

## Computing rotation numbers in open books

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ABSTRACT. We give explicit formulas and algorithms for the computation of the rotation number of a nullhomologous Legendrian knot on a page of a contact open book. On the way, we derive new formulas for the computation of the Thurston–Bennequin invariant of such knots and the Euler class and the  $d_3$ -invariant of the underlying contact structure.

### 1. Introduction

The *classical invariants*, the *Thurston–Bennequin invariant*  $tb$  and the *rotation number*  $rot$ , are the two most fundamental invariants of nullhomologous Legendrian knots in contact 3-manifolds. They carry a lot of information about the contact structure (for example the contact structure is overtwisted if and only if there exists a Legendrian unknot with  $tb = 0$ , see [7]) and the topological knot type (for example the classical invariants give obstructions to sliceness of a knot in  $S^3$ , see [20]).

According to Giroux, there is a deep connection between contact manifolds and open books (cf. [8]). In particular, to every open book presenting a 3-manifold there exists a (up to isotopy) unique contact structure on this 3-manifold with contact planes arbitrarily close to the pages of the open book outside a neighbourhood of the binding. Throughout the paper, we use the expression *contact open book* to emphasize that we are in fact considering the contact 3-manifold associated with the abstract open book in this way.

In this paper we continue in the spirit of [6] and consider Legendrian knots sitting on the page of a contact open book. In [6] we explained how to check if such a knot is nullhomologous and if so, how to compute its Thurston–Bennequin invariant. Here we concentrate on the second classical invariant, the rotation number, and give a formula to compute it.

**Theorem 1.1.** *Let  $K$  be a Legendrian knot sitting on the page of a contact open book  $(\Sigma, \phi)$  with monodromy  $\phi$  given as a concatenation of Dehn twists along non-isolating curves. Then there exists an arc basis of  $\Sigma$  such that the intersection behaviour of  $K$  and the Dehn twist curves with the arcs give criteria and formulas to*

- (a) *decide whether  $K$  is (rationally) nullhomologous,*
- (b1) *compute the (rational) Thurston–Bennequin invariant of  $K$  if  $K$  is (rationally) nullhomologous,*

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- (b2) *compute the (rational) rotation number of  $K$  if  $K$  is (rationally) nullhomologous,*
  - (b3) *compute the (rational) self-linking number of a transverse push-off of  $K$  if  $K$  is (rationally) nullhomologous,*
  - (c) *compute the Poincaré dual of the Euler class of the contact structure,*
  - (d) *decide whether the Euler class of the contact structure is torsion and if so, compute its  $d_3$ -invariant*
- (see Algorithm 6.1).

Readers only interested in using the resulting formulas can proceed to Algorithm 6.1 and the following discussion in Section 6.

Several results in this direction have been obtained earlier. Etnyre and Ozbagci [10] gave a formula to compute the Euler class and the  $d_3$ -invariant of a contact open book, which in many cases can be easier to compute than the one given in this paper. In order to do so, they also developed a method to calculate the rotation number of a Legendrian knot on the page of an open book (see also the explicit calculations in [16]). In [6] a method for checking if a Legendrian knot sitting on a page is nullhomologous is developed, and if so, a formula for its Thurston–Bennequin invariant is provided. On the other hand, Gay and Licata [11] studied Legendrian knots in open books which in general are not contained in a page by a generalisation of the front projection, where it is possible to compute tb as well.

Throughout this paper, all homology groups are understood to be integral unless indicated otherwise. We will also, by abuse of notation, use the same symbol for a curve, the homology class and the positive Dehn twist it represents.

We will first generalise an example of [17] to compute the rotation number of a Legendrian knot sitting on the page of a specific planar open book of  $(S^3, \xi_{\text{st}})$ . Afterwards we use the method of [2] to find an embedding of a more general non-planar abstract open book into  $(S^3, \xi_{\text{st}})$  and give formulas for computing the rotation number in these cases.

For the general case, we first use Avdek’s algorithm [2] for transforming a contact open book into a contact surgery diagram along a Legendrian link and then compute the invariants from the resulting contact surgery diagram via [5], which builds on [17, 13, 3, 15].

We begin with an application of our results on the binding number of Legendrian knots, which we propose to study in analogy to the binding number of a contact manifold as introduced in [10].

### **Application to the binding number of Legendrian knots**

Let  $K$  be a Legendrian knot in a contact 3-manifold  $(M, \xi)$ . Then the support genus  $\text{sg}(K)$  is defined to be the minimal genus of the page of a contact open book decomposition of  $(M, \xi)$  in which  $K$  is contained in a single page, i.e.

$$\text{sg}(K) = \min \{g(\Sigma) \mid K \subset \Sigma\},$$

where  $g(\Sigma)$  is the genus of the surface  $\Sigma$  (see [18]).

In analogy to the binding number of a contact manifold as introduced by [10], we propose to define the binding number  $\text{bn}$  of  $K$  to be the minimal number of boundary components of the pages of contact open book decompositions with minimal genus containing  $K$  in a page, i.e.

$$\text{bn}(K) := \min \{ |\partial\Sigma| : K \subset \Sigma \text{ with } g(\Sigma) = \text{sg}(K) \}.$$

**Corollary 1.2.** *Let  $K$  be a Legendrian knot with non-vanishing rotation number and support genus  $\text{sg}(K) = 1$  in an arbitrary contact 3-manifold. Then the binding number of  $K$  is at least two.*

Note that this also becomes evident from the proof of Lemma 6.1 in [10].

*Proof.*

Suppose that  $K$  has support genus and binding number both equal to one, then one can easily check using Theorem 1.1 or via the explicit formulas given in Algorithm 6.1 that the rotation number of  $K$  vanishes.  $\square$

**Example 1.3.** It is known that all Legendrian realizations of torus knots  $T_{2,2n+1}$ ,  $n \in \mathbb{N}$ , with Thurston–Bennequin invariant at least one and non-vanishing rotation number have support genus equal to one (see Theorem 1.3 in [16]) and thus binding number at least two.

## 2. Background: Legendrian curves on open books

For the basics in low-dimensional contact topology we refer the reader for example to [8, 19, 12]. Nevertheless, we will briefly recall some well-known facts about which curves sitting on the page of a contact open book represent Legendrian knots.

Let  $L$  be a simple closed curve on a convex surface  $S$ . We call  $L$  **non-isolating** if every component of  $S \setminus L$  has non-empty intersection with the dividing set  $\Gamma$  of  $S$ .

**Lemma 2.1.** *A simple closed curve  $L$  on  $S$  represents a Legendrian knot (i.e. can be realised as a Legendrian knot by a small perturbation of  $S$  through convex surfaces) if and only if  $L$  is non-isolating.*

*Proof.* A non-isolating simple closed curve always represents a Legendrian knot by the Legendrian realisation principle (see [14, Theorem 3.7]). So let  $L$  be not non-isolating, i.e. there is a component  $S_0$  of  $S \setminus L$  with  $S_0 \cap \Gamma = \emptyset$ , and assume that  $L$  represents a Legendrian knot. Without loss of generality, we have  $\text{div}_\Omega(X) > 0$  on  $\bar{S}_0$ , where  $\Omega$  is a volume form on  $S$  and  $X$  the vector field defining the characteristic foliation. Hence,

$$0 < \int_{\bar{S}_0} \text{div}_\Omega(X) = \int_{\bar{S}_0} d(i_X \Omega) = \int_L i_X \Omega = \int_L \alpha = 0,$$

where  $\alpha$  denotes the contact form and the last equality holds because  $L$  is Legendrian.  $\square$

Here we are interested in the special case of the page  $\Sigma$  of an open book, which is convex with  $\Gamma = \partial\Sigma$ . In particular, if  $\partial\Sigma$  is connected,  $L$  represents a Legendrian knot if and only

if  $L$  is non-separating. Note also, that for every Legendrian link in a contact manifold there exists a compatible open book decomposition such that the link is contained in a page (cf. [1] or [8, Corollary 4.23]), i.e. our assumption of a Legendrian knot sitting on the page of an open book is not exotic at all.

### 3. A special planar case

We begin by discussing a method to compute the rotation number in an easy planar case which is based on the idea presented in [17, Lemma 4.1].

Suppose that  $\Sigma$  is *planar*, i.e.  $\Sigma$  is a disc with  $k$  holes

$$\Sigma \cong D^2 \setminus \left( \bigsqcup_{i=1}^k D_i^2 \right),$$

and the monodromy is given by  $\phi = \beta_k^{+1} \circ \dots \circ \beta_1^{+1}$ , where  $\beta_i^{+1}$  denotes a positive Dehn twist along a curve  $\beta_i$  parallel to the inner boundary  $\partial D_i^2$ . We furthermore assume that the curves  $\beta_i$  are oriented consistently with the boundary orientation induced by  $\Sigma$  (see Figure 1). In particular, by destabilising the open book, we see that  $(\Sigma, \phi)$  describes the standard contact 3-sphere  $(S^3, \xi_{\text{st}})$  and from this it also follows that every Legendrian knot  $K$  sitting on the page of this contact open book is some Legendrian unknot.

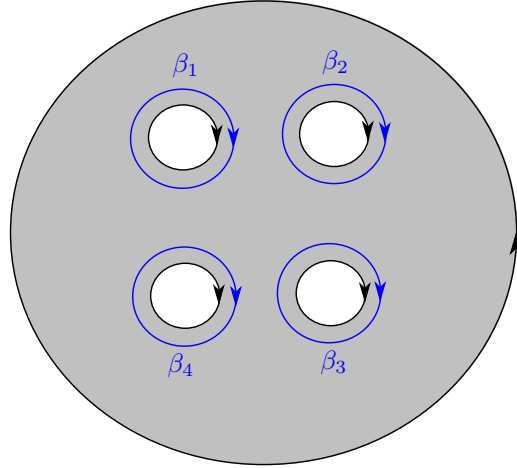


FIGURE 1. A planar open book decomposition of  $(S^3, \xi_{\text{st}})$

**Proposition 3.1.** *Let  $K$  be a Legendrian knot sitting on the page of a planar open book  $(\Sigma, \phi)$  with  $\phi$  as described above. Then the following holds:*

- (1)  $K = \sum_{i=1}^k b_i \beta_i \in H_1(\Sigma)$  such that either all  $b_i \in \{+1, 0\}$  or all  $b_i \in \{-1, 0\}$ ,

(2) *the rotation number of  $K$  computes as*

$$\text{rot}(K) = \sum_{i=1}^k b_i - \text{sign} \left( \sum_{i=1}^k b_i \right).$$

*Proof.*

(1) First note, that a simple closed curve cannot have  $|b_i| > 1$  or it would have self-intersections. With orientations chosen as above, one also observes that all non-vanishing  $b_i$  have to be equal.

(2) By the first statement, we can glue small oriented rectangular bands connecting the  $b_i\beta_i$  with non-vanishing coefficients  $b_i$  inside  $\Sigma$  in such a way that the oriented boundary of the resulting region is isotopic to  $K$  in  $\Sigma$  (cf. Figure 2). The orientation of these rectangles coincides with the orientation of the page  $\Sigma$  exactly if the  $b_i$  are positive.

Note that the  $\beta_i$  are unknots with Thurston–Bennequin invariant  $-1$  and vanishing rotation number. Indeed,  $\beta_i$  can be assumed to be parallel to a Dehn twist curve arising by a stabilisation. These curves bound a disc in the complement and by the Dehn twist, the Seifert framing differs by one from the contact framing given by the page. So  $\beta_i$  is a  $tb = -1$  unknot, i.e. the rotation number is zero. Furthermore, a Seifert surface for  $K$  is given by the union of the discs bounded by the non-vanishing  $b_i\beta_i$  (in the complement of the page) and the attached bands in the page. The rotation number computes as the sum of the indices of a vector field in the contact structure extending the positive tangent of  $K$  over  $\Sigma$ . As  $\text{rot}(\beta_i) = 0$ , an extension without zeros is possible over the discs bounded by  $\beta_i$  and we only have to study the bands. As the contact framing and the page framing coincide, this reduces the problem to extending the positive tangent vector field to the boundary of the bands over the bands in  $\Sigma$ . This is  $\pm 1$  for each band by Poincaré–Hopf, depending on whether the orientation of the band agrees with the orientation of the page  $\Sigma$  or not. Hence, the rotation number of  $K$  is a signed count of the number of bands, i.e.  $\text{rot}(K) = \sum_{i=1}^k b_i - \text{sign} \left( \sum_{i=1}^k b_i \right)$ .  $\square$

**Remark 3.2.** The formula from Proposition 3.1 can also be obtained by observing that a curve enclosing  $k$ -holes is the result of  $(k - 1)$ -times stabilising a curve running around a single hole. The latter has Thurston–Bennequin invariant  $-1$  and vanishing rotation number.

**Example 3.3.** Consider the Legendrian knot  $L$  on the planar open book of  $(S^3, \xi_{\text{st}})$  as depicted in Figure 2. The class in the first homology group of  $\Sigma$  represented by  $L$  can be written as

$$L = \sum_{i=1}^4 b_i\beta_i = \beta_2 + \beta_3 + \beta_4.$$

By Proposition 3.1, the rotation number of  $L$  is

$$\text{rot}(L) = \sum_{i=1}^4 b_i - \text{sign} \left( \sum_{i=1}^4 b_i \right) = 2.$$

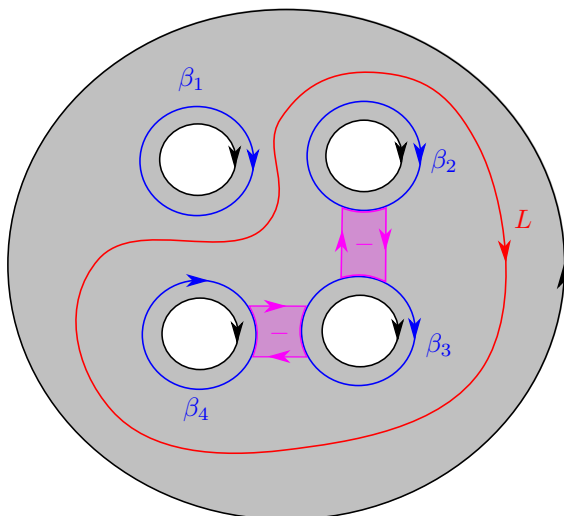


FIGURE 2. A Legendrian knot on the page of a planar open book of  $(S^3, \xi_{st})$

This method is not known to generalise to non-planar open books. One reason is, that on surfaces of higher genus, the isotopy class of a curve is not determined by its homology class.

#### 4. Another special case

Next we consider knots on open books  $(\Sigma, \phi)$  of the standard contact 3-sphere with an arbitrary page but a special monodromy. Denote the genus of  $\Sigma$  by  $g$  and the number of boundary components by  $h + 1$ . Suppose that the monodromy is given by

$$\phi = \beta_{g+h}^{+1} \circ \dots \circ \beta_{g+1}^{+1} \circ \beta_g^{+1} \circ \alpha_g^{+1} \circ \dots \circ \beta_1^{+1} \circ \alpha_1^{+1}$$

as indicated in Figure 3. We also choose orientations of  $\alpha_i$  and  $\beta_i$  as in the picture. In particular, the signed count  $\alpha_i \bullet \beta_j$  of intersection points between  $\alpha_i$  and  $\beta_j$  is  $\delta_{ij}$ . Let  $r_i, i = 1, \dots, g + h - 1$ , be the depicted reducing arcs, which do not intersect the  $\alpha$ - and  $\beta$ -curves. I.e. when cutting along them the page  $\Sigma$  decomposes into a collection of tori with a disc removed and annuli. Let  $a_i$  and  $b_i$  be arcs on the page  $\Sigma$  representing a basis of  $H_1(\Sigma, \partial\Sigma)$  dual to  $\{\alpha_i, \beta_i\}$  with respect to the intersection product (oriented such that  $\alpha_i \bullet a_i = 1, \beta_j \bullet b_j = 1$ ).

The following algorithm will be applied to a word corresponding to the knot  $K$  in Proposition 4.2. Note that the conventions presented below for labelling vertical tangencies in this setting by  $\rho_+$  and  $\lambda_+$  do not agree with those for counting cusps of a Legendrian front projection as in [12, Proposition 3.5.19].

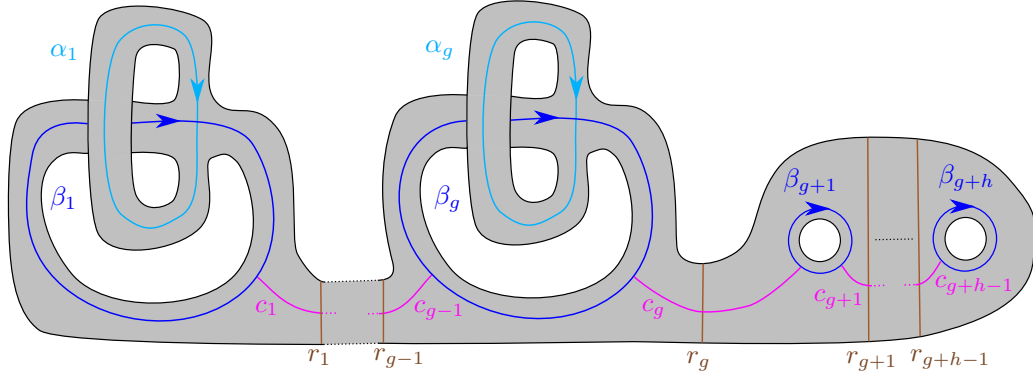


FIGURE 3. A non-planar open book of  $(S^3, \xi_{\text{st}})$  with arbitrarily many boundary components

**Algorithm 4.1.** Let  $w$  be a word in  $\langle \alpha_i, \beta_i \mid i = 1 \dots, k \rangle$ . Set  $\lambda_+$  to be the number of times a  $\beta^{-1}$  is followed by an  $\alpha^{-1}$  of the same index also considering the step from the last to the first letter, and similarly, set  $\rho_+$  equal to the number of times an  $\alpha^{-1}$  is followed by a  $\beta^{-1}$  of the same index.

Denote places where the index changes by  $r_u$  ( $r_d$ ) if the index increases (decreases) – including the last position if the index of the last letter is not equal to the index of the first letter. For instance, in the word

$$\alpha_1 \beta_2 \alpha_2 \beta_4^{-1} \alpha_3^{-1} \beta_2$$

we have five positions of index changes:

$$\alpha_1 r_u \beta_2 \alpha_2 r_u \beta_4^{-1} r_d \alpha_3^{-1} r_d \beta_2 r_d.$$

Now run through the index changes and increase  $\lambda_+$  and  $\rho_+$  according to the following rule:

- increase  $\lambda_+$  by 1 for
  - a  $\beta^{-1}$  followed by  $r_u$
  - $r_d$  followed by an  $\alpha^{-1}$
- increase  $\rho_+$  by 1 for
  - an  $\alpha^{-1}$  followed by  $r_d$
  - a  $\beta$  followed by  $r_d$ .

In the example sequence above, we have  $\lambda_+ = 0 = \rho_+$  as starting values and then get  $\lambda_+ = 1$  and  $\rho_+ = 2$  as the final result after following the rules for the increments.

**Proposition 4.2.** Let  $K$  be an oriented non-isolating knot on the abstract open book  $(\Sigma, \phi)$  of  $(S^3, \xi_{\text{st}})$  specified above. Choose a starting point on  $K$  and write  $K$  as a word in the  $\alpha_i$  and  $\beta_i$  by noting intersections with  $a_i$  and  $b_i$  when traversing along  $K$ . Then

the rotation number of  $K$  is

$$\text{rot}(K) = \rho_+ - \lambda_+$$

with  $\rho_+$  and  $\lambda_+$  calculated from the presentation of  $K$  as described in Algorithm 4.1.

*Proof.*

First note that without loss of generality, we can assume that the page  $\Sigma$  has only a single boundary component by stabilising the open book along arcs not intersecting the  $r_i$  connecting a hole to the outer boundary component. Then the open book  $(\Sigma, \phi)$  can be embedded into  $(S^3, \xi_{\text{st}})$  with the front projection shown in Figure 4 (in lightly shaded regions the orientation of  $\Sigma$  agrees with the blackboard orientation, in darkly shaded regions the orientations disagree) – the embedded page  $\Sigma$  is the ribbon of the Legendrian graph displayed in the upper half of Figure 4 (see [2] for details). Note that in particular, the contact vector field  $\partial_z$  is transverse to the embedded page. Furthermore, after rescaling the embedding can be assumed to be such that in  $\mathbb{R}^3 \subset S^3$  we have

$$\begin{aligned} [-1, 1] \times \Sigma &\longrightarrow (\mathbb{R}^3, \xi_{\text{st}} = \ker(xdy + dz)), \\ (t, p) &\longmapsto p + (0, 0, t). \end{aligned}$$

I.e. for a point  $p$  in the interior of the embedded page  $\{0\} \times \Sigma$  the line through  $p$  parallel to the  $z$ -axis hits every page in  $[-1, 1] \times \Sigma$  exactly once. So we can relate a specific page in  $[-1, 1] \times \Sigma$  to its shift in the  $z$ -direction.

The rotation number of a nullhomologous Legendrian knot with respect to a Seifert surface  $S$  is given by the rotation of its tangent vector with respect to a fixed trivialisation of the contact planes over  $S$ . If the contact structure is globally trivialisable, one can instead fix a global trivialisation. The standard contact structure  $\xi_{\text{st}}$  on  $\mathbb{R}^3 \subset (S^3, \xi_{\text{st}})$  can be trivialised globally by  $\partial_x$  and  $\partial_y - x\partial_z$ . As the contact vector field  $\partial_z$  is transverse to the embedded page  $\Sigma$  of the open book, this trivialisation also induces a trivialisation of the tangent planes to  $\Sigma$ . Then the rotation number of the Legendrian realisation of a curve sitting on the page agrees with the rotation of the original curve on the page with respect to the induced trivialisation.

The projection of  $\partial_x$  to  $\Sigma$  along  $\partial_z$  lies in the  $xz$ -plane. Observe that the  $\partial_z$ -component changes sign when passing from a lightly shaded region to a darkly shaded region and vice-versa. To compute the rotation of a curve on the embedded page which is non-singular in the front projection diagram, we thus have to count vertical tangencies in the front projection according to the rule described in Figure 5. The rotation then equals  $\rho_+ - \lambda_+$ . Alternatively, we can also compute it as  $\lambda_- - \rho_-$  where  $\lambda_-$  and  $\rho_-$  are defined analogously to  $\lambda_+$  and  $\rho_+$ .

In fact, we do not even have to count all vertical tangencies, but we can ignore those cancelling each other. To this end, we write  $K$  as a word in the  $\alpha_i$  and  $\beta_i$  by noting intersections with  $a_i$  and  $b_i$  when traversing along  $K$ . Observe that the  $\alpha$ - and  $\beta$ -curves have vanishing rotation, as they have two vertical tangencies cancelling each other. Changing from  $\alpha_i$  to  $\beta_i$  accounts for a  $\lambda_-$ , changing from  $\beta_i$  to  $\alpha_i$  for a  $\rho_-$ . Likewise, the change from  $\alpha_i^{-1}$  to  $\beta_i^{-1}$  gives a  $\rho_+$ : the one from  $\beta_i^{-1}$  to  $\alpha_i^{-1}$  a  $\lambda_+$ . It is easily verified that



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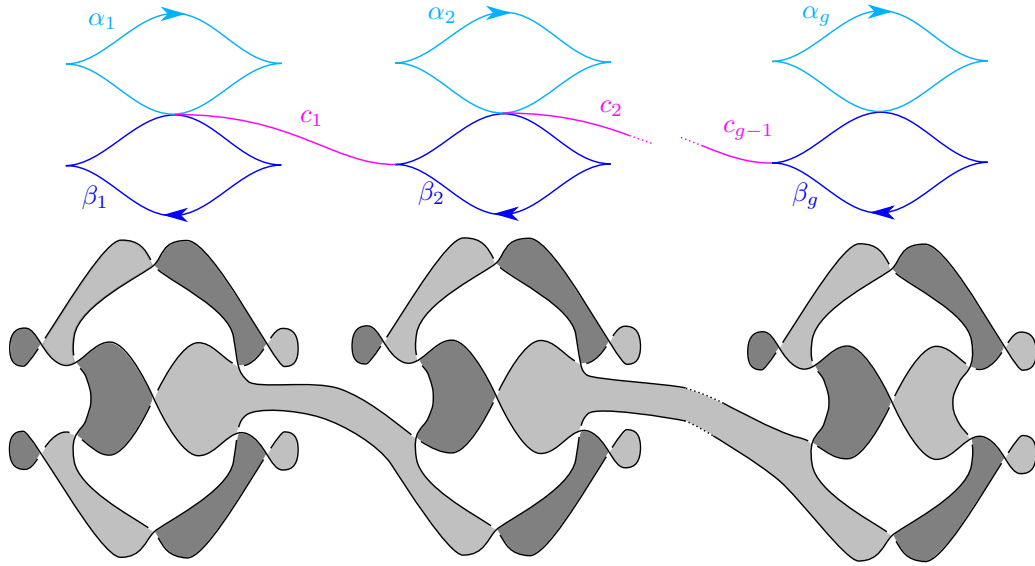


FIGURE 4. An embedding into  $(S^3, \xi_{\text{st}})$  of the (stabilised) abstract open book from Figure 3 and the Legendrian graph shown in the front projection

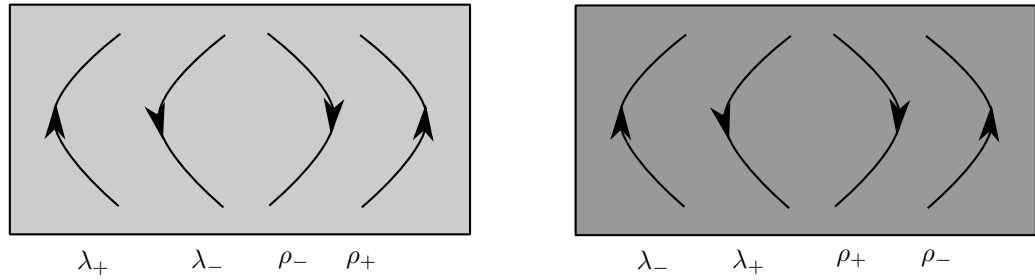


FIGURE 5. The labelling of the vertical tangencies.

all other changes with fixed index do not introduce vertical tangencies. In particular, a knot not intersecting any of the reducing arcs has vanishing rotation number, since it has  $\lambda_+ = \rho_+$ . It thus remains to inspect those tangencies occurring before or after an intersection with a reducing arc. These intersections happen when the index of the letters change. The vertical tangencies occurring in these cases are summarised in Table 1.

Hence, the rotation number can be computed from the word according to the rule given in Algorithm 4.1.  $\square$

leaving to the right from	count
$\alpha$	$\lambda_-$
$\alpha^{-1}$	-
$\beta$	-
$\beta^{-1}$	$\lambda_+$
coming from the left to	count
$\alpha$	$\rho_-$
$\alpha^{-1}$	-
$\beta$	-
$\beta^{-1}$	$\rho_+$
leaving to the left from	count
$\alpha$	-
$\alpha^{-1}$	$\rho_+$
$\beta$	$\rho_+$
$\beta^{-1}$	-
coming from the right to	count
$\alpha$	-
$\alpha^{-1}$	$\lambda_+$
$\beta$	$\lambda_-$
$\beta^{-1}$	-

TABLE 1

**Example 4.3.** Consider the knot on the embedded page of the open book of  $(S^3, \xi_{st})$  given in Figure 6. The knot corresponds to the word  $\alpha_1\beta_2\alpha_2\alpha_3^{-1}\beta_3\beta_2$ . The vertical tangencies corresponding to the  $\alpha$ - and  $\beta$ -curves which immediately cancel are marked in green. The remaining vertical tangencies are marked blue and labelled. We have  $\rho_+ = 2$ ,  $\lambda_+ = 0$ ,  $\rho_- = 1$ ,  $\lambda_- = 3$ . I.e. the rotation number of the Legendrian knot represented by  $K$  is

$$\text{rot}(K) = \rho_+ - \lambda_+ = \lambda_- - \rho_- = 2.$$

We will now apply Algorithm 4.1 to the word  $\alpha_1\beta_2\alpha_2\alpha_3^{-1}\beta_3\beta_2$ .

As neither a  $\beta^{-1}$  is followed by an  $\alpha^{-1}$  of the same index, nor an  $\alpha^{-1}$  by a  $\beta^{-1}$ , we set  $\lambda_+ = 0 = \rho_+$  as initial values. Next, we consider the index changes:

$$\alpha_1r_u\beta_2\alpha_2r_u\alpha_3^{-1}\beta_3r_d\beta_2r_d.$$

The positions  $\beta_3r_d$  and  $\beta_2r_d$  both increase  $\rho_+$  by one. All other positions leave the counts unchanged. Hence, the algorithm yields

$$\text{rot}(K) = \rho_+ - \lambda_+ = 2.$$

Note that we could also adapt the rules specified in the algorithm to consider  $\rho_-$  and  $\lambda_-$  instead using the proof of the preceding proposition.

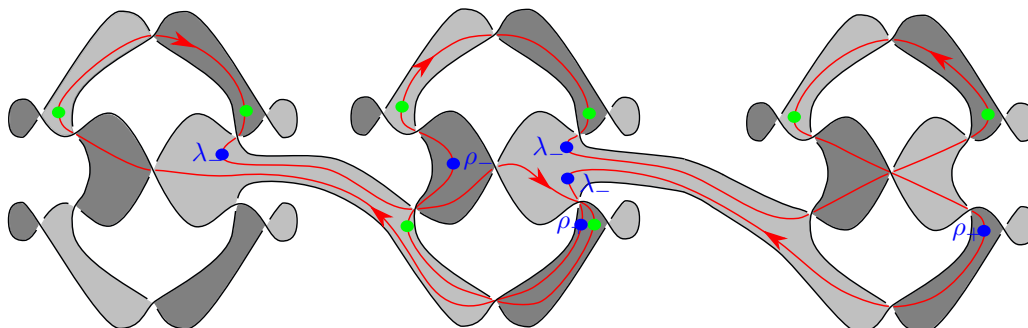


FIGURE 6. A knot on an embedded page in  $(S^3, \xi_{st})$ . Vertical tangencies cancelling each other are marked green, other vertical tangencies are marked blue and labelled.

## 5. The general case

Now we are prepared to deal with a Legendrian knot in a general contact open book. The idea is to change the open book to the special case discussed in the previous section by a sequence of surgeries, then compute the rotation number in  $(S^3, \xi_{st})$  as above and finally use [5] with the inverse surgeries to get the rotation number of the Legendrian in the original open book. The result will be presented in a formula that can be directly computed with the data of the original open book.

In the following remark, we will briefly recall how to compute the rotation number in contact surgery diagrams.

**Remark 5.1** (Computing rot in a surgery diagram (see [5])). For an oriented Legendrian link  $L = L_1 \sqcup \dots \sqcup L_k$  in  $(S^3, \xi_{st})$  let  $(M, \xi)$  be the contact manifold obtained from  $(S^3, \xi_{st})$  by contact  $(1/n_i)$ -surgeries ( $n_i \in \mathbb{Z}$ ) along  $L_i$ . Denote the topological surgery coefficients by  $p_i/q_i$ . I.e.

$$\frac{p_i}{q_i} = \frac{n_i \text{tb}(L_i) + 1}{n_i}.$$

Let  $L_0$  be an oriented Legendrian knot in the complement of  $L$  and define the vector  $\mathbf{l}$  with components  $l_i = l_{0i}$  and the generalised linking matrix

$$Q = \begin{pmatrix} p_1 & q_2 l_{12} & \cdots & q_n l_{1k} \\ q_1 l_{21} & p_2 & & \\ \vdots & & \ddots & \\ q_1 l_{k1} & & & p_k \end{pmatrix},$$

where  $l_{ij} := \text{lk}(L_i, L_j)$ . The knot  $L_0$  is (rationally) nullhomologous in  $M$  if and only if there is an integral (rational) solution  $\mathbf{a}$  of the equation  $\mathbf{l} = Q\mathbf{a}$ , in which case its (rational) rotation number in  $(M, \xi)$  with respect to the Seifert class  $\widehat{\Sigma}$  constructed in [5]

is equal to

$$\text{rot}_M(L_0, \widehat{\Sigma}) = \text{rot}_{S^3}(L_0) - \sum_{i=1}^k a_i n_i \text{rot}_{S^3}(L_i).$$

*Proof of Theorem 1.1.*

Let  $K \subset (M, \xi)$  be a Legendrian knot sitting on the page of a compatible open book

$$(\Sigma, \phi = T_l^{\pm n_l} \circ \dots \circ T_1^{\pm n_1})$$

with monodromy encoded in a concatenation of Dehn twists, where  $T^{\pm n}$  denotes  $n$  positive (resp. negative) Dehn twists along the non-isolating oriented curve  $T$  ( $n \in \mathbb{N}$ ). We denote the genus of  $\Sigma$  by  $g$  and the number of boundary components by  $h + 1$ .

In the following, we want to choose a *special* arc basis of  $\Sigma$  to exactly mimic the setting from Proposition 4.2 (also see Remark 5.2). Together with a suitable monodromy yielding  $(S^3, \xi_{\text{st}})$ , this will enable us to use the proposition to compute the invariants first in  $(S^3, \xi_{\text{st}})$  and then to apply the surgery formulas to obtain the desired result.

Choose reducing arcs  $r_1, \dots, r_{g+h-1}$  such that when cutting along  $r_i$

- $\Sigma$  decomposes into a surface  $\Sigma_i$  of genus  $i$  containing  $r_1, \dots, r_{i-1}$  with one boundary component and a surface of genus  $g - i$  with  $h + 1$  boundary components for  $i = 1, \dots, g$ ,
- $\Sigma$  decomposes into a surface  $\Sigma_i$  of genus  $g$  containing  $r_1, \dots, r_{i-1}$  with  $i + 1$  boundary components and a disk with  $h - i$  holes for  $i = g + 1, \dots, g + h - 1$ .

Then choose an arc basis of  $\Sigma_i \setminus \Sigma_{i-1}$ . Orient and label the arcs by  $a_i, b_i$  such that when travelling along the oriented boundary of  $\Sigma$  from

- $r_1$  to  $r_1$ 
  - first  $a_1$  is met pointing outwards, then  $b_1$  is met pointing inwards if  $g \geq 1$
  - $b_1$  is met and pointing outwards if  $g = 0$
- $r_{i-1}$  to  $r_i$  only  $b_i$  is met and pointing outwards ( $i = 2, \dots, g + h - 2$ )
- $r_{g+h-1}$  to  $r_{g+h-1}$ 
  - first  $b_g$  is met pointing outwards, then  $a_g$  is met pointing outwards if  $h = 0$
  - $b_{g+h}$  is met and pointing outwards if  $h > 0$ .

Choose non-trivial oriented simple closed curves  $\alpha_i, \beta_i$  representing a basis of  $H_1(\Sigma)$  dual to the arcs with respect to the intersection product on  $\Sigma$  oriented such that  $\alpha_i \bullet a_i = 1$ ,  $\beta_j \bullet b_j = 1$  and  $\alpha_i \bullet \beta_i = 1$  (i.e. the situation is as in Figure 3).

**Remark 5.2.** Note that the arc basis cannot be chosen arbitrarily, as we will use it to write the knot as a word in  $\alpha_i, \beta_i$  as above and use the formula from Proposition 4.2 to compute the rotation from this word. For this to work with the given formula, we have to ensure that the word we get in the abstract setting is the same as the word we get in the embedded case, which coincides with the specific abstract open book depicted in Figure 3. In particular, the word obtained from the oriented boundary of the page is

$$\alpha_1^{-1} \beta_1 \alpha_1 \beta_1^{-1} \beta_2^{-1} \dots \beta_{g+h}^{-1} \alpha_g^{-1} \beta_g \alpha_g \dots \alpha_2^{-1} \beta_2 \alpha_2.$$

A different arc basis would require a different formula to compute the rotation number from the word (see also Example 6.4).

This is only important for calculating the rotation number in  $(S^3, \xi_{st})$  which is not determined by the class of the knot in the homology of the page – the linking information required to compute the rotation number via the surgery formula is purely homological and does not depend on the specific ordering. In particular, we can use an arbitrary arc basis in a planar open book if we use Proposition 3.1 to compute the rotation number of the involved curves in  $(S^3, \xi_{st})$ .

Observe that we can get from the open book

$$(\Sigma, \phi_{S^3} = \beta_{g+h}^{+1} \circ \cdots \circ \beta_{g+1}^{+1} \circ \beta_g^{+1} \circ \alpha_g^{+1} \circ \cdots \circ \beta_1^{+1} \circ \alpha_1^{+1})$$

to the open book  $(\Sigma, \phi)$  by a sequence of contact surgeries along Legendrian knots corresponding to the Dehn twist curves.

By the algorithm presented in [2], the surgery link is as follows: every component corresponding to a Dehn twist sits on a page of the embedded open book. The shift in the  $z$ -direction of the respective page relates to the position of the Dehn twists in the monodromy factorisation – the later the Dehn twist is performed, the higher the level of the page. Using Avdek’s convention, we will denote a knot  $K$  sitting on the page with level  $t$  by  $K(t)$ .

Observe that the  $\alpha_i(s), \beta_i(s)$  are unknots with rotation number zero and Thurston–Bennequin invariant  $-1$  and that for  $t \neq s$  we have

$$\begin{aligned} \text{lk}(\alpha_i(t), \beta_j(s)) &= \begin{cases} 0, & \text{if } i \neq j \text{ or } t > s, \\ -1, & \text{if } i = j \text{ and } t < s, \end{cases} \\ \text{lk}(\alpha_i(t), \alpha_j(s)) &= \begin{cases} 0, & \text{if } i \neq j, \\ -1, & \text{if } i = j, \end{cases} \\ \text{lk}(\beta_i(t), \beta_j(s)) &= \begin{cases} 0, & \text{if } i \neq j, \\ -1, & \text{if } i = j. \end{cases} \end{aligned}$$

If  $s = t$ , the curves form a Legendrian graph on a single page with  $\alpha$  and  $\beta$  joined by a single transverse intersection point.

The first homology class represented by a knot  $K$  on  $\Sigma$  can then be written as

$$K = \sum_{i=1}^{g+h} ((K \bullet a_i)\alpha_i + (K \bullet b_i)\beta_i)$$

and hence

$$K(t) = \sum_{i=1}^{g+h} ((K \bullet a_i)\alpha_i(t) + (K \bullet b_i)\beta_i(t)),$$

where  $(K \bullet a_i)$  is defined to be zero for  $k > g$ . The linking number of two knots  $K_1(t)$  and  $K_2(s)$  behaves linearly and distributively with respect to this decomposition. I.e. the

linking number is easily computable with the linking behaviour of the  $\alpha$  and  $\beta$  curves specified above.

It is well-known that the surgery link in  $S^3$  to obtain  $(\Sigma, \phi)$  is the link  $L = L_1 \sqcup \dots \sqcup L_{2g+h+l}$  as specified in Table 2 (e.g. see Avdek's algorithm [2]).

name	knot	contact surgery coefficient
$L_1$	$\beta_1(-1)$	+1
$\vdots$	$\vdots$	$\vdots$
$L_{g+h}$	$\beta_{g+h}(-1)$	+1
$L_{g+h+1}$	$\alpha_1(0)$	+1
$\vdots$	$\vdots$	$\vdots$
$L_{2g+h}$	$\alpha_g(0)$	+1
$L_{2g+h+1}$	$T_1(1/l)$	$\mp 1/n_1$
$\vdots$	$\vdots$	$\vdots$
$L_{2g+h+l}$	$T_l(l/l)$	$\mp 1/n_l$

TABLE 2

To compute the rotation number of a knot on a page of  $(\Sigma, \phi)$  using the method explained in Remark 5.1, we need the generalised linking matrix  $Q$  – which requires us to know  $\text{tb}$  for deducing the topological surgery coefficient from the contact one as well as all linking numbers – and the rotation numbers in  $(S^3, \xi_{\text{st}})$ .

For a knot  $K(t)$ , we have

$$\text{tb}_{S^3}(K(t)) = \text{lk}(K(t), K(t + \varepsilon))$$

and hence, for  $i = 1, \dots, l$ ,

$$\text{tb}_{S^3}(L_{2g+h+i}) = - \sum_{k=1}^{g+h} ((T_i \bullet a_k)^2 + (T_i \bullet a_k)(T_i \bullet b_k) + (T_i \bullet b_k)^2).$$

Therefore, the topological surgery coefficient of  $L_{2g+h+i}$  is

$$\frac{p_{2g+h+i}}{q_{2g+h+i}} = \frac{n_i \text{tb}_{S^3}(T_i) \mp 1}{n_i}.$$

Furthermore, the linking behaviour with  $L_j = \beta_j$ ,  $j = 1, \dots, g + h$  is

$$\text{lk}(L_{2g+h+i}, L_j) = -(T_i \bullet b_j)$$

and similarly, for  $L_{g+h+j} = \alpha_j$ ,  $j = 1, \dots, g$

$$\text{lk}(L_{2g+h+i}, L_{g+h+j}) = -((T_i \bullet a_j) + (T_i \bullet b_j)).$$

The linking number of two surgery knots  $L_{2g+h+i}$  and  $L_{2g+h+j}$  with  $i < j$  can be computed to be

$$\text{lk}(L_{2g+h+i}, L_{2g+h+j}) = - \sum_{k=1}^{g+h} ((T_i \bullet a_k)(T_j \bullet a_k) + (T_i \bullet a_k)(T_j \bullet b_k) + (T_i \bullet b_k)(T_j \bullet b_k)).$$

Note that the knot  $K$  can be put on the page with the lowest as well as the highest level. Depending on which is chosen, the class of Seifert surface with respect to which the rotation number is given in Remark 5.1 might change, and hence the rotation numbers may differ. However, if the Euler class of  $\xi$  vanishes, the rotation number of a nullhomologous is independent of the Seifert surface. If we choose the knot  $L_0 = K(\text{low})$  to sit on a lower page than the surgery link, we get the following linking numbers

$$\begin{aligned} \text{lk}(L_0, L_j) &= -((K \bullet a_j) + (K \bullet b_j)), \quad j = 1, \dots, g+h, \\ \text{lk}(L_0, L_{g+h+j}) &= -(K \bullet a_j), \quad j = 1, \dots, g, \\ \text{lk}(L_0, L_{2g+h+j}) &= - \sum_{k=1}^{g+h} ((K \bullet a_k)(T_j \bullet a_k) + (K \bullet a_k)(T_j \bullet b_k) \\ &\quad + (K \bullet b_k)(T_j \bullet b_k)), \quad j=1, \dots, l. \end{aligned}$$

If on the other hand  $L_0 = K(\text{high})$  is assumed to sit on a page with the highest level, we get

$$\begin{aligned} \text{lk}(L_0, L_j) &= -(K \bullet b_j), \quad j = 1, \dots, g+h, \\ \text{lk}(L_0, L_{g+h+j}) &= -((K \bullet a_j) + (K \bullet b_j)), \quad j = 1, \dots, g, \\ \text{lk}(L_0, L_{2g+h+j}) &= - \sum_{k=1}^{g+h} ((K \bullet a_k)(T_j \bullet a_k) + (K \bullet b_k)(T_j \bullet a_k) \\ &\quad + (K \bullet b_k)(T_j \bullet b_k)), \quad j = 1, \dots, l. \end{aligned}$$

The only data that is left to compute are the rotation numbers in  $S^3$  of the  $L_i$ , but this can be done as in Proposition 4.2. Observe that using the formula from [15] also allows us to calculate the Thurston–Bennequin invariant, which is an alternative to the method presented in [6]. Similarly, one can directly calculate the Poincaré-dual of the Euler class and the  $d_3$ -invariant of the contact structure (see [5, Theorem 5.1]).

Thus, we have proved Theorem 1.1.  $\square$

## 6. Algorithm and examples

We summarise the process and all required formulas in the following algorithm and illustrate them by giving examples. This section is meant as a self-contained guideline to do actual computations and can be used independently from the rest of the paper.

**Algorithm 6.1.** *The setting.*

Given is a non-isolating curve  $K$  on the page of an open book

$$(\Sigma_{g,h+1}, \phi = T_l^{\pm n_l} \circ \dots \circ T_1^{\pm n_1})$$

with  $n_i \in \mathbb{N}$  and  $\Sigma_{g,h+1}$  a surface of genus  $g$  with  $h + 1$  boundary components. The monodromy is given as a sequence of Dehn twists along non-isolating oriented curves  $T_i$ .

*The choices.*

Choose reducing arcs  $r_1, \dots, r_{g+h-1}$  such that when cutting along  $r_i$

- $\Sigma$  decomposes into a surface  $\Sigma_i$  of genus  $i$  with one boundary component containing  $r_1, \dots, r_{i-1}$  and a surface of genus  $g - i$  with  $h + 1$  boundary components for  $i = 1, \dots, g$ ,
- $\Sigma$  decomposes into a surface  $\Sigma_i$  of genus  $g$  with  $i + 1$  boundary components containing  $r_1, \dots, r_{i-1}$  and a disk with  $h - i$  holes for  $i = g + 1, \dots, g + h - 1$ .

Then choose an arc basis of  $\Sigma_i \setminus \Sigma_{i-1}$  and label it by  $a_i, b_i$  and orient it such that when travelling along the oriented boundary of  $\Sigma$  from

- $r_1$  to  $r_1$ 
  - first  $a_1$  is met pointing outwards, then  $b_1$  is met pointing inwards if  $g \geq 1$
  - $b_1$  is met and pointing outwards if  $g = 0$
- $r_{i-1}$  to  $r_i$  only  $b_i$  is met and pointing outwards ( $i = 2, \dots, g + h - 2$ )
- $r_{g+h-1}$  to  $r_{g+h-1}$ 
  - first  $b_g$  is met pointing outwards, then  $a_g$  is met pointing outwards if  $h = 0$
  - $b_{g+h}$  is met and pointing outwards if  $h > 0$ .

Choose non-trivial oriented simple closed curves  $\alpha_i, \beta_i$  representing a basis of  $H_1(\Sigma)$  dual to the arcs with respect to the intersection product on  $\Sigma$  oriented such that  $\alpha_i \bullet a_i = 1$ ,  $\beta_j \bullet b_j = 1$  and  $\alpha_i \bullet \beta_i = 1$  (i.e. the situation is as in Figure 3).

*The definitions.*

Define an integral vector  $\mathbf{l} \in \mathbb{Z}^{2g+h+l}$  with entries:

$$\begin{aligned} \mathbf{l}_j &= -(K \bullet b_j), \\ &\text{for } j = 1, \dots, g + h, \\ \mathbf{l}_{g+h+j} &= -((K \bullet a_j) + (K \bullet b_j)), \\ &\text{for } j = 1, \dots, g, \\ \mathbf{l}_{2g+h+j} &= -\sum_{k=1}^{g+h} ((K \bullet a_k)(T_j \bullet a_k) + (K \bullet b_k)(T_j \bullet a_k) \\ &\quad + (K \bullet b_k)(T_j \bullet b_k)), \\ &\text{for } j = 1, \dots, l, \end{aligned}$$



Define an integral  $(2g + h + l) \times (2g + h + l)$ -matrix  $Q$  with entries:

$$\begin{aligned}
 Q_{i,j} &= 0, \\
 &\text{for } i, j = 1, \dots, 2g + h, \\
 Q_{2g+h+i, 2g+h+i} &= \mp 1 - n_i \sum_{k=1}^{g+h} ((T_i \bullet a_k)^2 + (T_i \bullet a_k)(T_i \bullet b_k) + (T_i \bullet b_k)^2), \\
 &\text{for } i = 1, \dots, l, \\
 Q_{2g+h+i, j} &= -(T_i \bullet b_j), \\
 &\text{for } i = 1, \dots, l, j = 1, \dots, g + h, \\
 Q_{2g+h+i, j} &= -((T_i \bullet a_j) + (T_i \bullet b_j)), \\
 &\text{for } i = 1, \dots, l, j = g + h + 1, \dots, 2g + h, \\
 Q_{i, 2g+h+j} &= -n_j(T_j \bullet b_i), \\
 &\text{for } i = 1, \dots, g + h, j = 1, \dots, l, \\
 Q_{g+h+i, 2g+h+j} &= -n_j((T_j \bullet a_i) + (T_j \bullet b_i)), \\
 &\text{for } i = 1, \dots, g, j = 1, \dots, l, \\
 Q_{2g+h+i, 2g+h+j} &= -n_j \sum_{k=1}^{g+h} ((T_i \bullet a_k)(T_j \bullet a_k) + (T_i \bullet a_k)(T_j \bullet b_k) \\
 &\quad + (T_i \bullet b_k)(T_j \bullet b_k)), \text{ for } i < j, i, j = 1, \dots, l, \\
 Q_{2g+h+i, 2g+h+j} &= -n_i \sum_{k=1}^{g+h} ((T_i \bullet a_k)(T_j \bullet a_k) + (T_i \bullet b_k)(T_j \bullet a_k) \\
 &\quad + (T_i \bullet b_k)(T_j \bullet b_k)), \text{ for } i > j, i, j = 1, \dots, l.
 \end{aligned}$$

For an oriented non-isolating curve  $L$  we define the quantity  $r(L)$  as follows: choose a starting point on  $L$  and write  $L$  as a word in the  $\alpha_i$  and  $\beta_i$  by noting intersections with  $a_i$  and  $b_i$  when traversing along  $L$ . Set  $\lambda_+$  to be the number of times a  $\beta^{-1}$  is followed by an  $\alpha^{-1}$  of the same index also considering the step from the last to the first letter, and similarly, set  $\rho_+$  equal to the number of times an  $\alpha^{-1}$  is followed by a  $\beta^{-1}$  of the same index. Denote places where the index changes by  $r_u$  ( $r_d$ ) if the index increases (decreases) – including the last position if the index of the last letter is not equal to the index of the first letter. Now run through the index changes and increase  $\lambda_+$  and  $\rho_+$  according to the following rule:

- increase  $\lambda_+$  by 1 for
  - a  $\beta^{-1}$  followed by  $r_u$
  - $r_d$  followed by an  $\alpha^{-1}$
- increase  $\rho_+$  by 1 for

- an  $\alpha^{-1}$  followed by  $r_d$
- a  $\beta$  followed by  $r_d$ .

Then define

$$r(L) := \rho_+ - \lambda_+.$$

*The results.*

Then the following holds:

- (a)  $K$  is nullhomologous if and only if there is an integral solution  $\mathbf{a}$  of the equation  $\mathbf{l} = Q\mathbf{a}$ .
- (a')  $K$  is rationally nullhomologous in the manifold if and only if there is a rational solution  $\mathbf{a}$  of the equation  $\mathbf{l} = Q\mathbf{a}$ .
- (b1) If  $K$  is (rationally) nullhomologous, the (rational) Thurston–Bennequin invariant of  $K$  is

$$\begin{aligned} \text{tb}(K) = & - \sum_{k=1}^{g+h} ((K \bullet a_k)^2 + (K \bullet a_k)(K \bullet b_k) + (K \bullet b_k)^2) \\ & - \sum_{j=1}^{2g+h} a_j l_j - \sum_{j=1}^l a_{2g+h+j} n_j l_{2g+h+j}. \end{aligned}$$

- (b2) If  $K$  is (rationally) nullhomologous, the (rational) rotation number with respect to some special Seifert surface  $S$  of  $K$  is

$$\text{rot}(K, S) = r(K) - \sum_{j=1}^l a_{2g+h+j} n_j r(T_j).$$

- (b3) Denote by  $K^\pm$  the positive (resp. negative) transverse push-off of a (rationally) nullhomologous Legendrian  $K$ . Then its (rational) self-linking number with respect to the Seifert surface  $S$  from (b2) is

$$\text{sl}(K^\pm, S) = \text{tb}(K) \mp \text{rot}(K, S).$$

- (c) The Poincaré-dual of the Euler class is given by

$$\text{PD}(e(\xi)) = \sum_{i=1}^l n_i r(T_i) \mu_{T_i} \in H_1(M).$$

The first homology group  $H_1(M)$  of  $M$  is generated by the meridians  $\mu$  of the  $\alpha_i, \beta_i$  and  $T_i$  and the relations are given by the generalized linking matrix  $Q\mu = 0$ .

- (d) The Euler class  $e(\xi)$  is torsion if and only if there exists a rational solution  $\mathbf{b}$  of  $Q\mathbf{b} = \mathbf{r}$  with  $\mathbf{r}_i = 0$  for  $i = 1, \dots, 2g + h$  and  $\mathbf{r}_{2g+h+i} = r(T_i)$  for  $i = 1, \dots, l$ . In this case, the  $d_3$ -invariant of  $\xi$  computes as

$$d_3(\xi) = g + \frac{h}{2} + \frac{1}{4} \left( \sum_{i=1}^l n_i b_{2g+h+i} r(T_i) - (3 - n_i) \text{sign}_i \right) - \frac{3}{4} \sigma(Q) - \frac{1}{2},$$

where  $\text{sign}_i$  denotes the sign of the power of the Dehn twist  $T_i^{\pm n_i}$ .

**Remark 6.2.** In the algorithm above, we implicitly assumed that the knot  $K$  sits on the page with a higher level than the monodromy curves. As described in Section 5,  $K$  could also be assumed to sit on the lowest level, which would change the formulas defining the vector  $\mathbf{l}$ . Note that in general, if  $e(\xi) \neq 0$ , the resulting rotation number might differ, as it is computed with respect to a different class of Seifert surface. However, if the open book is planar or  $e(\xi) = 0$ , we get the same values for both cases.

**Remark 6.3.** In the planar case, the formulas simplify to

$$\begin{aligned} \mathbf{l}_j &= -(K \bullet b_j), \text{ for } j = 1, \dots, h, \\ \mathbf{l}_{h+j} &= -\sum_{k=1}^h (K \bullet b_k)(T_j \bullet b_k), \text{ for } j = 1, \dots, l, \\ Q_{i,j} &= 0, \text{ for } i, j = 1, \dots, h, \\ Q_{h+i, h+i} &= \mp 1 - n_i \sum_{k=1}^h (T_i \bullet b_k)^2, \text{ for } i = 1, \dots, l, \\ Q_{h+i, j} &= -(T_i \bullet b_j), \text{ for } i = 1, \dots, l, j = 1, \dots, h, \\ Q_{i, h+j} &= -n_j (T_j \bullet b_i), \text{ for } i = 1, \dots, h, j = 1, \dots, l, \\ Q_{h+i, h+j} &= -n_j \sum_{k=1}^h (T_i \bullet b_k)(T_j \bullet b_k), \text{ for } i \neq j, i, j = 1, \dots, l. \end{aligned}$$

If furthermore all  $n_i = 1$ , we have that

$$Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}$$

with  $Q_1 = 0_{h \times h}$  the zero  $(h \times h)$ -matrix,

$$Q_2 = Q_3^t = -(T_j \bullet b_i)_{i=1, \dots, h; j=1, \dots, l}$$

and

$$Q_4 = Q_3 Q_2 \mp \text{diag}(\text{sign}(T_1), \dots, \text{sign}(T_l)).$$

**Example 6.4.** In this example we want to reconsider the planar open book of  $(S^3, \xi_{\text{st}})$  discussed in Example 3.3, where we calculated the rotation number to be 2 using Proposition 3.1. If we choose the arc basis as described above, the knot is encoded by the word  $\beta_2 \beta_4 \beta_3$ . This yields  $\lambda_+ = 0$  and  $\rho_+ = 2$ , i.e.  $\text{rot} = 2$  as expected.

Note that if we choose a different arc basis, e.g. such that the word is  $\beta_2 \beta_3 \beta_4$ , then the formula does not give the desired result, as the knot would be represented by a different word. In fact, the word  $\beta_2 \beta_3 \beta_4$  does not even encode a simple closed curve on the embedded page.

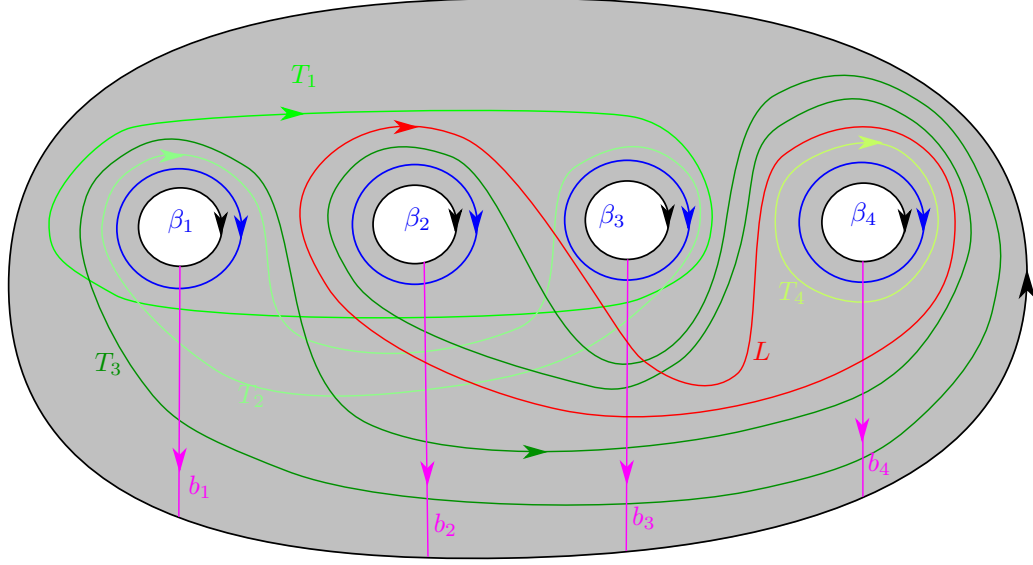


FIGURE 7. The open book  $(\Sigma, \phi = T_3^{+1} \circ T_2^{+1} \circ T_1^{+1})$  of  $(S^3, \xi_{st})$ .

**Example 6.5.** Consider the open book  $(\Sigma, \phi = T_3^{+1} \circ T_2^{+1} \circ T_1^{+1})$  and knot  $L$  as specified in Figure 7. This is an example of a non-destabilisable planar open book of  $(S^3, \xi_{st})$  taken from [9].

By the formulas to compute  $\text{rot}$  in the special planar case, it follows directly that

$$\mathbf{r} = (0, 0, 0, 0, 2, 1, 1, 0)^t$$

and  $r(L) = 1$ .

Using the simplified formulas for planar open books given in Remark 6.3, we obtain

$$\mathbf{l} = (0, -1, 0, -1, -1, 0, -1, -1)^t$$

and

$$Q_2 = - \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As the manifold is  $S^3$ , it follows that  $Q$  is invertible and thus the equation  $\mathbf{l} = Q\mathbf{a}$  admits a unique solution, which is easily computed to be

$$\mathbf{a} = (2, -2, -1, -1, 1, -1, 0, 1)^t$$

(in particular, the calculation shows that  $L$  is nullhomologous). The Thurston–Bennequin invariant of  $L$  then computes to be

$$\text{tb}(L) = - \sum_{k=1}^4 (L \bullet b_k)^2 - \langle \mathbf{a}, \mathbf{l} \rangle = -3$$

and the rotation number is

$$\text{rot}(L) = r(L) - \langle \mathbf{a}, \mathbf{r} \rangle = 0.$$

The self-linking number of both the positive and the negative transverse push-off of  $L$  is  $-3$ .

Since  $Q$  is invertible, we have  $H_1 = 0$ . I.e. the Poincaré dual to the Euler class of the contact structure  $\xi$  vanishes. As expected, our formula then returns

$$d_3(\xi) = -\frac{1}{2}.$$

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