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Exceptional Dehn surgeries along the Mazur link

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Abstract. The Mazur manifold is known as the first example of a cork, that is, a contractible 4-manifold that can change differential structures of 4-manifolds by cut and reglue with a twisting map. The Mazur link is a two-component link that describes the Mazur manifold. Akbulut-Yasui generalized them and constructed a sequence of corks. We name their links Akbulut-Yasui links and make a complete list of exceptional, i.e., non-hyperbolic integral Dehn surgeries along them. We use Martelli-Petronio-Roukema's theorem on exceptional Dehn surgeries along the minimally twisted four chain link.

1. Introduction

The Mazur link MZ is a hyperbolic, two-component two-bridge link ([Maz]), see Figure 1. Each component is unknotted, and the linking number of the components is ±1. By these properties, the Mazur link describes the Mazur manifold, which is contractible but not a 4-ball. In fact, its boundary (denoted by $(MZ; 0, 0)$ below, up to orientation) is not homeomorphic to the 3-sphere S^3 but an integral homology sphere that admits a hyperbolic structure. The Mazur manifold and similarly constructed manifolds have played important roles in the theory of 4-manifolds, and have been considered, for example, in [AKi, Ak, Mat], and more recently in [O], et al.

The Mazur manifold is now known as the Akbulut cork, where a *cork* is a compact contractible 4-manifold that can change differential structures of closed 4-manifolds by cut and reglue with a twisting map over the boundary. Corks are supposed to admit a Stein structure (see [AY] for the definition). The twisting map of the Mazur manifold is induced from the symmetry τ that switches the components. We define the Mazur link MZ as $C(2, 1, 4)$ by Conway's notation of two-bridge links, see Figure 1 again, though the original link (in [Maz, Mat]) was its mirror image. This is more convenient for recent study in the theories of Stein structures, Legendre diagram descriptions, positive allowable Lefschetz fibrations (PALF), and so on (see [AM, AY, O, U]). Akbulut-Yasui generalized the (first) Akbulut cork and constructed a sequence of corks [AY] which we call "Akbulut-Yasui corks". They are also described by two-bridge links, which we name the *Akbulut-Yasui link*: $AY_m = C(2m, 1, 2(m + 1))$ with $m \geq 1$ (see Figure 2). Each AY_m has a symmetry like τ. In the figures of the present paper, a boxed integer means right-handed full-twists.

Key words and phrases. Dehn surgery, 3-manifold, 4-manifold.

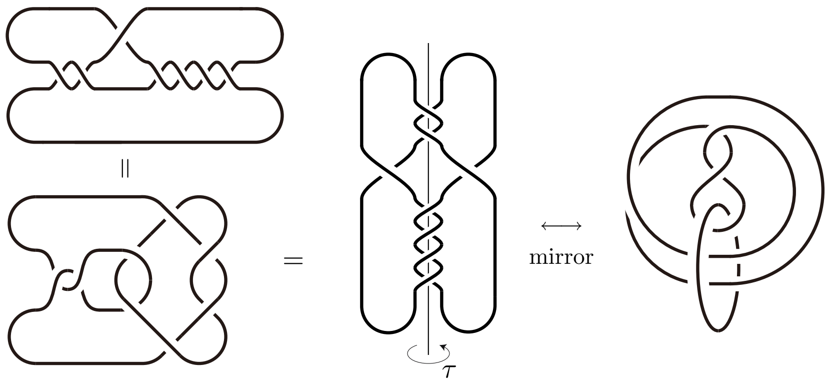


FIGURE 1. The Mazur link: $MZ = C(2, 1, 4)$

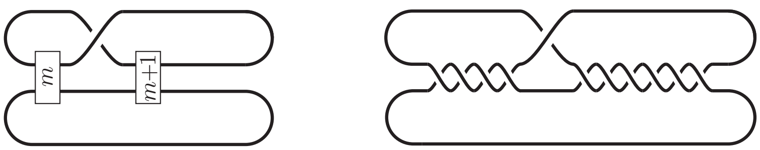


FIGURE 2. The Akbulut-Yasui link $AY_m = C(2m, 1, 2(m + 1))$ (ex. AY_2)

Let $L = K_1 \cup K_2 \cup \cdots \cup K_n$ be an orderd n-component link in S^3 and $r_1, r_2, \ldots, r_n$ integers or rational numbers (or $\infty = 1/0$). By $(L; r_1, r_2, \ldots, r_n)$, we denote the 3-manifold obtained by the Dehn surgery, or the surgery itself. We say that the surgery $(L; r_1, r_2, \ldots, r_n)$ is *hyperbolic* if the resulting manifold of the surgery admits a hyperbolic structure. In the present paper, we are interested in integral Dehn surgeries $(AY_m; p, q)$ along the Mazur link $MZ(= AY_1)$ and Akbulut-Yasui links AY_m, the distribution of *exceptional (i.e., non-hyperbolic) Dehn surgeries*, especially in lens space surgeries and reducible surgeries (surgeries whose results are connected sums of 3-manifolds). By the symmetry τ, it holds that $(AY_m; q, p) = (AY_m; p, q)$, thus we often assume that $p \leq q$.

We summarize our results as follows: Roughly speaking, we will see that, among all Akbulut-Yasui links, $AY_1 = MZ$ is very special and AY_2 is a little special, from the view point of exceptional Dehn surgery.

Theorem 1.1. *There exist some sequences of exceptional (i.e., non-hyperbolic) integral Dehn surgeries along Akbulut–Yasui link AY_m with $m \geq 1$. In fact,*

$$(AY_m; 2m + 1, q), \quad (AY_m; 2m + 2, q), \quad (AY_m; 2m + 3, q) \quad \textit{and} \quad (AY_m; 2m, 2m + 4)$$

are exceptional Dehn surgeries, for any integers q. Furthermore,

(1) *For Akbulut–Yasui links AY_m with $m \geq 3$, up to the symmetry*

$$(AY_m; q, p) = (AY_m; p, q),$$

the above list is complete. The other surgeries are hyperbolic.

(2) *For Akbulut–Yasui links AY_2 ($m = 2$), in addition to the list above, one more exceptional surgery $(AY_2; 4, 9)$ (and $(AY_2; 9, 4)$ by the symmetry) exists.*

(3) *For the Mazur link MZ ($= AY_1$, $m = 1$), up to the symmetry*

$$(MZ; q, p) = (MZ; p, q),$$

has more exceptional surgeries. The following is a complete list of exceptional surgeries:

$$(MZ; 3, q), \quad (MZ; 4, q), \quad (MZ; 5, q), \quad (MZ; 2, q) \quad \text{and} \quad (MZ; 1, 1),$$

for any integers q.

See graphic Figure 18 for the distribution (geography) of exceptional Dehn surgeries. In the next section, in Theorem 2.1, 2.3, 2.4 and Corollary 2.7, we will study the resulting manifolds of all exceptional Dehn surgeries above, and prove them by Kirby calculus, in Section 4.

To show that many, almost all, surgeries $(AY_m; p, q)$ are hyperbolic, we use Martelli-Petronio-Roukema's theorem [MPR, Corollary 3.6] as a criterion (see Theorem 3.2). This theorem motivates the author to study our classification in the present paper. Using the software `SnapPy` by Culler-Dunfield-Weeks [CDW], Martelli-Petronio-Roukema observed all exceptional Dehn surgeries with rational coefficients along the minimally twisted i-component chain links M_i with $i \leq 5$ (see [MPR, MP]). See Figure 3 for the link M_4. Note that, in the diagram, only one clasp (at $-$ in Figure 3) is opposite from the others. The link M_{i+1} is obtained from M_i by a blow-up: M_1 is the figure-eight knot, M_2 is the Whitehead link. The results in [MPR] are an extension of those along "the magic link" M_3 in [MP]. See [KiKoT] and [KiT] for the magic link M_3. Yoshida proved that M_4 has the minimal volume among four-component hyperbolic links [Yo]. We use the fact

$$(AY_m; 2m + a, 2m + b) = (M_4; a - 1, -1/m, b - 1, -1/(m + 1)).$$

In our context, *Martelli-Petronio-Roukema's Criterion* (MPR Criterion) is as follows: Let $\overline{\mathbb{Q}}$ denote $\mathbb{Q} \cup \{\infty = 1/0\}$. For $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \overline{\mathbb{Q}}^4$, they study the Dehn surgery $(M_4; \boldsymbol{\alpha}) = (M_4; \alpha_1, \alpha_2, \alpha_3, \alpha_4)$. MPR Criterion consists of two steps:

(1) They ([MPR]) defined some deformations (from $\boldsymbol{\alpha}$ to $\boldsymbol{\alpha}'$) on $\overline{\mathbb{Q}}^4$ that do not change the resulting manifold of the surgeries ($(M_4; \boldsymbol{\alpha}) \cong (M_4; \boldsymbol{\alpha}')$) up to orientation. The most basic one is the dihedral deformations. See Definition 3.3 in the present paper.

(2) They ([MPR]) made a list of exceptional Dehn surgeries along M_4, which consists of three families and three concrete manifolds. They claim that every $(M_4; \boldsymbol{\alpha})$

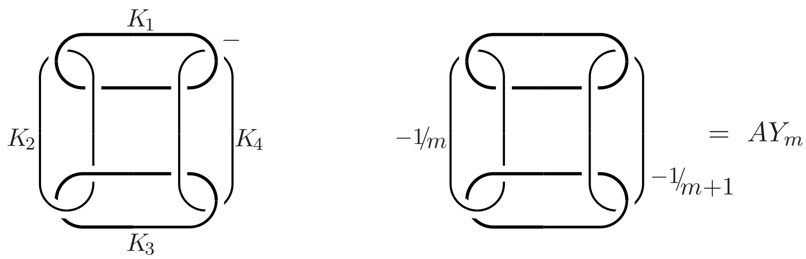


FIGURE 3. The minimally twisted four chain link M_4

is hyperbolic except the case where $\boldsymbol{\alpha}$ can be deformed to one in the list. See Theorem 3.2.

Recently, Hoffman-Ichihara-Kashiwagi-Masai-Oishi-Takayasu [HIKMOT] made a software program `HIKMOT`, supported by *Verified computations*. One can say "if `HIKMOT` says that the manifold is hyperbolic, then the manifold is really hyperbolic". According to the author's knowledge, no counter examples of MPR Criterion (examples of hyperbolic surgeries that `HIKMOT` does not determine hyperbolic) have appeared.

2. Resulting manifolds

We start with notations for Seifert manifolds and graph manifolds.

Notation. ([MPR]) We let $X(b;(a_1,b_1),\ldots,(a_r,b_r))$ denote a Seifert manifold (or a Seifert piece) over a sphere ($X = S$), a disk ($X = D$) or an annulus ($X = A$). We omit b as $X((a_1,b_1),\ldots,(a_r,b_r))$ in the case $b = 0$. The indices admit the following deformation:

$$X(b;\ldots,(a_i,b_i),\ldots) \;=\; X(b-1;\ldots,(a_i,a_i+b_i),\ldots).$$

Let X_1, X_2 be a pair of Seifert pieces with torus boundaries, and M a matrix in $GL(2;\mathbb{Z})$. By $X_1 \cup_M X_2$, we denote a *graph manifold* obtained by pasting X_1 and X_2 along their boundary tori by a homeomorphism defined by the matrix M, with respect to the basis $\{\text{a regular fiber, a section}\}$ in the first homology. Similarly, by $A(b;(a_1,b_1))/_M$, we denote a graph manifold obtained from a Seifert manifold $A(b;(a_1,b_1))$ over an annulus by pasting their boundary tori by a homeomorphism defined by the matrix M. We often use the matrix

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Even if the obtained manifold degenerates to a Seifert manifold, a lens space or a connected sum of two lens spaces, we we say that the manifold is a graph manifold.

Our convention about orientations of lens spaces is "the p/q Dehn surgery along an unknot is $-L(p,q)$".

The resulting manifolds of exceptional (i.e., non-hyperbolic) Dehn surgeries along Akbulut-Yasui links AY_ms with ($m \geq 1$), are as follows:

Theorem 2.1. *The resulting manifolds of the exceptional integral Dehn surgeries along Akbulut–Yasui link AY_m with $m \geq 1$ are as follows:*

$$\begin{aligned}(AY_m; 2m+1, 2m+b) &= D(2; (m,1), (m+1,1)) \cup_H D(0; (2,1), (b-3,1)),\\ (AY_m; 2m+2, 2m+b) &= S(-1; (2m+3, m+2), (2m+1, m+1), (b-2,1)),\\ (AY_m; 2m+3, 2m+b) &= D(-2; (m+1,1), (m+2,1)) \cup_H D(-1; (2,1), (b-1,1)),\\ (AY_m; 2m, 2m+4) &= L(4m^2+8m-1, 2m^2+3m-2).\end{aligned}$$

See the diagrams in Figure 7. For $m = 1$ (i.e., the Mazur link $MZ = AY_1$), see Theorem 2.4.

Remark 2.2. Maruyama pointed out that $(AY_m; 2m+1, 0)$ is a graph manifold [Mar]. Akbulut-Karakurt used this fact to calculate its Heegaard Floer homolgy [AKa].

Theorem 2.3. *As a list of exceptional integral Dehn surgeries along Akbulut–Yasui link AY_m with $m \geq 2$, up to the symmetry $(AY_m; q, p) = (AY_m; p, q)$, the list in Theorem 2.1 is almost complete. More precisely,*

(1) *For $m \geq 3$, the list in Theorem 2.1 is complete.*
(2) *For $m = 2$, the list in Theorem 2.1 is complete except for one more example*

$$(AY_2; 4, 9) = D(-1; (2,1), (3,1)) \cup_H D(-1; (2,1), (3,1)),$$

(and $(AY_2; 9, 4)$ by the symmetry), see Figure 7.

All other integral Dehn surgeries are hyperbolic.

Theorem 2.4. *As a list of exceptional integral Dehn surgeries along the Mazur link MZ ($= AY_1$) and their resulting manifolds, up to the symmetry $(MZ; q, p) = (MZ; p, q)$, the list below is complete:*

$$\begin{aligned}(MZ; 3, q) &= S(-1; (7,5), (2,1), (q-5,1)),\\ (MZ; 4, q) &= S(-1; (5,3), (3,2), (q-4,1)),\\ (MZ; 5, q) &= D(-2; (2,1), (3,1)) \cup_H D(-1; (2,1), (q-3,1)),\\ (MZ; 2, q) &= S(-1; (2,1), (3,2), (2q-13, 2)),\\ (MZ; 1, 1) &= A((2,3))/_H.\end{aligned}$$

All other integral Dehn surgeries are hyperbolic.

We will prove these theorems in Section 4 by Kirby calculus [Ki2, Ro].

Remark 2.5. Surgeries $(MZ; 3, q), (MZ; 4, q), (MZ; 5, q)$ and $(MZ; 2, 6)$ follow from Theorem 2.1 by substitution $m = 1$. Surgeries $(MZ; 2, q)$ with any q (except 6) and $(MZ; 1, 1)$ are special cases, see Figure 7 and 5(3).

Remark 2.6. Akbulut-Kirby have already shown that $(MZ; 2, 0), (MZ; 3, 0)$ and $(MZ; 4, 0)$ are the Brieskorn homology spheres $\Sigma(2,3,13), \Sigma(2,5,7)$ and $\Sigma(3,4,5)$, respectively [AKi].

Corollary 2.7. *On lens space surgeries and reducible surgeries, up to the symmetry* $(AY_m; q, p) = (AY_m; p, q)$*, the list below is complete:*

(1) *(Case* $m \geq 2$*) Lens space and lens ♯ lens surgeries along Akbulut–Yasui link* AY_m *with* $m \geq 2$*.*

$$\begin{aligned}
(AY_m; 2m, 2m+4) &= L(4m^2+8m-1, 2m^2+3m-2),\\
(AY_m; 2m+1, 2m+2) &= L(4m^2+6m+1, 4m^2+2m),\\
(AY_m; 2m+2, 2m+3) &= L(4m^2+10m+5, 4m^2+6m+2),
\end{aligned}$$

$$\begin{aligned}
(AY_m; 2m+1, 2m+3) &= L(2,1)\sharp L(2m^2+4m+1, 2m^2+2m),\\
(AY_m; 2m+2, 2m+2) &= L(2m+1,2)\sharp L(2m+3, 2m+1).
\end{aligned}$$

(2) *(Case* $m = 1$*) Lens space and lens ♯lens surgeries along the Mazur link* MZ *(*$= AY_1$*).*

$$\begin{aligned}
(MZ; 2, 6) &= L(11, 3), & (MZ; 3, 5) &= L(2, 1)\sharp L(7, 2),\\
(MZ; 3, 4) &= L(11, 2), & (MZ; 4, 4) &= L(3, 2)\sharp L(5, 2),\\
(MZ; 4, 5) &= L(19, 8), & &
\end{aligned}$$

$$(MZ; 2, 7) = L(13, 5), \quad (MZ; 3, 6) = L(17, 10).$$

Remark 2.8. Comparing cases (1) and (2) in the corollary, the lens space surgeries $(MZ; 2, 7)$ and $(MZ; 3, 6)$ in (2) are special cases. The other surgeries follow from the general case (1) by substitution $m = 1$.

Remark 2.9. On exceptional Dehn surgeries $(MZ; 1, q)$, we can use the fact $(MZ; 1, q) = (C(-2, 4); q-1)$, see Figure 17, [Ak2] and the results in [BW] on exceptional Dehn surgeries on two-bridge knots, where $C(-2, 4)$ is the two-bridge knot in our convention.

3. Hyperbolic cases

Using the results by Martelli-Petronio-Roukema [MPR] on exceptional Dehn surgeries along the minimally twisted four chain link M_4 as a criterion, we show that many surgeries $(AY_m; p, q)$ are hyperbolic.

Since $-1/n$-surgery ($n \in \mathbb{Z}$) along an unknot in S^3 (or in any 3-manifold) acts on links in the complement as n full-twists without changing the manifold, we have $(M_4; *, -\frac{1}{m}, *, -\frac{1}{m+1}) = S^3$ and that the union of the first and the third components (at $*$s) becomes an Akbulut-Yasui link AY_m in the resulting S^3, see Figure 3. Considering framings, we have the fact

$$(AY_m; 2m+a, 2m+b) = (M_4; a-1, -1/m, b-1, -1/(m+1)). \qquad (1)$$

3.1. Martelli-Petronio-Roukema Criterion

Let $\overline{\mathbb{Q}}$ denote $\mathbb{Q} \cup \{\infty\}$, where $\infty = 1/0$.

Definition 3.1. We let $j, k, i : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ denote linear fractional transformations

$$j(x) = \frac{x}{x-1}, \quad k(x) = 2 - x \quad \text{and} \quad i(x) = \frac{x-2}{x-1},$$

respectively. We also define $j(\infty) = 1, k(\infty) = \infty$ and $i(\infty) = 1$. They are involutions: $j^2 = k^2 = i^2 = 1$ (1 means the identity map) and satisfy $jk = kj = i$, see Figure 4. They generate a group $\langle j, k | j^2 = k^2 = 1, jk = kj (= i)\rangle$, isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

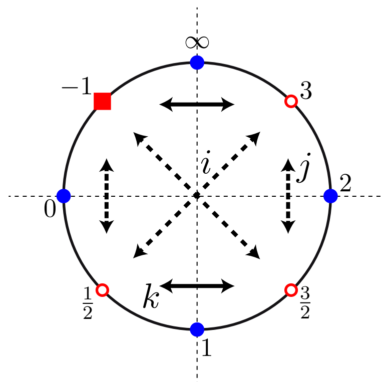


FIGURE 4. The involutions i, j and k

Next, we recall several deformations on $\overline{\mathbb{Q}}^4$ from [MPR].

(0) Dihedral deformations generated by the following two deformations

$$\begin{aligned} C: \ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) &\mapsto (\alpha_2, \alpha_3, \alpha_4, \alpha_1) \\ R: \ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) &\mapsto (\alpha_1, \alpha_4, \alpha_3, \alpha_2) \end{aligned}$$

It holds that $C^4 = R^2 = 1, CR = RC^{-1}$.

(1) Deformations J, K and I

$$\begin{aligned} J(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (j(\alpha_1), k(\alpha_2), j(\alpha_3), k(\alpha_4)) \\ K(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (k(\alpha_1), j(\alpha_2), k(\alpha_3), j(\alpha_4)) \\ I(\alpha_1, \alpha_2, \alpha_3, \alpha_4) &= (i(\alpha_1), i(\alpha_2), i(\alpha_3), i(\alpha_4)) \end{aligned}$$

Note that they are involutions $J^2 = K^2 = I^2 = 1$ and $JK = KJ = I$, because of the relations among i, j and k. There are some symmetries: $CJ = KC$, $KR = RK, CI = IC$, and so on.

(2) (-1)-deformation $\quad (-1, \alpha, \beta, \gamma) \mapsto (-1, \beta - 1, \alpha + 1, \gamma)$

(3) A rare deformation $\quad (-1, -2, -2, \alpha) \mapsto (-1, -2, -2, -\alpha - 4)$

From now on, for elements in $\overline{\mathbb{Q}}^4$, we use "$=$" for the equivalence relation defined by the dihedral deformations.

Theorem 3.2. (MPR Criterion in Martelli-Petronio-Roukema [MPR]) *Every filling on M_4 is hyperbolic, except those listed below, and those obtained from them via composition of the maps (deformations) in Definition 3.1.*

(1) $(M_4;\ \infty, a/b, c/d, e/f) \ = \ S((a,b),(d,-c),(e,f))$
(2) $(M_4;\ 0, a/b, c/d, e/f)$

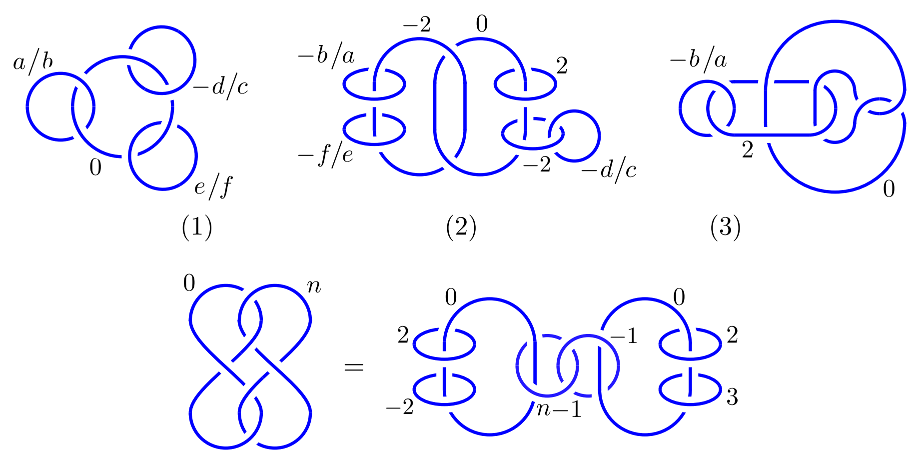
$$= \ D(2;(b,-a),(f,-e)) \cup_H D((2,1),(c-2d,d))$$

(3) $(M_4;\ -1,-2,-1,a/b) \ = \ A((b,-a))/_H$
(4) $(M_4;\ -1,-2,-3,-4) \ = \ D((2,1),(2,-1)) \cup_{M(2)} D((2,1),(3,1))$
(5) $(M_4;\ -1,-3,-2,-3) \ = \ D((2,1),(2,-1)) \cup_{M(3)} D((2,1),(3,1))$
(6) $(M_4;\ -2,-2,-2,-2) \ = \ D((2,1),(2,-1)) \cup_{M(4)} D((2,1),(3,1))$

Here, the matrices H and $M(n)$ with $n = 2,3,4$ are as follows:

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M(n) = \begin{bmatrix} -1 & n \\ 1 & -(n-1) \end{bmatrix}.$$

See Figure 5, for the resulting manifolds.



(4),(5),(6) correspond to $n = 2,3,4$ respectively

FIGURE 5. The resulting manifolds of Martelli-Petronio-Roukema theorem

Definition 3.3. For a given $\alpha \in \overline{\mathbb{Q}}$, by $\langle jk \rangle(\alpha)$, we denote the orbit set of the group action of α by the group $\langle j,k | j^2 = k^2 = 1, jk = kj (= i) \rangle$ in Definition 3.1.

$$\langle jk \rangle(\alpha) \ = \{\alpha, j(\alpha), k(\alpha), i(\alpha)\} \ \subset \overline{\mathbb{Q}}.$$

Most important examples are

$$\langle jk\rangle(\infty) = \{\infty, 1\}\,, \quad \langle jk\rangle(0) = \{0, 2\}\,, \quad \langle jk\rangle(-1) = \{-1, 1/2, 3, 3/2\}.$$

Furthermore, for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \overline{\mathbb{Q}}^n$ (with $n = 4$ mainly, or $n = 3$), by $\langle jk\rangle(\boldsymbol{\alpha})$, we also denote the union of the orbit sets of the entries:

$$\langle jk\rangle(\boldsymbol{\alpha}) \; = \langle jk\rangle(\alpha_1, \ldots, \alpha_n) \; = \langle jk\rangle(\alpha_1) \cup \cdots \cup \langle jk\rangle(\alpha_n).$$

Lemma 3.4. *Let $\boldsymbol{\alpha} = (\alpha, \beta, \gamma, \delta) \in \overline{\mathbb{Q}}^4$. On the Dehn surgery $(M_4; \boldsymbol{\alpha}) = (M_4; \alpha, \beta, \gamma, \delta)$, we have:*

(1) *If $\langle jk\rangle(\boldsymbol{\alpha}) \cap \{\infty, 0\} \neq \emptyset$, then $(M_4; \alpha, \beta, \gamma, \delta)$ is an exceptional surgery.*
(2) *If $\langle jk\rangle(\boldsymbol{\alpha}) \cap \langle jk\rangle(-1) \neq \emptyset$, equivalently $\langle jk\rangle(\boldsymbol{\alpha}) \ni -1$, then we have a chance of (-1)-deformation from $\boldsymbol{\alpha}, J(\boldsymbol{\alpha}), K(\boldsymbol{\alpha})$ or $I(\boldsymbol{\alpha})$.*

Proof. Existence of the intersection means that one of $\boldsymbol{\alpha}, J(\boldsymbol{\alpha}), K(\boldsymbol{\alpha}), I(\boldsymbol{\alpha})$ contains ∞ or 0 for the case (1), -1 for (2), respectively. The lemma follows from MPR Criterion, Theorem 3.2. □

Lemma 3.5. *For general $\alpha, \beta, \gamma \in \overline{\mathbb{Q}}$, the set of elements in $\overline{\mathbb{Q}}^4$ obtained from $(-1, \alpha, \beta, \gamma)$ by combination of (-1)- and dihedral deformations has only three elements:*

$$\{(-1,\, \alpha, \beta, \gamma), (-1,\, \beta - 1, \alpha + 1, \gamma), (-1,\, \alpha, \gamma + 1, \beta - 1)\}$$

up to dihedral deformations.

Definition 3.6. We call the set (-1)-triple of $(-1, \alpha, \beta, \gamma)$ and write it as

$$\begin{cases} (-1,\, \alpha, \beta, \gamma) \\ (-1,\, \beta - 1, \alpha + 1, \gamma) \\ (-1,\, \alpha, \gamma + 1, \beta - 1) \end{cases}.$$

Proof (of Lemma 3.5). The (-1)-deformation is involutive. For mixed deformations, see the following:

$$\begin{aligned}(-1,\, \alpha, \beta, \gamma) &\xrightarrow{R} (-1,\, \gamma, \beta, \alpha) \xrightarrow{(-1)} (-1,\, \beta - 1, \gamma + 1, \alpha) \xrightarrow{R} (-1,\, \alpha, \gamma + 1, \beta - 1) \\ &\xrightarrow{(-1)} (-1,\, \gamma, \alpha + 1, \beta - 1) \xrightarrow{R} (-1,\, \beta - 1, \alpha + 1, \gamma) \xrightarrow{(-1)} (-1,\, \alpha, \beta, \gamma)\end{aligned}$$

We find that "general α, β, γ" in the statement means that

$$\{\alpha, \beta, \gamma,\; \alpha + 1, \beta - 1, \gamma + 1\} \not\ni -1.$$

□

Lemma 3.7. *For an integer n (with $n \neq 0$ for (2) and (3)), we have:*

(1) *$\langle jk\rangle(n) \cap \{\infty, 0\} \neq \emptyset$ iff $n \in \{0, 1, 2\}$, and $\langle jk\rangle(n) \ni -1$ iff $n \in \{-1, 3\}$.*
(2) *$\langle jk\rangle(1/n) \cap \{\infty, 0\} \neq \emptyset$ iff $n = 1$, and $\langle jk\rangle(1/n) \ni -1$ iff $n \in \{-1, 2\}$.*
(3) *$\langle jk\rangle((n \pm 1)/n) \cap \{\infty, 0\} \neq \emptyset$ iff $n \in \{-1, 1\}$, and $\langle jk\rangle((n \pm 1)/n) \ni -1$ iff $n = \pm 2$, for each sign at $\pm$.*

Proof. (1) and (2) follow from $\langle jk\rangle(n) = \{n, n/(n-1), 2-n, (n-2)/(n-1)\}$ and $\langle jk\rangle(1/n) = \{1/n, 1/(1-n), (2n-1)/n, (2n-1)/(n-1)\}$, respectively. For (3), we use

$$\langle jk\rangle((n-1)/n) = \{(n-1)/n, 1-n, (n+1)/n, n+1\} = \langle jk\rangle((n+1)/n) = \langle jk\rangle(n+1)$$

and (1). □

3.2. Proof of Theorem 1.1 (Case $m \geq 2$)

Condition. (Case $m \geq 2$) In this subsection, we study Dehn surgeries

$$(AY_m; 2m+a, 2m+b)$$

with $m \geq 2$. We assume $a \leq b$. In Section 4, we will verify that, if a or b equals to $1, 2$ or 3, the surgeries $(AY_m; 2m+a, 2m+b)$ are exceptional. Thus we assume that

$$\{a, b\} \cap \{1, 2, 3\} = \emptyset. \tag{2}$$

To prove Theorem 2.3, we have to show

Claim. The surgery $(AY_m; 2m+a, 2m+b)$ is hyperbolic except when $(a, b) = (0, 4)$ and "$m = 2$ and $(a, b) = (0, 5)$".

For elements $\boldsymbol{\alpha} \in \overline{\mathbb{Q}}^4$, we are always concerned with $\langle jk\rangle(\boldsymbol{\alpha}) \cap \{\infty, 0\}$ and $\langle jk\rangle(\boldsymbol{\alpha}) \cap \langle jk\rangle(-1)$, to use Lemma 3.4. We take *effective deformations*, that is, deformations whose results contain -1 as entries, among $\boldsymbol{\alpha}$ itself, $J(\boldsymbol{\alpha}), K(\boldsymbol{\alpha})$ and $I(\boldsymbol{\alpha})$.

We start with $\boldsymbol{\alpha} = (a-1, -1/m, b-1, -1/(m+1)) \in \overline{\mathbb{Q}}^4$, with $a, b, m \in \mathbb{Z}$, since $(AY_m; 2m+a, 2m+b) = (M_4; \boldsymbol{\alpha})$ by fact (1). We divide the proof into some cases as Table 1.

Case	Condition	Conclusion (hyp. = hyperbolic)
pre	$\{a, b\} \cap \{1, 2, 3\} \neq \emptyset$	exceptional
1	$\{a, b\} \cap \{0, 1, 2, 3, 4\} = \emptyset$	hyp.
2	$a = 0$	hyp. except $m = 2$ or $b = 4, 5$
	$(a, b) = (0, 4)$ with any m	exceptional
	$m \geq 3$, $a = 0$ and $b \neq 5$	hyp.
2-1	$(a, b) = (0, 0)$	hyp.
2-2	$m = 2$ and $a = 0$	hyp. except $(a, b) = (0, 5)$
2-3	$m \geq 3$ and $(a, b) = (0, 5)$	hyp.
3	$b = 0$	hyp.
4	$b = 4$	hyp.
5	$a = 4$	hyp.

TABLE 1. Organization of cases ($m \geq 2$, $a \leq b$)

(Case 1: $\{a,b\} \cap \{0,4\} = \emptyset$) First, we have

$$\langle jk\rangle(a-1,-1/m,b-1,-1/(m+1)) \cap \{\infty,0\} = \emptyset.$$

Thus, there exist no deformations to one that contains ∞ or 0 among $\boldsymbol{\alpha}$ itself, $J(\boldsymbol{\alpha}), K(\boldsymbol{\alpha})$ and $I(\boldsymbol{\alpha})$. Second, we have

$$\langle jk\rangle(a-1,-1/m,b-1,-1/(m+1)) \not\ni -1.$$

Here, Lemma 3.7 is helpful: $\langle jk\rangle(a-1)\cap\{\infty,0\} \neq \emptyset$ iff $a-1 \in \{0,1,2\}$, $\langle jk\rangle(1/(-m)) \ni -1$ iff $-m \in \{-1,2\}$, and so on. Thus, we have no chance to use (-1)-deformation on $\boldsymbol{\alpha}, J(\boldsymbol{\alpha}), K(\boldsymbol{\alpha})$ nor $I(\boldsymbol{\alpha})$. This means that there exist no *sequences* of deformations to one that contains ∞ or 0.

It is easy to see that neither $\boldsymbol{\alpha}$ itself, $J(\boldsymbol{\alpha}), K(\boldsymbol{\alpha})$ nor $I(\boldsymbol{\alpha})$ agree with the (3)(4)(5)(6) in MPR list in Theorem 3.2. (In what follows, we sometimes omit this sentence.) By MPR Criterion, the corresponding surgeries are hyperbolic.

(Case 2: $a = 0$) The (-1)-triple of $\boldsymbol{\alpha} = (-1,-1/m,b-1,-1/(m+1))$ is

$$\begin{cases} (-1,\ -1/m, b-1, -1/(m+1)) \\ (-1,\ b-2, (m-1)/m, -1/(m+1)) = \boldsymbol{\alpha}_1 \\ (-1,\ -1/m, m/(m+1), b-2) \qquad = \boldsymbol{\alpha}_2 \end{cases}.$$

On $\boldsymbol{\alpha}$, we have that $\langle jk\rangle(-1/m, b-1, -1/(m+1)) \cap \{\infty,0\} = \emptyset$ and that $\langle jk\rangle(-1/m, b-1, -1/(m+1)) \ni -1$ iff $b=0$, otherwise it does not have an effective deformation.

On $\boldsymbol{\alpha}_1$, we have $\langle jk\rangle\,(b-2,(m-1)/m,-1/(m+1)) \cap \{\infty,0\} \neq \emptyset$ iff $b=4$ (under the condition (2) and $m \geq 2$). Here we find that "$(a,b) = (0,4)$" with any $m \geq 2$ are exceptional surgeries, because $k(b-2) = k(2) = 0$ if $b=4$, by Lemma 3.4 and MPR Criterion. On the other hand, $\langle jk\rangle\,(b-2,(m-1)/m,-1/(m+1)) \ni -1$ iff $m=2$ or $b=5$.

On $\boldsymbol{\alpha}_2$, we have that $\langle jk\rangle\,(-1/m, m/(m+1), b-2) \cap \{\infty,0\} \neq \emptyset$ iff $b=4$, and $\langle jk\rangle\,(-1/m, m/(m+1), b-2) \ni -1$ iff $b=5$. We treat $(a,b) = (0,0)$, "$m=2$ and $a=0$" and $(a,b) = (0,5)$ as subcases.

(Case 2-1: $(a,b) = (0,0)$) We start with the (-1)-triple of

$$\boldsymbol{\alpha} = (-1,-1/m,-1,-1/(m+1)),$$

which has a symmetry under the deformation R. Its (-1)-triple is, up to dihedral deformations,

$$\begin{cases} (-1,-1/m,-1,-1/(m+1)) \\ (-1,-2,(m-1)/m,-1/(m+1)) = \boldsymbol{\alpha}_1 \\ (-1,-1/m,m/(m+1),-2) \end{cases}.$$

For each $(-1,x,y,z)$ above, using Lemma 3.7 (Here Lemma 3.7(2),(3) and $\langle jk\rangle((m-1)/m) = \langle jk\rangle(m+1)$ are convenient), we can check $\langle jk\rangle(x,y,z) \cap \{\infty,0\} = \emptyset$ and that $\langle jk\rangle(x,y,z) \ni -1$ only if $m=2$, at $\boldsymbol{\alpha}_1 = (-1,\ -2,1/2,-1/3)$. If $m \geq 3$, then there are no deformations to one that contains ∞ or 0. These surgeries are hyperbolic.

If $m = 2$, then since $j(1/2) = -1$, we take $J(\boldsymbol{\alpha}_1) = (1/2, 4, -1, 7/3)$. Its (-1)-triple is

$$\begin{cases} (-1,\ 4, 1/2, 7/3) \\ (-1,\ -1/2, 5, 7/3) \\ (-1,\ 4, 10/3, -1/2) \end{cases}.$$

They have no effective deformations except going back. Thus, this surgery "$m = 2$ and $(a, b) = (0, 0)$" is hyperbolic.

(Case 2-2: $m = 2$ and $a = 0$) We are interested in the condition on b. We start with the (-1)-triple of $\boldsymbol{\alpha} = (-1, -1/m, b-1, -1/(m+1))$

$$\begin{cases} (-1,\ -1/2, b-1, -1/3) \\ (-1,\ b-2, 1/2, -1/3) = \boldsymbol{\alpha}_1 \\ (-1,\ -1/2, 2/3, b-2) = \boldsymbol{\alpha}_2 \end{cases}.$$

On $\boldsymbol{\alpha}_2$, we have $\langle jk \rangle\,(-1/2, 2/3, b-2) \ni -1$ iff $b = 5$ (under the condition (2)), included in the following. On $\boldsymbol{\alpha}_1$, since $\{b-2, 1/2, -1/3\} \cap \langle jk \rangle(-1) \ni 1/2$ (and $3 = b-2$ if $b = 5$) and $j(1/2) = -1$, we take $J(\boldsymbol{\alpha}_1) = (1/2, 4-b, -1, 7/3)$ whose (-1)-triple is

$$\begin{cases} (-1,\ 7/3, 1/2, 4-b) \\ (-1,\ -1/2, 10/3, 4-b) = \boldsymbol{\beta}_1 \\ (-1,\ 7/3, 5-b, -1/2)\ \ = \boldsymbol{\beta}_2 \end{cases}.$$

By applying Lemma 3.4 to $\boldsymbol{\beta}_2$, we find that the corresponding surgery "$m = 2$ and $(a, b) = (0, 5)$" is an exceptional surgery.

From now on, we assume $b \neq 0, 1, 2, 3, 4, 5$. There are no effective deformations on $\boldsymbol{\beta}_1$, since $\langle jk \rangle\,(-1/2, 10/3, 4-b) \not\ni -1$. On $\boldsymbol{\beta}_2$, we have $\langle jk \rangle\,(7/3, 5-b, -1/2) \ni -1$ iff $b = 6$, where $-1 = 5 - b$ if $b = 6$. It is proved that surgeries "$m = 2$ and $a = 0$" are hyperbolic except $b = 1, 2, 3, 4, 5$ and 6.

Now we consider $(a, b) = (0, 6)$. Then $\boldsymbol{\beta}_2$ has two (-1)s. We change the first -1 of (-1)-deformation by cyclic deformations twice: $C^2(\boldsymbol{\beta}_2) = (-1, -1/2, -1,\ 7/3)$. Its (-1)-triple is

$$\begin{cases} (-1,\ -1/2, -1, 7/3) \\ (-1,\ -2, 1/2, 7/3) = \boldsymbol{\gamma} \\ (-1,\ -1/2, 10/3, -2) \end{cases}.$$

Only $\boldsymbol{\gamma}$ has an effective deformation J and $J(\boldsymbol{\gamma}) = (1/2, 4, -1, -1/3)$, whose (-1)-triple is

$$\begin{cases} (-1,\ 4, 1/2, -1/3) \\ (-1,\ -1/2, 5, -1/3) \\ (-1,\ 4, 2/3, -1/2) \end{cases}.$$

There are no effective deformations, since

$$\langle jk \rangle(-1/2, 5, -1/3) \cap \big(\{\infty, 0\} \cup \langle jk \rangle(-1)\big) = \emptyset,$$
$$\langle jk \rangle(4, 2/3, -1/2) \cap \big(\{\infty, 0\} \cup \langle jk \rangle(-1)\big) = \emptyset.$$

This surgery "$m = 2$ and $(a, b) = (0, 6)$" is hyperbolic.

(Case 2-3: $(a,b)=(0,5)$) We are interested in the condition on m. We assume that $m \geq 3$, since the case $m=2$ is already done in Case 2-2. We start with the (-1)-triple of $\boldsymbol{\alpha}=(-1,-1/m,b-1,-1/(m+1))$ from Case 1.

$$\begin{cases}(-1,\ -1/m,4,-1/(m+1)) \\ (-1,\ 3,(m-1)/m,-1/(m+1)) = \boldsymbol{\alpha}_1 \\ (-1,\ -1/m,m/(m+1),3) \qquad\quad = \boldsymbol{\alpha}_2\end{cases}.$$

Since $\{3,(m-1)/m,-1/(m+1)\} \cap \langle jk\rangle(-1)=\{3\}$, we take

$$J(\boldsymbol{\alpha}_1)=(1/2,-1,1-m,(2m+3)/(m+1)).$$

For the same reason, we take $J(\boldsymbol{\alpha}_2)=(1/2,(2m+1)/m,-m,-1)$. These (-1)-triples are

$$\begin{cases}(-1,\ 1/2,(2m+3)/(m+1),1-m) = \boldsymbol{\gamma}_1 \\ (-1,\ (m+2)/(m+1),3/2,1-m) \ \ = \boldsymbol{\gamma}_2 \\ (-1,\ 1/2,2-m,(m+2)/(m+1)) \ \ = \boldsymbol{\gamma}_3\end{cases}, \quad \begin{cases}(-1,\ 1/2,(2m+1)/m,-m) \ \ = \boldsymbol{\delta}_1 \\ (-1,\ (m+1)/m,3/2,-m) \quad\ = \boldsymbol{\delta}_2 \\ (-1,\ 1/2,1-m,(m+1)/m) = \boldsymbol{\delta}_3\end{cases}.$$

For each $(-1,x,y,z)$ above, using Lemma 3.7, we study whether $\langle jk\rangle(x,y,z) \ni -1$ or not, and if it holds, we take their effective deformations. There are only the following six possibilities:

$$\begin{array}{lcl} K(\boldsymbol{\gamma}_1)=(3,-1,-1/(m+1),(m-1)/m) & & K(\boldsymbol{\delta}_1)=(3,-1,-1/m,m/(m+1)) \\ I(\boldsymbol{\gamma}_2)=(3/2,-m,-1,(m+1)/m) & , & I(\boldsymbol{\delta}_2)=(3/2,1-m,-1,(m+2)/(m+1)). \\ K(\boldsymbol{\gamma}_3)=(3,-1,m,m+2) & & K(\boldsymbol{\delta}_3)=(3,-1,m+1,m+1) \end{array}$$

Here we ignore $K(\boldsymbol{\gamma}_1)$, since it goes back to $\boldsymbol{\alpha}_1$ up to dihedral deformations. (More precisely, $K\boldsymbol{\gamma}_1=KRCJ\boldsymbol{\alpha}_1=RKCJ\boldsymbol{\alpha}_1=RCJ^2\boldsymbol{\alpha}_1=RC\boldsymbol{\alpha}_1$.) For the same reason, we ignore $K(\boldsymbol{\delta}_1)$. Next, it holds that $I(\boldsymbol{\gamma}_2)=\boldsymbol{\delta}_2$ and $I(\boldsymbol{\delta}_2)=\boldsymbol{\gamma}_2$, which generate *closed loops of deformations*. Finally, (-1)-deformations from $K(\boldsymbol{\gamma}_3)$ and $K(\boldsymbol{\delta}_3)$ are

$$\begin{cases}(-1,3,m+2,m) \\ (-1,m+1,4,m)=\boldsymbol{\epsilon} \\ (-1,3,m+1,m+1)=K(\boldsymbol{\delta}_3)\end{cases}, \quad \begin{cases}(-1,3,m+1,m+1) \\ (-1,m,4,m+1)=\boldsymbol{\epsilon} \\ (-1,3,m+2,m)=K(\boldsymbol{\gamma}_3)\end{cases},$$

respectively. If $m \geq 4$, then they are all closed loops of deformations. If $m=3$, then $\boldsymbol{\epsilon}=(-1,3,4,4)=K(\boldsymbol{\delta}_3)$ and $K(\boldsymbol{\gamma}_3)=(-1,3,5,3)$ have other deformations using $k(3)=-1$, but they are $J(\boldsymbol{\epsilon})=(1/2,-1,4/3,-2)=\boldsymbol{\delta}_3$ and

$$J(-1,3,5,3)=(1/2,-1,5/4,-1)=\boldsymbol{\gamma}_3,$$

respectively. Thus, they are also closed loops of deformations, which means that they have no sequence of deformations to one that contains ∞ or 0. These surgeries "$m \geq 3$ and $(a,b)=(0,5)$" are hyperbolic.

(Case 3: $b=0$) We assume $a \leq b=0$ and $a \neq 0$ also, since we have already studied $(a,b)=(0,0)$ in Case 2-1. We deform $\boldsymbol{\alpha}=(a-1,-1/m,-1,-1/(m+1))$ to

$$(-1,-1/m,a-1,-1/(m+1)),$$

whose (-1)-triple is

$$\begin{cases}(-1,\ -1/m, a-1, -1/(m+1)) \\ (-1,\ a-2, (m-1)/m, -1/(m+1)) = \boldsymbol{\alpha}_1 \\ (-1,\ -1/m, m/(m+1), a-2)\end{cases}.$$

For each $(-1,x,y,z)$ above, using Lemma 3.7, we can check $\langle jk\rangle(x,y,z)\cap\{\infty,0\}=\emptyset$ and that $\langle jk\rangle(x,y,z)\ni -1$ only if $m=2$, at $\boldsymbol{\alpha}_1=(-1,\ a-2, 1/2, -1/3)$. Since $j(1/2)=-1$, we take $J(\boldsymbol{\alpha}_1)=(1/2, 4-a, -1, 7/3)$ whose (-1)-triple is

$$\begin{cases}(-1,\ 7/3, 1/2, 4-a) \\ (-1,\ -1/2, 10/3, 4-a) \\ (-1,\ 7/3, 5-a, -1/2)\end{cases}.$$

They have no effective deformations except going back, and there are no deformations to one that contains ∞ or 0. These surgeries are hyperbolic.

(Case 4: $b=4$) We assume $a\le b=4$, $a\ne 1,2,3$ and $a\ne 0$, since $(a,b)=(0,4)$ is already studied in Case 2. We start with $\boldsymbol{\alpha}=(a-1,-1/m,3,-1/(m+1))$. First, we take $K(\boldsymbol{\alpha})=(3-a, 1/(m+1), -1, 1/(m+2))$ and deform it to

$$\boldsymbol{\alpha}'=(-1, 1/(m+1), 3-a, 1/(m+2)),$$

whose (-1)-triple is

$$\begin{cases}(-1,\ 1/(m+1), 3-a, 1/(m+2)) = \boldsymbol{\alpha}' \\ (-1,\ 2-a, (m+2)/(m+1), 1/(m+2)) = \boldsymbol{\alpha}_1 \\ (-1,\ 1/(m+1), (m+3)/(m+2), 2-a) = \boldsymbol{\alpha}_2\end{cases}.$$

For each $(-1,x,y,z)$ above, using Lemma 3.7, we can check $\langle jk\rangle(x,y,z)\cap\{\infty,0\}=\emptyset$, but $\langle jk\rangle(x,y,z)\ni -1$ holds in the following cases: (i) $a=4$ at $\boldsymbol{\alpha}'$, (ii) $a=-1$ at both $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$.

First, in the case (i) $(a,b)=(4,4)$. Then $\boldsymbol{\alpha}'=(-1,\ 1/(m+1), -1, 1/(m+2))$ has two (-1)s. We can change the first -1 by cyclic deformations, but its (-1)-triples are unchanged as

$$\begin{cases}(-1,\ 1/(m+2), -1, 1/(m+1)) \\ (-1,\ -2, (m+3)/(m+2), 1/(m+1)) \\ (-1,\ 1/(m+2), (m+2)/(m+1), -2)\end{cases}.$$

They have no effective deformations except going back. These surgeries are hyperbolic.

Next, in the case (ii) $(a,b)=(-1,4)$. Then $\boldsymbol{\alpha}_1=(-1,\ 3, (m+2)/(m+1), 1/(m+2))$ and $\boldsymbol{\alpha}_2=(-1,\ 1/(m+1), (m+3)/(m+2), 3)$. Since $k(3)=-1$, we take

$$J(\boldsymbol{\alpha}_1)=(1/2, -1, m+2, (2m+3)/(m+2)),$$
$$J(\boldsymbol{\alpha}_2)=(1/2, (2m+1)/(m+1), m+3, -1).$$

Their (-1)-triples and their effective deformations are as follows:

$$\begin{cases}(-1,\ 1/2,(2m+3)/(m+2),m+2)\\(-1,\ (m+1)/(m+2),3/2,m+2)=\boldsymbol{\beta}_1 \overset{I}{\to} (3/2,m+3,-1,m/(m+1))=\boldsymbol{\gamma}_1\\(-1,\ 1/2,m+3,(m+1)/(m+2))=\boldsymbol{\beta}_2 \overset{K}{\to} (3,-1,-m-1,-m-1)=\boldsymbol{\delta}\end{cases}.$$

$$\begin{cases}(-1,\ 1/2,(2m+1)/(m+1),m+3)\\(-1,\ m/(m+1),3/2,m+3)=\boldsymbol{\gamma}_1 \overset{I}{\to} (3/2,m+2,-1,(m+1)/(m+2))=\boldsymbol{\beta}_1\\(-1,\ 1/2,m+4,m/(m+1))=\boldsymbol{\gamma}_2 \overset{K}{\to} (3,-1,-m-2,-m)=\boldsymbol{\epsilon}\end{cases}.$$

We have closed loops $I(\boldsymbol{\beta}_1)=\boldsymbol{\gamma}_1$ and $I(\boldsymbol{\gamma}_1)=\boldsymbol{\beta}_1$. We also find that the (-1)-triple of $\boldsymbol{\delta}$ and that of $\boldsymbol{\epsilon}$ agree as

$$\begin{cases}(-1,\ 3,-m-1,-m-1)\\(-1,\ -m-2,4,-m-1)\\(-1,\ 3,-m,-m-2)\end{cases}.$$

They are only closed loops of deformations and there are no deformations to one that contains ∞ or 0. These surgeries are hyperbolic.

(Case 5: $a=4$) We assume $a=4\le b$ and also $b\ne 4$, since $(a,b)=(4,4)$ is already studied in the last case. We start with $\boldsymbol{\alpha}=(3,-1/m,b-1,-1/(m+1))$. First, we take $K(\boldsymbol{\alpha})=(-1,1/(m+1),3-b,1/(m+2))$, whose (-1)-triple is

$$\begin{cases}(-1,\ 1/(m+1),3-b,1/(m+2))\\(-1,\ 2-b,(m+2)/(m+1),1/(m+2))\\(-1,\ 1/(m+1),(m+3)/(m+2),2-b)\end{cases}.$$

For each $(-1,x,y,z)$ above, we can check $\langle jk\rangle(x,y,z)\cap\{\infty,0\}=\emptyset$ and that

$$\langle jk\rangle(x,y,z)\not\ni -1.$$

They have no effective deformations except going back. Thus these surgeries are hyperbolic. The proof of the case $m\ge 2$ is completed. □

3.3. Proof of Theorem 1.1 (Case $m=1$)

Condition. (Case $m=1$) In this subsection, we study Dehn surgeries $(MZ;2+a,2+b)$ with $a,b\in\mathbb{Z}$. In contrast to the case $m\ge 2$, we do not assume $a\le b$ for a while. In Section 4, we will verify that, if a or b equals to $0,1,2$ or 3, the surgeries $(MZ;2+a,2+b)$ are exceptional, thus we assume that

$$\{a,b\}\cap\{0,1,2,3\}=\emptyset. \tag{3}$$

Then, to prove Theorem 2.4, we have to show

Claim. The surgery $(MZ;2+a,2+b)$ is hyperbolic except when $(a,b)=(-1,-1)$.

By the fact (1), we study $\boldsymbol{\alpha} = (a-1, -1, b-1, -1/2) \in \overline{\mathbb{Q}}^4$. It contains a -1 (as $-1/m$), thus we take a dihedral deformation $RC(\boldsymbol{\alpha}) = (-1, a-1, -1/2, b-1)$, whose (-1)-triple is

$$\begin{cases} (-1, a-1, -1/2, b-1) = \boldsymbol{\alpha}_1 \\ (-1, -3/2, a, b-1) \qquad = \boldsymbol{\alpha}_2 \\ (-1, a-1, b, -3/2) \qquad = \boldsymbol{\alpha}_3 \end{cases}.$$

For each $(-1, x, y, z)$ above, we can check $\langle jk\rangle(x, y, z) \cap \{\infty, 0\} = \emptyset$ under condition (3). Here we use $\langle jk\rangle(-1/2) = \{-1/2, 1/3, 5/2, 5/3\}$. On the other hand, we have

(i) On $\boldsymbol{\alpha}_1$, $\langle jk\rangle(a-1, -1/2, b-1) \ni -1$ iff "$a = 4$ or $b = 4$",
(ii) On $\boldsymbol{\alpha}_2$, $\langle jk\rangle(-3/2, a, b-1) \ni -1$ iff "$a = -1$ or $b = 4$", and
(iii) On $\boldsymbol{\alpha}_3$, $\langle jk\rangle(a-1, b, -3/2) \ni -1$ iff "$a = 4$ or $b = -1$",

respectively. Here, it is proved that if $\{a, b\} \cap \{-1, 0, 1, 2, 3, 4\} = \emptyset$, the surgeries are hyperbolic. We start with the case $a = 4$ ($a = 4$ or $b = 4$, more precisely).

(Case 1: $a = 4$ and $b \neq -1, 4$) We recall $\boldsymbol{\alpha}_i$ ($i = 1, 2, 3$) with $a = 4$ and take the effective deformations:

$$\begin{cases} (-1, 3, -1/2, b-1) = \boldsymbol{\alpha}_1 \overset{J}{\to} (1/2, -1, 1/3, 3-b) = \boldsymbol{\beta} \\ (-1, -3/2, 4, b-1) = \boldsymbol{\alpha}_2 \\ (-1, 3, b, -3/2) \qquad = \boldsymbol{\alpha}_3 \overset{J}{\to} (1/2, -1, b/(b-1), 7/2) = \boldsymbol{\gamma} \end{cases}.$$

The (-1)-triples of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are

$$\begin{cases} (-1,\ 1/2, 3-b, 1/3) = \boldsymbol{\beta}_1 \\ (-1,\ 2-b, 3/2, 1/3) = \boldsymbol{\beta}_2 \\ (-1,\ 1/2, 4/3, 2-b) = \boldsymbol{\beta}_3 \end{cases}, \quad \begin{cases} (-1, 1/2, 7/2, b/(b-1)) \qquad = \boldsymbol{\gamma}_1 \\ (-1, 5/2, 3/2, b/(b-1)) \qquad = \boldsymbol{\gamma}_2 \\ (-1, 1/2, (2b-1)/(b-1), 5/2) = \boldsymbol{\gamma}_3 \end{cases}.$$

We do not have to go back from $\boldsymbol{\beta}_1$ to $K(\boldsymbol{\beta}_1) = \boldsymbol{\alpha}_1$, and from $\boldsymbol{\gamma}_1$ to $K(\boldsymbol{\gamma}_1) = \boldsymbol{\alpha}_2$. Their effective deformations are only the following four:

$$I(\boldsymbol{\beta}_2) = (3/2, b/(b-1), -1, 5/2), \qquad I(\boldsymbol{\gamma}_2) = (3/2, 1/3, -1, 2-b),$$
$$K(\boldsymbol{\beta}_3) = (3, -1, 2/3, (b-2)/(b-1)), \quad K(\boldsymbol{\gamma}_3) = (3, -1, -1/(b-1), 5/3).$$

The (-1)-triples of $I(\boldsymbol{\beta}_2)$ (and $I(\boldsymbol{\gamma}_2)$, respectively) are related to those of $\boldsymbol{\gamma}_i$s (and $\boldsymbol{\beta}_i$s) as below. They are closed loops of deformations, up to dihedral deformation:

$$\begin{cases} (-1,\ 5/2, 3/2, b/(b-1)) \qquad = \boldsymbol{\gamma}_2 \\ (-1,\ 1/2, 7/2, b/(b-1)) \qquad = \boldsymbol{\gamma}_1 \\ (-1,\ 5/2, (2b-1)/(b-1), 1/2) = \boldsymbol{\gamma}_3 \end{cases}, \quad \begin{cases} (-1,\ 1/3, 3/2, 2-b) = \boldsymbol{\beta}_2 \\ (-1,\ 1/2, 4/3, 2-b) = \boldsymbol{\beta}_3 \\ (-1,\ 1/3, 3-b, 1/2) = \boldsymbol{\beta}_1 \end{cases}.$$

On the other hand, the (-1)-triples of $K(\boldsymbol{\beta}_3)$ and $K(\boldsymbol{\gamma}_3)$ are related to each other

$$\begin{cases} (-1,\ 3, (b-2)/(b-1), 2/3) \\ (-1,\ -1/(b-1), 4, 2/3) \\ (-1,\ 3, 5/3, -1/(b-1)) \end{cases}, \quad \begin{cases} (-1,\ 3, 5/3, -1/(b-1)) \\ (-1,\ 2/3, 4, -1/(b-1)) \\ (-1,\ 3, (b-2)/(b-1), 2/3) \end{cases}.$$

They are also closed loops of deformations. It is proved that surgeries "$a = 4$ and $b \neq -1, 4$" are hyperbolic.

Before starting the next case, we survey the complicated network of deformations as *a circuit of deformations* (see the diagram and the table in Figure 8). It starts from $(-1,\, 3, -1/2, b-1)$ as [0]. First, we name the (-1)-triple of [0] as [0-0], [0-1], [0-2]. Here, [0-0]=[0], up to dihedral deformation. Second, we study all possible effective deformations (by J, K and I) of them, and number them consecutively as [1], [2],…,[n_1]. We draw an arrow with a symbol J, K or I in the diagram. Note that arrows are reversible. Third, for the ones which contains -1 as entries among the results of effective deformations, we study their (-1)-deformations, which we name the (-1)-triple of [i] as [i-0] (=[i]), [i-1], [i-2]. Next, we study all effective deformations and number them consecutively as [n_1+1], [n_1+2],…,[n_2]. Here, we ignore going back (from [i-0]=[i]) to older ones. We repeat these steps and make the diagram. When the same element (up to dihedral deformation) appears twice, we connect them by a thin curve (with a symbol =, in Figure 8). The symbol • means a terminal point of deformations, i.e., no effective deformation (except going back) from it.

(Case 2: $(a,b) = (4,4)$) Then $\boldsymbol{\alpha} = (3, -1, 3, -1/2) \in \overline{\mathbb{Q}}^4$. Since the calculation is the same as the last case, we get them by putting $b = 4$ to each entry b in the the last case. We underline such entries. We are interested in *extra deformations*, i.e., deformations that appear only if $b = 4$.

The first dihedral deformation is $RC(\boldsymbol{\alpha}) = (-1, 3, -1/2, \underline{3})$, whose (-1)-triple and effective deformations are

$$\begin{cases} (-1, 3, -1/2, \underline{3}) = \boldsymbol{\alpha}_1 & \xrightarrow{J} (1/2, -1, 1/3, \underline{-1}) = \boldsymbol{\beta} \\ (-1, -3/2, 4, \underline{3}) = \boldsymbol{\alpha}_2 & \\ (-1, 3, \underline{4}, -3/2) = \boldsymbol{\alpha}_3 & \xrightarrow{J} (1/2, -1, \underline{4/3}, 7/2) = \boldsymbol{\gamma} \end{cases}.$$

It holds that $\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_3$ in this case. The (-1)-triples of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are

$$\begin{cases} (-1,\, 1/2, \underline{-1}, 1/3) = \boldsymbol{\beta}_1 \\ (-1,\, \underline{-2}, 3/2, 1/3) = \boldsymbol{\beta}_2 \\ (-1,\, 1/2, 4/3, \underline{-2}) = \boldsymbol{\beta}_3 \end{cases}, \quad \begin{cases} (-1, 1/2, 7/2, \underline{4/3}) = \boldsymbol{\gamma}_1 \\ (-1, 5/2, 3/2, \underline{4/3}) = \boldsymbol{\gamma}_2 \\ (-1, 1/2, \underline{7/3}, 5/2) = \boldsymbol{\gamma}_3 \end{cases}.$$

There are four effective deformations as in the last case:

$$\begin{aligned} I(\boldsymbol{\beta}_2) &= (3/2, \underline{4/3}, -1, 5/2), & I(\boldsymbol{\gamma}_2) &= (3/2, 1/3, -1, \underline{-2}), \\ K(\boldsymbol{\beta}_3) &= (3, -1, 2/3, \underline{2/3}), & K(\boldsymbol{\gamma}_3) &= (3, -1, \underline{-1/3}, 5/3). \end{aligned}$$

As in the last case, (-1)-triple of $I(\boldsymbol{\beta}_2)$ is equal to $\{\boldsymbol{\gamma}_2, \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_3\}$. that of $I(\boldsymbol{\gamma}_2)$ is equal to $\{\boldsymbol{\beta}_2, \boldsymbol{\beta}_3, \boldsymbol{\beta}_1\}$. The (-1)-triples of $K(\boldsymbol{\beta}_3)$ and $K(\boldsymbol{\gamma}_3)$ are both equal to

$$\begin{cases} (-1,\, 3, \underline{2/3}, 2/3) \\ (-1,\, \underline{-1/3}, 4, 2/3) \\ (-1,\, 3, 5/3, \underline{-1/3}) \end{cases}.$$

Here we study the underlined entries (those obtained by putting $b = 4$ in the calculus of the last case) above, and search for ones which possibly cause extra deformations. They are only $\boldsymbol{\beta} = (-1,\, 1/2, \underline{-1}, 1/3)$ $(= C^2(\boldsymbol{\beta}_1))$, but its deformation is absorbed by the dihedral symmetry. There exist only closed loops of deformations. The surgery $(a,b)=(4,4)$ is hyperbolic.

(Case 3: $(a,b)=(4,-1)$ or $(-1,4)$) Then $\boldsymbol{\alpha}=(3,-1,-2,-1/2)\in\overline{\mathbb{Q}}^4$. The method is same as in Case 2. Since the calculation is same as Case 1, we put $b=-1$ to each entry b there, and underline such entries. We are interested in extra deformations that appear only if $b=-1$.

The first dihedral deformation is $RC(\boldsymbol{\alpha})=(-1,3,-1/2,\underline{-2})$, whose (-1)-triple is

$$\begin{cases}(-1,3,-1/2,\underline{-2})=\boldsymbol{\alpha}_1 & \xrightarrow{J}(1/2,-1,1/3,\underline{4})=\boldsymbol{\beta}\\ (-1,-3/2,4,\underline{-2})=\boldsymbol{\alpha}_2 & \\ (-1,3,\underline{-1},-3/2)=\boldsymbol{\alpha}_3 & \xrightarrow{J}(1/2,-1,\underline{1/2},7/2)=\boldsymbol{\gamma}\end{cases}.$$

This $\boldsymbol{\gamma}$ has a dihedral symmetry in this case. The (-1)-triples of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are

$$\begin{cases}(-1,\,1/2,\underline{4},1/3)=\boldsymbol{\beta}_1\\ (-1,\,\underline{3},3/2,1/3)=\boldsymbol{\beta}_2\\ (-1,\,1/2,4/3,\underline{3})=\boldsymbol{\beta}_3\end{cases},\quad \begin{cases}(-1,1/2,7/2,\underline{1/2})=\boldsymbol{\gamma}_1\\ (-1,5/2,3/2,\underline{1/2})=\boldsymbol{\gamma}_2\\ (-1,1/2,\underline{3/2},5/2)=\boldsymbol{\gamma}_3\end{cases}.$$

In contrast to Case 1 and 2, there exist eight effective deformations as below:

$$\begin{aligned}&I(\boldsymbol{\beta}_2)=(3/2,\underline{1/2},-1,5/2), && I(\boldsymbol{\gamma}_2)=(3/2,1/3,-1,\underline{3}),\\ &K(\boldsymbol{\beta}_3)=(3,-1,2/3,\underline{3/2}), && K(\boldsymbol{\gamma}_3)=(3,-1,\underline{1/2},5/3),\end{aligned}$$

$$\begin{aligned}&J(\boldsymbol{\beta}_2)=(1/2,-1,3,5/3), && K(\boldsymbol{\gamma}_2)=(3,5/3,1/2,-1),\\ &J(\boldsymbol{\beta}_3)=(1/2,3/2,4,-1), && I(\boldsymbol{\gamma}_3)=(3/2,3,-1,1/3)=\boldsymbol{\beta}_2.\end{aligned}$$

In fact, not only the first four but also four more effective deformations exist. It holds that $J(\boldsymbol{\beta}_2)=K(\boldsymbol{\gamma}_2)=K(\boldsymbol{\gamma}_3)$, up to dihedral deformations, and that $I(\boldsymbol{\gamma}_3)=\boldsymbol{\beta}_2$, which appeared before.

As we saw in Case 1, (-1)-triple of $I(\boldsymbol{\beta}_2)$ is equal to $\{\boldsymbol{\gamma}_2,\boldsymbol{\gamma}_1,\boldsymbol{\gamma}_3\}$, and that of $I(\boldsymbol{\gamma}_2)$ is equal to $\{\boldsymbol{\beta}_2,\boldsymbol{\beta}_3,\boldsymbol{\beta}_1\}$. The (-1)-triples of $K(\boldsymbol{\beta}_3)$ and $K(\boldsymbol{\gamma}_3)$ are both equal to

$$\begin{cases}(-1,\,3,\underline{3/2},2/3)=K(\boldsymbol{\beta}_3)\\ (-1,\,\underline{1/2},4,2/3)=\boldsymbol{\delta}\\ (-1,\,3,5/3,\underline{1/2})=K(\boldsymbol{\gamma}_3)\end{cases}.$$

The new $\boldsymbol{\delta}$ has an effective deformation, but we suspend it. Finally, we take (-1)-triple of $J(\boldsymbol{\beta}_3)$ and their effective deformations (we ignore going back):

$$\begin{cases}(-1,\,4,3/2,1/2) & \\ (-1,\,1/2,5,1/2) & \xrightarrow{K}(3,-1,-3,-1)\\ (-1,\,4,3/2,1/2) & \end{cases}.$$

We take (-1)-triples of $(3,-1,-3,-1)$ and their effective deformations:

$$\begin{cases}(-1,\,3,-1,-3) & \\ (-1,\,-2,4,-3) & \\ (-1,\,3,-2,-2) & \xrightarrow{J}(1/2,-1,2/3,4)=\boldsymbol{\delta}\end{cases}.$$

We reach the $\boldsymbol{\delta}$ that appeared before.

There exists only closed loops of deformations (see the circuit of deformations and the table in Figure 9, where we omit symbols "="). Note that the circuit is more complicated than Figure 8 because of the extra deformations. The surgery $(a, b) = (4, -1)$ is (and, by symmetry, $(-1, 4)$ are also) hyperbolic.

(Case 4: $a = -1$ and $b \neq -1, 4$) We recall $\boldsymbol{\alpha}_i$ $(i = 1, 2, 3)$ with $a = -1$.

$$\begin{cases} (-1, -2, -1/2, b-1) = \boldsymbol{\alpha}_1 \\ (-1, -3/2, -1, b-1) = \boldsymbol{\alpha}_2 \\ (-1, -2, b, -3/2) \quad\quad = \boldsymbol{\alpha}_3 \end{cases}.$$

Though $\boldsymbol{\alpha}_2$ has two (-1)s, the (-1)-triple of the second (-1) is equal to that of the first (-1) by the symmetry. By the assumption (3) and "$b \neq -1, 4$", there are no effective deformations. These surgeries are hyperbolic.

(Case 5: $(a, b) = (-1, -1)$) In this case $\boldsymbol{\alpha}_3 = (-1, -2, -1, -3/2)$ as $(-1, a-1, b, -3/2)$. This is a rare case (3) in MPR list in Theorem 3.2. The proof is completed. □

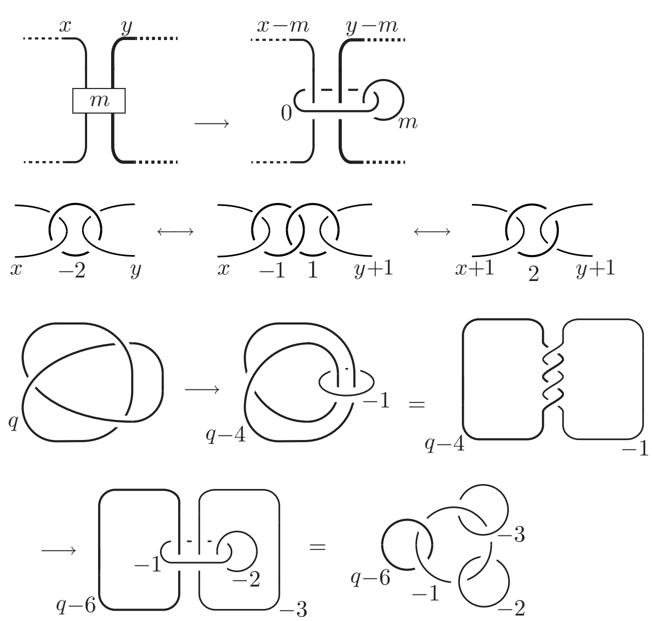


Figure 6. Some handle calculus

4. Nonhyperbolic case

We prove non-hyperbolic surgeries in Theorems 2.1, 2.3(2) and 2.4. In each figure, we stop drawing when the rest of calculus is obvious. Parts of the calculus in Figure 6 may help the readers.

Proof of Theorem 2.1. The proof is given by Kirby calculus in Figures 10, 11, 12 and 13. □

Proof of Theorem 2.3. Completeness follows from Theorem 1.1. For the resulting manifold of $(AY_2; 4, 9)$ in (2) in the theorem, see the calculus in Figure 14. □

Proof of Theorem 2.4. Completeness follows from Theorem 1.1. By Remark 2.5, we only have to consider the resulting manifold of $(MZ; 2, q)$ and $(MZ; 1, 1)$. The proof is given by the calculus in Figures 15, 16 and 17. □

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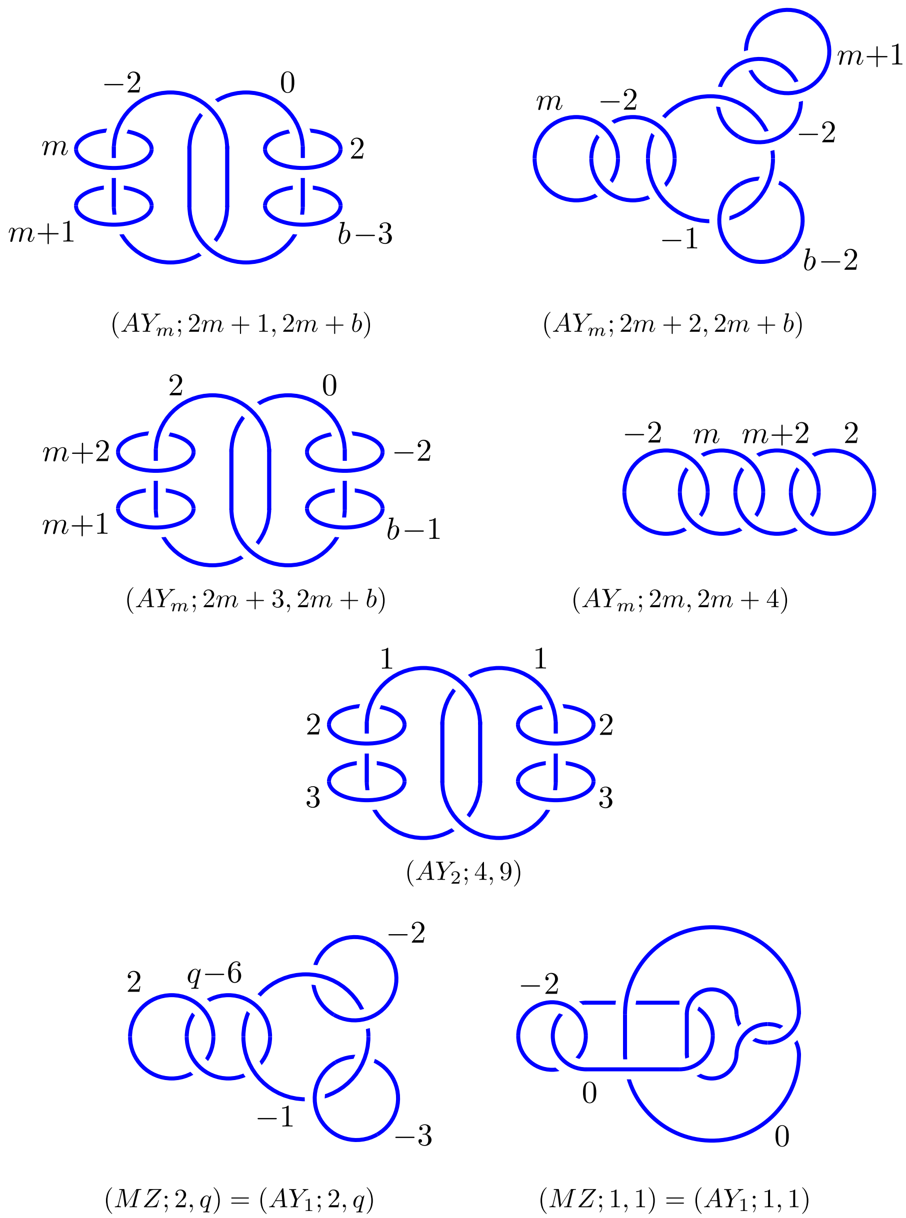
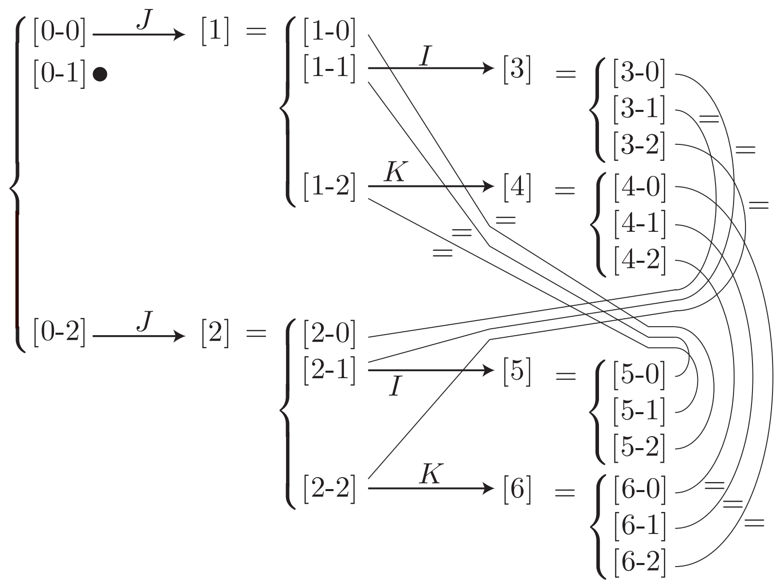


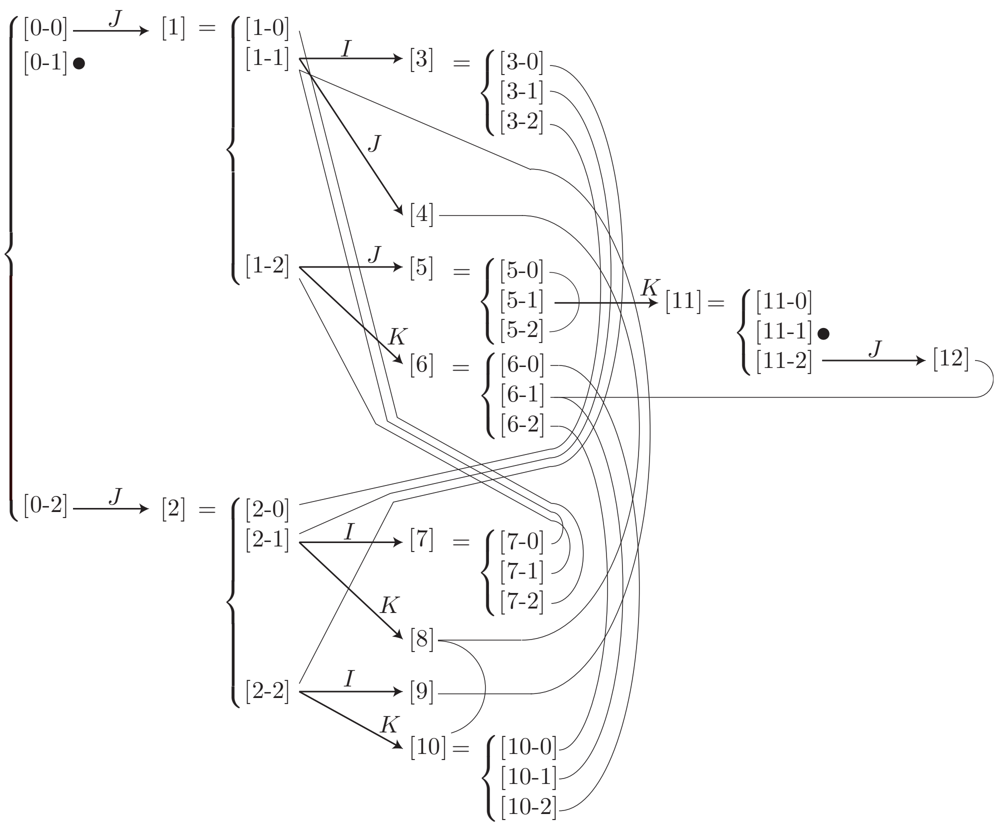
FIGURE 7. Exceptional surgeries along $(AY_m; p, q)$



[0-0]	$(-1,\ 3, -1/2, b-1) = \boldsymbol{\alpha}_1$
[0-1]	$(-1,\ -3/2, 4, b-1) = \boldsymbol{\alpha}_2$
[0-2]	$(-1,\ 3, b, -3/2) = \boldsymbol{\alpha}_3$
[1]	$(1/2, -1, 1/3, 3-b) = \boldsymbol{\beta}$
[2]	$(1/2, -1, b/(b-1), 7/2) = \boldsymbol{\gamma}$
[1-0]	$(-1,\ 1/2, 3-b, 1/3) = \boldsymbol{\beta}_1$
[1-1]	$(-1,\ 2-b, 3/2, 1/3) = \boldsymbol{\beta}_2$
[1-2]	$(-1,\ 1/2, 4/3, 2-b) = \boldsymbol{\beta}_3$
[2-0]	$(-1, 1/2, 7/2, b/(b-1)) = \boldsymbol{\gamma}_1$
[2-1]	$(-1, 5/2, 3/2, b/(b-1)) = \boldsymbol{\gamma}_2$
[2-2]	$(-1, 1/2, (2b-1)/(b-1), 5/2) = \boldsymbol{\gamma}_3$
[3]	$(3/2, b/(b-1), -1, 5/2)$
[4]	$(3, -1, 2/3, (b-2)/(b-1))$
[5]	$(3/2, 1/3, -1, 2-b)$

[6]	$(3, -1, -1/(b-1), 5/3)$
[3-0]	$(-1,\ 5/2, 3/2, b/(b-1))$
[3-1]	$(-1,\ 1/2, 7/2, b/(b-1))$
[3-2]	$(-1,\ 5/2, (2b-1)/(b-1), 1/2)$
[4-0]	$(-1,\ 3, (b-2)/(b-1), 2/3)$
[4-1]	$(-1,\ -1/(b-1), 4, 2/3)$
[4-2]	$(-1,\ 3, 5/3, -1/(b-1))$
[5-0]	$(-1,\ 1/3, 3/2, 2-b)$
[5-1]	$(-1,\ 1/2, 4/3, 2-b)$
[5-2]	$(-1,\ 1/3, 3-b, 1/2)$
[6-0]	$(-1,\ 3, 5/3, -1/(b-1))$
[6-1]	$(-1,\ 2/3, 4, -1/(b-1))$
[6-2]	$(-1,\ 3, (b-2)/(b-1), 2/3)$

FIGURE 8. Circuit of deformations: Case 1 ($m = 1, a = 4$ and $b \neq -1, 4$)



[0-0]	$(-1,\ 3, -1/2, -2) = \boldsymbol{\alpha}_1$
[0-1]	$(-1,\ -3/2, 4, -2) = \boldsymbol{\alpha}_2$
[0-2]	$(-1,\ 3, -1, -3/2) = \boldsymbol{\alpha}_3$
[1]	$(1/2, -1, 1/3, 4) = \boldsymbol{\beta}$
[2]	$(1/2, -1, 1/2, 7/2) = \boldsymbol{\gamma}$
[1-0]	$(-1,\ 1/2, 4, 1/3) = \boldsymbol{\beta}_1$
[1-1]	$(-1,\ 3, 3/2, 1/3) = \boldsymbol{\beta}_2$
[1-2]	$(-1,\ 1/2, 4/3, 3) = \boldsymbol{\beta}_3$
[2-0]	$(-1, 1/2, 7/2, 1/2) = \boldsymbol{\gamma}_1$
[2-1]	$(-1, 5/2, 3/2, 1/2) = \boldsymbol{\gamma}_2$
[2-2]	$(-1, 1/2, 3/2, 5/2) = \boldsymbol{\gamma}_3$
[3]	$(3/2, 1/2, -1, 5/2)$
[4]	$(1/2, -1, 3, 5/3)$
[5]	$(1/2, 3/2, 4, -1)$
[6]	$(3, -1, 2/3, 3/2)$
[7]	$(3/2, 1/3, -1, 3)$
[8]	$(3, 5/3, 1/2, -1)$
[9]	$(3/2, 3, -1, 1/3)$
[10]	$(3, -1, 1/2, 5/3)$

[3-0]	$(-1,\ 1/2, 3/2, 5/2)$
[3-1]	$(-1,\ 1/2, 3/2, 2/3)$
[3-2]	$(-1,\ 1/2, 7/2, 1/2)$
[5-0]	$(-1,\ 4, 3/2, 1/2)$
[5-1]	$(-1,\ 1/2, 5, 1/2)$
[5-2]	$(-1,\ 4, 3/2, 1/2)$
[6-0]	$(-1,\ 3, 3/2, 2/3)$
[6-1]	$(-1,\ 1/2, 4, 2/3) = \boldsymbol{\delta}$
[6-2]	$(-1,\ 3, 5/3, 1/2)$
[7-0]	$(-1,\ 3, 3/2, 1/3)$
[7-1]	$(-1,\ 1/2, 4, 1/3)$
[7-2]	$(-1,\ 3, 4/3, 1/2)$
[10-0]	$(-1,\ 3, 5/3, 1/2)$
[10-1]	$(-1,\ 2/3, 4, 1/2)$
[10-2]	$(-1,\ 3, 3/2, 2/3)$
[11]	$(3, -1, -3, -1)$
[11-0]	$(-1,\ 3, -1, -3)$
[11-1]	$(-1,\ -2, 4, -3)$
[11-2]	$(-1,\ 3, -2, -2)$
[12]	$(1/2, -1, 2/3, 4)$

Figure 9. Circuit of deformations: Case 3 $(m = 1, (a, b) = (4, -1))$

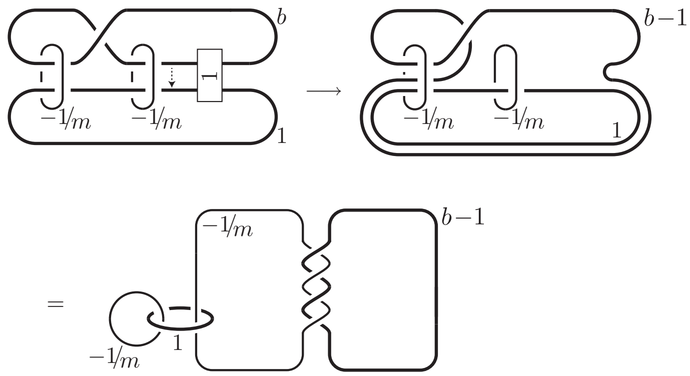


FIGURE 10. $(AY_m; 2m+1, 2m+b)$

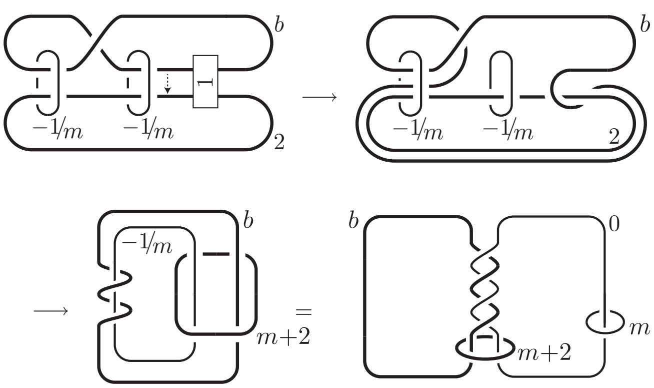


FIGURE 11. $(AY_m; 2m+2, 2m+b)$

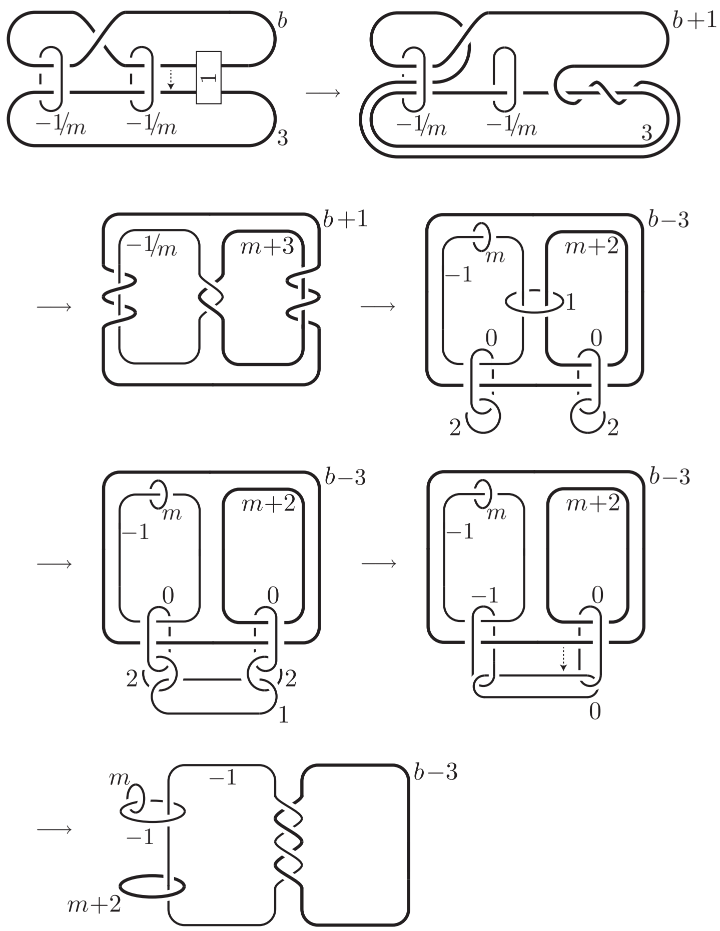


Figure 12. $(AY_m; 2m+3, 2m+b)$

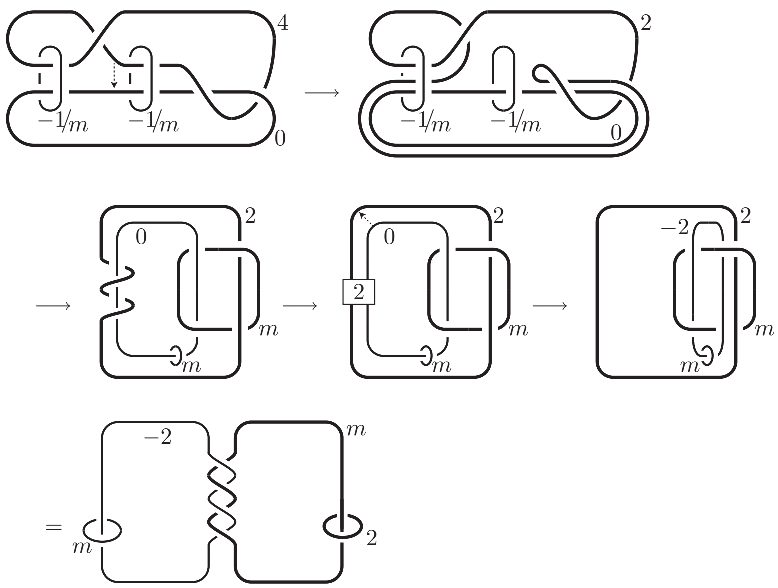


FIGURE 13. $(AY_m; 2m, 2m+4)$

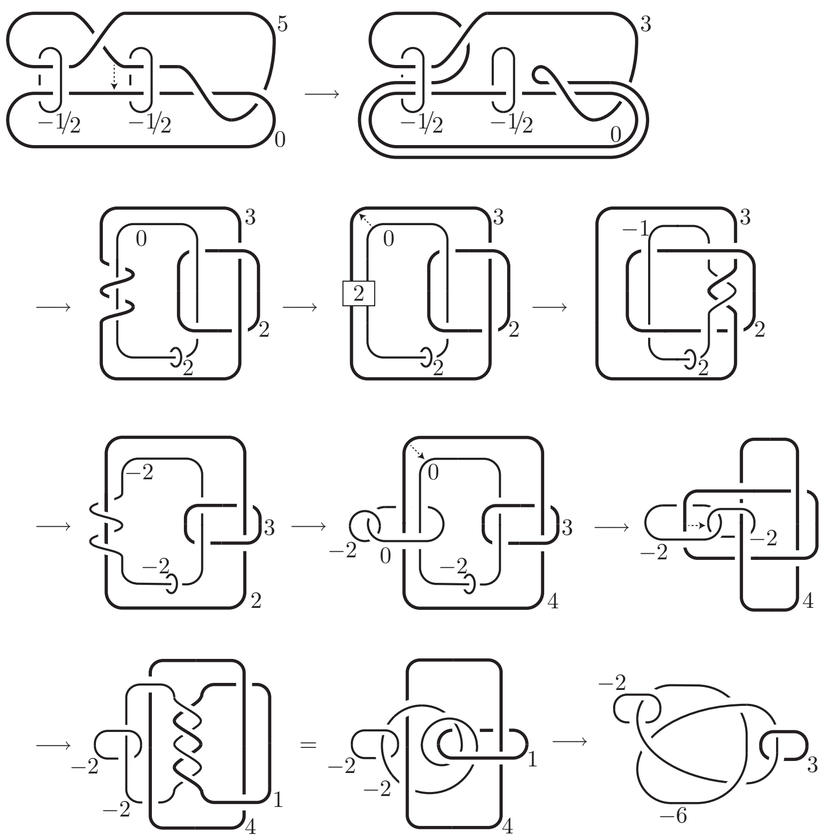


Figure 14. $(AY_2; 4, 9)$

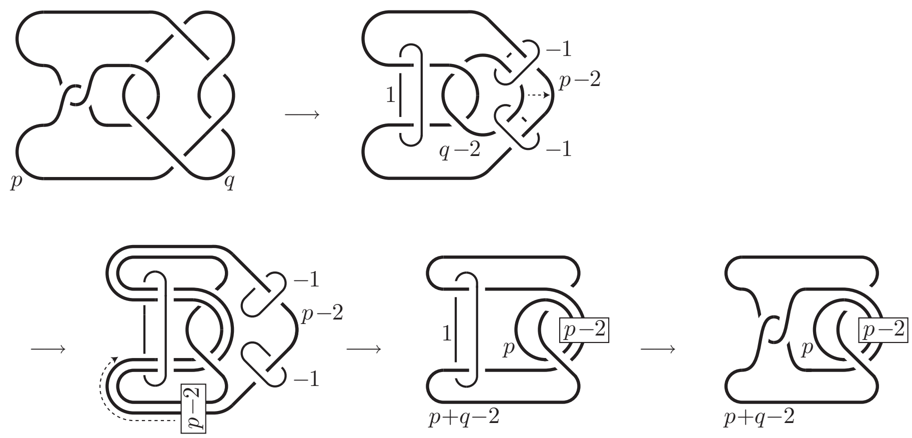


FIGURE 15. $(MZ; p, q)$ the first step

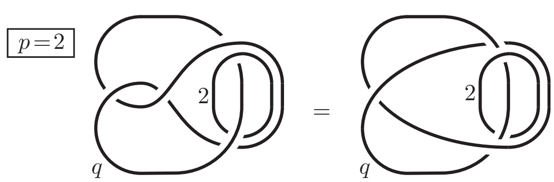


FIGURE 16. $(MZ; 2, q)$

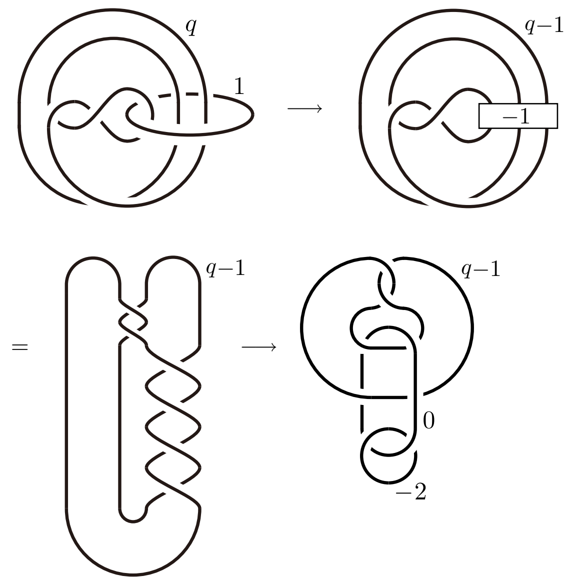
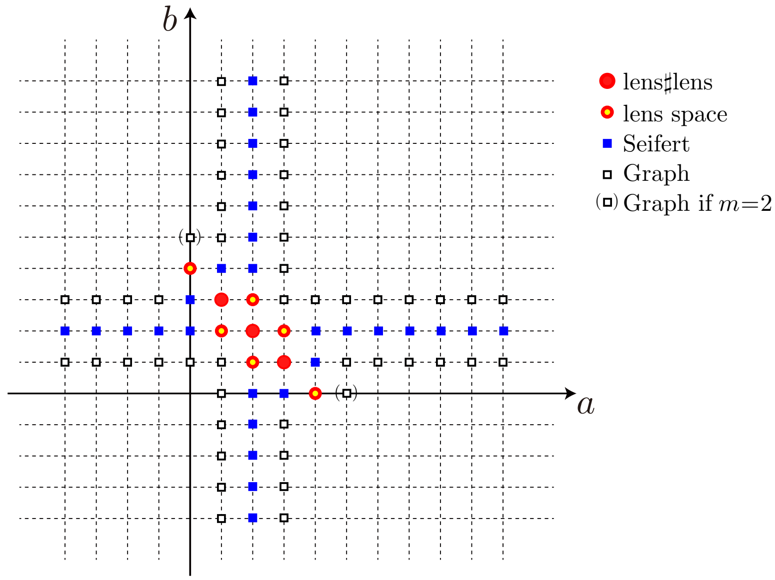
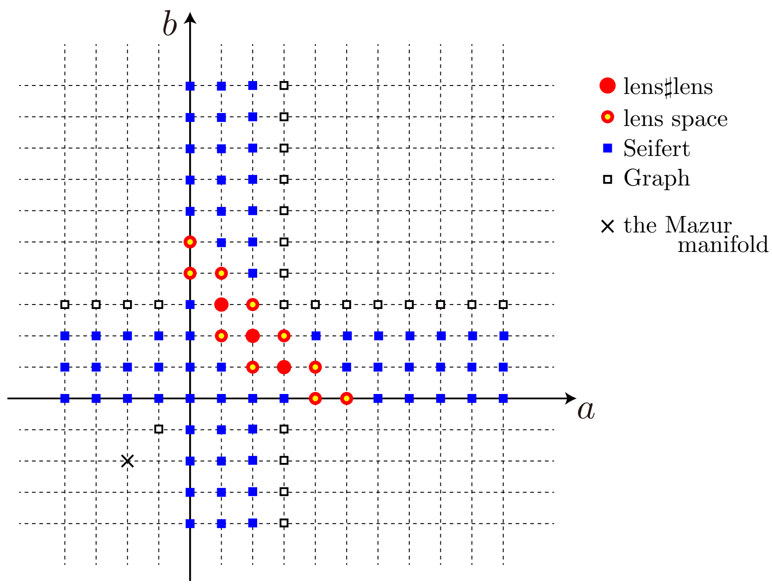


FIGURE 17. $(MZ; 1, q)$ (see [Ak2])



along $AY_m(2m+a, 2m+b)$ with $m \geq 2$



along $MZ(2+a, 2+b)$, where $MZ = AY_1$.

FIGURE 18. Geography of exceptional Dehn surgeries