Knot concordances in $S^1 \times S^2$ and exotic smooth 4-manifolds

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ABSTRACT. It is known that there is a unique concordance class in the free homotopy class of $S^1 \times pt \subset S^1 \times S^2$. The constructive proof of this fact is given by the second author. It turns out that all the concordances in this construction are invertible. The knots $K \subset S^1 \times S^2$ with hyperbolic complements and trivial symmetry group are of special interest here, because they can be used to generate absolutely exotic compact 4-manifolds by the recipe given by Akbulut and Ruberman. Here we built absolutely exotic 4-manifold pairs by this construction, and show that this construction keeps the Stein property of the 4-manifolds we start out with. By using this we establish the existence of an absolutely exotic contractible Stein manifold pair, and an absolutely exotic homotopy $S^1 \times B^3$ Stein manifold pair.

1. Introduction

Here we will prove the following theorems which strengthens [3]:

Theorem 1. *There is a pair of compact contractible Stein 4-manifolds W_1, W_2, which are homeomorphic but not diffeomorphic to each other.*

As in the examples of [3], W_1 and W_2 are related to each other by cork twisting along a cork $(W, f) \subset W_i$, $i = 1, 2$ (e.g. [1]). It follows from the construction that W_1 and W_2 can not be corks.

Theorem 2. *There is a pair of compact Stein 4-manifolds Q_1, Q_2 homotopy equivalent to $S^1 \times B^3$, which are homeomorphic but not diffeomorphic to each other.*

Similarly, Q_1 and Q_2 are related to each other by "anticork" twisting (e.g. [1]) along an anticork contained in (W, f). The important feature of our theorems is that the manifolds constructed are all Stein. This property enables us to imbed them into closed symplectic manifolds and relate their exoticity to the non-vanishing gauge theory invariants of closed symplectic manifolds ([13]). The proofs of these theorems will be using:

Theorem 3. *Any knot K in $S^1 \times S^2$, which is freely homotopic to $S^1 \times pt$, is invertibly concordant to $S^1 \times pt$.*

2. Background

Let us recall some basic definitions about knot concordances:

Definition 1. *Two oriented knots K_1 and K_2 are said to be* concordant *if there is a smooth proper embedding of an annulus $F : S^1 \times [0, 1] \hookrightarrow Y \times [0, 1]$, such that its boundary is $\partial F(S^1 \times [0, 1]) = K_1 \times 0 \sqcup (-K_2) \times 1$, where $-K_2$ is the knot K_2 with the reversed orientation.*

Definition 2. [12] *A concordance F between knots K_1 and K_2 is said to be* invertible *if there is a concordance F' from K_2 to K_1 such that $F \cup F' : S^1 \times [0, 1] \hookrightarrow Y \times [0, 1]$ is the product concordance $K_1 \times [0, 1] \subset Y \times [0, 1]$. In this case, we say K_1 is invertibly concordant to K_2, and K_2 splits $K_1 \times [0, 1]$. In particular when $Y = S^3$ and K_1 is the unknot then K_2 is called* doubly slice.

Definition 3. *An invertible cobordism X from M to N is a smooth manifold with $\partial X = M \sqcup -N$, such that there is a cobordism X' with $\partial X' = N \sqcup -M$ and $X \underset{N}{\cup} X'$ is diffeomorphic to $M \times I$.*

Proposition 4. *The knot $K \subset S^1 \times S^2$ given in the Figure 1 is hyperbolic with trivial symmetry group.*
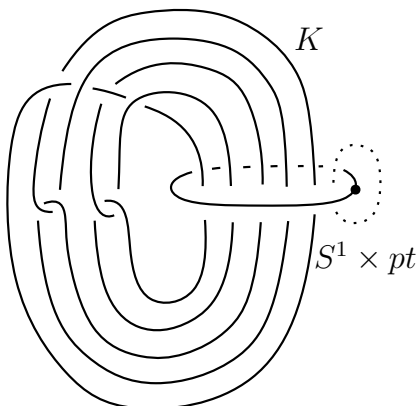

FIGURE 1. The knot $K \subset S^1 \times S^2$ has hyperbolic complement and trivial symmetry group

Proposition 4 was proved by computations using SnapPy [4], and verifications of these computations are due to [5] and [8].

By using Theorem 3 and Proposition 4 we will convert relatively exotic structures on compact 4-manifolds to absolutely exotic smooth structures, by using the construction given in [3]. Our goal is to generate Stein exotic examples that can not be obtained from [3]. Now recall the construction of absolutely exotic smooth structures on compact 4-manifolds, from relatively exotic structures:

Theorem 5 ([3]). *Let $F : W \to W$ be self homeomorphism of a compact smooth 4-manifold, whose restriction to ∂W is a diffeomorphism which does not extend to a self diffeomorphism of W. Then W contains a pair of homeomorphic smooth 4-manifolds V and V' homotopy equivalent to W, but V and V' are not diffeomorphic to each other.*

The construction of [3] relies on finding a knot in S^3 which is doubly slice, hyperbolic, and with no symmetry. Here we will remind the original construction, and show that we can use invertible knot concordances in $S^1 \times S^2$ between a knot K and $S^1 \times pt$, where K is a hyperbolic knot with trivial symmetry group, splitting the concordance.

Lemma 6. *Let K be a knot in $S^1 \times S^2$, which is freely homotopic to $S^1 \times pt$ with exterior X, then $H_*(X; \mathbb{Z}) = H_*(S^1 \times D^2; \mathbb{Z})$.*

Proof. Follows from Mayer-Vietoris sequence. $\qquad\square$

Corollary 7. *Let K_1, K_2 be knots in the free homotopy class of the $S^1 \times pt$ in $S^1 \times S^2$, and C be a concordance between K_1 and K_2. Then:*

$$H_*(S^1 \times S^2 \times I - \nu(C); \mathbb{Z}) = H_*(S^1 \times D^2 \times I; \mathbb{Z})$$

Note that a tubular neighbourhood $\nu(C)$ of C in $S^1 \times S^2 \times I$ is an embedding of $S^1 \times D^2 \times I$ in $S^1 \times S^2 \times I$, where the image of $S^1 \times 0 \times I$ is C. The image of $pt \times \partial D^2 \times t$ is called a meridian of C, which bounds a disk in the tubular neighbourhood of C, and it is homologous to zero in the exterior. Similarly we call the image of $S^1 \times pt \times t$ as a longitude.

The following lemma is adapted from [3]. We restate it here in a slightly different way to show that we can use appropriate concordances of knots in $S^1 \times S^2$ instead of in S^3. The only difference here is that the gluing map ϕ identifies meridian to meridian not to longitude.

Lemma 8. *Suppose that γ is a framed knot in a closed 3-manifold M, and C is an invertible concordance from the $S^1 \times pt$ to the knot K in $S^1 \times S^2 \times I$. Define*

$$H = \left(M \times I - \left(\gamma \times D^2 \times I \right) \right) \bigcup_{\phi} \left(S^1 \times S^2 \times I - \left(C \times D^2 \right) \right)$$

where $\phi : T^2 \times I \to T^2 \times I$ is a diffeomorphism which sends the meridian of K to the meridian of the knot γ. Then H is an invertible homology cobordism from M to a 3-manifold N. If $\pi_1(S^1 \times S^2 \times I - C \times D^2) \cong \mathbb{Z}$, then the inclusion $\iota : M \hookrightarrow H$ induces an isomorphism on fundamental groups.

3. Proof of Theorem 3

Proof of Theorem 3. Start with a knot $K \subset S^1 \times S^2$ in the concordance class of $S^1 \times pt$. Built a concordance F between K and $S^1 \times pt$ as it is explained in [15, Theorem 2]. F is properly embedded in $S^1 \times S^2 \times I$, the last coordinate is the time, and we have $S^1 \times S^2$ at every level. From the construction we see the knot K at the top level, and we perform genus zero cobordism, i.e. we attach bunch of bands $\{b_i\}_{i=1}^m$, turning K to $S^1 \times pt$ union disjoint unknots $\{u_i\}_{i=1}^m$ linking $S^1 \times pt$. For $1 > t_2 > t_1 > 0$, following levels are depicted partially in Figure 2

- $F \cap (S^1 \times S^2 \times 1) = K$
- $F \cap (S^1 \times S^2 \times t_2) = K \cup b_1$
- $F \cap (S^1 \times S^2 \times t_1) = u_1 \sqcup (S^1 \times pt)$
- $F \cap (S^1 \times S^2 \times 0) = S^1 \times pt$
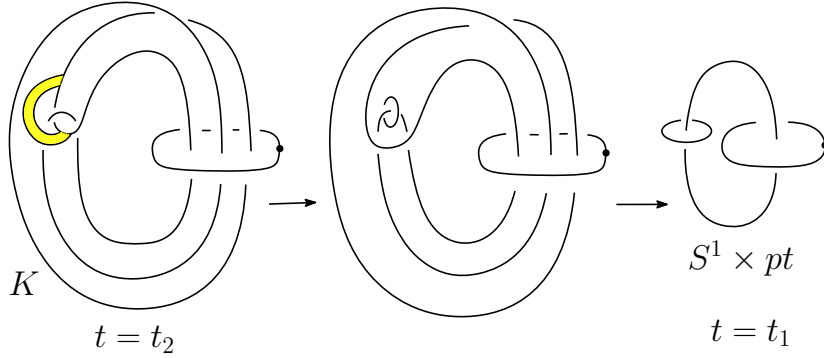
FIGURE 2. The construction of the concordance

where each $u_i = \partial D^2$ is an unknot in $S^1 \times S^2$ which is the boundary of the corresponding 0-handle of F, and b_i represents a band which is a 1-handle attached to the surface. F' in the Definition 2 is $-F$ here.

Next will be to construct a handlebody decomposition for the concordance complement $S^1 \times S^2 \times I - \nu(F \cup (-F))$ from the handlebody decomposition of the surface F. For a detailed discussion of handlebody decomposition of surface complement in 4-manifolds one can consult Chapter 1.4 of [1]. To construct the complement of F in $S^1 \times S^2 \times I$ start from the bottom. First we see complement of $S^1 \times pt \times [0, t]$ in $S^1 \times S^2 \times [0, t]$ which is $S^1 \times D^2 \times [0, t]$. Then as t increase, 0-handles of the surface F appears, in the complement which corresponds to carving properly embedding of disks from B^4. So we have connected sum of $m + 1$ copies of $S^1 \times B^3$. At the last step the complement gains m many 4-dimensional 2-handles for the 1-handles b_i of the surface, as in the middle picture

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of Figure 3. Alternatively the reader should compare this to [15], where the cobordism was constructed from top to bottom. For example, Figure 3 describes the concordance from the Mazur knot to $S^1 \times pt$.

To double the concordance complement, we attach upside-down handles along the dual of the 2-handles (with zero framing), and dual 3-handles as upside-down 1-handles. We use the dual 2-handles to get rid of the self-linking of the original 2-handle, therefore ending up with cancelling 1/2 and 2/3 handle pairs. After cancellations, the complement becomes a product. In Figure 3, reader can verify that sliding over the dual 2-handle (0-framed circle linking the 2-handle) will make the 2-handle trivially link the 1-handle.
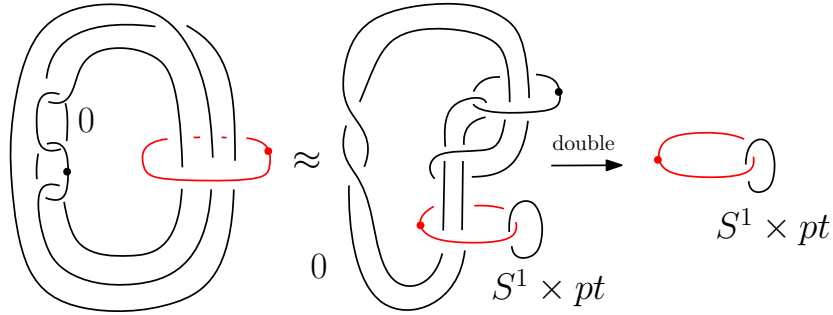
FIGURE 3. The construction of the concordance complement

All these cancellations happens away from the endpoints
$S^1 \times S^2 \times \{0\} - \nu(S^1 \times \{pt\})$ and $S^1 \times S^2 \times \{1\} - \nu(S^1 \times \{pt\})$, provided that the attaching circles of the two handles are away from the $S^1 \times pt$. Note that anything links to $S^1 \times pt$ can be unlinked by sliding over the middle (red) 1-handle. So far we have constructed a diffeomorphism from $S^1 \times S^2 \times I - \nu(F \cup (-F))$ to $S^1 \times S^2 \times I - \nu(S^1 \times pt \times I) = S^1 \times B^2 \times I$, which is identity near the endpoints. Next, we prove that this diffeomorphism extends to self diffeomorphism of $S^1 \times S^2 \times I$ which takes the surface $F \cup -F$ to the product $S^1 \times \{pt\} \times I$.

The diffeomorphism above induces a boundary diffeomorphism on $\partial(S^1 \times B^2 \times I) = S^1 \times S^1 \times I \cup (S^1 \times B^2 \times 0 \sqcup S^1 \times B^2 \times 1)$. When we restrict this to the partial boundary we get a diffeomorphism $\tau : T^2 \times I \to T^2 \times I$ which is identity near $T^2 \times \{0, 1\}$. By Lemma 3.5 of Waldhausen [14], τ is isotopic to the identity map rel boundary hence it extends to tubular neighbourhoods of the surfaces. Hence we have a self diffeomorphism of $S^1 \times S^2 \times I$ taking $\nu(S^1 \times pt \times I)$ to $\nu(F \cup (-F))$. So diffeomorphic complements uniquely determine the surfaces. $\square$

Theorem 3 is related to light bulb theorems, that is after doubling the concordance, we have a surface in $S^1 \times S^2 \times I$ which is bounded by $S^1 \times pt \times i$ $(i = 0, 1)$. By capping off both boundaries with $D^2 \times S^2$ s, we get a surface in $S^2 \times S^2$ intersecting $pt \times S^2$ at one point, then apply [11] and [6]. The advantage of this construction is, it gives a rel boundary diffeomorphism $(S^1 \times S^2 \times I, F \cup -F) \to (S^1 \times S^2 \times I, S^1 \times pt \times I)$.

By applying the technique of Section 3 to the knot $K \subset S^1 \times S^2$ of Figure 1, we get the Figure 4, where the knot K now looks like the standard linking circle to the 1-handle (dotted curve in the figure).
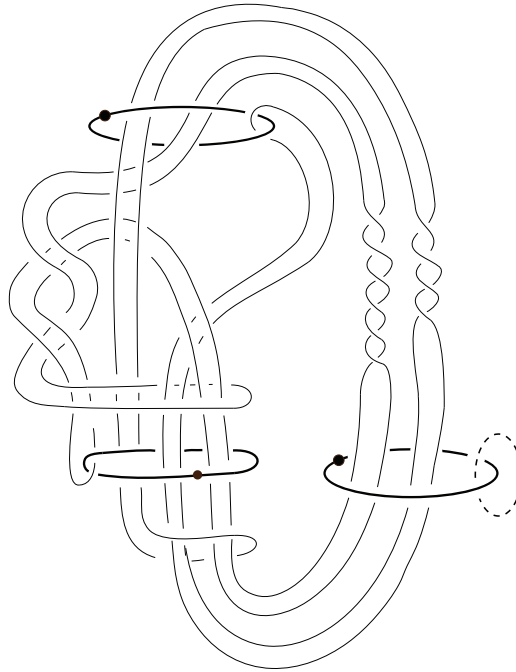
FIGURE 4. The concordance complement in $S^1 \times S^2 \times I$

In Figure 4 the concordance from $S^1 \times pt$ to K is fully visible (i.e. the handles of the complement of K are visible). Clearly cancelling the handles of Figure 4 gives Figure 1 (the dotted circle corresponds to K).

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3.1. Proof of Theorem 1

First we proceed as in [3], i.e. glue the homology product cobordism H obtained from Figure 4 to the boundary of the cork W. From this we get a new contractible manifold W_1 containing W, such that performing the cork twisting $W \subset W_1$ gives us an absolutely exotic copy W_2 of W_1. Recall $W_1 = W \cup H$ by gluing W and the homology product cobordism H along the longitudes of $\eta \subset W$ and $K \subset S^1 \times S^2$, which is the Figure 5. Here we are using the *roping* technique of [1] to draw the handlebody of W_1. From the discussion above, Figure 5 is equivalent to Figure 6. By zero and dot exchanges to W inside W_1 gives W_2.The only remaining issue is to put Stein structure on W_1 and W_2 (they are homeomorphic by Freedman's theorem).
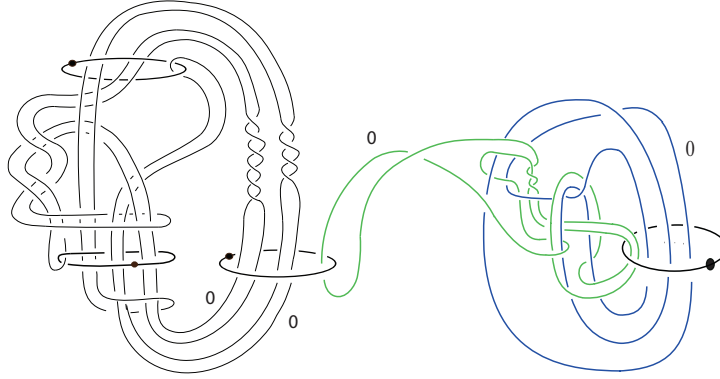
FIGURE 5. W_1
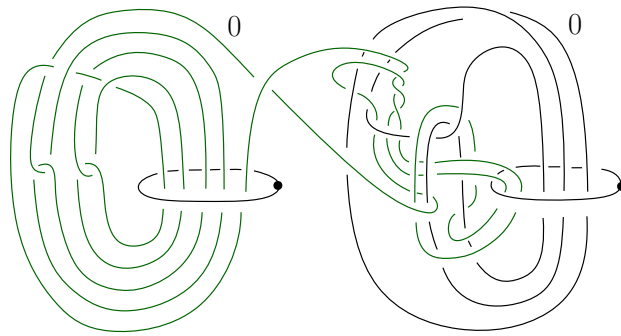
FIGURE 6. W_1

For this we will work with the picture of W_1, given in Figure 6. We first put Figure 1 in Legendrian position and get Figure 7, where the 2-handle has $tb = 4$.
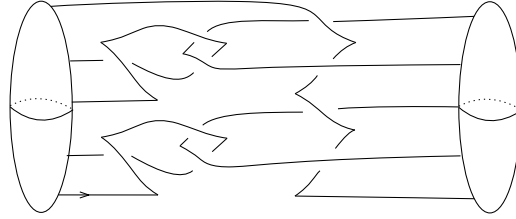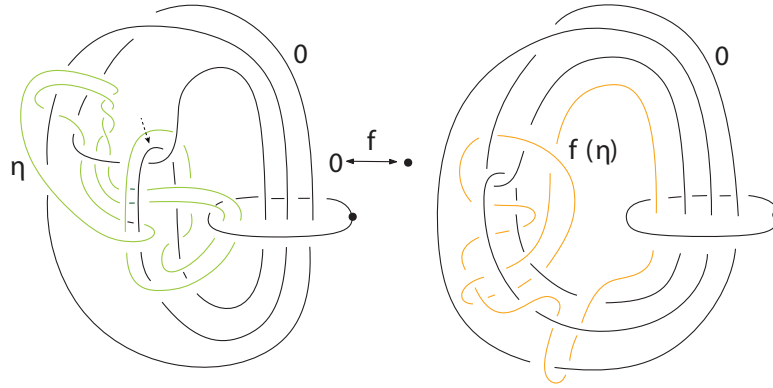

FIGURE 7. tb=4

FIGURE 8

Two pictures of Figure 8 are related to each other by zero-dot exchanges. Then we put both handlebodies of Figure 8 in Legendrian position, and connected sum with Figure 7. This gives Figures 11 and 12, which are pictures of W_1 and W_2 where both are Stein. $\square$
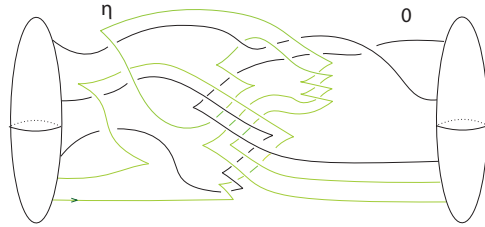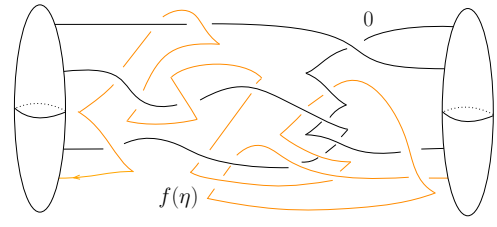
Knot concordances in $S^1 \times S^2$ and exotic smooth 4-manifolds

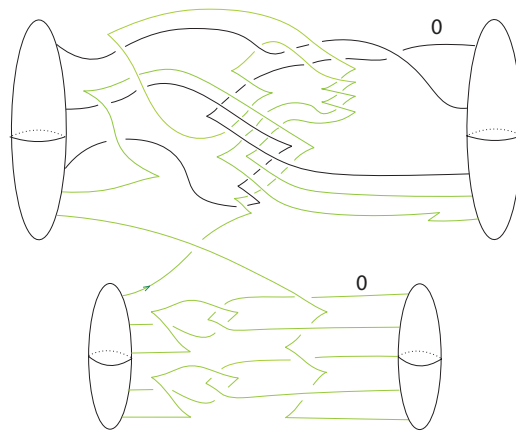FIGURE 9. $tb(\eta) = -3$

FIGURE 10. $tb(f(\eta)) = -3$
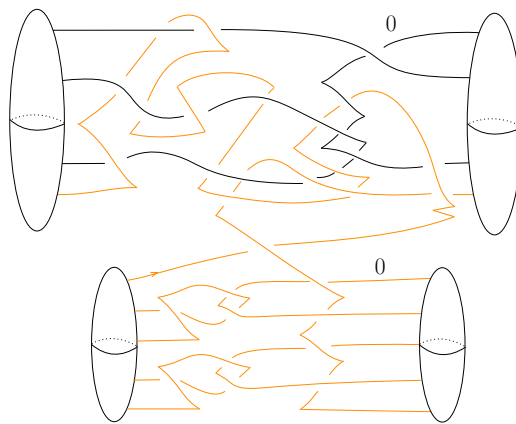

FIGURE 11. W_1, as a Stein handlebody

FIGURE 12. W_2, an exotic copy of W_1, as a Stein handlebody

3.2. Proof of Theorem 2

As above, we will use the construction of [3] in such a way that the end result will be a pair of Stein manifolds, homotopy equivalent to S^1, which are absolutely exotic copies of each other. Now recall the construction of [2] (and [1]): Knot $R \subset S^3$ on the left of Figure 13 bounds a ribbon $D^2 \subset B^4$ in two different ways. Both of these ribbon complements $C^4 = B^4 - N(D)$ are diffeomorphic to each other, where $N(D)$ denotes the tubular neighborhood of D. Furthermore, C^4 is homotopy equivalent to S^1, and the diffeomorphism $f : \partial C \to \partial C$ is described by the involution of Figure 13 (the "zero-dot exchange" of the two middle pictures of Figure 13) does not extend to a diffeomorphism $C \to C$, but extends to a homeomorphism. f is the anticork involution in 10.2 of [1].
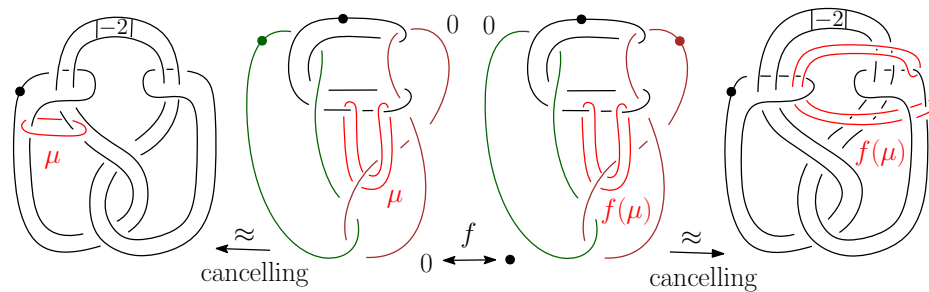

Figure 13. $f : \partial C \to \partial C$

Next we observe that C^4 is a Stein manifold by drawing the first picture of Figure 13 as in Figure 14 (this was first verified by Luke Williams). Then pick a knot $\mu \subset \partial C^4$ with hyperbolic complement and trivial symmetry (again verifying by SnapPy [4]). The first and last pictures of Figure 13 becomes Figures 14 and Figure 15.
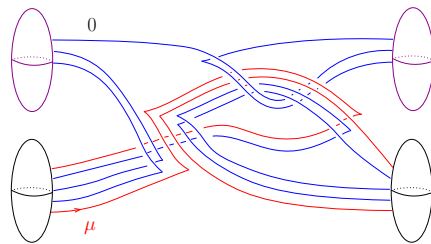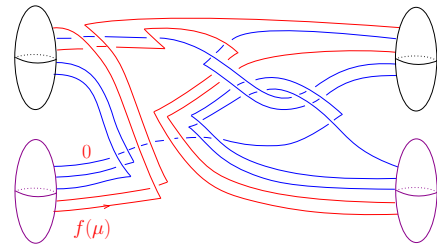

Figure 14. $tb(\mu) = -1$

Figure 15. $tb(f(\mu)) = -3$

Finally by the roping technique of [1], we glue H to C by first identifying K and μ, then identifying K and $f(\mu)$ getting the manifolds Q_1 and Q_2 in Figures 16 and 17. Clearly both are Stein manifolds, and are homeomorphic to each other by Freedman's theorem (calculations $\pi_1(Q_i) \cong \mathbb{Z}$ for $i = 1, 2$ can be checked from the pictures). $\square$
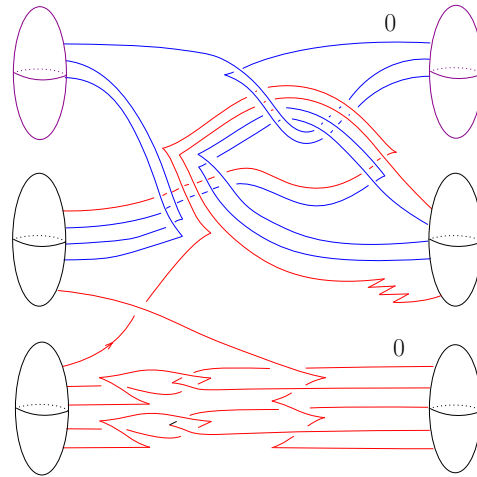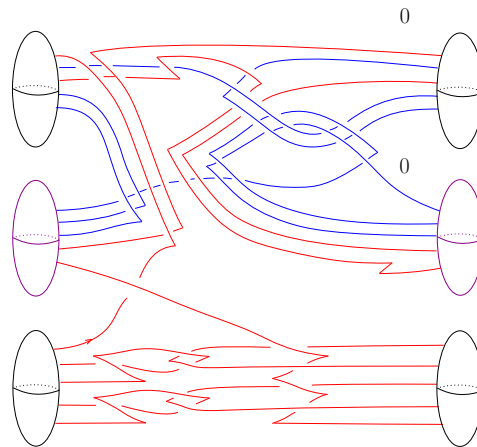

FIGURE 16. Q_1

FIGURE 17. Q_2

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Remark 1. *Notice each of the contractible manifolds W_1 and W_2 can be represented by a single 1/2-handle pairs. To see this in Figure 12 cancel the 1-handle, at top of the picture, with the 2-handle which goes through it geometrically once. To see this in Figure 11, we first slide the colored 2-handle over the other 2-handle (which corresponds the handle slide indicated by the arrow in the first picture of Figure 8) then the resulting 2-handle in Figure 11 goes through the 1-handle once. This means that if γ_i are the linking circles of the 2-handles of W_i, for $i = 1, 2$, there is no diffeomorphism $f : \partial W_1 \to \partial W_2$ taking γ_1 to γ_2; if there was, we can extend it to a diffeomorphism $W_1 \to W_2$ by using "carving" (Section 2.5 of [1]). It was pointed out to us that this provides an answer to Problem 1.16 in [10], because surgering each ∂W_i along γ_i results $S^1 \times S^2$ (since posting of our paper this fact has also been pointed out in [9] and [7] as well).*

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