H ,

with $L(g, M) = 2$, it follows easily that M/G has 6 isolated singular points, where two of them have isotropy of order 8, one of isotropy of order 4, and three of isotropy of order 2. Now we determine the action of g at the two fixed points. We note that the minimal resolution of a singular point of order 8 of type (1, 3) in M_G is a pair of (-3) -spheres intersecting transversely and positively at one point. Its contribution to $c_1(K_{M_G})^2$ is easily seen to be -1 . All other types of singular points of M/G are Du Val singularities, so make zero contribution. On the other hand, the minimal resolution of a singular point of order 8 of type (1, 7) in M_G is a linear chain of seven (-2) -spheres, so its contribution to $\chi(M_G)$ is 7. With $c_1(K_{M_G})^2 = 12 - \chi(M_G)$, it follows easily that there cannot be any fixed point of g of type (1, 7). This finishes the discussion on case (i).

The analysis for case (ii) of Lemma 2.11, where M_H is rational, is completely analogous, hence omitted. This finishes the proof. \square

3. Symplectic surfaces in a rational 4-manifold

Let (X, ω) be a symplectic rational 4-manifold where $X = \mathbb{C}\mathbb{P}^2 \# N \overline{\mathbb{C}\mathbb{P}^2}$. We shall denote the canonical line bundle of (X, ω) by K_ω to indicate the dependence on ω . We also use K_X when the dependence on ω needs not to be emphasized.

We begin with the definition of reduced bases of (X, ω) . To this end, let \mathcal{E}_X be the set of classes in $H^2(X)$ which can be represented by a smooth (-1) -sphere, and let $\mathcal{E}_\omega := \{E \in \mathcal{E}_X \mid c_1(K_\omega) \cdot E = -1\}$. Then each class in \mathcal{E}_ω can be represented by a symplectic (-1) -sphere (cf. [29]); in particular, $\omega(E) > 0$ for any $E \in \mathcal{E}_\omega$.

Definition 3.1. A basis H, E_1, \dots, E_N of $H^2(X)$ is called a **reduced basis** of (X, ω) if the following are true:

- it has a standard intersection form, i.e., $H^2 = 1$, $E_i^2 = -1$ and $H \cdot E_i = 0$ for any i , and $E_i \cdot E_j = 0$ for any $i \neq j$;
- $E_i \in \mathcal{E}_\omega$ for each i , and moreover, if $N \geq 3$, the following area conditions are satisfied: $\omega(E_N) = \min_{E \in \mathcal{E}_\omega} \omega(E)$, and for any $2 < i < N$, $\omega(E_i) = \min_{E \in \mathcal{E}_i} \omega(E)$, where $\mathcal{E}_i := \{E \in \mathcal{E}_\omega \mid E \cdot E_j = 0 \ \forall j > i\}$ for any $i < N$;
- $c_1(K_\omega) = -3H + E_1 + \dots + E_N$.

Without loss of generality, we assume $\omega(E_1) \geq \omega(E_2)$. Then the following constraints on the symplectic areas are straightforward from Definition 3.1.

- $\omega(H) > 0$, and $\omega(E_i) \geq \omega(E_j)$ for any $i < j$;
- for any $i \neq j$, $H - E_i - E_j \in \mathcal{E}_\omega$, so that $\omega(H - E_i - E_j) > 0$;
- $\omega(H - E_i - E_j - E_k) \geq 0$ for any distinct i, j, k .

Reduced bases always exist, see [31] for more details. We remark that a reduced basis is not necessarily unique, however, the symplectic areas of its classes

$$(\omega(H), \omega(E_1), \dots, \omega(E_N))$$

uniquely determine the symplectic structure ω up to symplectomorphisms, cf. [25].

Secondly, we recall the following technical result concerning reduced bases, which will be used in Section 5.

Lemma 3.2. (cf. [25]) *Let $N \geq 2$. Then for any ω -compatible almost complex structure J , any class $E \in \mathcal{E}_\omega$ of minimal symplectic area can be represented by an embedded J -holomorphic sphere. In particular, for $N \geq 3$, the class E_N in a reduced basis H, E_1, \dots, E_N can be represented by a J -holomorphic (-1) -sphere for any J .*

With the preceding understood, we fix a reduced basis H, E_1, \dots, E_N of (X, ω) . Then for any $A \in H^2(X)$, we can write

$$A = aH - \sum_{i=1}^N b_i E_i, \text{ where } a, b_i \in \mathbb{Z}.$$

We first derive some general constraints on the coefficients a and b_i when A is represented by a connected, embedded symplectic surface, particularly, when A is the class of a symplectic $(-\alpha)$ -sphere for $\alpha > 1$. These constraints are consequences of the fundamental work of Li-Liu [30] and Li-Li [26] on symplectic rational 4-manifolds.

First of all, a few useful facts. For a generic ω -compatible almost complex structure J , the class H and any class $E \in \mathcal{E}_\omega$ can be represented by a J -holomorphic sphere (cf. [29]). In particular, this implies that for any $E \in \mathcal{E}_\omega$, where $E \neq E_i$, $1 \leq i \leq N$, the coefficients in $E = aH - \sum_{i=1}^N b_i E_i$ satisfy $a > 0$, $b_i \geq 0$ for all i by the positivity of intersection of J -holomorphic curves. Similarly, if $A = aH - \sum_{i=1}^N b_i E_i$ is the class of a connected, embedded symplectic surface with $A^2 \geq 0$, then by choosing an ω -compatible almost complex structure J such that the symplectic surface is J -holomorphic, we see easily that $a > 0$ and $b_i \geq 0$ for all i .

The situation is more subtle when $A^2 < 0$ and A is not a class in \mathcal{E}_ω . We begin with the following lemma.

Lemma 3.3. *Suppose $A = aH - \sum_{i=1}^N b_i E_i$ is the class of a connected, embedded symplectic surface of genus g .*

- (1) *If $a > 0$, then $b_i \geq 0$ for all i .*
- (2) *The a -coefficient of A satisfies the following inequality: $(a-1)(a-2) \geq 2g$, with “=” if and only if $b_i = 0$ or 1 for all i .*

Proof. For part (1), we begin by noting that the genus g of the symplectic surface representing A is given by the adjunction formula

$$g = \frac{1}{2}(A^2 + c_1(K_\omega) \cdot A) + 1.$$

Suppose to the contrary that $a > 0$ but $b_k < 0$ for some k . Then we consider the reflection $R(E_k)$ on $H^2(X)$ defined by the class E_k , where

$$R(E_k)\beta = \beta + 2(\beta \cdot E_k)E_k, \quad \forall \beta \in H^2(X).$$

If we let \tilde{A} be the image of A under $R(E_k)$ and write $\tilde{A} = \tilde{a}H - \sum_{i=1}^N \tilde{b}_i E_i$, then $\tilde{a} = a$, $\tilde{b}_k = -b_k > 0$, and $\tilde{b}_i = b_i$ for all $i \neq k$. It follows easily that $\tilde{A}^2 = A^2$ and

$c_1(K_\omega) \cdot \tilde{A} - c_1(K_\omega) \cdot A = 2\tilde{b}_k > 0$. Finally, since $R(E_k)$ is induced by an orientation-preserving diffeomorphism of X (cf. [26]), the class \tilde{A} is represented by a smoothly embedded, connected surface of genus g .

Now the condition $a > 0$ enters the argument. Pick a sufficiently small $\epsilon > 0$, and let $e := H - \sum_{i=1}^N \epsilon E_i \in H^2(X, \mathbb{R})$. Then $a > 0$ implies that $e \cdot \tilde{A} = a - \sum_{i=1}^N \epsilon \tilde{b}_i > 0$ for sufficiently small $\epsilon > 0$. On the other hand, we claim that e lies in the symplectic cone associated to the symplectic canonical class $c_1(K_\omega)$. To see this, we only need to verify that (i) $e^2 = 1 - N\epsilon^2 > 0$, which is obviously true when $\epsilon > 0$ is sufficiently small, and (ii) $e \cdot E > 0$ for any class $E \in \mathcal{E}_\omega$ (cf. [30]). To see (ii) is true, we write $E = uH - \sum_{i=1}^N v_i E_i$. Then $u^2 = \sum_i v_i^2 - 1$ and $u \geq 0$, and $e \cdot E = u - \epsilon \sum_i v_i$. If $E = E_l$ for some l , then $e \cdot E = \epsilon > 0$. If $u > 0$, then $e \cdot E = \sqrt{\sum_i v_i^2 - 1} - \epsilon \sum_i v_i > 0$ when $\epsilon > 0$ is sufficiently small. Hence the claim that e lies in the symplectic cone associated to the symplectic canonical class $c_1(K_\omega)$.

Now the fact that $e \cdot \tilde{A} > 0$ together with the fact that e lies in the symplectic cone associated to the symplectic canonical class $c_1(K_\omega)$ imply the following inequality on the symplectic genus $\eta(\tilde{A})$ of \tilde{A} (cf. [26], Definition 3.1, p. 130):

$$\eta(\tilde{A}) \geq \frac{1}{2}(\tilde{A}^2 + c_1(K_\omega) \cdot \tilde{A}) + 1.$$

On the other hand, the minimal genus is bounded from below by the symplectic genus (cf. [26], Lemma 3.2). Thus $g \geq \eta(\tilde{A})$, which implies that $c_1(K_\omega) \cdot A \geq c_1(K_\omega) \cdot \tilde{A}$, a contradiction. This finishes off part (1) of the lemma.

For part (2), the adjunction formula $A^2 + c_1(K_\omega) \cdot A + 2 = 2g$ gives

$$a^2 - \sum_{i=1}^N b_i^2 - 3a + \sum_{i=1}^N b_i + 2 = 2g.$$

With $\sum_{i=1}^N b_i^2 - \sum_{i=1}^N b_i = \sum_{i=1}^N b_i(b_i - 1) \geq 0$, we obtain easily $(a-1)(a-2) \geq 2g$, with “=” if and only if $b_i = 0$ or 1 for all i . This finishes off part (2), and the proof of the lemma is complete. \square

The following lemma deals with the case where the a -coefficient of A is negative.

Lemma 3.4. *Let $A = aH - \sum_{i=1}^N b_i E_i$ be the class of a connected, embedded symplectic surface of genus g such that $a < 0$. Then*

- (1) *the symplectic surface representing A must be a symplectic $(-\alpha)$ -sphere where $\alpha > 2$, i.e., $g = 0$ and $A^2 < -2$, and*
- (2) *the expression $A = aH - \sum_{i=1}^N b_i E_i$ must be in the following form:*

$$A = aH + (|a| + 1)E_{j_1} - E_{j_2} - \cdots - E_{j_s}, \text{ where } s = \alpha - 2|a|,$$

in particular, $2|a| < \alpha$. Moreover, $E_{j_1} = E_1$ and $\omega(E_1) > \omega(E_i)$ for any $i > 1$.

Proof. Let $b_i^- = \max(0, -b_i)$ and $b_i^+ = \max(0, b_i)$, and consider the class

$$\tilde{A} = |a|H - \sum_{i=1}^N (b_i^- + b_i^+)E_i.$$

Since $b_i^- = |b_i|$ when $b_i < 0$ and equals 0 otherwise, and $b_i^+ = b_i$ when $b_i > 0$ and equals 0 otherwise, it follows easily that \tilde{A} is the image of $-A$ under the action of the composition of the reflections $R(E_k)$, where k is running over the set of indices such that $b_k > 0$. In particular, \tilde{A} is represented by a smoothly embedded surface of genus g . As in the proof of the previous lemma, $e := H - \sum_{i=1}^N \epsilon E_i$ lies in the symplectic cone associated to the symplectic canonical class $c_1(K_\omega)$ when $\epsilon > 0$ is sufficiently small. Furthermore, as $a \neq 0$, we have $e \cdot \tilde{A} > 0$, so that

$$g \geq \eta(\tilde{A}) \geq \frac{1}{2}(\tilde{A}^2 + c_1(K_\omega) \cdot \tilde{A}) + 1,$$

where $\eta(\tilde{A})$ denotes the symplectic genus of \tilde{A} (cf. [26]). The above inequality is equivalent to

$$-3|a| + \sum_{i=1}^N (b_i^- + b_i^+) \leq -A^2 + 2g - 2.$$

On the other hand, the adjunction formula for A gives the equation $-3a + \sum_{i=1}^N b_i = -A^2 + 2g - 2$, which implies easily, when combined with the above inequality, that $\sum_{i=1}^N b_i^+ \leq -A^2 + 2g - 2$. It follows that $\sum_{i=1}^N b_i^- \leq 3|a|$.

Note that the adjunction formula $A^2 + c_1(K_\omega) \cdot A + 2 = 2g$ also implies easily that

$$2g + \sum_{i=1}^N b_i(b_i - 1) = a^2 - 3a + 2 = (a - 1)(a - 2) = (|a| + 1)(|a| + 2).$$

(The last equality is due to the assumption that $a < 0$.) It follows that $b_i^- \leq |a| + 1$ for each i , and moreover, if $b_i^- = |a| + 1$ for some i , then $g = 0$, and for any $j \neq i$, $b_j = 0$ or 1. With this understood, we shall next exclude the possibility that $b_i^- \leq |a|$ for any i , using the constraints of symplectic areas for a reduced basis.

Suppose to the contrary that $b_i^- \leq |a|$ for all i . Then we will write A as follows:

$$A = -(|a|H - \sum_{i=1}^N b_i^- E_i) - \sum_{i=1}^N b_i^+ E_i.$$

Since $b_i^- \leq |a|$ for all i and $\sum_{i=1}^N b_i^- \leq 3|a|$, the class $|a|H - \sum_{i=1}^N b_i^- E_i$ can be written as a sum of classes of the form H , $H - E_i$, $H - E_i - E_j$, or $H - E_i - E_j - E_k$, where distinct indices stand for distinct classes. Since all these classes have non-negative symplectic areas, it follows that $\omega(A) \leq 0$, which is a contradiction. Hence

$$A = aH + (|a| + 1)E_{j_1} - E_{j_2} - \cdots - E_{j_s}, \text{ where } s = -A^2 - 2|a|.$$

In particular, $A^2 = -2|a| - s < -2$, and moreover, $2|a| < -A^2$. Finally, if there is a class E_i such that $\omega(E_{j_1}) \leq \omega(E_i)$, then

$$\omega(aH + (|a| + 1)E_{j_1}) \leq -(|a| - 1)\omega(H - E_{j_1}) - \omega(H - E_{j_1} - E_i) < 0,$$

which implies that $\omega(A) < 0$. It follows easily that $E_{j_1} = E_1$, and $\omega(E_1) > \omega(E_i)$ for any $i > 1$. This finishes the proof. \square

In the rest of this section, we shall be focusing on the possible homological expressions of a symplectic $(-\alpha)$ -sphere, in particular, for $\alpha = 2$ and 3 . The constraints in Lemmas 3.3 and 3.4 allow us to easily determine all the possible expressions of the class A of a symplectic $(-\alpha)$ -sphere in terms of the reduced basis H, E_1, \dots, E_N when the a -coefficient of A is relatively small, say $a \leq 3$.

To this end, write $A = aH - \sum_{i=1}^N b_i E_i$, and observe that in the following equation

$$\sum_{i=1}^N b_i(b_i - 1) = a^2 - 3a + 2 = (a - 1)(a - 2)$$

which is satisfied by the coefficients a, b_i of A , the left-hand side is always a nonnegative, even integer. In particular, when $a = 1$ or 2 , b_i must be either 0 or 1 . For $a = 0$, the area condition $\omega(A) > 0$ implies that exactly one of the b_i 's equals -1 and the rest are either 0 or 1 . For $a = 3$, exactly one of the b_i 's equals either 2 or -1 , however, the latter possibility is ruled out by Lemma 3.3. The rest of the b_i 's are either 0 or 1 . We summarize the discussions in the following

Observation: *Let $A = aH - \sum_{i=1}^N b_i E_i$ be the class of a symplectic $(-\alpha)$ -sphere where $a \leq 3$. Then A must take the following expression*

$$A = aH - (a - 1)E_{j_1} - E_{j_2} - \dots - E_{j_{2a+\alpha}}.$$

If $a > 3$ but is small, the possibilities for the values of b_i can be easily determined. However, when a is large, though there are only finitely many solutions for the b_i 's for a fixed value of a , it is in general impossible to determine all the possible solutions for the b_i 's. Finally, note that there is no a priori upper bound for the a -coefficient in terms of N and α .

With this understood, the following technical lemma plays a key role in determining the expression of A when the a -coefficient is large, for the case where $\alpha = 2$ or 3 .

Lemma 3.5. *Let $A = aH - \sum_{i=1}^N b_i E_i$ be any class which satisfies*

$$A^2 = -\alpha, \quad c_1(K_\omega) \cdot A = \alpha - 2, \quad \text{where } \alpha = 2, 3.$$

If $a > 3$, then there are at least $\alpha + 7$ terms in A with non-zero b_i -coefficient.

Proof. We begin by recalling a reduction procedure useful in this kind of problems. For any distinct indices i, j, k , we set $H_{ijk} := H - E_i - E_j - E_k$. Then H_{ijk} satisfies the following conditions:

$$H_{ijk}^2 = -2, \quad c_1(K_\omega) \cdot H_{ijk} = 0, \quad \text{and } \omega(H_{ijk}) \geq 0.$$

Furthermore, there is a reflection R_{ijk} on $H^2(M)$ associated to H_{ijk} , which is defined by the following formula:

$$R_{ijk}(A) := A + (A \cdot H_{ijk})H_{ijk}, \quad \forall A \in H^2(X).$$

To ease the notation, let $\tilde{A} := R_{ijk}(A)$. Then it is easy to see that

$$\tilde{A}^2 = A^2, \quad c_1(K_\omega) \cdot \tilde{A} = c_1(K_\omega) \cdot A, \quad \text{and} \quad \tilde{A} \cdot H_{ijk} = -A \cdot H_{ijk}.$$

The last equality implies that

$$A = R_{ijk}(\tilde{A}) = \tilde{A} + (\tilde{A} \cdot H_{ijk})H_{ijk}.$$

Finally, note that the operation R_{ijk} will decrease (resp. increase) the a -coefficient in the expression of A if and only if $A \cdot H_{ijk} < 0$ (resp. $A \cdot H_{ijk} > 0$), where $A \cdot H_{ijk} = a - (b_i + b_j + b_k)$. See [31] or [3] for further discussions on this reduction procedure.

With the preceding understood, let $A = aH - \sum_{i=1}^N b_i E_i$ be any class satisfying the conditions in the lemma, i.e., $A^2 = -\alpha$, $c_1(K_\omega) \cdot A = \alpha - 2$, where $\alpha = 2, 3$, and assume $a > 3$. Suppose to the contrary that A has no more than $\alpha + 6$ terms in the expression with non-zero b_i -coefficient.

Claim: *There are distinct indices i, j, k such that (i) b_i, b_j, b_k are positive, and (ii) $A \cdot H_{ijk} = a - (b_i + b_j + b_k) < 0$.*

Proof of Claim: We shall prove by contradiction. But first, we observe that there are at least 3 terms in A with the b_i -coefficient positive. To see this, note that the conditions $A^2 = -\alpha$, $c_1(K_\omega) \cdot A = \alpha - 2$ easily imply that

$$\sum_{i=1}^N b_i(b_i - 1) = (a - 1)(a - 2).$$

Since $a > 3$, it follows that for any i , if $b_i > 0$, then $b_i \leq a - 1$ must be true. Therefore, if there were at most 2 terms in A with the b_i -coefficient positive, then $\sum_{i=1}^N b_i \leq 2(a - 1)$, which contradicts $-3a + \sum_{i=1}^N b_i = c_1(K_\omega) \cdot A = \alpha - 2$.

With the preceding understood, suppose the claim is not true. Then it follows that $b_i + b_j + b_k \leq a$ holds true for any distinct indices i, j, k , where b_i, b_j, b_k are not necessarily positive or non-zero. Consider first the case where $\alpha = 2$. Pick a b_i -coefficient, say b_s , such that $b_s > 0$. Then we have

$$\sum_{i=1}^N b_i = \sum_{i=1}^N b_i + b_s - b_s \leq 3a - b_s \leq 3a - 1,$$

which is a contradiction to $-3a + \sum_{i=1}^N b_i = \alpha - 2 = 0$. A similar argument also confirms the claim for $\alpha = 3$. This finishes off the proof of the claim.

Now going back to the proof of the lemma, we pick the indices i, j, k given by the claim above, and perform the operation R_{ijk} to reduce A to $\tilde{A} := R_{ijk}(A)$, which continues to

obey the conditions on A , i.e.,

$$\tilde{A}^2 = -\alpha \text{ and } c_1(K_\omega) \cdot \tilde{A} = \alpha - 2.$$

Set $c := b_i + b_j + b_k - a$. We shall derive an upper bound on c . To this end, note that

$$b_i(b_i - 1) + b_j(b_j - 1) + b_k(b_k - 1) \leq (a - 1)(a - 2).$$

Using the inequality $3(b_i^2 + b_j^2 + b_k^2) \geq (b_i + b_j + b_k)^2$, we obtain

$$\frac{b_i + b_j + b_k}{3} \left(\frac{b_i + b_j + b_k}{3} - 1 \right) \leq \frac{b_i^2 + b_j^2 + b_k^2}{3} - \frac{b_i + b_j + b_k}{3} \leq \frac{1}{3}(a - 1)(a - 2).$$

Since $a > 3$, this gives $\frac{b_i + b_j + b_k}{3} - 1 \leq \frac{1}{\sqrt{3}}(a - 2)$, and consequently, $c \leq \sqrt{3}(a - 2) + 3 - a$.

It follows that the a -coefficient of \tilde{A} , denoted by \tilde{a} , will be at least 2, because

$$\tilde{a} = a - c \geq (2 - \sqrt{3})a + 2\sqrt{3} - 3 \geq (2 - \sqrt{3}) \times 4 + 2\sqrt{3} - 3 = 5 - 2\sqrt{3} > 1.$$

Finally, because b_i, b_j, b_k are non-zero, this operation does not introduce any new terms with non-zero b_i -coefficient, so \tilde{A} continues to have no more than $\alpha + 6$ terms in its expression with non-zero b_i -coefficient.

After finitely many steps, we will arrive at a class, continuing to be denoted by \tilde{A} , whose a -coefficient lies in the range $2 \leq \tilde{a} \leq 3$. We may assume \tilde{A} is the first class whose a -coefficient lies in this range; in particular, the a -coefficient of the previous class, denoted by A , obeys $a > 3$. We shall examine \tilde{A} according to the value of \tilde{a} below. To this end, we denote by \tilde{b}_i the b_i -coefficients of \tilde{A} . Then it is helpful to observe that $\tilde{b}_i + \tilde{b}_j + \tilde{b}_k - \tilde{a} = -c < 0$, because of the relation $\tilde{A} \cdot H_{ijk} = -A \cdot H_{ijk}$.

Suppose $\tilde{a} = 2$. Then $\tilde{A} = 2H - E_{j_1} - \cdots - E_{j_{\alpha+4}}$. The condition $a > 3$ requires that in this case we must have $c \geq 2$, and consequently, $\tilde{b}_i + \tilde{b}_j + \tilde{b}_k - \tilde{a} = -c \leq -2$. Since the b_i -coefficients of \tilde{A} are non-negative and $\tilde{a} = 2$, it follows that $\tilde{b}_i = \tilde{b}_j = \tilde{b}_k = 0$ and $c = \tilde{a} = 2$ must be true. In particular, the indices i, j, k are not appearing in the expression of \tilde{A} , and it follows that A takes the form

$$A = 4H - 2E_i - 2E_j - 2E_k - E_{j_1} - \cdots - E_{j_{\alpha+4}}, \text{ where } j_s \neq i, j, k,$$

which has $\alpha + 7$ terms with non-zero b_i -coefficient, contradicting the assumption.

Suppose $\tilde{a} = 3$. Then the expressions for \tilde{A} are

$$\tilde{A} = 3H - 2E_{j_1} - \cdots - E_{j_{\alpha+6}} \text{ or } \tilde{A} = 3H - E_{j_1} - \cdots - E_{j_{\alpha+8}} + E_{j_{\alpha+9}}.$$

The latter case is ruled out immediately as \tilde{A} has $\alpha + 9$ many terms with non-zero b_i -coefficient. For the former case, we note that with $\tilde{a} = 3$, $c \geq 1$, $\tilde{b}_i + \tilde{b}_j + \tilde{b}_k \leq 3 - 1 = 2$. It follows easily that the following are the only possibilities for $\tilde{b}_i, \tilde{b}_j, \tilde{b}_k$:

$$(\tilde{b}_i, \tilde{b}_j, \tilde{b}_k) = (2, 0, 0), (1, 1, 0), (1, 0, 0), (0, 0, 0).$$

With this understood, note that $b_l = \tilde{b}_l + c$ for $l = i, j, k$. Since at least one of $\tilde{b}_i, \tilde{b}_j, \tilde{b}_k$ is zero, it follows that the number of terms in the expression of A with non-zero b_i -coefficient is at least 1 more than the number of terms with non-zero b_i -coefficient in \tilde{A} .

Now \tilde{A} has $\alpha + 6$ many terms of non-zero b_i -coefficient, so A must have at least $\alpha + 7$ many terms, which is a contradiction. This completes the proof of the lemma. \square

With the preceding understood, we now state a lemma which is of fundamental importance for our project on symplectic Calabi-Yau 4-manifolds. The key observation is that, when combined with Lemma 3.5, the area condition $\omega(A) < -c_1(K_\omega) \cdot [\omega]$ will give severe constraints on the a , b_i -coefficients of A ; in particular, it implies an upper bound on the a -coefficient of A in terms of N for the case of $\alpha = 2$ or 3.

Lemma 3.6. *Let $A = aH - \sum_{i=1}^N b_i E_i$ be the class of a symplectic $(-\alpha)$ -sphere where $\alpha = 2$ or 3, such that $\omega(A) < -c_1(K_\omega) \cdot [\omega]$. Then A must be of the following form*

$$A = aH - (a-1)E_{j_1} - E_{j_2} - \cdots - E_{j_{2a+\alpha}}.$$

In particular, $a \leq \frac{1}{2}(N - \alpha)$.

Proof. It suffices to only consider the situation where $a > 3$. First, note that by Lemma 3.3, $b_i \geq 0$ for all i . If we let $b_i^+ = \max(1, b_i)$, then $b_i^+ = b_i$ when $b_i > 0$ and $b_i^+ = 1$ when $b_i = 0$. Next we observe that $-3a + \sum_{i=1}^N b_i = \alpha - 2$. Let n be the number of b_i 's which are non-zero. Then because $n \geq \alpha + 7$ by Lemma 3.5, we have

$$\sum_{i=1}^N (b_i^+ - 1) = \sum_{i=1}^N b_i - n = 3a + \alpha - 2 - n \leq 3(a-3).$$

On the other hand, we claim that there must be one b_i such that $b_i = a - 1$. Suppose to the contrary that this is not true. Then for each i , $b_i^+ - 1 \leq a - 3$ must be true. With this understood, note that the class $(a-3)H - \sum_{i=1}^N (b_i^+ - 1)E_i$ can be written as a sum of classes of the form H , $H - E_i$, $H - E_i - E_j$, or $H - E_i - E_j - E_k$, where distinct indices stand for distinct classes, because $\sum_{i=1}^N (b_i^+ - 1) \leq 3(a-3)$, and for each i , $b_i^+ - 1 \leq a - 3$. However, observe that we can write

$$A = -c_1(K_\omega) + (a-3)H - \sum_{i=1}^N (b_i^+ - 1)E_i + \sum_{i=1}^N \max(0, 1 - b_i)E_i,$$

from which it follows easily that $\omega(A) \geq -c_1(K_\omega) \cdot [\omega]$, contradicting the area assumption in the lemma. Hence the claim.

Now we observe that in the equation $\sum_{i=1}^N b_i(b_i - 1) = (a-1)(a-2)$ which is satisfied by the a , b_i -coefficients of A , if $b_i = a - 1$ for some i , then the rest of the b_i 's are all equal to either 0 or 1. With this understood, the equation $-3a + \sum_{i=1}^N b_i = \alpha - 2$ implies that the number of b_i 's equaling 1 must be $2a + \alpha - 1$. It follows immediately that A must take the expression

$$A = aH - (a-1)E_{j_1} - E_{j_2} - \cdots - E_{j_{2a+\alpha}}.$$

\square

We remark that if A is the class of a symplectic $(-\alpha)$ -sphere whose a -coefficient satisfies $a > 3$ and there are at least $\alpha + 7$ terms in the expression of A having non-zero b_i -coefficients, then the same proof shows that the condition $\omega(A) < -c_1(K_\omega) \cdot [\omega]$ would imply that A also takes the special expression in Lemma 3.6. However, in general it is not true that there are always at least $\alpha + 7$ terms having non-zero b_i -coefficients in the expression of a symplectic $(-\alpha)$ -sphere. For example, the following class, which has only 10 terms with non-zero b_i -coefficients, can be represented by a symplectic (-4) -sphere (cf. [13]): $A = 6H - 2E_{j_1} - 2E_{j_2} - \cdots - 2E_{j_{10}}$.

4. Nonexistence of certain symplectic configurations

In this section, we give several results concerning nonexistence of certain configurations of symplectic surfaces in rational 4-manifolds. To prove these results, we examine the possible homological expressions of the components in the configurations in a certain reduced basis, using the constraints established in Section 3, and show that the configurations can not exist even at the homology level. These nonexistence results will then be used in Section 5 to eliminate several possibilities of the fixed-point set structure obtained in Section 2 concerning the 2-dimensional fixed components, which have resisted all the known obstructions available so far.

First, we prove a lemma which allows us to impose certain auxiliary area conditions.

Lemma 4.1. *Let (X, ω) be a symplectic 4-manifold, and let $D = \sqcup_i D_i \subset X$, where each $D_i = \cup_j C_{ij}$ is a configuration of symplectic surfaces intersecting transversely and positively according to a negative definite plumbing graph Γ_i . Then for any given collection of positive real numbers $\{a_{ij}\}$, there exists a $\delta_0 > 0$, such that for any choice of $\{\delta_i\}$ where $0 < \delta_i < \delta_0$, there is a symplectic 4-manifold $(\tilde{X}, \tilde{\omega})$ with $D \subset \tilde{X}$, which has the following significance:*

- $D = \sqcup_i D_i$ is a set of symplectic configurations in $(\tilde{X}, \tilde{\omega})$, and there is a diffeomorphism $\psi : \tilde{X} \rightarrow X$ which is identity on D , such that $\psi^* c_1(K_\omega) = c_1(K_{\tilde{\omega}})$,
- the $\tilde{\omega}$ -symplectic area of each surface C_{ij} equals $\delta_i a_{ij}$, i.e., $\tilde{\omega}(C_{ij}) = \delta_i a_{ij}$.

Proof. First of all, we may assume without loss of generality that the intersections of C_{ij} are ω -orthogonal, because we can always slightly perturb the symplectic surfaces to achieve this (cf. [22]). With this understood, since the plumbing graph Γ_i is negative definite, each configuration D_i has a regular neighborhood U_i such that $L_i := \partial U_i$ is a convex contact boundary (in the strong sense), cf. [20]. Furthermore, by a theorem of Park and Stipsicz [36], the contact structure on L_i is the Milnor fillable contact structure (cf. [4]). We denote by α_i the contact form on L_i , where $\omega = d\alpha_i$ on L_i . It is clear that we can arrange so that $\{U_i\}$ are disjoint in X .

Now for any given collection of positive real numbers $\{a_{ij}\}$, let (U'_i, ω'_i) be a convex regular neighborhood of $D_i = \cup_j C_{ij}$ constructed in [20] such that $\omega'_i(C_{ij}) = a_{ij}$. Fixing an identification $\partial U'_i = L_i$, we let α'_i denote the contact form on L_i such that $\omega'_i = d\alpha'_i$ on L_i . Then by [36], $\alpha'_i = e^{f_i} \alpha_i$ for some smooth function f_i on L_i . With this understood, we set $\delta_0 > 0$ by the condition $\delta_0^{-1} := \max_i \{\sup_{x \in L_i} e^{f_i(x)}\}$.

Given any $\{\delta_i\}$ where $0 < \delta_i < \delta_0$, we set $C_i := \log \delta_i$. Then it is easy to see that $C_i + f_i(x) < 0$ for any $x \in L_i$. With this understood, we let

$$W_i := \{(x, t) \in L_i \times \mathbb{R} \mid C_i + f_i(x) \leq t \leq 0\},$$

given with the symplectic structure $d(e^t \alpha_i)$. We define $(\tilde{U}_i, \tilde{\omega}_i)$ to be the symplectic 4-manifold obtained by gluing $(U'_i, \delta_i \omega'_i)$ to W_i via the contactomorphism sending $x \in L_i = \partial U'_i$ to $(x, C_i + f_i(x)) \in W_i$. Note that each $(\tilde{U}_i, \tilde{\omega}_i)$ has a convex contact boundary $\partial \tilde{U}_i = L_i$ where $\tilde{\omega}_i = d\alpha_i$ on L_i . With this understood, we define $(\tilde{X}, \tilde{\omega})$ to be the symplectic 4-manifold obtained by removing $\cup_i U_i$ from X and then gluing back $\cup_i \tilde{U}_i$ along $\cup_i L_i$. It is easy to see that there is a diffeomorphism $\psi : \tilde{X} \rightarrow X$ which is identity on D , such that $\psi^* c_1(K_\omega) = c_1(K_{\tilde{\omega}})$, and the $\tilde{\omega}$ -symplectic area of each surface C_{ij} equals $\delta_i a_{ij}$. This finishes the proof of the lemma. \square

The second lemma contains two useful observations. In particular, the first observation implies that in a configuration of symplectic surfaces there is at most one symplectic sphere with negative a -coefficient.

Lemma 4.2. (1) *Let A_1, A_2 be the classes of two symplectic spheres whose a -coefficients are negative. Then $A_1 \cdot A_2 < 0$.*

(2) *Let $B = aH - \sum_{i=1}^N b_i E_i$ be a nonzero class satisfying $B^2 = c_1(K_\omega) \cdot B = 0$. If $a \geq 0$, then $a \geq 3$. Moreover, for $a = 3$, the following are the only possible expressions for B :*

$$B = 3H - E_{j_1} - \cdots - E_{j_9}.$$

Proof. For (1), let a_1, a_2 be the a -coefficients of A_1, A_2 respectively, which are negative by assumption. Then it follows easily from the expression in Lemma 3.4 that

$$A_1 \cdot A_2 \leq a_1 a_2 - (|a_1| + 1)(|a_2| + 1) = -(|a_1| + |a_2| + 1) < 0.$$

For (2), we first note that $B \neq 0$ and $B^2 = 0$ imply easily that $a \neq 0$ in B . With this understood, we note that the conditions $B^2 = c_1(K_\omega) \cdot B = 0$ are equivalent to

$$a^2 - \sum_{i=1}^N b_i^2 = -3a + \sum_{i=1}^N b_i = 0.$$

It follows easily that $a(a-3) = \sum_{i=1}^N b_i(b_i-1) \geq 0$. With the assumption that $a \geq 0$, it follows immediately that $a \geq 3$. Moreover, if $a = 3$, each b_i must be either 0 or 1, from which the expression of B follows easily. This finishes the proof of the lemma. \square

With these preparations, we now prove the aforementioned nonexistence results.

Proposition 4.3. *Let $\{B_i\}$ be a nonempty set of disjoint symplectic surfaces in $X = \mathbb{C}\mathbb{P}^2 \# 10\overline{\mathbb{C}\mathbb{P}^2}$, where there is at most one spherical component, and F_1, F_2, F_3 be a disjoint union of symplectic (-3) -spheres in the complement of B_i , such that*

$$c_1(K_X) = -\frac{2}{3} \sum_i B_i - \frac{1}{3} (F_1 + F_2 + F_3).$$

Suppose $F_{4,1}, F_{4,2}$ are a pair of symplectic (-2) -spheres in the complement of B_i and F_1, F_2, F_3 , such that $F_{4,1}, F_{4,2}$ intersect transversely and positively at one point. Then $\{B_i\}$ must consist of one component which is a torus.

Proof. First of all, since $c_1(K_X)$ is represented by F_1, F_2, F_3 and B_i , which are disjoint from the two (-2) -spheres $F_{4,1}, F_{4,2}$, it is clear that, by Lemma 4.1, we may assume without loss of generality that the following area condition holds:

$$\omega(F_{4,1}) = \omega(F_{4,2}) < -c_1(K_X) \cdot [\omega].$$

Then by Lemma 3.6, the a -coefficients of $F_{4,1}, F_{4,2}$ lie in the range $0 \leq a \leq 4$, and moreover, their classes take the special form in Lemma 3.6. Furthermore, again by Lemma 4.1, we can also arrange so that F_1, F_2, F_3 have the same area, which is sufficiently small, so that $\omega(F_k) < \omega(B_i)$ for each i, k .

With this understood, we next derive some basic information about B_i . First, $c_1(K_X) = -\frac{2}{3} \sum_i B_i - \frac{1}{3}(F_1 + F_2 + F_3)$ implies that $c_1(K_X)^2 = \frac{4}{9} \sum_i B_i^2 - 1$, and with $X = \mathbb{C}\mathbb{P}^2 \# 10\mathbb{C}\mathbb{P}^2$, it follows easily that $\sum_i B_i^2 = 0$. On the other hand, if we denote by g_i the genus of B_i , then the adjunction formula applied to each B_i gives us $-\frac{2}{3}B_i^2 + B_i^2 = 2g_i - 2$, which is equivalent to $B_i^2 = 6(g_i - 1)$ for each i . In particular, $B_i^2 < 0$ if and only if B_i is spherical, hence by our assumption, there is at most one component B_i with $B_i^2 < 0$, and such a component must be a (-6) -sphere.

With the preceding understood, we observe that the proposition follows readily if there is no B_i such that $B_i^2 < 0$. Under this condition, it is easy to see that each B_i must be a torus. To see that there is only one component in $\{B_i\}$, we note that by Lemma 4.2(2), the a -coefficient of each B_i is at least 3. On the other hand, each B_i contributes at least $\frac{2}{3} \times 3 = 2$ to the a -coefficient of $-c_1(K_X)$, which equals 3, while the total contribution from F_1, F_2, F_3 to the a -coefficient of $-c_1(K_X)$ is at least $\frac{1}{3} \times (-1) = -\frac{1}{3}$ by Lemmas 3.4(2) and 4.2(1). Hence the claim. Therefore, it boils down to show that there is no B_i such that $B_i^2 < 0$.

Suppose to the contrary that there is a component, call it B_1 , such that $B_1^2 < 0$. Since $b_2^+(X) = 1$, there must be exactly one B_i , call it B_2 , such that $B_2^2 > 0$, and the rest of the B_i 's have $B_i^2 = 0$ hence are tori if there is any. Furthermore, as B_1 is a (-6) -sphere, B_2 must be a genus-2 surface with $B_2^2 = 6$. By a similar argument analyzing the contributions of B_i to the a -coefficient of $-c_1(K_X)$, using Lemmas 3.3 and 4.2, it follows easily that B_1, B_2 are the only components in $\{B_i\}$. Finally, note that the sum of the a -coefficients of F_1, F_2, F_3 is at most 3.

Case (1): Suppose $a = -2$ in B_1 . Then by Lemma 3.4(2), we can write $B_1 = -2H + 3E_1 - E_p$ for some E_p . We consider the possibilities for the classes of F_1, F_2, F_3 . Note that by Lemma 4.2(1), $a \geq 0$ in F_1, F_2, F_3 . Consequently, $a \leq 3$ in F_1, F_2, F_3 . Suppose $a = 3$ in one of them, say F_1 . Then $B_1 \cdot F_1 = 0$ easily implies that

$$F_1 = 3H - 2E_1 - E_{i_1} - \cdots - E_{i_s},$$

where E_p does not show up in F_1 . But this is a contradiction:

$$\omega(F_1 - B_1) = \omega(5H - 5E_1 - E_{i_1} - \cdots - E_{i_8}) + \omega(E_p) > 0$$

as the class $5H - 5E_1 - E_{i_1} - \cdots - E_{i_8}$ can be written as a sum of classes of the form $H - E_i - E_j$ and $H - E_i - E_j - E_k$, which all have nonnegative areas. If $a = 2$ in F_1 , then one can check easily that $F_1 \cdot B_1 < 0$ is always true. If $a = 1$ in F_1 , then F_1 must take the form $F_1 = H - E_1 - E_p - E_q - E_r$ for some E_q, E_r . In particular, since F_2, F_3 are disjoint from F_1 , we must have $a \neq 1$ in F_2, F_3 . It follows that both F_2, F_3 should have $a = 0$. Since the sum of the a -coefficients of F_1, F_2, F_3 is always an odd number, it follows that the sum must equal 1. Consequently, we must have

$$F_1 = H - E_1 - E_p - E_q - E_r,$$

and both F_2, F_3 have zero a -coefficients. It follows that the sum of the a -coefficients of B_1, B_2 equals 4, so that $a = 6$ in B_2 .

To proceed further, we write $B_2 = 6H - \sum_{i=1}^{10} b_i E_i$. Note that B_2 has genus 2, so that $c_1(K_X) \cdot B_2 + B_2^2 = 2 \times 2 - 2 = 2$. With $B_2^2 = 6$, this implies that

$$-18 + \sum_{i=1}^{10} b_i + 6 = 2, \quad 36 - \sum_{i=1}^{10} b_i^2 = 6.$$

Consequently, $\sum_{i=1}^{10} b_i(b_i - 1) = 16$, and as a result, note that $b_i \leq 4$ for each i . On the other hand, $B_2 \cdot B_1 = 0$, which gives $-12 + 3b_1 - b_p = 0$. Since $b_1 \leq 4$, we must have $b_p = 0$ and $b_1 = 4$. Then $\sum_{i=2}^{10} b_i(b_i - 1) = 16 - 4 \times 3 = 4$ implies that in b_2, \dots, b_{10} , there are exactly two of them equaling 2; the rest are either 1 or 0. With $F_1 \cdot B_2 = 0$, it follows easily that

$$B_2 = 6H - 4E_1 - E_q - E_r - 2E_{i_1} - 2E_{i_2} - E_{i_3} - \cdots - E_{i_6}.$$

With this understood, we note that

$$2(B_1 + B_2) + F_1 = 9H - 3E_1 - 3E_p - 3E_q - 3E_r - 4E_{i_1} - 4E_{i_2} - 2E_{i_3} - \cdots - 2E_{i_6}.$$

This implies that without loss of generality,

$$F_2 = E_{i_1} - E_{i_3} - E_{i_4}, \quad F_3 = E_{i_2} - E_{i_5} - E_{i_6}.$$

With the preceding understood, let A be the class of any of the (-2) -spheres $F_{4,1}, F_{4,2}$. Then recall that because of the area condition we imposed at the beginning, the a -coefficient of A lies in the range $0 \leq a \leq 4$, and its expression must be of the form specified in Lemma 3.6. With this understood, if $a = 4$ in A , then

$$A = 4H - 3E_{j_1} - E_{j_2} - \cdots - E_{j_{10}},$$

containing all 10 E_i -classes. It is easy to see that $A \cdot F_2 \neq 0$, which rules out this possibility. If $a = 3$ in A , then we can write $A = 3H - 2E_{j_1} - E_{j_2} - \cdots - E_{j_8}$. Then $B_1 \cdot A = 0$ implies that $E_{j_1} = E_1$ must be true, and E_p is not contained in A . With this understood, $A \cdot F_2 = A \cdot F_3 = 0$ implies that one of the E_i -classes in each pair $(E_{i_3}, E_{i_4}), (E_{i_5}, E_{i_6})$ can not appear in A . Together with E_p , there are 3 E_i -classes not contained in A , which

is a contradiction as there are only 10 E_i -classes in total. If $a = 2$ in A , then it is easy to see that $A \cdot B_1 < 0$. Hence we must have either $a = 1$ or $a = 0$ in A . If $a = 1$ in A , then $A \cdot B_1 = 0$ implies that A contains both E_1 and E_p . But this leads to $A \cdot F_1 < 0$, which is a contradiction. This shows that $A = E_s - E_t$ for some E_s, E_t . It is easy to check that there are only 3 possibilities: $E_q - E_r$, $E_{i_3} - E_{i_4}$, and $E_{i_5} - E_{i_6}$. We just showed that the classes of $F_{4,1}$, $F_{4,2}$ must be from the three classes above. But they mutually intersect trivially with each other, contradicting the fact that $F_{4,1} \cdot F_{4,2} = 1$. Hence Case (1) is ruled out.

Case (2): Suppose $a = -1$ in B_1 . Then $B_1 = -H + 2E_1 - E_x - E_y - E_z$ for some E_x, E_y, E_z . Again by Lemma 4.2(1), $a \geq 0$ in F_1, F_2, F_3 . If $a = 3$ in F_1 , then it is easy to see from $F_1 \cdot B_1 = 0$, that E_1 must appear in F_1 with coefficient -2 , and two of E_x, E_y, E_z can not appear in F_1 . But F_1 contains 9 E_i -classes and there are totally 10 E_i -classes, which is a contradiction. If $a = 2$ in F_1 , then $F_1 \cdot B_1 = 0$ implies that $F_1 = 2H - E_1 - E_{i_1} - \dots - E_{i_6}$. But this gives a contradiction

$$\omega(F_1 - B_1) = \omega(3H - 3E_1 - E_{i_1} - \dots - E_{i_6}) + \omega(E_x + E_y + E_z) > 0,$$

as the class $3H - 3E_1 - E_{i_1} - \dots - E_{i_6}$ can be written as a sum of classes of the form $H - E_i - E_j - E_k$, which all have nonnegative areas. Consequently, $a = 1$ in F_1 and $a = 0$ in F_2, F_3 , where

$$F_1 = H - E_1 - E_x - E_u - E_v$$

for some E_u, E_v . By the same argument as in Case (1), the sum of the a -coefficients of B_1, B_2 equals 4, so that $a = 5$ in B_2 .

Let $B_2 = 5H - \sum_{i=1}^{10} b_i E_i$. Then $c_1(K_X) \cdot B_2 + B_2^2 = 2$ and $B_2^2 = 6$ imply that

$$-15 + \sum_{i=1}^{10} b_i + 6 = 2, \quad 25 - \sum_{i=1}^{10} b_i^2 = 6.$$

As we argued in Case (1), B_2 must have the following expression:

$$B_2 = 5H - 3E_1 - E_y - E_u - E_v - 2E_{i_1} - E_{i_2} - E_{i_3} - E_{i_4}.$$

After computing $2(B_1 + B_2) + F_1$, we see that E_y, E_{i_1} must be the E_i -classes in F_2, F_3 which has a $(+1)$ -coefficient. It follows then

$$F_2 = E_y - E_z - E_{i_4}, \quad F_3 = E_{i_1} - E_{i_2} - E_{i_3}$$

without loss of generality.

With the preceding understood, let A be the class of any of the (-2) -spheres $F_{4,1}, F_{4,2}$. If $a = 4$ in A , we have $A \cdot F_2 \neq 0$ which is not allowed as in Case (1). If $a = 3$ in A , then we can write $A = 3H - 2E_{j_1} - E_{j_2} - \dots - E_{j_8}$. Then $B_1 \cdot A = 0$ implies that $E_{j_1} = E_1$ must be true, and exactly one of E_x, E_y, E_z appears in A . With $F_2 \cdot A = 0$, we see that E_x is contained in A . But this leads to $A \cdot F_1 = -2$, which is a contradiction. To proceed further, we rule out $a = 2$ in A by a similar argument as in Case (1). Now suppose $a = 1$ in A . Then $B_1 \cdot A = 0$ implies that E_1 and exactly one of E_x, E_y, E_z appears in A . Then $A \cdot F_2 = 0$ implies A must contain E_x . But we then get $A \cdot F_1 < 0$ which is a contradiction. This leaves only two possibilities for A : $E_u - E_v, E_{i_2} - E_{i_3}$.

But these two classes intersect trivially, contradicting $F_{4,1} \cdot F_{4,2} = 1$. Hence Case (2) is also eliminated.

Case (3): Suppose $a = 0$ in B_1 . Then since $a \geq 4$ in B_2 , we see immediately that the sum of the a -coefficients of F_1, F_2, F_3 is either 1 or -1 . In the former case, $a = 4$ in B_2 . If we write $B_2 = 4H - \sum_{i=1}^{10} b_i E_i$, then $c_1(K_X) \cdot B_2 + B_2^2 = 2$ and $B_2^2 = 6$ give

$$B_2 = 4H - 2E_{j_1} - E_{j_2} - \cdots - E_{j_7}.$$

But B_1 takes the form of $B_1 = E_{i_1} - E_{i_2} - \cdots - E_{i_6}$. The fact that there are totally 10 E_i -classes implies easily that $B_1 \cdot B_2 < 0$. In the latter case, $a = 5$ in B_2 . But then by Lemma 3.4(2), exactly one of F_1, F_2, F_3 has $a = -1$. Suppose it is F_1 . Then $F_1 = -H + 2E_1$. It is easy to see that $F_1 \cdot B_2$ is always odd because the a -coefficient of B_2 is 5. This rules out Case (3).

Case (4): Suppose $a = 1$ in B_1 . Then $a = 4$ in B_2 and $F_1 = -H + 2E_1$. But note that $B_1 \cdot F_1$ is always odd, hence this is not possible. This rules out Case (4).

Case (5): Suppose $a > 1$ in B_1 . Then with $a \geq 4$ in B_2 , the total contribution of B_1, B_2 to the a -coefficient of $-3c_1(K_X)$ is at least 12. But the a -coefficient of $-3c_1(K_X)$ is 9, so F_1, F_2, F_3 must contribute -3 to a -coefficient of $-3c_1(K_X)$. This is not possible by Lemmas 3.4(2) and 4.2(1). Hence Case (5) is eliminated.

The above discussions show that there is no component B_i with $B_i^2 < 0$. Hence the proposition is proved. \square

Proposition 4.4. *Let F_1, F_2, \dots, F_9 be a disjoint union of symplectic (-3) -spheres in a rational 4-manifold X , and let $\{B_i\}$ be a set of disjoint symplectic surfaces, possibly empty, which lie in the complement of F_1, F_2, \dots, F_9 , such that*

$$c_1(K_X) = -\frac{1}{3}(F_1 + F_2 + \cdots + F_9) - \frac{2}{3} \sum_i B_i.$$

Then $\{B_i\}$ must be empty if each B_i is a torus of self-intersection zero.

Proof. We shall prove by contradiction. Suppose $\{B_i\} \neq \emptyset$, where each B_i is a torus with $B_i^2 = 0$. We first note that $c_1(K_X)^2 = -3 + \frac{4}{9} \sum_i B_i^2 = -3$, so that $X = \mathbb{C}\mathbb{P}^2 \# 12\overline{\mathbb{C}\mathbb{P}^2}$. Again, by analyzing the contributions of B_i to the a -coefficient of $-c_1(K_X)$, it follows easily that there is only one component in $\{B_i\}$, and moreover, the sum of the a -coefficients of F_1, \dots, F_9 can be at most 3.

With the preceding understood, the following is the key observation:

The maximal number of disjoint symplectic (-3) -spheres in $\mathbb{C}\mathbb{P}^2 \# 12\overline{\mathbb{C}\mathbb{P}^2}$ with a -coefficient equaling 0 is six, and moreover, such six (-3) -spheres must be of the form:

- $E_{i_1} - E_{i_2} - E_{i_3}, E_{i_2} - E_{i_3} - E_{i_4},$
- $E_{j_1} - E_{j_2} - E_{j_3}, E_{j_2} - E_{j_3} - E_{j_4},$
- $E_{k_1} - E_{k_2} - E_{k_3}, E_{k_2} - E_{k_3} - E_{k_4},$

where $i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4, k_1, k_2, k_3, k_4$ are distinct indices.

To see this, let $A = E_i - E_j - E_k$, $A' = E_r - E_s - E_t$ be two distinct symplectic (-3) -spheres such that $A \cdot A' = 0$. Then it is easy to see that if E_r is not contained in A and E_i not in A' , the indices i, j, k, r, s, t must be distinct. On the other hand, without loss of generality, assume that E_r appears in A , say $r = j$, then $k = s$ or t must be true. The above claim follows easily from the fact that we only have these two alternatives.

With the preceding understood, note that by Lemma 3.4(2), $a \geq -1$ in each F_k . Moreover, if $a = -1$, the class must be $-H + 2E_1$, and there is at most one such (-3) -sphere in F_1, \dots, F_9 by Lemma 4.2(1).

We claim that the class $A = -H + 2E_1$ can not be represented by any of the (-3) -spheres F_k . To see this, note that if A' is the class of one of F_k which has positive a -coefficient, then $A \cdot A' \neq 0$ unless the a -coefficient of A' is an even number. Now with the fact that the sum of the a -coefficients of F_1, \dots, F_9 can be at most 3, it follows easily that at least six of the nine (-3) -spheres F_1, \dots, F_9 have zero a -coefficient. But this is a contradiction because it is easy to see that $A = -H + 2E_1$ intersects nontrivially with one of the six (-3) -spheres. Hence the claim that the class $A = -H + 2E_1$ can not occur. It follows easily that six of the nine (-3) -spheres F_1, \dots, F_9 have zero a -coefficient, and three of them have a -coefficient equaling 1. Moreover, note that the a -coefficient of B must be 3.

To proceed further, we denote the single component of $\{B_i\}$ by B . Note that as B is disjoint from the six (-3) -spheres with zero a -coefficient, it must be the class:

$$B = 3H - E_{i_1} - E_{i_2} - E_{i_4} - E_{j_1} - E_{j_2} - E_{j_4} - E_{k_1} - E_{k_2} - E_{k_4}.$$

In other words, the three E_i -classes which are missing from B are $E_{i_3}, E_{j_3}, E_{k_3}$. With this understood, let $A = H - E_{l_1} - E_{l_2} - E_{l_3} - E_{l_4}$ be any of the three (-3) -spheres whose a -coefficient equals 1. Then $A \cdot B = 0$ implies that exactly three of the four E_i -classes $E_{l_1}, E_{l_2}, E_{l_3}, E_{l_4}$ must appear in B . Without loss of generality, let E_{l_4} be the one not contained in B , and without loss of generality, assume $E_{l_4} = E_{i_3}$. Then since A intersects trivially with the (-3) -sphere $E_{i_2} - E_{i_3} - E_{i_4}$, it is easy to see that A must also contain the class E_{i_2} . Now with both E_{i_2}, E_{i_3} contained in A , the intersection of A with the (-3) -sphere $E_{i_1} - E_{i_2} - E_{i_3}$ must be negative. This is a contradiction, hence the proposition is proved. \square

Proposition 4.5. *Let $F_{j,1}, F_{j,2}$, where $1 \leq j \leq 5$, be a disjoint union of five pairs of symplectic (-3) -sphere and (-2) -sphere intersecting transversely and positively at one point in a rational 4-manifold X , and let $\{B_i\}$ be a set of disjoint symplectic surfaces, possibly empty, lying in the complement of $F_{j,1}, F_{j,2}$, such that*

$$c_1(K_X) = - \sum_{j=1}^5 \left(\frac{2}{5} F_{j,1} + \frac{1}{5} F_{j,2} \right) - \frac{4}{5} \sum_i B_i.$$

Then $\{B_i\}$ must be empty if each B_i is a torus of self-intersection zero.

Proof. We prove by contradiction. Suppose to the contrary that $\{B_i\}$ is nonempty, with each B_i being a torus of self-intersection zero. Then again, there can be only one component in $\{B_i\}$. We call it B . Moreover, the a -coefficient of B is either 4 or 3.

Before we proceed further, note that $c_1(K_X)^2 = -2$, so that $X = \mathbb{C}\mathbb{P}^2 \# 11\overline{\mathbb{C}\mathbb{P}^2}$. In particular, there are only 11 E_i -classes in X .

Case (1): Suppose $a = 4$ in B . Then if we write $B = 4H - \sum_{i=1}^{11} b_i E_i$, the b_i 's satisfy the following equation: $4(4-3) = \sum_{i=1}^{11} b_i(b_i-1)$ (see the proof of Lemma 4.2). It follows easily that

$$B = 4H - 2E_{j_1} - 2E_{j_2} - E_{j_3} - \cdots - E_{j_{10}}.$$

With this understood, note that since the contribution of B to the a -coefficient of $-5c_1(K_X)$ is $16 > 15$, it follows easily that there must be a (-3) -sphere F_{j_1} having $a = -1$, with the remaining four (-3) -spheres having $a = 0$. By Lemma 3.4(2), the class of the (-3) -sphere with $a = -1$ must be $-H + 2E_1$, and since its intersection with B is zero, either $E_1 = E_{j_1}$ or $E_1 = E_{j_2}$ must be true. Without loss of generality, assume $E_{j_1} = E_1$. Then it is clear that none of the four (-3) -spheres with $a = 0$ can contain the class $E_1 = E_{j_1}$.

With the preceding understood, it is easy to see that the expressions of the four (-3) -spheres with $a = 0$ fall into the following two possibilities without loss of generality:

$$\begin{aligned} (!) & E_{i_1} - E_{i_2} - E_{i_3}, E_{i_2} - E_{i_3} - E_{i_4}, E_{i_5} - E_{i_6} - E_{i_7}, E_{i_6} - E_{i_7} - E_{i_8}, \\ (!!)& E_{i_1} - E_{i_2} - E_{i_3}, E_{i_2} - E_{i_3} - E_{i_4}, E_{i_5} - E_{i_6} - E_{i_7}, E_{i_8} - E_{i_9} - E_{i_{10}}. \end{aligned}$$

Suppose we are in case (!). Consider the pair of (-3) -spheres $E_{i_1} - E_{i_2} - E_{i_3}$ and $E_{i_2} - E_{i_3} - E_{i_4}$. If the class E_{i_1} is not contained in the expression of B , then it is easy to see that none of the four classes $E_{i_1}, E_{i_2}, E_{i_3}, E_{i_4}$ are contained in B . But this contradicts the fact that there are only 11 E_i -classes in total. Hence E_{i_1} must be contained in B . We know that $E_{i_1} \neq E_{j_1}$. If $E_{i_1} = E_{j_2}$, then both E_{i_2}, E_{i_3} are contained in B , and it follows that E_{i_4} does not show up in the expression of B . On the other hand, if $E_{i_1} = E_{j_s}$ for some $s > 2$, then it is easy to see that E_{i_3} can not show up in B . In any event, one of E_{i_3}, E_{i_4} does not appear in the expression of B . With this understood, the same argument shows that one of E_{i_7}, E_{i_8} also does not appear in the expression of B . But this clearly contradicts the fact that there are totally only 11 E_i -classes, hence case (!) is not possible. The argument for case (!!) is similar. First, note that one of E_{i_3}, E_{i_4} does not appear in B as we have argued in case (!). Secondly, consider the pair of (-3) -spheres $E_{i_5} - E_{i_6} - E_{i_7}$ and $E_{i_8} - E_{i_9} - E_{i_{10}}$. We observe that one of the classes E_{i_5}, E_{i_8} is not equal to E_{j_2} . Without loss of generality, assume $E_{i_5} \neq E_{j_2}$. Then one of E_{i_6}, E_{i_7} can not be contained in B . So totally there are at least 2 E_i -classes not contained in B , which contradicts the fact that there are only 11 E_i -classes. Hence case (!!) is also not possible. This rules out Case (1).

Case (2): Suppose $a = 3$ in B . Then by Lemma 4.2(2), $B = 3H - E_{j_1} - \cdots - E_{j_9}$. With this understood, we first observe that the class $-H + 2E_1$ intersects nontrivially with B , so none of the five (-3) -spheres can have $a < 0$. On the other hand, from the proof of Proposition 4.4, it is easy to see that the five (-3) -spheres can not all have $a = 0$.

Now observe that the contribution of B to the a -coefficient of $-5c_1(K_X)$ is 12. It follows easily that exactly one of the five (-3) -spheres has $a = 1$, and the other four all have $a = 0$. The possible expressions of the four (-3) -spheres with $a = 0$ are given in either (!) or (!!) listed in Case (1). In the second case (!!), it is easy to see that there are three E_i -classes in the four (-3) -spheres with $a = 0$ which do not show up in B . This contradicts the fact that there are only 11 E_i -classes, hence (!!) is not possible. In case (!), it is easy to see that E_{i_3}, E_{i_7} are precisely the two E_i -classes that are not in the expression of B . To derive a contradiction, we consider the (-3) -sphere with $a = 1$. We write its class as $A = H - E_{l_1} - E_{l_2} - E_{l_3} - E_{l_4}$. Then we note that one of E_{i_1} and E_{i_5} , say E_{i_1} , must appear in the above expression. It follows that $E_{i_1}, E_{i_2}, E_{i_4}$ must all appear in A , but not E_{i_3} . Without loss of generality, assume $\{E_{i_1}, E_{i_2}, E_{i_4}\} = \{E_{l_1}, E_{l_2}, E_{l_3}\}$. Then $A \cdot B = 0$ implies easily that E_{l_4} can not show up in B . It follows that $E_{l_4} = E_{i_7}$ must be true. But this implies that A has nonzero intersection with the (-3) -sphere $E_{i_5} - E_{i_6} - E_{i_7}$, which is a contradiction. Hence (!) is also not possible. This rules out Case (2) as well, and the proof of the proposition is complete. \square

5. The proof of main theorems

We begin with the key technical lemma, which classifies the possible homological expressions of a disjoint union of 8 symplectic (-2) -spheres in $\mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$ under a very delicately chosen assumption on the symplectic structure.

Lemma 5.1. *Let F_1, F_2, \dots, F_8 be a disjoint union of 8 symplectic (-2) -spheres in $X = \mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$. Suppose the symplectic structure ω obeys the following constraints:*

- one of F_k has ω -area δ_1 , the remaining seven have ω -area δ_2 ;
- $\delta_2 < \delta_1 < 2\delta_2$, and $7\delta_i < -c_1(K_X) \cdot [\omega]$ for $i = 1, 2$.

Then for any given reduced basis H, E_1, E_2, \dots, E_9 of (X, ω) , there are three possibilities for the classes of F_1, F_2, \dots, F_8 :

- (a) $F_1 = 3H - 2E_{i_1} - E_{i_2} - \dots - E_{i_7} - E_{i_8}$, and $F_2 = H - E_{i_2} - E_{i_3} - E_{i_4}$, $F_3 = H - E_{i_2} - E_{i_5} - E_{i_6}$, $F_4 = H - E_{i_2} - E_{i_7} - E_{i_8}$, $F_5 = H - E_{i_3} - E_{i_5} - E_{i_7}$, $F_6 = H - E_{i_3} - E_{i_6} - E_{i_8}$, $F_7 = H - E_{i_4} - E_{i_5} - E_{i_8}$, $F_8 = H - E_{i_4} - E_{i_6} - E_{i_7}$.
- (b) $F_1 = H - E_{l_1} - E_{l_2} - E_{l_3}$, $F_2 = H - E_{l_1} - E_{l_4} - E_{l_5}$, $F_3 = H - E_{l_1} - E_{l_6} - E_{l_7}$, $F_4 = H - E_{l_2} - E_{l_4} - E_{l_6}$, $F_5 = H - E_{l_3} - E_{l_5} - E_{l_6}$, $F_6 = H - E_{l_2} - E_{l_5} - E_{l_7}$, $F_7 = H - E_{l_3} - E_{l_4} - E_{l_7}$, and $F_8 = E_{l_8} - E_{l_9}$.
- (c) $F_1 = H - E_{l_1} - E_{l_2} - E_{l_3}$, $F_2 = H - E_{l_1} - E_{l_4} - E_{l_5}$, $F_3 = H - E_{l_1} - E_{l_6} - E_{l_7}$, $F_4 = H - E_{l_1} - E_{l_8} - E_{l_9}$, $F_5 = E_{l_2} - E_{l_3}$, $F_6 = E_{l_4} - E_{l_5}$, $F_7 = E_{l_6} - E_{l_7}$, $F_8 = E_{l_8} - E_{l_9}$.

Proof. By Lemma 3.6, $a \leq 3$ in each F_k .

Case (1): Suppose there is a F_k whose a -coefficient equals 3. We may assume without loss of generality that it is F_1 , and write

$$F_1 = 3H - 2E_{i_1} - E_{i_2} - \dots - E_{i_7} - E_{i_8}.$$

Furthermore, we denote by E_{i_9} the unique E_i -class that is missing in F_1 .

Let A be the class of any of the remaining (-2) -spheres, i.e., F_2, F_3, \dots, F_8 . Our first observation is that $a \neq 3$ in A . To see this, we note that if the a -coefficient of A equals 3, then $A \cdot F_1 = 0$ implies that A must take the following form without loss of generality:

$$A = 3H - E_{i_1} - 2E_{i_2} - \dots - E_{i_7} - E_{i_9}.$$

With this understood, we observe that

$$F_1 + A + c_1(K_X) = 3H - 2E_{i_1} - 2E_{i_2} - E_{i_3} - E_{i_4} - \dots - E_{i_7},$$

which can be written as a sum of three terms of the form $H - E_i - E_j - E_k$. It follows that $\omega(A + F_1) \geq -c_1(K_X) \cdot [\omega]$, which is a contradiction. Hence the claim.

To proceed further, we first examine the classes A whose a -coefficient equals 1. Note that if A is a class with $a = 1$, then $A \cdot F_1 = 0$ implies that if E_{i_1} appears in A , then so does E_{i_9} . This allows us to divide the classes A with $a = 1$ into two types:

$$(\alpha) A = H - E_{i_1} - E_{i_9} - E_x, \quad (\beta) A = H - E_r - E_s - E_x,$$

where $E_x, E_r, E_s \in \{E_{i_2}, E_{i_3}, \dots, E_{i_8}\}$.

Claim: *There are no classes A with $a = 2$.*

Proof of Claim: We first observe that if A is a class with $a = 2$, then E_{i_1} is not contained in A . This is because if E_{i_1} is contained in A , then $A \cdot F_1 = 0$ implies that E_{i_9} must also be contained in A , and A takes the following form

$$A = 2H - E_{i_1} - E_{i_9} - E_{k_1} - E_{k_2} - E_{k_3} - E_{k_4}.$$

But this would lead to a contradiction

$$\omega(F_1 + A) + c_1(K_X) \cdot [\omega] = \omega(2H - 2E_{i_1} - E_{k_1} - E_{k_2} - E_{k_3} - E_{k_4}) \geq 0,$$

as $2H - 2E_{i_1} - E_{k_1} - E_{k_2} - E_{k_3} - E_{k_4}$ is a sum of terms of the form $H - E_i - E_j - E_k$.

With the preceding understood, suppose to the contrary that there is a class A with $a = 2$. Then without loss of generality, we may write it as

$$A_1 = 2H - E_{i_2} - E_{i_3} - E_{i_4} - E_{i_5} - E_{i_6} - E_{i_7}.$$

Moreover, if A is another class of F_2, F_3, \dots, F_8 with $a = 2$, then it is easy to check that $A_1 \cdot A < 0$. Hence A_1 is the only one with $a = 2$.

Next we examine the possible classes of A with $a = 1$, which intersects trivially with F_1 and A_1 . It is easy to see that if A is a class with $a = 1$ and $A \cdot A_1 = 0$, then A can not be of type (α) , and for a type (β) class, A must contain E_{i_8} . It is easy to see that maximally, there are three such type (β) classes that are mutually disjoint, i.e.,

$$A_2 = H - E_{i_2} - E_{i_3} - E_{i_8}, A_3 = H - E_{i_4} - E_{i_5} - E_{i_8}, A_4 = H - E_{i_6} - E_{i_7} - E_{i_8}$$

without loss of generality. The remaining three classes of A must all have a -coefficient equaling 0, and it is easy to see that, without loss of generality, they are

$$A_5 = E_{i_2} - E_{i_3}, A_6 = E_{i_4} - E_{i_5}, A_7 = E_{i_6} - E_{i_7}.$$

To derive a contradiction, we appeal to the area constraints. First, we observe that the area of F_1 must be greater than the area of any of A_5, A_6, A_7 . For example,

$$\omega(F_1 - A_5) = \omega(3H - 2E_{i_1} - 2E_{i_2} - E_{i_4} - E_{i_5} - \cdots - E_{i_8}) \geq 0$$

as $3H - 2E_{i_1} - 2E_{i_2} - E_{i_4} - E_{i_5} - \cdots - E_{i_8}$ is a sum of terms of the form $H - E_i - E_j - E_k$. Furthermore, note that if $\omega(F_1 - A_5) = 0$, then $\omega(H - E_x - E_y - E_z) = 0$ for any three classes E_x, E_y, E_z from the set $\{E_{i_1}, E_{i_2}, E_{i_4}, E_{i_5}, \dots, E_{i_8}\}$. In particular, $E_{i_4}, E_{i_5}, E_{i_6}, E_{i_7}$ have the same area, contradicting $\omega(A_6) > 0, \omega(A_7) > 0$. It follows that $\omega(F_1) = \delta_1$ and the remaining classes have the same area equaling $\delta_2 < \delta_1$. With this understood, we note that $\omega(F_1 - A_5 - A_4) = \omega(2H - 2E_{i_1} - 2E_{i_2} - E_{i_4} - E_{i_5}) \geq 0$ as $2H - 2E_{i_1} - 2E_{i_2} - E_{i_4} - E_{i_5}$ is a sum of terms of the form $H - E_i - E_j - E_k$, contradicting the constraint $\delta_1 < 2\delta_2$. This finishes off the proof of the Claim.

Now back to the discussion on Case (1), we claim that no type (α) classes can occur. Suppose to the contrary that there is a type (α) class, call it A_1 . It is easy to see that any other type (α) class has a negative intersection with A_1 , hence A_1 is the only type (α) class. Without loss of generality, let $A_1 = H - E_{i_1} - E_{i_9} - E_{i_8}$. Now let A be any type (β) class such that $A \cdot A_1 = 0$. Then A must contain E_{i_8} , and furthermore, it is easy to see that maximally, there are three such type (β) classes which are mutually disjoint. Without loss of generality, they are

$$A_2 = H - E_{i_2} - E_{i_3} - E_{i_8}, A_3 = H - E_{i_4} - E_{i_5} - E_{i_8}, A_4 = H - E_{i_6} - E_{i_7} - E_{i_8}$$

The remaining three classes of A must all have a -coefficient equaling 0, and it is easy to see that, without loss of generality, they are

$$A_5 = E_{i_2} - E_{i_3}, A_6 = E_{i_4} - E_{i_5}, A_7 = E_{i_6} - E_{i_7}.$$

This possibility can be ruled out using the area constraints as we did in the proof of the Claim. Hence no type (α) classes can occur.

With the preceding understood, we further observe that no class A with $a = 0$ can be realized by F_2, F_3, \dots, F_8 . Suppose, without loss of generality, $A_1 = E_{i_7} - E_{i_8}$ is realized. Let A be a type (β) class which intersects trivially with A_1 . Then it is easy to see that either A contains both E_{i_7}, E_{i_8} , or A contains neither E_{i_7} nor E_{i_8} . It is clear that there can be at most one type (β) class which contains both E_{i_7}, E_{i_8} . Without loss of generality, we let it be $A_2 = H - E_{i_2} - E_{i_7} - E_{i_8}$. Then any other type (β) classes which intersect trivially with A_1, A_2 must contain E_{i_2} , and there are maximally two such classes: $H - E_{i_2} - E_{i_3} - E_{i_4}, H - E_{i_2} - E_{i_5} - E_{i_6}$. With this understood, note that there are at most two other classes, both having $a = 0$, that are allowed, i.e., $E_{i_3} - E_{i_4}, E_{i_5} - E_{i_6}$, bringing total number of allowable classes for F_2, F_3, \dots, F_8 to 6. But apparently, there are not enough many classes, hence our claim.

The above discussions show that the classes of F_2, F_3, \dots, F_8 are all of type (β) . With this understood, we first rule out the possibility that no triple of F_2, F_3, \dots, F_8 shares a common E_i -class. Suppose to the contrary that this is the case. Then without loss of

generality, we write

$$F_2 = H - E_{i_2} - E_{i_3} - E_{i_4}, \quad F_3 = H - E_{i_2} - E_{i_5} - E_{i_6}.$$

Note that by our assumption, F_4 can not contain E_{i_2} . With this understood, $F_4 \cdot F_2 = F_4 \cdot F_3 = 0$ implies that we may write $F_4 = H - E_{i_3} - E_{i_5} - E_{i_7}$ without loss of generality. Now observe that F_5 can not contain $E_{i_2}, E_{i_3}, E_{i_5}$. Hence $F_5 = H - E_{i_4} - E_{i_6} - E_{i_7}$ must be true. Now examining the class of F_6 , by our assumption it can not contain any of $E_{i_2}, E_{i_3}, \dots, E_{i_7}$. This is clearly a contradiction. Hence the claim.

With the preceding understood, we may write without loss of generality that

$$F_2 = H - E_{i_2} - E_{i_3} - E_{i_4}, \quad F_3 = H - E_{i_2} - E_{i_5} - E_{i_6}, \quad F_4 = H - E_{i_2} - E_{i_7} - E_{i_8}.$$

With this given, it is easy to see that the other four (-2) -spheres must be

$$F_5 = H - E_{i_3} - E_{i_5} - E_{i_7}, \quad F_6 = H - E_{i_3} - E_{i_6} - E_{i_8},$$

and

$$F_7 = H - E_{i_4} - E_{i_5} - E_{i_8}, \quad F_8 = H - E_{i_4} - E_{i_6} - E_{i_7}.$$

This possibility of classes of F_1, F_2, \dots, F_8 is listed as Case (a) of the lemma.

Case (2): Suppose $a \leq 2$ in all eight (-2) -spheres F_1, F_2, \dots, F_8 .

(i): Assume at least two of F_1, F_2, \dots, F_8 have a -coefficient equaling 2. Without loss of generality, let F_1, F_2 be such two (-2) -spheres. It is easy to see from $F_1 \cdot F_2 = 0$ that F_1, F_2 must have exactly 4 E_i -classes in common. Hence without loss of generality, we may write them as

$$F_1 = 2H - E_{j_1} - E_{j_2} - E_{j_3} - E_{j_4} - E_{j_5} - E_{j_6}, \quad F_2 = 2H - E_{j_1} - E_{j_2} - E_{j_3} - E_{j_4} - E_{j_7} - E_{j_8}.$$

With this understood, we denote by E_{j_9} the unique E_i -class that is missing in F_1, F_2 . Moreover, we denote by A the class of any of the remaining (-2) -spheres, i.e., F_3, F_4, \dots, F_8 .

Claim: *There are no classes A which contains E_{j_9} .*

Proof of Claim: First, it is easy to see that if A is a class with $a = 0$ which contains E_{j_9} , the intersection of A with one of F_1, F_2 will be nonzero. Now suppose A is a class with $a = 1$ which contains E_{j_9} . Then $A \cdot F_1 = A \cdot F_2 = 0$ implies that A must be of the form $A = H - E_x - E_y - E_{j_9}$ for some $E_x, E_y \in \{E_{j_1}, \dots, E_{j_4}\}$. With this understood, we note that

$$F_1 + F_2 + A + c_1(K_X) = 2H - E_{j_1} - E_{j_2} - E_{j_3} - E_{j_4} - E_x - E_y,$$

which is a sum of terms of the form $H - E_i - E_j - E_k$, leading to a contradiction in areas: $\omega(F_1 + F_2 + A) \geq -c_1(K_X) \cdot [\omega]$. Finally, suppose A is a class with $a = 2$ which contains E_{j_9} . Then $A \cdot F_1 = A \cdot F_2 = 0$ implies that, without loss of generality,

$$A = 2H - E_{j_1} - E_{j_2} - E_{j_3} - E_{j_5} - E_{j_7} - E_{j_9}.$$

In this case, we have $F_1 + F_2 + A + c_1(K_X) = 3H - 2E_{j_1} - 2E_{j_2} - 2E_{j_3} - E_{j_4} - E_{j_5} - E_{j_7}$, which by the same reason also leads to the contradiction in areas: $\omega(F_1 + F_2 + A) \geq -c_1(K_X) \cdot [\omega]$. Hence the Claim.

Now back to the discussion on Case (2), it is easy to see that there are two other classes A with $a = 2$ and trivial mutual intersection, which intersect trivially with F_1, F_2 ; we denote them by A_1, A_2 , where

$$A_1 = 2H - E_{j_1} - E_{j_2} - E_{j_5} - E_{j_6} - E_{j_7} - E_{j_8}, A_2 = 2H - E_{j_3} - E_{j_4} - E_{j_5} - E_{j_6} - E_{j_7} - E_{j_8}.$$

On the other hand, let A be a class with $a = 1$ which intersects trivially with F_1, F_2 . Then A must be of the form $A = H - E_r - E_s - E_t$, where $E_r \in \{E_{j_1}, E_{j_2}, E_{j_3}, E_{j_4}\}$, $E_s \in \{E_{j_5}, E_{j_6}\}$, and $E_t \in \{E_{j_7}, E_{j_8}\}$.

With the preceding understood, if both of A_1, A_2 are realized by the (-2) -spheres, then it is easy to see that no classes A with $a = 1$ can be realized. On the other hand, it is easy to see that there are maximally 4 classes A with $a = 0$:

$$E_{j_1} - E_{j_2}, E_{j_3} - E_{j_4}, E_{j_5} - E_{j_6}, E_{j_7} - E_{j_8}.$$

Hence all of them must be realized. With this understood, it is easy to see that three of F_1, F_2, A_1, A_2 and all of the classes with $a = 0$ must have the smaller area δ_2 . As a consequence, we may assume without loss of generality that $\omega(F_1) = \omega(E_{j_1} - E_{j_2})$. Then observe that $2H - 2E_{j_1} - E_{j_3} - E_{j_4} - E_{j_5} - E_{j_6}$ is a sum of terms of the form $H - E_i - E_j - E_k$, so that

$$\omega(2H - 2E_{j_1} - E_{j_3} - E_{j_4} - E_{j_5} - E_{j_6}) = \omega(F_1) - \omega(E_{j_1} - E_{j_2}) = 0$$

implies that $E_{j_3}, E_{j_4}, E_{j_5}, E_{j_6}$ have the same area. But this contradicts the fact that the classes $E_{j_3} - E_{j_4}, E_{j_5} - E_{j_6}$ are realized by the symplectic (-2) -spheres. It follows that A_1, A_2 can not be both realized.

Suppose only one of A_1, A_2 , say A_1 , is realized. Then there are four classes A with $a = 1$ that are possible, i.e.,

$$A_3 = H - E_{j_3} - E_{j_5} - E_{j_7}, A_4 = H - E_{j_3} - E_{j_6} - E_{j_8},$$

and

$$A_5 = H - E_{j_4} - E_{j_5} - E_{j_8}, A_6 = H - E_{j_4} - E_{j_6} - E_{j_7}.$$

If all of A_3, A_4, A_5, A_6 are realized, then the remaining (-2) -sphere must have a -coefficient equaling 0, and it must be the class $A_7 = E_{j_1} - E_{j_2}$ without loss of generality. But this leads to a contradiction in areas as follows: note that

$$\omega(F_1 - A_7) = \omega(2H - 2E_{j_1} - E_{j_3} - E_{j_4} - E_{j_5} - E_{j_6}) \geq 0$$

as $2H - 2E_{j_1} - E_{j_3} - E_{j_4} - E_{j_5} - E_{j_6}$ is a sum of terms of the form $H - E_i - E_j - E_k$. Furthermore, if $\omega(F_1 - A_7) = 0$, the four classes $E_{j_3}, E_{j_4}, E_{j_5}, E_{j_6}$ must have the same area. It follows easily that $\omega(A_7) = \delta_2 < \delta_1$. The same argument applies with F_1 being replaced by F_2 or A_1 . Note that at least two of F_1, F_2, A_1 must have the smaller area δ_2 . It follows easily that the six classes $E_{j_3}, E_{j_4}, E_{j_5}, E_{j_6}, E_{j_7}, E_{j_8}$ must have the same area. But this would imply that all the eight (-2) -spheres have the same area, which

is a contradiction. Finally, note that if any of A_3, A_4, A_5, A_6 is realized, A_7 is the only possible class with $a = 0$. If none of A_3, A_4, A_5, A_6 is realized, the allowable classes with $a = 0$ are $E_{j_3} - E_{j_4}, E_{j_5} - E_{j_6}, E_{j_7} - E_{j_8}$, in addition to A_7 . It follows that neither A_1 nor A_2 can be realized.

The above discussion shows that F_1, F_2 are the only two (-2) -spheres with $a = 2$. From the discussion, it is also clear that the maximal number of mutually disjoint classes with $a = 1$ which intersect trivially with F_1, F_2 is 4, which, without loss of generality, are given by A_3, A_4, A_5, A_6 . If any of them is realized, there is only one possible class with $a = 0$, i.e., $A_7 = E_{j_1} - E_{j_2}$. If none of the $a = 1$ classes are realized, then there are maximally 4 classes with $a = 0$ that are allowed. In any event, we do not have enough classes that can be realized. Thus (i) is eliminated.

(ii): Assume only one of F_1, F_2, \dots, F_8 has a -coefficient equaling 2. Without loss of generality, assume it is F_1 , and we write

$$F_1 = 2H - E_{k_1} - E_{k_2} - E_{k_3} - E_{k_4} - E_{k_5} - E_{k_6}.$$

We denote the remaining three E_i -classes by $E_{k_7}, E_{k_8}, E_{k_9}$, and denote by A the class of any of the (-2) -spheres F_2, F_3, \dots, F_8 .

Examining classes A with $a = 1$ which intersect trivially with F_1 , we note that A must be of the form

$$A = H - E_r - E_s - E_t, \text{ where } E_r, E_s \in \{E_{k_1}, \dots, E_{k_6}\} \text{ and } E_t \in \{E_{k_7}, E_{k_8}, E_{k_9}\}.$$

Consider first the case where amongst the classes A with $a = 1$, the E_i -classes $E_{k_1}, E_{k_2}, \dots, E_{k_6}$ can only appear once. It is easy to see that in this case, all the $a = 1$ classes must have a common E_i -class which must be one of $E_{k_7}, E_{k_8}, E_{k_9}$. It is clear that there are maximally three such classes with $a = 1$, i.e.,

$$H - E_{k_1} - E_{k_2} - E_{k_7}, \quad H - E_{k_3} - E_{k_4} - E_{k_7}, \quad H - E_{k_5} - E_{k_6} - E_{k_7}$$

without loss of generality. The remaining four (-2) -spheres must have a -coefficient equaling 0, and they must be

$$E_{k_1} - E_{k_2}, \quad E_{k_3} - E_{k_4}, \quad E_{k_5} - E_{k_6}, \quad E_{k_8} - E_{k_9}$$

without loss of generality. With this understood, we note that the area of F_1 must be the larger δ_1 , with the remaining seven (-2) -spheres having area δ_2 . However, as $2H - 2E_{k_1} - 2E_{k_3} - E_{k_5} - E_{k_6}$ is a sum of terms of the form $H - E_i - E_j - E_k$, it follows that

$$\omega(F_1) - \omega(E_{k_1} - E_{k_2}) - \omega(E_{k_3} - E_{k_4}) = \omega(2H - 2E_{k_1} - 2E_{k_3} - E_{k_5} - E_{k_6}) \geq 0,$$

which contradicts the constraint $\delta_1 < 2\delta_2$. Hence this first case is ruled out.

Next we assume that the E_i -classes $E_{k_1}, E_{k_2}, \dots, E_{k_6}$ can appear at most twice in the $a = 1$ classes, and at least one of them, say E_{k_1} , appeared twice. Then without loss of generality, we may assume

$$A_1 = H - E_{k_1} - E_{k_2} - E_{k_7}, \quad A_2 = H - E_{k_1} - E_{k_3} - E_{k_8}$$

are realized by the (-2) -spheres. Since there are at most 4 mutually disjoint classes with $a = 0$ that can possibly be realized by the (-2) -spheres, we must have another $a = 1$ class, call it A_3 . By our assumption, A_3 can not contain E_{k_1} . The fact that A_3 intersects trivially with A_1, A_2 implies that either $A_3 = H - E_{k_2} - E_{k_3} - E_{k_9}$, or without loss of generality, $A_3 = H - E_{k_3} - E_{k_4} - E_{k_7}$. In the former case, none of $E_{k_1}, E_{k_2}, E_{k_3}$ can appear anymore by our assumption, which implies easily that there can be no more $a = 1$ classes. On the other hand, there is only one possible $a = 0$ class, say $E_{k_5} - E_{k_6}$. Hence the former case is not possible. In the latter case, $E_{k_1}, E_{k_3}, E_{k_7}$ can no longer appear. We note that there is only one possible $a = 0$ class, i.e., $E_{k_5} - E_{k_6}$, so there must be three more $a = 1$ classes. Call them A_4, A_5, A_6 . Then observe that A_4, A_5, A_6 intersect trivially with A_2 , so all of them must contain E_{k_8} . Likewise, A_4, A_5, A_6 intersect trivially with A_1 , so that they must all contain E_{k_2} , which is clearly a contradiction. Thus this second case is also ruled out.

Finally, assume one of the E_i -classes $E_{k_1}, E_{k_2}, \dots, E_{k_6}$, say E_{k_1} , appears in the $a = 1$ classes three times. Without loss of generality, we assume

$$A_1 = H - E_{k_1} - E_{k_2} - E_{k_7}, \quad A_2 = H - E_{k_1} - E_{k_3} - E_{k_8}, \quad A_3 = H - E_{k_1} - E_{k_4} - E_{k_9}$$

are realized by the (-2) -spheres. Again, there is only one possible $a = 0$ class, i.e., $E_{k_5} - E_{k_6}$, so there must be three more $a = 1$ classes, which are denoted by A_4, A_5, A_6 . It is easy to see that the following are the only possibility:

$$A_4 = H - E_{k_3} - E_{k_4} - E_{k_7}, \quad A_5 = H - E_{k_2} - E_{k_4} - E_{k_8}, \quad A_6 = H - E_{k_2} - E_{k_3} - E_{k_9}.$$

In order to rule out this last case, we observe that

$$F_1 + \sum_{i=1}^6 A_i + c_1(K_X) = 5H - 3(E_{k_1} + \dots + E_{k_4}) - E_{k_7} - E_{k_8} - E_{k_9}.$$

The right-hand side is a sum of terms of the form $H - E_i - E_j - E_k$, hence has non-negative area. But this leads to a contradiction to the constraint $7\delta_i < -c_1(K_X) \cdot [\omega]$ for $i = 1, 2$. Hence (ii) is also eliminated.

(iii): It remains to consider the case where the a -coefficient of F_1, F_2, \dots, F_8 equals either 1 or 0. We begin by noting that there are at least four (-2) -spheres with $a = 1$.

The first possibility is that each E_i -class appears amongst the $a = 1$ classes at most three times. To analyze this case, we take two of the (-2) -spheres with $a = 1$, say F_1, F_2 , and we write them as

$$F_1 = H - E_{l_1} - E_{l_2} - E_{l_3}, \quad F_2 = H - E_{l_1} - E_{l_4} - E_{l_5}.$$

Assume F_3 also has a -coefficient equaling 1. Then there are two possibilities for F_3 : either $F_3 = H - E_{l_1} - E_{l_6} - E_{l_7}$ or $F_3 = H - E_{l_2} - E_{l_4} - E_{l_6}$ without loss of generality. There is at least one more (-2) -sphere with $a = 1$, say F_4 . Then if $F_3 = H - E_{l_1} - E_{l_6} - E_{l_7}$, we may assume without loss of generality that $F_4 = H - E_{l_2} - E_{l_4} - E_{l_6}$ because of our assumption that each E_i -class appears amongst the $a = 1$ classes at most three times. If $F_3 = H - E_{l_2} - E_{l_4} - E_{l_6}$ in the latter case, we may assume $F_4 = H - E_{l_3} - E_{l_5} - E_{l_6}$

(note that the other choice $F_4 = H - E_{l_1} - E_{l_6} - E_{l_7}$ is equivalent to the former case). In any event, with these choices for F_1, F_2, F_3, F_4 , there can be at most one (-2) -sphere with $a = 0$. Consequently, there must be three more (-2) -spheres with $a = 1$. One can check easily that without loss of generality, in this case the eight (-2) -spheres are

$$\begin{aligned} F_1 &= H - E_{l_1} - E_{l_2} - E_{l_3}, F_2 = H - E_{l_1} - E_{l_4} - E_{l_5}, F_3 = H - E_{l_1} - E_{l_6} - E_{l_7}, \\ F_4 &= H - E_{l_2} - E_{l_4} - E_{l_6}, F_5 = H - E_{l_3} - E_{l_5} - E_{l_6}, F_6 = H - E_{l_2} - E_{l_5} - E_{l_7}, \\ F_7 &= H - E_{l_3} - E_{l_4} - E_{l_7}, \text{ and } F_8 = E_{l_8} - E_{l_9}, \end{aligned}$$

which is listed as Case (b) of the lemma.

The remaining possibility is that one of the E_i -classes appears in the $a = 1$ classes four times. In this case, it is easy to check that without loss of generality, the eight (-2) -spheres are

$$\begin{aligned} F_1 &= H - E_{l_1} - E_{l_2} - E_{l_3}, F_2 = H - E_{l_1} - E_{l_4} - E_{l_5}, F_3 = H - E_{l_1} - E_{l_6} - E_{l_7}, \\ F_4 &= H - E_{l_1} - E_{l_8} - E_{l_9}, F_5 = E_{l_2} - E_{l_3}, F_6 = E_{l_4} - E_{l_5}, F_7 = E_{l_6} - E_{l_7}, F_8 = E_{l_8} - E_{l_9}. \end{aligned}$$

This is listed as Case (c) of the lemma. The proof of the lemma is complete. \square

In the following lemma, $D \subset \mathbb{C}$ is an open disc centered at the origin, with radius unspecified. Let $\Psi : D \times D \rightarrow \mathbb{C}^2$ be a diffeomorphism onto a neighborhood of $0 \in \mathbb{C}^2$, given by equations $z_1 = \psi(z, w)$, $z_2 = w$, where z_1, z_2 are the standard holomorphic coordinates on \mathbb{C}^2 and z, w are a local complex coordinate on the first and second factor in $D \times D$. Furthermore, assume Ψ satisfies the following conditions: $\psi(z, w)$ is holomorphic in $w \in D$ (but only C^∞ in $z \in D$), and $\psi(0, w) = 0$ for all $w \in D$.

Lemma 5.2. *Let $C \subset \mathbb{C}^2$ be an embedded holomorphic disc containing the origin, where C intersects the z_2 -axis with a tangency of order $n > 1$. Let $F : D \rightarrow \mathbb{C}^2$ be a holomorphic parametrization of C such that $F(0) = 0$. Then the map $\pi_1 \circ \Psi^{-1} \circ F : D \rightarrow D$ is an n -fold branched covering in a neighborhood of $0 \in D$, ramified at 0 , where $\pi_1 : D \times D \rightarrow D$ is the projection onto the first factor.*

Proof. Considering the parametrization $\Psi^{-1} \circ F$ of C in the coordinates (z, w) , it is clear that after a re-parametrization of the domain D if necessary, we may assume that $\Psi^{-1} \circ F$ is given by $z = f(\xi)$, $w = \xi$, where ξ is a local holomorphic coordinate on the domain D . We remark that $\Psi^{-1} \circ F$ is J -holomorphic with respect to the almost complex structure J on $D \times D$, where J is the pullback of the standard complex structure on \mathbb{C}^2 via Ψ .

We shall compute $\partial_{\bar{w}} f$ for the function f , where f is considered a function of w (as $w = \xi$). To this end, we set $z_k = x_k + \sqrt{-1}y_k$, $k = 1, 2$, and $z = s + \sqrt{-1}t$, $w = u + \sqrt{-1}v$. Then with respect to the coordinates (s, t, u, v) and (x_1, y_1, x_2, y_2) , the Jacobian of Ψ is given by the matrix

$$D\Psi = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix},$$

where $A = \begin{pmatrix} \frac{\partial x_1}{\partial s} & \frac{\partial x_1}{\partial t} \\ \frac{\partial y_1}{\partial s} & \frac{\partial y_1}{\partial t} \end{pmatrix}$, $B = \begin{pmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \end{pmatrix}$. Let $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be the matrix representing the standard complex structure. Then the assumptions that $\psi(z, w)$ is holomorphic in $w \in D$ and $\psi(0, w) = 0$ for all $w \in D$ imply that $J_0 B = B J_0$ and $B = 0$ along the disc $z = 0$.

With the preceding understood, we note that the almost complex structure J is given by the matrix

$$J = D\Psi \begin{pmatrix} J_0 & 0 \\ 0 & J_0 \end{pmatrix} (D\Psi)^{-1} = \begin{pmatrix} A J_0 A^{-1} & (-A J_0 A^{-1} + J_0) B \\ 0 & J_0 \end{pmatrix}.$$

Now the Jacobian of $\Psi^{-1} \circ F$ is $\begin{pmatrix} Df \\ I \end{pmatrix}$ where Df is the Jacobian of f . It follows easily that the J -holomorphic equation satisfied by $\Psi^{-1} \circ F$, i.e.,

$$J \begin{pmatrix} Df \\ I \end{pmatrix} = \begin{pmatrix} Df \\ I \end{pmatrix} J_0$$

is equivalent to the equation $Df + (A J_0 A^{-1}) \cdot Df \cdot J_0 = (A J_0 A^{-1} J_0 + I) B$. Intrinsically, this can be written as

$$\partial_{\bar{w}} f = \frac{1}{2} (A J_0 A^{-1} J_0 + I) B.$$

With the above understood, we note that since $B = 0$ along the disc $z = 0$, we have $\|B\| \leq C_1 |z|$ near $z = 0$ for some constant $C_1 > 0$. It follows easily that the function f obeys the inequality $|\partial_{\bar{w}} f| \leq C_2 |f|$ for some constant $C_2 > 0$. By the Carleman similarity principle (e.g. see Siebert-Tian [41], Lemma 2.9), there is a complex valued function g of class C^α and a holomorphic function ϕ , such that $f(w) = \phi(w)g(w)$, where $g(0) \neq 0$. Note that ϕ vanishes at $w = 0$ of order n because by the assumption, the holomorphic disc C intersects the z_2 -axis with a tangency of order n . After a further change of coordinate, we may assume that $f(w) = w^n g(w)$ for a C^α -class function g , where $w \in D$.

Our next goal is to show that for any $c \neq 0$, with $|c|$ sufficiently small, the equation

$$f(w) = c$$

has exactly n distinct solutions lying in a small neighborhood of $0 \in D$. To see this, we take $h(w)$ to be an n -th root of the function $g(w)$, i.e., $h(w)^n = g(w)$, which is also of C^α -class. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the n -th roots of c . For each $i = 1, 2, \dots, n$, we consider the equation

$$wh(w) = \lambda_i.$$

Set $P(w) := \frac{1}{h(0)} (\lambda_i - w(h(w) - h(0)))$. Then the above equation becomes $w = P(w)$. With this understood, let $B(r) \subset D$ be the closed disc of radius r . Then for $r > 0$ sufficiently small, $P : B(r) \rightarrow B(r)$ is a well-defined continuous map, as long as $|\lambda_i| \leq \frac{1}{2} |h(0)| \cdot r$. Now we pick any $w_1 \in B(r)$ and define inductively $w_{k+1} = P(w_k)$ for $k \geq 1$. Since $B(r)$ is compact, the sequence $\{w_k\}$ has a convergent subsequence. The limit $w_0 \in B(r)$ satisfies the equation $w_0 = P(w_0)$.

It follows easily that when $c \neq 0$ lies in the disc of radius $(\frac{1}{2}|h(0)| \cdot r)^n$, the equation $f(w) = c$ has at least n distinct solutions, all lying in the disc $B(r)$. The local intersection number of the holomorphic disc C with each holomorphic disc $z = c$ equals n . This implies that the equation $f(w) = c$ has precisely n distinct solutions in $B(r)$, and the intersection of C with each holomorphic disc $z = c \neq 0$ is transversal. It follows easily that the map $\pi_1 \circ \Psi^{-1} \circ F : D \rightarrow D$ is an n -fold branched covering in a neighborhood of $0 \in D$, ramified at 0 . This finishes the proof. \square

With these preparations, we now prove the main theorems.

Proof of Theorem 1.1:

We first consider the case where M_G is irrational ruled. It is easily seen that there is a subgroup H of prime order p such that M_H is irrational ruled. By Lemma 2.2 and Lemma 2.6(i), the fixed-point set of H consists of only tori of self-intersection zero. Moreover, from the proofs it is known that M_H is a \mathbb{S}^2 -bundle over T^2 , and M is simply a branched cover of M_H along the fixed-point set.

With this understood, we denote by $\{B_i\}$ the image of the fixed-point set of H in M_H , which is a disjoint union of symplectic tori of self-intersection zero. Let F be the fiber class of the \mathbb{S}^2 -fibration on M_H . Then we note that $c_1(K_{M_H}) = \frac{1-p}{p} \sum_i B_i$ (cf. Proposition 3.2 in [7]), and $c_1(K_{M_H}) \cdot F = -2$. It follows easily that $p = 2$ or 3 , and $(\sum_i B_i) \cdot F = 4$ or 3 accordingly.

To proceed further, we choose an ω -compatible almost complex structure J on M_H , where ω denotes the symplectic structure on M_H , such that J is integrable in a neighborhood of each B_i . Note that this is possible because ω admits a standard model near each B_i . Now by Gromov's theory, there exists a \mathbb{S}^2 -bundle structure on M_H , with base T^2 and each fiber J -holomorphic. We denote by $\pi : M_H \rightarrow T^2$ the corresponding projection onto the base. Then by Lemma 5.2, the restriction $\pi|_{B_i} : B_i \rightarrow T^2$ is a branched covering where the ramification occurs exactly at the non-transversal intersection points of B_i with the fibers. But each B_i is a torus, so that $\pi|_{B_i}$ must be unramified, or equivalently, B_i intersects each fiber transversely. With this understood, it follows easily that the pre-image of each fiber of the \mathbb{S}^2 -bundle in M is a symplectic torus (here we use the fact that $(\sum_i B_i) \cdot F = 4$ or 3 respectively according to whether $p = 2$ or 3), giving M a structure of a T^2 -bundle over T^2 with symplectic fibers. This finishes the proof for the case where M_G is irrational ruled.

Next we assume M_G is rational and $G = \mathbb{Z}_2$. By Lemma 2.3 and Lemma 2.6(ii), the fixed-point set M^G consists of 8 isolated points and a disjoint union of 2-dimensional components $\{Y_i\}$, where $\sum_i Y_i^2 = 2(1 - b_2^-(M/G))$, and $b_2^-(M/G) \in \{0, 1, 2\}$. We denote by B_i the image of Y_i in M_G . Then $B_i^2 = 2Y_i^2$ for each i , and $c_1(K_{M_G}) = -\frac{1}{2} \sum_i B_i$ (cf. [7], Proposition 3.2), so that

$$c_1(K_{M_G})^2 = \frac{1}{4} \sum_i B_i^2 = 1 - b_2^-(M/G).$$

It follows easily that $M_G = \mathbb{C}\mathbb{P}^2 \# N \overline{\mathbb{C}\mathbb{P}^2}$ where $N = 8, 9$ or 10 , corresponding to $b_2^-(M/G) = 0, 1$ or 2 respectively. Moreover, note that M_G contains 8 symplectic (-2) -spheres coming from the resolution of the 8 isolated singular points of M/G .

By Theorem 1.4, the case where $M_G = \mathbb{C}\mathbb{P}^2 \# 8 \overline{\mathbb{C}\mathbb{P}^2}$ is immediately ruled out. The case where $M_G = \mathbb{C}\mathbb{P}^2 \# 10 \overline{\mathbb{C}\mathbb{P}^2}$ is ruled out as follows. We consider the double branched cover Z of M_G with branch loci $\{B_i\}$. Then Z is easily seen a symplectic Calabi-Yau 4-manifold with $b_1 = 0$, which is an integral homology $K3$ surface (compare [8], Theorems 1.1 and 1.2). Note that Z contains 16 embedded (-2) -spheres in the complement of the branch set. Now observe that in the case of $M_G = \mathbb{C}\mathbb{P}^2 \# 10 \overline{\mathbb{C}\mathbb{P}^2}$, $\sum_i Y_i^2 = -2$, so that there must be one Y_i with $Y_i^2 < 0$. This Y_i gives rise to an embedded (-2) -sphere in Z , in addition to the 16 embedded (-2) -spheres, so that Z contains 17 disjointly embedded (-2) -spheres. But this contradicts a theorem of Ruberman in [37], which says that an integral homology $K3$ surface can contain at most 16 disjointly embedded (-2) -spheres. Hence $M_G = \mathbb{C}\mathbb{P}^2 \# 10 \overline{\mathbb{C}\mathbb{P}^2}$ is ruled out. Finally, we note that the same argument shows that in the case of $M_G = \mathbb{C}\mathbb{P}^2 \# 9 \overline{\mathbb{C}\mathbb{P}^2}$, the surfaces B_i must be tori of self-intersection zero.

We continue by analyzing the case of $M_G = \mathbb{C}\mathbb{P}^2 \# 9 \overline{\mathbb{C}\mathbb{P}^2}$ in more detail. First, we note that there are at most two components in $\{B_i\}$. This is because $c_1(K_{M_G}) = -\frac{1}{2} \sum_i B_i$, and the a -coefficient of each B_i with respect to a given reduced basis is at least 3 (cf. Lemma 4.2(2)). Next, we determine the homology classes of the 8 symplectic (-2) -spheres F_1, F_2, \dots, F_8 in M_G . By Lemma 4.1, we can choose a symplectic structure on M_G so that the area constraints in Lemma 5.1 are satisfied. (Note that this is possible because $-c_1(K_{M_G}) \cdot [\omega] = \frac{1}{2} \sum_i \omega(B_i) > 0$.) Then the classes of F_1, F_2, \dots, F_8 are given in 3 cases as listed in Lemma 5.1. We claim that case (a) and case (b) cannot occur. To see this, suppose we are in case (a). It is easy to check, with the area constraints in Lemma 5.1, that the class E_{i_9} has the smallest area among the E_i -classes in the reduced basis. With this understood, we choose an almost complex structure J such that each symplectic (-2) -sphere F_k is J -holomorphic. Then by Lemma 3.2, the class E_{i_9} can be represented by a J -holomorphic (-1) -sphere C . Symplectically blow down M_G along C , noting that C is disjoint from the (-2) -spheres F_k as $C \cdot F_k = 0$, we obtain 8 disjointly embedded symplectic (-2) -spheres in $\mathbb{C}\mathbb{P}^2 \# 8 \overline{\mathbb{C}\mathbb{P}^2}$, contradicting Theorem 1.4. Case (b) is similarly eliminated. Consequently, the homology classes of F_1, F_2, \dots, F_8 are given by case (c) of Lemma 5.1.

Our next step is to show that there is an embedded symplectic sphere with self-intersection zero, denoted by F , which lies in the complement of F_1, F_2, \dots, F_8 and intersects transversely and positively with B_i . This can be seen as follows. It is easy to check that in case (c) of Lemma 5.1, the class E_{l_1} has the largest area. By Lemma 3.2, we can choose ω -compatible almost complex structures J so that B_i and F_k are all J -holomorphic, and successively represent the classes E_{l_s} , $s \geq 2$, beginning with the one of the smallest area, by a J -holomorphic (-1) -sphere. By successively symplectically blowing down the classes E_{l_s} , $s \geq 2$, we reach $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$, with E_{l_1} being the (-1) -class

(see [8], Section 4, for a discussion on the general procedure). Note that the (-2) -spheres F_1, F_2, F_3, F_4 descend to 4 disjointly embedded symplectic spheres of self-intersection zero (they all have class $H - E_{i_1}$); in fact there is a symplectic \mathbb{S}^2 -fibration of $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$ containing them as fibers. With this understood, we can take a fiber F in the complement which intersects transversely and positively with the descendant of B_i in $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$. We then symplectically blow up $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$ successively, reversing the symplectic blowing down procedure, in order to go back to M_G . In this way, we recover the 8 symplectic (-2) -spheres F_1, F_2, \dots, F_8 and the tori B_i , although the symplectic structure on M_G may be different since we don't keep track of the sizes of the symplectic blowing up.

Now we symplectically blow down F_1, F_2, \dots, F_8 , which results in a symplectic 4-orbifold X with 8 isolated singular points, all of isotropy of order 2. In the complement of the singularities, there lies the embedded symplectic sphere F with $F^2 = 0$, and the tori B_i . By [22], we can assume that F and B_i intersect symplectically orthogonally without loss of generality.

With the preceding understood, we consider the set \mathcal{J} of ω -compatible almost complex structures on X which satisfy the following conditions: fix a sufficiently small regular neighborhood V of $\cup_i B_i$, not containing any singular points of X , and fix an integrable ω -compatible almost complex structure J_0 on V , then for each $J \in \mathcal{J}$, $J = J_0$ on V and F is J -holomorphic. With this understood, note that for any $J \in \mathcal{J}$, the deformation of the J -holomorphic sphere F is unobstructed (cf. [24]). We denote by \mathcal{M}_J the moduli space of J -holomorphic spheres having the homology class of F . Then $\mathcal{M}_J \neq \emptyset$ and is a smooth 2-dimensional manifold. In the present situation, \mathcal{M}_J is not compact, but can be compactified using the orbifold version of Gromov compactness theorem (cf. [6, 11]). The key issue here is to understand the compactification $\overline{\mathcal{M}_J}$ of \mathcal{M}_J , at least for a generic $J \in \mathcal{J}$.

Lemma 5.3. *Let $\{S_n\}$ be a sequence in \mathcal{M}_J which converges to a Gromov limit $\sum_i m_i C_i \in \overline{\mathcal{M}_J} \setminus \mathcal{M}_J$. Then for a generic $J \in \mathcal{J}$, $\{C_i\}$ consists of a single component of multiplicity 2, which is an embedded orbifold sphere containing exactly 2 singular points of X .*

Proof. Since J is generic, there is no J -holomorphic $(-\alpha)$ -sphere lying in the complement of the singular points of X for any $\alpha > 1$. Moreover, just as in the smooth case, $\{S_n\}$ can not split off a J -holomorphic (-1) -sphere lying entirely in the smooth locus of X . It follows easily that in the Gromov limit $\sum_i m_i C_i \in \overline{\mathcal{M}_J} \setminus \mathcal{M}_J$, each component C_i must contain a singular point of X .

With this understood, we take an arbitrary component C_i . Suppose C_i contains $k > 0$ singular points of X . Then we can pick an orbifold Riemann sphere Σ with k orbifold points of order 2, which are denoted by z_1, z_2, \dots, z_k , and find a J -holomorphic map $f : \Sigma \rightarrow X$ parametrizing C_i . Recall that such a map f near an orbifold point z_j , assuming of order m_j , is given by a pair (\hat{f}_j, ρ_j) , where $\hat{f}_j : D \rightarrow \mathbb{C}^2$ is a local lifting of f near z_j to the uniformizing system at $f(z_j) \in X$, and $\rho_j : \mathbb{Z}_{m_j} \rightarrow G_{f(z_j)}$ is an injective homomorphism to the isotropy group $G_{f(z_j)}$ at $f(z_j) \in X$, with respect to which \hat{f}_j is equivariant. With this understood, we let $g \in \mathbb{Z}_{m_j}$ be the generator acting on D by a

rotation of angle $2\pi/m_j$, and let $(m_{j,1}, m_{j,2})$, $0 \leq m_{j,1}, m_{j,2} < m_j$, be the weights of the action of $\rho_j(g) \in G_{f(z_j)}$ on \mathbb{C}^2 . Then the dimension of the moduli space of J -holomorphic curves containing C_i equals $2d$, where $d \in \mathbb{Z}$ and is given by

$$d = c_1(TX) \cdot C_i + 2 - \sum_{j=1}^k \frac{m_{j,1} + m_{j,2}}{m_j} - (3 - k).$$

See [11, 6]. Note that in the present situation, $m_j = 2$ and $m_{j,1} = m_{j,2} = 1$ for each j . It follows easily that $d = c_1(TX) \cdot C_i - 1$; in particular, $c_1(TX) \cdot C_i \in \mathbb{Z}$. Moreover, since J is generic, we have $d \geq 0$, which implies that $c_1(TX) \cdot C_i \geq 1$.

As an immediate corollary, we note that $\{C_i\}$ either consists of two components, each with multiplicity 1, or a single component with multiplicity 2, and moreover, $c_1(TX) \cdot C_i = 1$ for each i . This is because $c_1(TX) \cdot F = 2$, and $F = \sum_i m_i C_i$. We can further rule out the possibility of two components as follows. Suppose there are two components C_1, C_2 in $\{C_i\}$. Then $C_1^2 + 2C_1 \cdot C_2 + C_2^2 = F^2 = 0$ implies that one of C_1^2, C_2^2 must be negative. Without loss of generality, assume $C_1^2 < 0$. Then $C_2^2 \geq 0$ because $b_2^-(X) = 1$. With this understood, we note that $C_1 \cdot C_2 \geq \frac{1}{2}$ by the orbifold intersection formula in [5] (see also [6]). This implies $C_1^2 \leq -1$. Now we apply the orbifold adjunction inequality (cf. [5, 6]) to C_1 , which gives

$$C_1^2 - c_1(TX) \cdot C_1 + 2 \geq k(1 - \frac{1}{2}).$$

With $C_1^2 \leq -1$ and $c_1(TX) \cdot C_1 = 1$, it follows that $k = 0$, which is a contradiction. Hence the claim that there is only one component in $\{C_i\}$.

Let C denote the single component which has multiplicity 2, and let $f : \Sigma \rightarrow X$ be a J -holomorphic parametrization of C . Then we note that $C^2 = 0$ and $c_1(TX) \cdot C = 1$. Applying the orbifold adjunction formula to C (cf. [5, 6]), we get

$$C^2 - c_1(TX) \cdot C + 2 = k(1 - \frac{1}{2}) + \sum k_{[z, z']} + \sum k_z,$$

where $k_{[z, z']}, k_z \in \mathbb{Q}$ are nonnegative and have the following significance. For any $z, z' \in \Sigma$, where $z \neq z'$, such that $f(z) = f(z')$, the number $k_{[z, z']} > 0$. Moreover, if $f(z) = f(z')$ is a smooth point of X , then $k_{[z, z']} \in \mathbb{Z}$. Likewise, for any $z \in \Sigma$, if f is not a local orbifold embedding near z , then $k_z > 0$. Moreover, if $f(z)$ is a smooth point of X , then $k_z \in \mathbb{Z}$. With this understood, it follows easily that $k \leq 2$, and if $k = 2$, then all $k_{[z, z']}, k_z = 0$, which means that C is an embedded 2-dimensional suborbifold. To rule out the possibility that $k = 1$, we first observe that in this case, $k_{[z, z']} \in \mathbb{Z}$. This is because as $k = 1$, we can not have a pair of points $z, z' \in \Sigma$, where $z \neq z'$, such that $f(z) = f(z')$ is a singular point of X . It follows easily that all $k_{[z, z']}$ must be zero, and $k_z = \frac{1}{2}$ at the unique singular point $f(z)$ on C . The number k_z is the local self-intersection number of C at the singular point, and $k_z = \frac{1}{2}$ means that in the uniformizing system near the singular point, C is given by a J -holomorphic (singular) disc with a local self-intersection 1 at the origin. It follows that the singularity at the origin must be a cusp singularity and the J -holomorphic disc is parametrized by a pair of functions $z_1 = t^2, z_2 = t^3 + \dots$, where

$t \in D$. However, it is clear that such defined J -holomorphic disc is not invariant under the \mathbb{Z}_2 -action $(z_1, z_2) \mapsto (-z_1, -z_2)$, which is a contradiction. Hence $k = 1$ is ruled out. This finishes the proof of the lemma. \square

It follows easily that the compactified moduli space $\overline{\mathcal{M}}_J$ gives rise to a J -holomorphic \mathbb{S}^2 -fibration on X , which contains 4 multiple fibers, each with multiplicity 2. We denote by $\pi : X \rightarrow B$ the \mathbb{S}^2 -fibration. It is easy to see that the base B is an orbifold sphere, with 4 orbifold points of order 2. Furthermore, note that for each i , $\pi|_{B_i} : B_i \rightarrow B$ is a branched covering in the complement of the multiple fibers by Lemma 5.2.

To proceed further, we note that $c_1(K_X) = -\frac{1}{2} \sum_i B_i$, so that $(\sum_i B_i) \cdot F = 4$. Let z_1, z_2, z_3, z_4 be the orbifold points of B , and let $w_1, \dots, w_k \in B$ be the points parametrizing those regular fibers which do not intersect transversely with $\cup_i B_i$. We denote by x_l the number of intersection points of $\cup_i B_i$ with the multiple fiber at z_l , $l = 1, 2, 3, 4$, and denote by y_j the number of intersection points of $\cup_i B_i$ with the regular fiber at w_j , where $j = 1, 2, \dots, k$. Then note that $x_l \leq 2$ and $y_j < 4$ for each l, j . On the other hand, we observe the following relation in Euler numbers:

$$\sum_i \chi(B_i) - \sum_{j=1}^k y_j - \sum_{l=1}^4 x_l = 4(\chi(|B|) - k - 4),$$

where $|B| = \mathbb{S}^2$ is the underlying space of B . With $x_l \leq 2$ and $y_j < 4$, it follows easily that k must be zero, and $x_l = 2$ for each l . This means that $\cup_i B_i$ intersects each regular fiber transversely at 4 points and intersects each multiple fiber at 2 points.

Finally, we observe that $X = |M/G|$, i.e., X is the symplectic 4-orbifold obtained by desingularizing M/G along the 2-dimensional singular components. With this understood, it is easy to see that under the projection $M \rightarrow X = |M/G|$, the pre-image of each regular fiber in the \mathbb{S}^2 -fibration on X is a symplectic T^2 in M , giving rise to a T^2 -fibration over B on M (here we use the fact that $\cup_i B_i$ intersects each regular fiber transversely at 4 points and the projection $M \rightarrow X$ is a double cover branched over $\cup_i B_i$). Moreover, the pre-image of each multiple \mathbb{S}^2 -fiber is a multiple T^2 -fiber of multiplicity 2 in the T^2 -fibration on M . It is known that such a 4-manifold M is diffeomorphic to a hyperelliptic surface or a secondary Kodaira surface, see [17]. Since $b_1(M) \neq 1$, M must be diffeomorphic to a hyperelliptic surface. This finishes the proof of Theorem 1.1.

Proof of Theorems 1.2 and 1.3:

Suppose G is of prime order p . The case where M_G has torsion canonical class is contained in Lemmas 2.1 and 2.8(2), and the case where M_G is irrational ruled is in Lemmas 2.2 and 2.6(i), with $p = 2$ or 3 from the proof of Theorem 1.1.

Suppose M_G is rational. Then by Lemmas 2.3, 2.4, 2.6 and 2.8, the order $p = 2, 3$ or 5. Concerning the fixed-point set structure, the case of $G = \mathbb{Z}_2$ follows readily from the proof of Theorem 1.1. For $G = \mathbb{Z}_3$, the fixed-point set structure for the isolated points is determined in Lemmas 2.4 and 2.9. Regarding the 2-dimensional fixed components, we explore the embedding $D \rightarrow M_G$. In order to determine M_G in each case, we use

the formula in Proposition 3.2 of [7] to determine $c_1(K_{M_G})$, based on the singular set structure of the quotient orbifold M/G , then we compute $c_1(K_{M_G})^2$. This allows us to determine the diffeomorphism type of M_G as M_G is a rational 4-manifold. In the case of $b_1(M) = 2$, it is easy to see that $M_G = \mathbb{C}\mathbb{P}^2 \# 10\overline{\mathbb{C}\mathbb{P}^2}$. If the set of 2-dimensional fixed components is nonempty, Proposition 4.3 implies that it must consist of a single torus. In the case of $b_1(M) = 4$, $M_G = \mathbb{C}\mathbb{P}^2 \# 12\overline{\mathbb{C}\mathbb{P}^2}$, and Proposition 4.4 implies that there are no 2-dimensional fixed components. For $G = \mathbb{Z}_5$ where $b_1(M) = 4$, the fixed-point set structure for the isolated points is determined in Lemma 2.10. The possible 2-dimensional fixed components are excluded by Proposition 4.5.

For the case where G is of non-prime order, the order of G and the fixed-point set structure are determined in Lemmas 2.11 and 2.12. This completes the discussion on Theorems 1.2 and 1.3.

Proof of Theorem 1.4:

First, consider the case of $N = 8$. We begin by showing that one can choose a symplectic structure on X such that the area constraints in Lemma 5.1 are fulfilled. To see this, by Lemma 4.1 we can choose symplectic structures ω on X such that one of the 8 symplectic (-2) -spheres has area δ_1 and the remaining 7 symplectic (-2) -spheres have area δ_2 , where $\delta_2 < \delta_1 < 2\delta_2$, and δ_1, δ_2 can be arbitrarily small. It remains to show that one can arrange so that $7\delta_i < -c_1(K_X) \cdot [\omega]$, $i = 1, 2$, hold true. For this, we recall the fact that for $X = \mathbb{C}\mathbb{P}^2 \# N\overline{\mathbb{C}\mathbb{P}^2}$, where $N \leq 8$, $-c_1(K_X)$ can be represented by pseudo-holomorphic curves, and moreover, one can require the pseudo-holomorphic curves to pass through any given point in X , see Taubes [42]. We pick a point $x_0 \in X$ in the complement of the 8 symplectic (-2) -spheres and require the pseudo-holomorphic curves representing $-c_1(K_X)$ to pass through x_0 . Then it is easy to see that no matter how small we choose the areas δ_1, δ_2 , $-c_1(K_X) \cdot [\omega] > \delta_0$ for some δ_0 independent of the choice of δ_1, δ_2 . It follows that we can arrange so that $7\delta_i < -c_1(K_X) \cdot [\omega]$, $i = 1, 2$, hold true.

With the preceding understood, by the same argument as in Lemma 5.1, we can show that the homology classes of the 8 symplectic (-2) -spheres must be given as in case (a) of Lemma 5.1. Then by Lemma 3.2, we can successively symplectically blow down the E_{i_s} classes for $s \geq 2$ and reach to the 4-manifold $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$, with E_{i_1} being the (-1) -class, such that the 7 symplectic (-2) -spheres F_2, F_3, \dots, F_8 descend to a configuration of symplectic spheres of the class H , which intersect transversely and positively according to the incidence relation of the Fano plane; that is, the 7 spheres intersect in 7 points, where each point is contained in 3 spheres. By a theorem of Ruberman and Starkston (cf. [38]), such a configuration cannot exist in $\mathbb{C}\mathbb{P}^2$. Thus to derive a contradiction, we need to represent the class E_{i_1} by a symplectic (-1) -sphere in the complement of the 7 symplectic spheres, to further blow down $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$.

To this end, we note that the configuration of 7 symplectic spheres in $\mathbb{C}\mathbb{P}^2 \# \overline{\mathbb{C}\mathbb{P}^2}$ is J -holomorphic with respect to some compatible almost complex structure J . On the other hand, the class E_{i_1} is represented by a finite set of J -holomorphic curves $\sum_i m_i C_i$ by Taubes' theorem (cf. [29]). Now the key observation is that if there are more than one

components in $\{C_i\}$, then one of them must have a negative a -coefficient in the reduced basis H, E_{i_1} . But such a component intersects negatively with any of the 7 J -holomorphic spheres in the configuration (which has class H). This is a contradiction, hence E_{i_1} must be represented by a single J -holomorphic curve, which is a (-1) -sphere and lies in the complement of the configuration of 7 symplectic spheres. This finishes the proof for the case of $N = 8$.

The argument for the case of $N = 7$ is similar. For $N = 9$, it is easy to see from Lemma 5.1 that the homology class for the 9-th symplectic (-2) -sphere does not exist. This completes the proof of Theorem 1.4.

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