

Bilinear-form invariants of Lefschetz fibrations over the 2-sphere

Takefumi Nosaka

ABSTRACT. We introduce invariants of, in general, Hurwitz equivalence classes with respect to arbitrary group G . The invariants are constructed from any right G -module M and any G -invariant bilinear function on M , and they are of bilinear forms. For instance, when G is the mapping class group, $\mathcal{M}_g$, of the closed surface Σ_g of genus g , we get an invariant of 4-dimensional Lefschetz fibrations over the 2-sphere. Moreover, the construction is applicable for the quantum representations of $\mathcal{M}_g$ derived from Chern-Simons field theory. We also see that our invariant is unstable with respect to fiber sum of Lefschetz fibrations.

1. Introduction

A (genus- g) Lefschetz fibration [Kas, Mat] is a smooth map $\pi : E \rightarrow S^2$ from a closed smooth 4-manifold E that is a Σ_g -fiber bundle projection away from finitely many singular points. Here, Σ_g is the closed surface of genus g , and the map π near the singular points is required to appear in appropriate oriented local complex coordinates as $\pi(z_1, z_2) = z_1 z_2$. Lefschetz fibrations provide a topological method available for studying 4-dimensional geometry as in complex surfaces, symplectic structures, and Stein surfaces. Further, a careful observation of regular neighborhoods around the singular fibers develops the close relation to 3-dimensional contact geometry and the mapping class group, $\mathcal{M}_g$, of Σ_g ; see an explanatory book [OS] for details.

We now explain an approach to Lefschetz fibrations, as a general setting for the Hurwitz equivalence problem. Let G be a group with identity 1_G , and let a subset $Z \subset G$ be closed under conjugation. For $m \in \mathbb{N}$, consider the quotient set, $\text{Hur}^m(Z)$, of the set

$$\{(z_1, \dots, z_m) \in Z^m \mid z_1 \cdots z_m = 1_G\}$$

modulo the following relations:

$$(z_1, z_2, \dots, z_m) \sim (z^{-1} z_1 z, z^{-1} z_2 z, \dots, z^{-1} z_m z), \quad (1)$$

$$(z_1, z_2, \dots, z_m) \sim (z_1, \dots, z_{i-1}, z_{i+1}, z_{i+1}^{-1} z_i z_{i+1}, z_{i+2}, \dots, z_m), \quad (2)$$

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for any $1 \leq i < m$ and $z \in G$. An element of this set $\text{Hur}^m(Z)$ is called a *Hurwitz equivalence class*, and is roughly viewed as a monodromy characterized by G over the 2-sphere S^2 with m -points removed. This viewpoint emphasizes the importance of studying $\text{Hur}^m(Z)$. As examples in topology, in the case where G is the mapping class group $\mathcal{M}_g$ (resp. the braid group B_n), the associated set $\text{Hur}^m(Z)$ is in bijective correspondence with the set of isomorphism classes of Lefschetz fibrations (resp. of simple surface-braids); see, e.g., [Kas, Mat, PY, Kam], Examples 3.4 and 3.5. Nevertheless, for any group G , there are few invariants with respect to the Hurwitz equivalence, although the definition is seemingly simple. For instance, even concerning Lefschetz fibrations related to symplectic geometry, most of useful approaches are (geometric) characteristic classes, group-theoretic study for the monodromy; see, e.g., [EK, EN, FS, Ozb, Sat, PY] (Refer to a recent work [EK] from “chart diagrams”).

In this paper, for any such pair $Z \subset G$, we generally introduce an algorithm to provide invariants of Hurwitz equivalence classes $\text{Hur}^m(Z)$. The idea follows from the Hopf fibration $S^3 \rightarrow S^2$. Since the preimage of m -points on S^2 is the (m, m) -torus link $T_{m,m}$ in S^3 , our invariants are roughly defined as link-invariants of the link $T_{m,m}$. To be precise, we show that (Lemma 2.2) any invariant of representations of link groups yields an invariant of $\text{Hur}^m(Z)$. Thus, for concrete computations and applications of such invariants, this paper mainly employs the link-invariant $\mathcal{Q}_\psi$ which was defined from cup products (see Section 3.1 for the reason why this paper does not use other standard link-invariants). The interesting point is that this $\mathcal{Q}_\psi$ is elementally constructed from any G -module M and any G -invariant bilinear function ψ , and is valued as a bilinear form. In conclusion, we obtain bilinear forms from $\text{Hur}^m(Z)$, as mentioned in the abstract above.

For their applications, this paper mainly focuses on Lefschetz fibrations over S^2 . As mentioned above, we shall let G be the mapping class group $\mathcal{M}_g$. Then, by virtue of bilinear form theory (see [MH]) and the help of computer, we can compute the invariants of many Lefschetz fibrations of low genus fibers (see Section 4.2). Furthermore, in Sections 3.2–4.2, we study properties of the bilinear forms, and obtain some interesting results as follows.

As the simplest case, in Section 4.2, we study the resulting bilinear form $\mathcal{Q}_\psi$ obtained from the standard symplectic representation of $\mathcal{M}_g$, that is, we set $M := H^1(\Sigma_g; A)$ and the symplectic form ψ on M . We then analyze the rational part and torsion part in turn. First, in rational case $A = \mathbb{Z}$, one will show the unimodularity of the form $\mathcal{Q}_\psi$ (Theorem 3.13), and show that the signature of $\mathcal{Q}_\psi$ is essentially equal to the signature of the total space of the Lefschetz fibration. As a result, we give a simpler formula for computing 4-dimensional signature (This formula is a modification of Meyer 2-cocycle [Mey]; see Appendix B for details). On the other hand, if A is the torsion group $\mathbb{Z}/P$ for some prime $P \in \mathbb{Z}$, the resulting form $\mathcal{Q}_\psi$ is not topological, but capture some group theoretic fiber-structures of Lefschetz fibrations. A typical evidence is that the form $\mathcal{Q}_\psi$ seems something unstable: Precisely, it does not always have the additivity with respect to (twist) fiber sums of Lefschetz fibrations; see Table 1 and Proposition 5.3. Since the

sum is a powerful tool for constructing new symplectic 4-manifolds as in [FS, PY]), we hope a usefulness of the torsion invariant.

Moreover, we emphasize that the quantum representation is applicable to the setting of our invariants. Witten [Wi] had made a prophetic discovery that the Chern-Simons quantum field theory of level k produces 3-manifold invariants and representations of $\mathcal{M}_g$; afterward the prophecy in some cases are mathematically formulated from several branches, e.g., conformal field theory. Here we remark that the formulation has the difficulty that, by an obstacle of “ p_1 -structures (or 2-framing anomaly”); see [BHMV, Wi]), the resulting representation is a right module not of $\mathcal{M}_g$, but of a $\mathbb{Z}$ -central extension of $\mathcal{M}_g$. However, starting from the study of such extensions (see [N3] for its finite presentation), we will apply the quantum representation to our invariants. Furthermore, the associated bilinear forms of Lefschetz fibrations are computable in some cases. For instance, in Section 5, we compute the associated invariants with $SU(2)$ -gauge of level 2 (i.e., a quantization of the spin structure on Σ_g); see Table 1 for the computation. As a result, we can detect some pairs of non-isomorphic Lefschetz fibrations (Proposition 5.3); however, unfortunately, the invariants seem not so strong, in contrast to the Jones knot-polynomial.

With respect to other gauges of higher level, the bilinear forms $\mathcal{Q}_\psi$ arising from the quantum representations are entirely mysterious. However, compared with a TQFT-like physical interpretation as in the Donaldson invariant, it may be worth for the future to hope that the philosophy of Chern-Simons perturbative [Wi] would partially produce invariants of 4-manifolds with some structures, and that Lefschetz fibrations would be good models which are affected by few 3-dimensional quantum obstructions.

Conventional notation. Throughout this paper, we write G for a group, and do Z for a subset of G as above. Furthermore an m -tuple (of Z) is $(z_1, \dots, z_m) \in Z^m$ with $z_1 \cdots z_m = 1$, and is denoted by $\mathbf{z}$. We denote by $\Sigma_{g,r}$ the oriented closed surface of genus $g \geq 2$ with r -boundaries. By A we mean a commutative ring with identity and with involution $\bar{\cdot} : A \rightarrow A$.

2. From Hopf fibration to Hurwitz equivalence classes

We will describe explicitly an algorithm to get something invariant with respect to the Hurwitz equivalence relation; see the introduction for the definition of $\text{Hur}^m(Z)$.

For this purpose, we start with a short review of the Hopf fibration $\mu : S^3 \rightarrow S^2$ with fiber S^1 . The fibration is formulated by the restriction on S^3 of the following map:

$$\mu : \mathbb{C}^2 \longrightarrow \mathbb{C} \times \mathbb{R}, \quad (z, w) \longmapsto (2z\bar{w}, |z|^2 - |w|^2).$$

Then we can easily see, by definitions, that the preimage of m -points, $\{b_1, \dots, b_m\} \subset S^2$, is the (m, m) -torus link $T_{m,m}$. Considering the Euler class, we can find a trivialization of the S^1 -bundle restricted on the complementary space $S^2 \setminus \{b_1, \dots, b_m\}$, that is, a

homeomorphism

$$S^3 \setminus T_{m,m} \cong S^1 \times (S^2 \setminus \{b_1, \dots, b_m\}).$$

In particular, $\pi_1(S^3 \setminus T_{m,m}) \cong \mathbb{Z} \times F_{m-1}$, where F_{m-1} is the free group of rank $m-1$. Hence

Proposition 2.1. *The complementary space $S^3 \setminus T_{m,m}$ is an Eilenberg MacLane space of $\mathbb{Z} \times F_{m-1}$.*

Moreover, the Wirtinger presentation yields a presentation of $\pi_1(S^3 \setminus T_{m,m})$ as follows:

$$\langle a_1, \dots, a_m \mid a_1 \cdots a_m = a_2 \cdots a_m a_1 = \cdots = a_m a_1 a_2 \cdots a_{m-1} \rangle. \quad (3)$$

Here, a_i corresponds to the meridian associated with the arc α_i in Figure 1, and the product $a_1 \cdots a_m$ generates the summand $\mathbb{Z} \subset \mathbb{Z} \times F_{m-1}$.

Next, to explain Lemma 2.2 below, we now set up terminologies. Wirtinger presentation in knot theory implies that every homomorphism from every link group to the group G is formulated as a map $\{\text{arcs of } D\} \rightarrow Z$. Let us consider the set of such maps:

$$\mathfrak{L}\mathfrak{H}\mathfrak{om}_Z := \left\{ \gamma : \{\text{arcs of } D\} \rightarrow Z \mid \begin{array}{l} D \text{ is a link diagram of some link } L \subset S^3, \text{ and} \\ \text{this } \gamma \text{ defines a homomorphism } \pi_1(S^3 \setminus L) \rightarrow G. \end{array} \right\}.$$

Further, we can equip $\mathfrak{L}\mathfrak{H}\mathfrak{om}_Z$ with the equivalence relation by considering Reidemeister moves.

Furthermore, we discuss the link-diagram D of $T_{m,m}$ in the left hand side of Figure 1. Here, any entry of the linking matrix is 1, and the band in Figure 1 means the $(m-2)$ -parallel strands, and the blackline is the $(i+1)$ -th strand and the dotted line indicates the i -th strand. Then, given an m -tuple $(z_1, \dots, z_m) \in Z^m$ with $z_1 \cdots z_m = 1$, the assignment $f_{\mathbf{z}}(a_i) = z_i$ defines a homomorphism $f_{\mathbf{z}} : \pi_1(S^3 \setminus T_{m,m}) \rightarrow G$ according to (3). Hence we have a map

$$\mathcal{Z} : \{ \mathbf{z} = (z_1, \dots, z_m) \in Z^m \mid z_1 \cdots z_m = 1 \} \longrightarrow \mathfrak{L}\mathfrak{H}\mathfrak{om}_Z; \quad \mathbf{z} \mapsto f_{\mathbf{z}}.$$

Lemma 2.2. *Let $\mathcal{S}$ be a set, and take a map $\mathcal{I} : \mathfrak{L}\mathfrak{H}\mathfrak{om}_Z \rightarrow \mathcal{S}$. Assume that $\mathcal{I}$ is invariant with respect to Reidemeister moves of type II and III, and assume that if two homomorphisms $f_{\mathbf{z}}$ and $f'_{\mathbf{z}} : \pi_1(S^3 \setminus L) \rightarrow G$ coming from $\mathfrak{L}\mathfrak{H}\mathfrak{om}_Z$ are conjugate, $\mathcal{I}(f_{\mathbf{z}}) = \mathcal{I}(f'_{\mathbf{z}})$.*

Then, the composite $\mathcal{I} \circ \mathcal{Z}$ induces a map $\text{Hur}_m(Z) \rightarrow \mathcal{S}$ by passage to the relations (1) (2).

Proof. It suffices to check the invariance with respect to the relations (1) (2). Since the former (1) is clear by assumption, we will discuss the other, (2). To this end, consider another diagram D' obtained from the above D by exchanging the i -th strand for the $(i+1)$ -th one (see the right of Figure 1). Notice that D is related to D' by a finite sequence of Reidemeister moves of type II and III (see Figure 1). Therefore, if two m -tuples $\mathbf{z}$ and $\mathbf{z}'$ are related by (2), then the equality $\mathcal{I}(f_{\mathbf{z}}) = \mathcal{I}(f_{\mathbf{z}'})$ results from the assumption. Hence, we complete the proof. $\square$

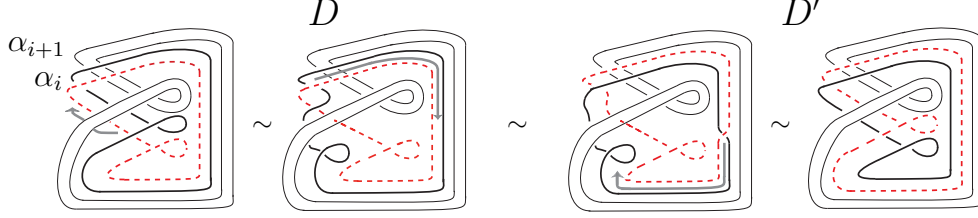


FIGURE 1. The exchange between the i -th strand and the $(i+1)$ -th one of the (m, m) -torus link.

To summarize, every invariant of representations of link groups produces that of $\text{Hur}_m(Z)$.

Remark 2.3. Here, we remark such invariants arising from group cohomology, as in twisted Alexander polynomials (modules) and the (complex) volume. However, Proposition 2.1 implies that the invariants factor through the group cohomology of $\mathbb{Z} \times F_{m-1}$ which is entirely simple. In fact, the author found no non-trivial examples in computer experience; in most cases, the resulting invariants of $\text{Hur}_m(Z)$ are expected to be trivial.

To solve such triviality, this paper will employ the link-invariants, which are obtained from cup products with quandle theory and were introduced in [N2]. In fact, we will see (Proposition A.3) that the Hopf fibration gives rise to an isomorphism between “the fundamental quandles” of $S^3 \setminus T_{m,m}$ and of $S^2 \setminus \{b_1, \dots, b_m\}$. Thus, some approach using quandle theory seem reasonable to establish Lemma 2.2. In summary, from the viewpoint of quandle theory, such invariants from $S^2 \setminus \{b_1, \dots, b_m\}$ are equivalent to those from $S^3 \setminus T_{m,m}$.

3. Invariants as bilinear forms

According to Remark 2.3 from quandle theory, Section 3.1 briefly formulates the invariants of linear forms. After that, in Section 3.2 and 3.3, we will state some properties of the bilinear forms.

3.1. Definition of the invariants as bilinear forms

In this subsection, we introduce invariants, which are of bilinear forms up to base change, with respect to the Hurwitz equivalence. As explained in Proposition 3.10, these invariants are roughly something like the cohomology pairing from cup products on D^2 .

We now review some results in the preceding paper [N2]. The author introduced invariants, which are bilinear forms, with respect to representations $\pi_1(S^3 \setminus L) \rightarrow G$. This invariant is generally constructed from any right $\text{As}(Z)$ -module M over a ring A and any $\text{As}(Z)$ -invariant bilinear function $\psi : M \times M \rightarrow A$. Here $\text{As}(Z)$ is the abstract

group generated by e_z with $z \in Z$ subject to the relation $e_{w^{-1}zw} = e_w^{-1}e_z e_w$, and is called *the associated group* [Joy]. Note that the inclusion $Z \hookrightarrow G$ gives rise to a group homomorphism

$$\mathcal{E} : \text{As}(Z) \longrightarrow G, \quad (4)$$

which makes any right G -module into a right $\text{As}(Z)$ -module.

We will explicitly formulate the invariants of the torus links as follows. For an m -tuple $\mathbf{z} = (z_1, \dots, z_m) \in Z^m$ and for $k \leq m$, we set up an A -linear map $\Gamma_{\mathbf{z},k} : M^m \rightarrow M$ defined by

$$\Gamma_{\mathbf{z},k}(x_1, \dots, x_m) := (x_{k-1} - x_k) + \sum_{j=k}^{m+k-2} (x_j - x_{j+1}) \cdot e_{z_{j+1}} e_{z_{j+2}} \cdots e_{z_{m+k-1}} \in M. \quad (5)$$

Here the indices are of period m , e.g., $x_{m+s} := x_s \in M$. Let us consider the sum $\oplus_k \Gamma_{\mathbf{z},k} : (M^m)^k \rightarrow M^k$, and denote the kernel $\text{Ker}(\oplus_k \Gamma_{\mathbf{z},k})$ by $\text{Ker}(\Gamma_{\mathbf{z}})$. Here we remark that $\text{Ker}(\Gamma_{\mathbf{z}})$ is not always an $\text{As}(Z)$ -submodule, but an A -submodule.

Definition 3.1 ([N2, Section 4.3]). Let $A, M, \mathbf{z} \in Z^m$ be as above. Let $\psi : M^2 \rightarrow A$ be an $\text{As}(Z)$ -invariant A -linear function. That is, ψ is bilinear over $\mathbb{Z}$ and satisfies the equalities

$$\psi(ax, by) = \bar{a}b\psi(x, y), \quad \psi(x \cdot e_z, y \cdot e_z) = \psi(x, y),$$

for any $x, y \in M$, $a, b \in A$ and $z \in Z$. Then, for $\ell \in \mathbb{Z}_{>0}$, we consider the A -bilinear form $\mathcal{Q}_{\psi,\ell} : \text{Ker}(\Gamma_{\mathbf{z}})^{\times 2} \rightarrow A$ that takes $(x_1, \dots, x_m) \otimes (y_1, \dots, y_m)$ to

$$\sum_{k=1}^{m-1} \psi(x_{k+\ell-1} - x_{k+\ell} + \sum_{j=1}^{k-1} (x_{j+\ell-1} - x_{j+\ell}) \cdot e_{z_{j+\ell}} e_{z_{j+\ell+1}} \cdots e_{z_{k+\ell-1}}, y_{k+\ell} \cdot (1 - e_{z_{k+\ell}}^{-1})) \in A. \quad (6)$$

It is worth noting that the linear forms are invariant with respect to conjugacy operations [N2, Corollary 2.4]. Hence, according to Lemma 2.2, we immediately conclude the following:

Theorem 3.2. *The correspondence from $(z_1, \dots, z_m) \in Z^m$ to the bilinear form $\mathcal{Q}_{\psi,\ell}$ up to base change gives rise to an invariant of the Hurwitz equivalence classes.*

Remark 3.3. This theorem can be proved by a direct computation in another way. Actually, according to the discussion of [N2, Section 2], if another tuple $\mathbf{z}' \in Z^m$ is Hurwitz equivalent to $\mathbf{z} \in Z^m$, then we can find an A -isomorphism $\mathcal{B}_{\mathbf{z},\mathbf{z}'} : \text{Ker}(\Gamma_{\mathbf{z}}) \rightarrow \text{Ker}(\Gamma_{\mathbf{z}'})$ satisfying the equality $\mathcal{Q}_{\psi,\ell} = \mathcal{Q}'_{\psi,\ell} \circ (\mathcal{B}_{\mathbf{z},\mathbf{z}'})^{\otimes 2}$. More precisely, regarding the relation (1), we define the map by $\mathcal{B}_{\mathbf{z},\mathbf{z}'}(x_i) := x_i \cdot e_z$. Further, concerning another (2), we can verify the equality by setting

$$\mathcal{B}_{\mathbf{z},\mathbf{z}'}(x_1, \dots, x_m) := (x_1, \dots, x_{i-1}, x_{i+1}, x_{i+1} + (x_i - x_{i+1}) \cdot e_{z_{i+1}}, x_{i+2}, \dots, x_m),$$

although the verifications are slightly complicated.

As a consequence, we get invariants of Lefschetz fibrations over S^2 and those of surface braids, thanks to the following facts:

Example 3.4 (Lefschetz fibration over S^2). We now review a well-known theorem in [Kas, Mat]. To describe this, let G be the mapping class group, $\mathcal{M}_g$, of the closed surface Σ_g with $g \geq 2$. Furthermore, we fix two subsets of G as follows:

$$\mathcal{D}_g := \{ \tau_\alpha \in \mathcal{M}_g \mid \alpha \text{ is an (unoriented) simple closed curve } \gamma \text{ in } \Sigma_g \}, \quad (7)$$

$$\mathcal{D}_g^{\text{ns}} := \{ \tau_\alpha \in \mathcal{D}_g \mid \alpha \text{ is a non-separating simple closed curve } \gamma \text{ in } \Sigma_g \}, \quad (8)$$

where the symbol τ_α is the (positive) Dehn twist along α . Let Z be the set $\mathcal{D}_g$ in (7). Then, given a Lefschetz fibration, we can observe that the associated monodromy is interpreted as an m -tuple. Moreover, it follows from [Kas] and [Mat, Theorems 2.6 and 2.8] that the interpretation gives a bijection between the Hurwitz equivalence classes $\text{Hur}^m(Z)$ and fiber-isomorphism classes of Lefschetz fibrations over S^2 with m -singular fibers.

Example 3.5 (Simple surface braids). Let G be the braid group B_n , and let Z be the set of all elements conjugate to either σ_i or σ_i^{-1} . Kamada [Kam] showed that the Hurwitz equivalence classes are in 1:1-correspondence with the isomorphism classes of “simple surface-braids with m -branch points of degree n ”; see [Kam] for the details.

Remark 3.6. Finally, we comment on the previous work on Lefschetz fibration invariants constructed from some quandles. Zablow [Zab, Theorem 5.3] combinatorially defined a certain invariant of Lefschetz fibrations over the 2-disk, which is valued in “the second quandle homology group” $H_2^Q(\mathcal{D}_g; \mathbb{Z})$ or $H_2^Q(\mathcal{D}_g^{\text{ns}}; \mathbb{Z})$ associated with the sets $\mathcal{D}_g$ and $\mathcal{D}_g^{\text{ns}}$. However the author showed that the homology $H_2^Q(\mathcal{D}_g^{\text{ns}}; \mathbb{Z})$ is $\mathbb{Z}/2$ for $g \geq 5$; [N4, Section 4.4]. Thus, it is sensible to conjecture that, in the case $X = \mathcal{D}_g$, such invariants might be almost trivial. Actually, there was no non-trivial example of such invariants with $X = \mathcal{D}_g$. In contrast, our invariants constructed from $\mathcal{M}_g$ -modules take non-trivial examples (see Section 4).

3.2. A reduction of the invariants of bilinear forms

In this paper, we focus on the bilinear forms $\mathcal{Q}_\psi$ by virtue of bilinear form theory. This subsection discusses some reductions to compute the invariants of bilinear forms.

First, we briefly make a reduction of the kernel $\text{Ker}(\Gamma_{\mathbf{z}})$ under an assumption.

Lemma 3.7. *Fix an m -tuple $(z_1, \dots, z_m) \in Z^m$. Assume the identity $e_{z_1} \cdots e_{z_m} = c \cdot \text{id}_M$ in $\text{End}(M)$ for some $c \in A$.*

If $c = 1$, then the kernel $\text{Ker}(\Gamma_{\mathbf{z}}) = \text{Ker}(\oplus_k \Gamma_{\mathbf{z},k})$ is equal to the kernel $\text{Ker}(\Gamma_{\mathbf{z},k})$ for any k .

If $(1-c) \cdot \text{id}_M$ is injective, then the kernel $\text{Ker}(\Gamma_{\mathbf{z}})$ is the diagonal set $\text{Diag}(M) \subset M^m$.

Proof. By definition, for any $\mathbf{x} = (x_1, \dots, x_m) \in M^m$, we easily see the equation

$$\Gamma_{\mathbf{z},k+1}(\mathbf{x}) = \Gamma_{\mathbf{z},k}(\mathbf{x}) \cdot e_{z_{m+k}} + (x_{k-1} - x_k) \cdot (1 - e_{z_k} \cdots e_{z_{m+k-1}}) \in M.$$

Hence the assumptions on $(1 - c) \cdot \text{id}_M$ deduce the required conclusions. $\square$

Therefore, to obtain non-trivial examples of $\mathcal{Q}_\psi$, it is reasonable to fix the assumption $e_{z_1} \cdots e_{z_m} = \text{id}_M$.

We now investigate this assumption. It is worth noticing that the kernel of the homomorphism $\mathcal{E} : \text{As}(Z) \rightarrow G$ in (4) is contained in the center, because of the equality

$$ge_zg^{-1} = e_{\mathcal{E}(g)z\mathcal{E}(g)^{-1}} \in \text{As}(Z) \quad \text{for any } z \in Z, g \in \text{As}(Z).$$

Thus, the product $e_{z_1} \cdots e_{z_m} \in \text{As}(Z)$ lies in the center, since $z_1 \cdots z_m = 1_G$. In summary, the assumption is detected by the image of the center in $\text{End}(M)$. However the assumption is not so strong in practice. For example,

Example 3.8. For any G -module M , we think of M as an $\text{As}(Z)$ -module via the map $\mathcal{E}$ in (4). Then the identity $e_{z_1} \cdots e_{z_m} = \text{id}_M$ follows from the condition $z_1 \cdots z_m = 1_G$.

Example 3.9. Let $G = \mathcal{M}_g$ and $Z = \mathcal{D}_g$ be as in Example 3.4. If $g \geq 3$, consider the two elements of the forms

$$\kappa_{3\text{-chain}} := (e_{c_1}e_{c_2}e_{c_3})^4e_d^{-1}e_{d'}^{-1}, \quad \kappa_{\text{lantern}} := e_{c_1}^{-1}e_{c_3}^{-1}e_{c_5}^{-1}e_{b_3}^{-1}e_{b_1}e_{b_2}e_{d'} \in \text{As}(\mathcal{D}_g),$$

where b_i , c_i , and $d^{(\cdot)}$ are the Dehn twists along the curves β_i , γ_i , and $\delta^{(\cdot)}$ in Figure 2, respectively. Otherwise, if $g = 2$, define $\kappa_{3\text{-chain}}$ to be $(e_{c_1}e_{c_2}e_{c_3})^4e_{c_5}^{-2}$, and κ_{lantern} to be $1_{\text{As}(\mathcal{D}_g)}$. As is well-known, $\mathcal{E}(\kappa_{3\text{-chain}})$ and $\mathcal{E}(\kappa_{\text{lantern}})$ are the identity in $\mathcal{M}_g$, which are commonly called the *3-chain relation* and the *lantern relation*, respectively. As is shown, the assumption $e_{z_1} \cdots e_{z_m} = \text{id}_M$ is detected by two elements $\kappa_{3\text{-chain}}$, κ_{lantern} and the signature of the Lefschetz fibrations; see the last section in [N3] for the details.

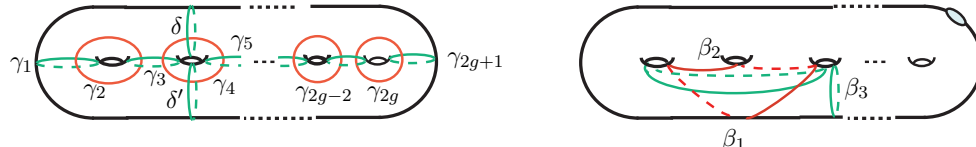


FIGURE 2. Generators of the group $\mathcal{M}_g$ with $g \geq 2$ and, the curves in the lantern relation with $g \geq 3$.

Under the assumption, we explain a topological interpretation of $\mathcal{Q}_{\psi,\ell}$ (see Proposition 3.10) which states a cohomological interpretation of the bilinear form $\mathcal{Q}_{\psi,\ell}$ in the context of cup products. Precisely, assume the identity $e_{z_1} \cdots e_{z_m} = \text{id}_M$ and fix a tubular neighborhood, $D_i (\cong \mathbb{R}^2)$, of the i -th point $\{b_i\}$. Let Y be the 2-sphere S^2 with the m -open discs $D_1 \sqcup \cdots \sqcup D_m$ removed. Considering the action of $\pi_1(Y)$ on M induced from that of $\pi_1(S^3 \setminus T_{m,m})$ and recalling (3), we can define the cohomology $H^*(Y, \partial Y; M)$ with local coefficients M . Furthermore, we write M^{op} for the A -module obtained by composing the module structure of M with the involution.

Proposition 3.10 ([N2, Proposition 4.7]). *Let $e_{z_1} \cdots e_{z_m} = \text{id} \in \text{End}(M)$. Then, the diagonal map $M \rightarrow \text{Ker}(\Gamma_{\mathbf{z}})$ is a splitting injection, and the cokernel is isomorphic to the relative cohomology $H^1(Y, \partial Y; M)$.*

Moreover, the bilinear form $\mathcal{Q}_{\psi, \ell}$ coincides with the following composite, up to sign:

$$H^1(Y, \partial Y; M)^{\times 2} \xrightarrow{\sim} H^2(Y, \partial Y; M^{\text{op}} \otimes M) \xrightarrow{\langle \bullet, [(Y, \partial Y)] \rangle} (M^{\text{op}} \otimes M)_{\text{As}(Z)} \xrightarrow{\psi(\bullet, \bullet)} A.$$

Here, the first map is the cup product, and the second is the pairing with the fundamental class $[(Y, \partial Y)] \in H_2(Y, \partial Y; \mathbb{Z}) \cong \mathbb{Z}$ in trivial coefficients, and the third is the evaluation with ψ .

As a result, the author [N2] showed some properties of the bilinear form $\mathcal{Q}_{\psi, \ell}$:

Proposition 3.11 ([N2, Section 4.3]). *Fix an m -tuple $(z_1, \dots, z_m) \in Z^m$ with the assumption $e_{z_1} \cdots e_{z_m} = \text{id}_M \in \text{End}(M)$.*

- (I) *For any ℓ with $1 \leq \ell \leq m$, the equality $\mathcal{Q}_{\psi, 1} = \mathcal{Q}_{\psi, \ell}$ holds as a map.*
- (II) *If ψ is hermitian, then the bilinear form $\mathcal{Q}_{\psi, \ell}$ is skew-hermitian. Furthermore, if ψ is skew-hermitian, then $\mathcal{Q}_{\psi, \ell}$ is hermitian.*

Since $\mathcal{Q}_{\psi, \ell}$ does not depend on ℓ under the assumption, we later only discuss $\mathcal{Q}_{\psi, 1}$ with $\ell = 1$, and denote it by $\mathcal{Q}_{\psi}$. We further review a reduction of $\mathcal{Q}_{\psi}$, which implies that a large part of $\text{Ker}(\Gamma_{\mathbf{z}})$ is contained in the kernel of $\mathcal{Q}_{\psi}$; Precisely,

Proposition 3.12 ([N2, Proposition 4.7]). *Consider an element $\mathbf{x} \in M^m$ of the form*

$$(x, \dots, x), \quad \text{or} \quad (x_1, \dots, x_m)$$

for some $x \in M$ and $x_i \in \text{Ker}(1 - e_{z_i}) \subset M$. Then, $\mathbf{x}$ lies in the kernel $\text{Ker}(\oplus_k \Gamma_{\mathbf{z}, k}) = \text{Ker}(\Gamma_{\mathbf{z}})$, and the vanishing $\mathcal{Q}_{\psi, \ell}(\mathbf{x}, \mathbf{y}) = \mathcal{Q}_{\psi, \ell}(\mathbf{y}, \mathbf{x}) = 0$ holds for any $\mathbf{y} \in \text{Ker}(\Gamma_{\mathbf{z}})$.

3.3. Unimodularity of the invariants of bilinear forms

Next, we will discuss the unimodularity of $\mathcal{Q}_{\psi}$. According to Proposition 3.12, we shall regard $\mathcal{Q}_{\psi}$ as a bilinear form on the quotient module of the form

$$\underline{M}_{\mathbf{z}} := \text{Ker}(\Gamma_{\mathbf{z}}) / (\text{Diag}(M) + (\bigoplus_{i: 1 \leq i \leq m} \text{Ker}(1 - e_{z_i} : M \rightarrow M))). \quad (9)$$

To analyse the unimodularity in more details, we assume that A is a Dedekind domain (because every known quantum representation is closed under a Dedekind domain; see [Gil, Theorem 1.1]), and M is a finitely generated projective A -module. Let $\text{Tor}(M)$ denote the torsion submodule of M . To the end, we suppose the following basic results on Dedekind domains:

- (i) Every finitely generated A -module M admits an ideal $\mathcal{I} \subset A$ and a unique integer $k \in \mathbb{N}$ for which the A -isomorphism $M \cong A^k \oplus \mathcal{I} \oplus \text{Tor}(M)$ holds.
- (ii) (Semi-heredity) Every A -submodule of a projective A -module is also projective.

In particular, the canonical projection $\text{Ker}(\Gamma_{\mathbf{z}}) \rightarrow \underline{M}_{\mathbf{z}}/\text{Tor}(\underline{M}_{\mathbf{z}})$ splits since the image is projective by (ii). Therefore, denoting by $M_{\mathbf{z}} \subset \text{Ker}(\Gamma_{\mathbf{z}})$ the summand obtained from the splitting, we have a decomposition

$$\text{Ker}(\Gamma_{\mathbf{z}}) \cong M_{\mathbf{z}} \oplus (\text{Ker}(\Gamma_{\mathbf{z}})/M_{\mathbf{z}})$$

as projective modules. Then, by Proposition 3.12, the vanishings $\mathcal{Q}_{\psi}(\mathbf{x}, \mathbf{y}) = \mathcal{Q}_{\psi}(\mathbf{y}, \mathbf{x}) = 0$ hold for any $\mathbf{x} \in \text{Ker}(\Gamma_{\mathbf{z}})$ and $\mathbf{y} \in \text{Ker}(\Gamma_{\mathbf{z}})/M_{\mathbf{z}}$. Thus we shall address non-degeneracy restricted on the summand $M_{\mathbf{z}}$:

Theorem 3.13. *Let A be a field or a Dedekind domain, and M a finitely generated free A -module. Fix an m -tuple $(z_1, \dots, z_m) \in Z^m$ with $e_{z_1} \cdots e_{z_m} = \text{id}_M$. Take the projective submodule $M_{\mathbf{z}} \subset \text{Ker}(\Gamma_{\mathbf{z}})$ explained above.*

- (I) *If the $\text{As}(Z)$ -invariant bilinear function $\psi : M^2 \rightarrow A$ is non-degenerate, so is the resulting bilinear form $\mathcal{Q}_{\psi}$ restricted on the direct summand $M_{\mathbf{z}}$.*
- (II) *Assume that ψ is unimodular. Then, the restriction $\mathcal{Q}_{\psi}$ on the summand $M_{\mathbf{z}}$ is unimodular if and only if the following map χ^* from a cokernel $\text{Coker}(\lambda^*)$ is injective:*

$$\chi^* : \text{Coker}\left(\text{Ext}_A^1\left(\frac{M}{\mathcal{A}_{\mathbf{z}}}, A\right) \xrightarrow{\lambda^*} \text{Ext}_A^1\left(\bigoplus_{1 \leq i \leq m} \frac{M}{\text{Ker}(1 - e_{z_i})}, A\right)\right) \longrightarrow \text{Ext}_A^1\left(\frac{M^{m-1}}{\mathcal{B}_{\mathbf{z}}}, A\right),$$

where the maps χ^* and λ^* are induced by the homomorphisms

$$\begin{aligned} \chi : M^{m-1} &\longrightarrow M^m; & (a_1, \dots, a_{m-1}) &\longmapsto (a_1, \dots, a_{m-1}, -a_1 - a_2 - \cdots - a_{m-1}), \\ \lambda : M^m &\longrightarrow M; & (b_1, \dots, b_m) &\longmapsto b_1 + \cdots + b_m, \end{aligned} \tag{10}$$

respectively, and the submodules $\mathcal{A}_{\mathbf{z}}$ and $\mathcal{B}_{\mathbf{z}}$ are defined to be the followings:

$$\mathcal{A}_{\mathbf{z}} := \{ b_1 \cdot (1 - e_{z_1}) + \cdots + b_{m-1} \cdot (1 - e_{z_{m-1}}) \in M \mid b_1, \dots, b_{m-1} \in M \},$$

$$\mathcal{B}_{\mathbf{z}} := \{ (b - b \cdot e_{z_1}, \dots, b - b \cdot e_{z_{m-1}}) \in M^{m-1} \mid b \in M \}.$$

The proof of Theorem 3.13 will appear in Section 6. Since the statement (II) seems a little complicated, we should mention an applicable corollary, which is immediately obtained from Theorem 3.13:

Corollary 3.14. *With notation as above, if ψ is unimodular and all the cokernel $\text{Coker}(1 - e_{z_i})$ with $i \leq m$ has no torsion, then the restriction $\mathcal{Q}_{\psi}$ on the summand $M_{\mathbf{z}}$ is unimodular.*

4. Invariants from symplectic representations

We will study the bilinear forms $\mathcal{Q}_{\psi}$ obtained from the standard symplectic representations, and give some applications. Section 4.2 examines the rational part. To this end, Section 4.1 explains an outline for computing $\mathcal{Q}_{\psi}$.

4.1. Preliminaries to compute the bilinear forms

We first describe an outline for computing the isomorphism class of $\mathcal{Q}_\psi$, on the basis of some knowledge in [MH]. This section assumes that M is a finitely generated free A -module.

Following Theorem 3.13, we will discuss $\mathcal{Q}_\psi$ over a Dedekind domain A . Recall the projective submodule $M_{\mathbf{z}} \subset M^m$ defined in Section 3.3. Then, as the properties (i) and (ii) of Dedekind domains, this $M_{\mathbf{z}}$ is a direct summand of M^m , and is isomorphic to $A^{N-1} \oplus \mathcal{I}$ for some $N \in \mathbb{N}$ and ideal $\mathcal{I} \subset A$. Therefore, we can choose a basis $v_1, \dots, v_N$, and we therefore formulate the bilinear form $\mathcal{Q}_\psi$ as the $N \times N$ -matrix, W , with entries $\mathcal{Q}_\psi(v_i, v_j) \in A$. Here we remark, thanks to Theorem 3.13 (i), that this W is equivalent to the original $\mathcal{Q}_\psi$, and that, if ψ is non-degenerate, then this W is non-degenerate, i.e., $\det(W) \in A \setminus \{0\}$. In practice, although the ranks of W are often large, we can list such W by the help of computer programs found in software such as Mathematica. Then, using the study of non-degenerate (or unimodular) bilinear forms over some rings (see, e.g., [MH]), we can sometimes determine some isomorphism classes of W and obtain something quantitative from the class (e.g., rank, signature) in some cases.

As an example, we particularly comment on the forms $\mathcal{Q}_\psi$ when $A = \mathbb{Z}$ and ψ is anti-symmetric and unimodular. Theorem 3.13 implies that the matrix W is symmetric and non-degenerate. Moreover, if the condition in Corollary 3.14 holds and the matrix W is indefinite, then we may compute only the eigenvalues of W and check whether the all diagonal entries $\mathcal{Q}_\psi(v_i, v_i)$ are even or not. Actually, by virtue of Serre's classification theorem (see [MH, Chapters I, II] for details), one can determine the bilinear form W up to bilinear isomorphisms over $\mathbb{Z}$. Furthermore, we also mention some definite matrices. Actually, we see later some definite W 's. However, the ranks are fortunately ≤ 16 . By the classifications of the definite unimodular bilinear forms of small rank (see [MH, Chapter II.6]), we later determine some isomorphism classes of $\mathcal{Q}_\psi$ following the above line (see Section 4.2 for concrete calculations).

For Section 5, we also comment on the coefficient rings of the forms $A = \mathbb{F}_p[T]/(T^p)$ for some prime $p \in \mathbb{N}$. In general, it is hard to determine isomorphism classes of $\mathcal{Q}_\psi$, even if $\mathcal{Q}_\psi$ is symmetric. However, as a special case, if the image of $\mathcal{Q}_\psi$ is over the ideal (T^{p-1}) , the class of $\mathcal{Q}_\psi$ is detected by the study of the Witt group $W(\mathbb{F}_p)$ (see [MH, Section I.7 and Appendixes]). For example, if $p = 2$, the class of $\mathcal{Q}_\psi$ is detected by the rank and the parity of diagonal entries $\mathcal{Q}_\psi(v_i, v_i)$ that is called Arf invariant.

4.2. Computations in symplectic case

Henceforth, we will compute the invariants of some Lefschetz fibrations by means explained in the preceding section. So, throughout this section, we assume that G is the mapping class group $\mathcal{M}_g$ with $g \geq 2$, and Z is the set $\mathcal{D}_g$ of the Dehn twists as in Example 3.4. This subsection discusses the standard symplectic representation of $\mathcal{M}_g$. Namely, $M = H_1(\Sigma_g; A) = A^{2g}$ and we set $\psi = \omega_A : M^2 \rightarrow A$ as the symplectic form.

We start by establishing some terminologies. For an m -tuple $(z_1, \dots, z_m) \in (\mathcal{D}_g)^m$, let $E_{\mathbf{z}}$ be the (4-dimensional) total space of the corresponding fibration ξ . The symbol $\sigma_{\mathbf{z}}$ denotes the signature of the closed 4-manifold $E_{\mathbf{z}}$, and the Euler characteristic of the space $E_{\mathbf{z}}$ is known to be $4 - 4g + m$ (see [EN, Ozb, Mat]). The fibration $E_{\mathbf{z}} \rightarrow S^2$ is said to be *of type* $(m_{\text{ns}}, m - m_{\text{ns}})$ if m_{ns} is the number of z_i 's contained in $\mathcal{D}_g^{\text{ns}}$ (8). Furthermore, for another fibration η associated with $(z'_1, \dots, z'_{m'}) \in (\mathcal{D}_g)^{m'}$, the (twist) fiber sum along $h \in \mathcal{M}_g$ is defined to be the fibration arising from the $(m + m')$ -tuples $(z_1, \dots, z_m, h^{-1}z'_1h, \dots, h^{-1}z'_{m'}h) \in (\mathcal{D}_g)^{m+m'}$, for which we write $\xi \#_h \eta$.

In addition, we will explain some examples of Lefschetz fibrations. Consider three tuples of the forms

$$\begin{aligned} \xi_1 &: (c_1, c_2, \dots, c_{2g}, c_{2g+1}, c_{2g+1}, c_{2g}, \dots, c_1)^2, \\ \xi_2 &: (c_1, c_2, \dots, c_{2g}, c_{2g+1})^{2g+2}, \\ \xi_3 &: (c_1, c_2, \dots, c_{2g})^{4g+2}. \end{aligned}$$

Here c_i is the Dehn twist of the curves γ_i in Figure 2. Then, they are well-known to be identities in $\mathcal{M}_g$ as the chain relation. (The associated Lefschetz fibrations are understood well. See [EN, Section 4.1].) Furthermore, as seen in Table 1, we also deal with other Lefschetz fibrations.

Next, we comment rationally the rank and unimodularity of the bilinear forms $\mathcal{Q}_{\omega_{\mathbb{Z}}}$, and show that $\mathcal{Q}_{\omega_{\mathbb{Z}}}$ includes rationally topological information of Lefschetz fibrations as follows:

Theorem 4.1. *Let $A = \mathbb{Z}$. Let $\xi_{\mathbf{z}}$ be a Lefschetz fibration $E_{\mathbf{z}} \rightarrow S^2$ associated with an m -tuple $\mathbf{z}$ of type $(m_{\text{ns}}, m - m_{\text{ns}})$. Take the homomorphism $\Gamma_{\mathbf{z}}$ in Definition 3.1 and the free module $\underline{M}_{\mathbf{z}}$ explained in Theorem 3.13. Then, the ranks are computed as*

$$\text{rk}(\underline{M}_{\mathbf{z}}) = m_{\text{ns}} - 4g + 2b_1(E_{\mathbf{z}}), \quad \text{rk}(\text{Ker } \Gamma_{\mathbf{z}}) = 2gm - 2g + b_1(E_{\mathbf{z}}),$$

where $b_1(E_{\mathbf{z}})$ is the first betti number of the 4-manifold $E_{\mathbf{z}}$.

Moreover, the signature of the resulting symmetric form $\mathcal{Q}_{\omega_{\mathbb{Z}}}$ is equal to $\sigma_{\mathbf{z}} - m + m_{\text{ns}}$.

We defer the proof until Section 6. As a corollary, by the unimodularity in Corollary 3.14, we immediately have the following:

Corollary 4.2. *The bilinear form $\mathcal{Q}_{\omega_{\mathbb{Z}}}$ is equivalent to a unimodular symmetric bilinear form on $\mathbb{Z}^{m_{\text{ns}} - 4g + 2b_1}$ of signature $\sigma_{\mathbf{z}} - m + m_{\text{ns}}$.*

Accordingly, following the outline mentioned in Section 4.1, if knowing whether $\mathcal{Q}_{\omega_{\mathbb{Z}}}$ is of type I or II, we succeeded in determining the isomorphism class of $\mathcal{Q}_{\omega_{\mathbb{Z}}}$. In fact, below is Table 1 on the resulting forms of some Lefschetz fibrations with $g = 2$ and $g = 3$. Especially, the column on the symbol $\mathcal{Q}_{\omega_{\mathbb{Z}}}$ is with respect to isomorphism classes of the bilinear forms associated with the symplectic form $\omega_{\mathbb{Z}}$. Here E_8 is the famous Cartan matrix as a unimodular quadratic form of rank 8 and of signature -8 , and $\mathcal{H}_c$ is the hyperbolic form $\begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$ for some $c \in A$. As seen in the column, the type of $\mathcal{Q}_{\omega_{\mathbb{Z}}}$ is

frequently useful: Actually, the pairs (ξ_{PJ}, ξ_{RI}) and $(\xi_1 \#_{c_1} \xi_1, \xi_3)$ with $g = 2$ are shown to be non-isomorphic by the types (though these results are proved also by considering the images of the tuple $(z_1, \dots, z_m)$ in the modular group $Sp(2g; \mathbb{Z})$). Here we shall cite a stronger result of Y. Sato [Sat]: Precisely, the total spaces associated with ξ_{PJ} and ξ_{RI} are non-diffeomorphic as a result in studying the (relative) Kodaira dimensions.

Fibration	g	Type	σ	$\mathcal{Q}_{\omega_{\mathbb{Z}}}$	$\mathcal{Q}_{\omega_{\text{spin}}^{\text{odd}}}$	$\mathcal{Q}_{\omega_{\text{spin}}^{\text{ev}}}$
ξ_1	2	(20,0)	-12	$(-1)^{12}0^{64}$	$\mathcal{H}_y^{28}0^{87}$	$\mathcal{H}_y^{50}0^{141}$
ξ_2	2	(30,0)	-18	$(-1)^{20}1^20^{94}$	$\mathcal{H}_y^{48}0^{127}$	$\mathcal{H}_y^{80}0^{211}$
ξ_3	2	(40,0)	-24	$\mathcal{H}_1^4(E_8)^30^{124}$	$\mathcal{H}_y^{70}0^{166}$	$\mathcal{H}_y^{108}0^{283}$
$\xi_1 \#_{\text{id}} \xi_1$	2	(40,0)	-24	$(-1)^{28}1^40^{124}$	$\mathcal{H}_y^{68}0^{169}$	$\mathcal{H}_y^{110}0^{281}$
ξ_{RI}	2	(28,1)	-17	$1^2(-1)^{18}0^{92}$	$\mathcal{H}_y^{56}0^{112}$	$\mathcal{H}_y^{74}0^{207}$
ξ_{PJ}	2	(28,1)	-17	$\mathcal{H}_1^2(E_8)^20^{92}$	$\mathcal{H}_y^{54}0^{115}$	$\mathcal{H}_y^{72}0^{209}$
ξ_1	3	(28,0)	-16	$(E_8)^20^{146}$	$\mathcal{H}_y^{114}0^{643}$	$\mathcal{H}_y^{238}0^{735}$
$\xi_1 \#_{\text{id}} \xi_1$	3	(56,0)	-32	$\mathcal{H}_1^6(E_8)^40^{286}$	$\mathcal{H}_y^{282}0^{1259}$	$\mathcal{H}_y^{546}0^{1436}$
$\xi_1 \#_d \xi_1$	3	(56,0)	-32	$1^6(-1)^{38}0^{286}$	$\mathcal{H}_y^{282}0^{1259}$	$\mathcal{H}_y^{544}0^{1437}$
ξ_2	3	(56,0)	-32	$\mathcal{H}_1^6(E_8)^40^{286}$	$\mathcal{H}_y^{282}0^{1259}$	$\mathcal{H}_y^{546}0^{1436}$
ξ_3	3	(84,0)	-48	$\mathcal{H}_1^{12}(E_8)^60^{426}$	$\mathcal{H}_y^{452}0^{1874}$	$\mathcal{H}_y^{854}0^{2136}$
$\xi_2 \#_{\text{id}} \xi_1$	3	(84,0)	-48	$\mathcal{H}_1^{12}(E_8)^60^{426}$	$\mathcal{H}_y^{450}0^{1875}$	$\mathcal{H}_y^{854}0^{2136}$
$\xi_2 \#_d \xi_1$	3	(84,0)	-48	$1^{12}(-1)^{60}0^{426}$	$\mathcal{H}_y^{450}0^{1876}$	$\mathcal{H}_y^{852}0^{2137}$
ξ_{CK}	3	(16,0)	-8	E_80^{84}	$\mathcal{H}_y^{54}0^{373}$	$\mathcal{H}_y^{110}0^{434}$
M_Q	3	(44,1)	-25	$\mathcal{H}_1^4(E_8)^30^{232}$	$\mathcal{H}_y^{210}0^{1023}$	$\mathcal{H}_y^{414}0^{1172}$
M_R	3	(44,1)	-25	$\mathcal{H}_1^4(E_8)^30^{232}$	$\mathcal{H}_y^{210}0^{1023}$	$\mathcal{H}_y^{414}0^{1172}$

TABLE 1. The bilinear-form invariants associated with some g -genus Lefschetz fibrations with $g = 2, 3$. Here global monodromies of the fibrations ξ_{PJ} , ξ_{RI} with $g = 2$ and ξ_{CK} , M_Q , M_R with $g = 3$ are formulated in [Endo, Section 4.1-2]. Furthermore, the sixth and seventh columns $\mathcal{Q}_{\omega_{\text{spin}}^{\text{ev}}}$ and $\mathcal{Q}_{\omega_{\text{spin}}^{\text{odd}}}$ will be explained in Section 5.1. We write $\#_d$ for the Dehn twist along the curve in Figure 2.

To end the integral symplectic case, we demonstrate from Table 1 that the bilinear form $\mathcal{Q}_{\omega_{\mathbb{Z}}}$ does not coincide with the usual cohomology ring structures with integral coefficients. First, the form of the fibration ξ_{CK} with $g = 3$ is E_8 (cf. Donaldson's Theorem A which implies that no differential closed 4-manifold admits the cohomology ring of the form E_8). Furthermore, while the simply-connected space ξ_3 with $g = 2$ has no spin structure (use the Rokhlin theorem), the form $\mathcal{Q}_{\omega_{\mathbb{Z}}}$ contain the matrix E_8 . Thus, we pose a problem:

Problem 4.3. Describe a sufficient and necessary condition for which the unimodular form $\mathcal{Q}_{\omega_{\mathbb{Z}}}$ is of type II.

As seen in the difference of the types between $\xi_2 \#_{\text{id}} \xi_1$ and $\xi_2 \#_d \xi_1$ in Table 1, we see that the torsion parts are not always topological. Here, the type of $\mathcal{Q}_{\mathbb{Z}}$ can be determined by computing a matrix presentation of $\mathcal{Q}_{\mathbb{Z}}$ and by checking whether its diagonal entries include an odd integer or not. To summarize, we hope a usefulness of the torsion invariant.

5. Invariants from the quantum representations

In this section, we will explain that the formulation of our invariants is applicable for the quantum representations.

For this purpose, we briefly review the quantum representations. First we fix a semi-simple Lie group $\mathcal{G}$ and an integer $k \in \mathbb{Z}$ called *level*. Then E. Witten [Wi] has predicted that the Chern-Simons quantum field theory of level k yields a (2+1)-topological quantum field theory $\mathcal{Q}_{\mathcal{G},k}$ (with ambiguity of framings). As a result, the restriction of $\mathcal{Q}_{\mathcal{G},k}$ on the single object Σ_g is regarded as a linear representation $\mathcal{M}_g$ of some $\mathbb{C}$ -vector space $V_{\mathcal{G},k}$, which is commonly called the *quantum representation*. However, to strictly formulate the TQFT in mathematical terms, we have to equip the TQFT with “ p_1 -structure” (see [BHMV, Appendix B] for the definition). Thus, due to the obstruction of p_1 , the associated representations are often right modules not of $\mathcal{M}_{g,r}$, but of $\mathcal{T}_{g,r}$. Here, $\mathcal{T}_{g,r}$ is a central $\mathbb{Z}$ -extension of $\mathcal{M}_{g,r}$,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{T}_{g,r} \xrightarrow{\text{proj.}} \mathcal{M}_{g,r} \longrightarrow 0 \quad (\text{central extension}). \quad (11)$$

Further, if $g \geq 3$, this $\mathcal{T}_{g,r}$ is universal in the sense of the group cohomology $H^2(\mathcal{M}_{g,r}; \mathbb{Z}) \cong \mathbb{Z}$; see [FM]. Moreover, since the $\mathcal{T}_{g,r}$ -module $V_{\mathcal{G},k}$ is unitary by the topological invariance of the TQFT, we get a non-singular $\mathcal{T}_{g,r}$ -invariant Hermite bilinear form on $V_{\mathcal{G},k}$. Furthermore, there is a conjecture that the space $V_{\mathcal{G},k}$ would admit a lattice over the ring of integers in some number field $F \subset \mathbb{C}$ under which the action of $\mathcal{T}_{g,r}$ is closed. Although we only refer the reader to [BHMV, GMW, Ker, Wi] for more details, we remark that the dimension of $V_{\mathcal{G},k}$ becomes exponentially more complicated as k -increase, that is, Verlinde formula.

Thus, to specialize to $\mathcal{Q}_{\psi}$ of small ranks, we shall mention only easy examples of the TQFT. First, if $\mathcal{G}$ is the unitary group $U(1)$, then the resulting module is the symplectic representation on $H_1(\Sigma_g)$, and the bilinear form is the symplectic form, which we used in Section 4. In addition, if $\mathcal{G}$ is $SU(2)$ or $SO(3)$, the Kauffman bracket gives a description of the TQFT explicitly (see [BHMV]). Further, for any prime $p \in \mathbb{N}$, the $SO(3)$ -quantum representation of level $(p-1)/2$ is over the ring of integers in the cyclotomic field $\mathbb{Q}(\zeta_p)$ (see, e.g., [Gil, GMW] and references therein), though the computations in $\mathbb{Q}(\zeta_p)$ are often hardly sophisticated as usual in quantum topology. However, if the TQFT is closed under the integral closed ring $\mathbb{Z}[\zeta_{p^t}]$ for some $t \in \mathbb{N}$ such as $SO(3)$ -case, the reduced TQFT via a base change

$$\mathbb{Z}[\zeta_{p^t}] \longrightarrow \mathbb{F}_p[y]/(y^p); \quad \zeta_{p^t} \longmapsto 1 + y \quad (12)$$

is more manageable, together with a relation to Casson invariant modulo p (see [Ker]). Such being the case, the next subsection will mainly focus on such reduced TQFTs.

5.1. With the $SU(2)$ -gauge of level 2.

We will return to study Lefschetz fibrations. Let $G = \mathcal{M}_g$ and $Z = \mathcal{D}_g$ as in Example 3.4. Then, as was shown [N3], $\text{As}(\mathcal{D}_g)$ includes the universal central extension (11) as follows:

Theorem 5.1 ([N3]). *(I) If $g \geq 3$, there is a group isomorphism $\text{As}(\mathcal{D}_g) \cong \mathcal{T}_g \times \mathbb{Z}^{[g/2]+2}$. (II) If $g = 2$, there are a $\mathbb{Z}$ -central extension $\mathcal{T}_2 \rightarrow \mathcal{M}_2$ and an isomorphism $\text{As}(\mathcal{D}_g) \cong \mathcal{T}_2 \times \mathbb{Z}^2$.*

As a result, for $g \geq 2$, the category of $\mathcal{T}_g$ -modules is essentially equivalent to that of $\text{As}(\mathcal{D}_g)$ -modules. Hence, we can apply the quantum representations to Lefschetz fibrations.

In this section, we will focus on the quantum field with $SU(2)$ -gauge of level 2¹, and classify some examples of Lefschetz fibrations by the associated bilinear forms. As is known from Verlinde formula [BHMV, Theorems 1.6 (iii) and 7.17], the quantum field produces two representations $V_{\text{spin}}^{\text{ev}}$ and $V_{\text{spin}}^{\text{odd}}$ of dimension

$$\dim(V_{\text{spin}}^{\text{ev}}) = (2^g + 1)2^{g-1}, \quad \dim(V_{\text{spin}}^{\text{odd}}) = (2^g - 1)2^{g-1}.$$

These $V_{\text{spin}}^{\text{ev}}$ and $V_{\text{spin}}^{\text{odd}}$ are modules of $\mathcal{T}_g$ and of $\mathcal{T}_{g,1}$, and are referred to as a quantization of spin structures on Σ_g of even parity and of odd parity, respectively.

We now roughly explain the reason that the space $V_{\text{spin}}^{\text{ev}}$ is reduced to a free module over the ring $\mathbb{Z}[\zeta] \subset \mathbb{C}$, where ζ is the 16-th root of unity: Namely $\zeta := \exp(\sqrt{-1}/8\pi)$. To begin, when $g = 1$, we can see that the representation is conjugate to the following matrix presentation with coefficients in $\mathbb{Z}[\zeta]$:

$$\rho(c_1) := \begin{pmatrix} 1 & 0 & (\zeta + \zeta^8)/(1 + \zeta) \\ 0 & -\zeta^3 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\rho(c_2) := \begin{pmatrix} 0 & \zeta^2 & 0 \\ -\zeta^6 & 0 & 0 \\ \zeta^4(1 - \zeta^7)/(1 - \zeta) & \zeta^3(1 - \zeta^7)/(1 - \zeta) & -\zeta^3 \end{pmatrix}.$$

For larger $g > 1$, it follows from the line of the paper [GMW] that the matrix presentation of $\rho(c_i)$ with $g > 1$ is formulated as a direct sum of that of $\rho(c_1)$ with $g = 1$ via “some change of coordinate system” by the 6j-symbols. Then, thanks to the integrality of the 6j-symbols, the quantum representation is closed over the ring $\mathbb{Z}[\zeta]$ as required. However, the details are unnecessary for this paper. Actually, since we work out only $g \leq 3$ in this section, it is enough only to describe explicitly two matrix presentations of $\rho(c_1)$, $\rho(c_2)$ over $\mathbb{Z}[\zeta]$.

¹In terminologies of [BHMV], the TQFT is written as “ $V_8(\Sigma, q)$ with $r = 4$ ”, where by q we mean the set of all spin structures on Σ either of odd parity or of even parity.

We mention some properties of the representation. First, the ring $\mathbb{Z}[\zeta]$ is a PID, since the class number of the fraction field $\mathbb{Q}(\zeta)$ is well-known to be 1. However, we remark that, for $g > 2$, the $\mathcal{T}_g$ -invariant Hermitian inner product $W_{\text{spin}}^{\text{ev}}$ obtained from the TQFT is not always unimodular. In fact, even if $g = 3$, we can check $\det(W_{\text{spin}}^{\text{ev}}) = 2^{84} \cdot 23 \cdot 7583 \cdot 14110027847$.

Hence, for simplicity, let us consider the reduction on the Artin ring $\mathbb{F}_2[y]/y^2$ as in (12). Then, we can see that the $\mathcal{T}_{g,*}$ -modules $V_{\text{spin}}^\dagger$ are reduced to modules of $\mathcal{M}_g$. Furthermore, we can find two bilinear forms from reduced representations as follows:

Lemma 5.2. *Let $g = 2$ or $= 3$. Consider the $\mathbb{F}_2[y]/y^2$ -space of $\mathcal{M}_g$ -invariant bilinear forms on the reduced representation $V_{\text{spin}}^\dagger$ in coefficients $\mathbb{F}_2[y]/y^2$. Then, the space is spanned by two symmetric bilinear forms $\omega_{\text{spin}}^\dagger$ and $\nu_{\text{spin}}^\dagger$ such that $\omega_{\text{spin}}^\dagger$ is non-singular and $\nu_{\text{spin}}^\dagger$ is singular.*

Though we omit describing the proof, it can be done by a (long) direct calculation, thank to the matrix presentation of $V_{\text{spin}}^\dagger$.

However, Theorem 3.13 indicates that the bilinear form $\mathcal{Q}$ from the singular form $\nu_{\text{spin}}^\dagger$ enjoys more interesting properties than that of $\omega_{\text{spin}}^\dagger$. In fact, in computer experience, the image of $\mathcal{Q}_{\nu_{\text{spin}}^\dagger}$ lies in the ideal $(y) \subset \mathbb{F}_2[y]/y^2$; Hence, we can easily determine the class of $\mathcal{Q}_{\nu_{\text{spin}}^\dagger}$ as mentioned in Section 4.1. In doing so, here are the fifth and sixth columns in Table 1 concerning the classes of Lefschetz fibrations. As a result, we obtain the following:

Proposition 5.3. *Fix $s, t \in \mathbb{Z}$. The triples of the fibrations $(\xi_2 \#_{c_0}^{2s} \xi_1, \xi_2 \#_{c_0}^{2t-1} \xi_1, \xi_3)$ are non-isomorphic.*

Proof. For any Dehn twist τ_α , twice $(\rho_{SU(2),2}(e_\alpha))^2$ is identity. Hence, the bilinear forms of $\xi_2 \#_{c_0}^{2s} \xi_1$ and $\xi_2 \#_{c_0}^{2t-1} \xi_1$ are independent of $s, t \in \mathbb{Z}$. According to Table 1, the above triple with $s = t = 0$ are non-isomorphic. Hence, we have the conclusion as stated. $\square$

However, as seen in Table 1, the invariants $\mathcal{Q}_{\omega_{\text{spin}}^\dagger}$ of lower level seem not so strong, in contrast to the Jones knot-polynomial and with a recent result [EK]. For example, in the paper [EK], the two pairs $(\xi_1 \# \xi_1, \xi_2)$ and (ξ_P, ξ_Q) are shown to be non-isomorphic Lefschetz fibrations, while the associated bilinear forms can not detect them.

To conclude, it is a problem for the future to compute and analyse the bilinear forms $\mathcal{Q}$ arising from the quantum representations with respect to other gauge of other level.

6. Proofs of theorems in Sections 3.3-4.2

We will make the proofs of the statements in Sections 3.3-4.2 that remain to be proved. Since the proof is based on Proposition 3.10, the reader should reread it.

Furthermore, we prepare the universal coefficient theorem over Dedekind domains:

Lemma 6.1. *Let A be a Dedekind domain, and M be a finitely generated A -module. Then there is an exact sequence*

$$0 \longrightarrow \text{Ext}_A^1(H_{n-1}(Y; M), A) \longrightarrow H^n(Y; M) \xrightarrow{\mathcal{I}} \text{Hom}(H_n(Y; M), A) \longrightarrow 0.$$

Here $\mathcal{I}$ is the canonical map on the (co)chain group of Y . Furthermore, $\mathcal{I}$ induces an isomorphism $H^n(Y; M)/\text{Tor} \cong \text{Hom}(H_n(Y; M), A)$.

Proof. Note that Ext^n with $n \geq 2$ are zero since A is of Krull dimension 1. Thus, the universal coefficient spectral sequence (see [McC, Section 12.1]) collapses at E_2 -term; hence we have the exact sequence. For the latter part, by the reason of (i) explained in Section 3.3, $\text{Hom}(H_n(Y; M), A)$ is projective and $\text{Ext}_A^1(H_{n-1}(Y))$ is torsion. Thereby, we immediately obtain the isomorphism. $\square$

Proof of Theorem 3.13. (I) We start by considering the long exact sequence associated with the pair $(Y, \partial Y)$:

$$0 \longrightarrow H^0(Y; M) \longrightarrow H^0(\partial Y; M) \xrightarrow{\delta^*} H^1(Y, \partial Y; M) \xrightarrow{p^*} H^1(Y; M) \xrightarrow{\iota^*} H^1(\partial Y; M). \quad (13)$$

As is seen in [N2, Proposition 4.7], the cokernel of δ^* can be identified with the quotient $\underline{M}_{\mathbf{z}}$ in (9). In particular, the module $M_{\mathbf{z}}$ is, by construction, regarded as $\text{Coker}(\delta^*)/\text{Tor}$.

Based on the identifications, we will reformulate the adjoint map of bilinear form $\mathcal{Q}_\psi$, and will show its injectivity. Define a homomorphism

$$\Upsilon : H^1(Y, \partial Y; M) \longrightarrow \text{Hom}(H^1(Y; M), (M \otimes M)_{\pi_1(Y)})$$

by setting $\Upsilon(u) := \langle u \smile \bullet, [Y, \partial Y] \rangle \in (M \otimes M)_{\pi_1(Y)}$. Hence, composing Υ with the pairing of the bilinear form ψ yields a homomorphism

$$\Upsilon_\psi : H^1(Y, \partial Y; M) \longrightarrow \text{Hom}(H^1(Y; M), A). \quad (14)$$

Here, we should notice that the map $\langle \bullet, [(Y, \partial Y)] \rangle$ is an isomorphism by Poincaré duality on the first (co)homology in local coefficients; thus, this Υ_ψ is injective and the cokernel is torsion, thanks to the Poincaré duality and the non-degeneracy of ψ . Here, recall an elementary equality on the cup product:

$$\langle u \smile p^*(v), [Y, \partial Y] \rangle = 0 \in (M \otimes M)_{\pi_1(Y)}, \quad \text{for any } u \in \text{Im}(\delta^*), \ v \in H^1(Y; M).$$

Hence, by considering the restriction on $\text{Im}(p^*) = \text{Coker}(\delta^*)$, the map Υ_ψ in (14) induces

$$\Upsilon'_\psi : \text{Coker}(\delta^*)/\text{Tor} \longrightarrow \text{Hom}(\text{Im}(p^*), A). \quad (15)$$

By constructions and Proposition 3.10, this Υ'_ψ is the adjoint map of $\mathcal{Q}_\psi$. Letting K be the fractional field of A , since $\Upsilon_\psi \otimes \text{id}_K$ is injective as mentioned in (14), so is $\Upsilon'_\psi \otimes \text{id}_K$. Hence, we have the injectivity of Υ'_ψ , which completes the proof for the non-degeneracy of $\mathcal{Q}_\psi$.

(II) Assuming the unimodularity of ψ , we will observe that of $\mathcal{Q}_\psi$. To this end, consider the following commutative diagram:

$$\begin{array}{ccccccc}
 H^1(Y, \partial Y, M) & \longrightarrow & \text{Im}(p^*)/\text{Tor} & \longrightarrow & 0 \\
 \Upsilon_\psi \downarrow & & \downarrow \Upsilon'_\psi & & \downarrow \\
 \text{Hom}(H^1(Y; M), A) & \longrightarrow & \text{Hom}(\text{Im}(p^*), A) & \longrightarrow & \text{Ext}^1(\text{Im}(\iota^*), A) & \xrightarrow{\iota^*} & \text{Ext}^1(H^1(Y; M), A),
 \end{array}$$

where the bottom sequence is the Ext-functor obtained from (13), and the horizontal sequences are exact. Notice that, by Poincaré duality, the map Υ_ψ mentioned in (14) is an isomorphism. Therefore the snake lemma implies that the cokernel of Υ'_ψ is identified with the kernel of the bottom right map ι^* .

We will analyse the kernel $\text{Ker}(\iota^*)$, because the unimodularity of $\mathcal{Q}_\psi$ is equivalent to the injectivity of the map ι^* . Here, note that, by 1-dimensional duality of Y , we can regard the image $\text{Im}(\iota^*)$ as the kernel of the pushforward $\iota_* : H_0(\partial Y; M) \rightarrow H_0(Y; M)$.

Therefore, we now analyze this ι_* by computing $H_0(\partial Y; M)$ and $H_0(Y; M)$. A standard resolution of the free group $\pi_1(Y)$ after tensoring with M is written as

$$0 \longrightarrow M \otimes \mathbb{Z}[\pi_1(Y)]^{m-1} \xrightarrow{\partial_1} M \otimes \mathbb{Z}[\pi_1(Y)] \longrightarrow 0.$$

Here, the map ∂_1 sends $a \otimes (g_1, \dots, g_{m-1})$ to $\sum_{i=1}^{m-1} (ag_i) \otimes (1 - g_i)$. Thereby, the zeroth and first cohomologies are given by

$$H^0(Y; M) \cong M/\mathcal{A}_\mathbf{z}, \quad H^1(Y; M) \cong M^{m-1}/\mathcal{B}_\mathbf{z},$$

where the submodules $\mathcal{A}_\mathbf{z}$ and $\mathcal{B}_\mathbf{z}$ were defined in Theorem 3.13. Here, notice by definitions that the zeroth homology of the boundary forms $H_0(\partial Y; M) \cong \oplus_{i=1}^m \text{Coker}(1 - e_{z_i})$. Then, the map $\iota_* : H_0(\partial Y; M) \rightarrow H_0(Y; M)$ is identified with the map λ from the definition (10). Hence, in the Ext^1 terms, we have an isomorphism between $\text{Ext}^1(\text{Im}(\iota^*), A)$ and the cokernel of χ^* mentioned in Theorem 3.13. In a similar fashion, we can identify the kernel of the map $\text{Ext}^1(\text{Im}(\iota^*), A) \rightarrow \text{Ext}^1(H^1(Y; M), A)$ with the kernel of χ^* . In particular, the cokernel of Υ'_ψ is zero if and only if the kernel of χ^* is zero. Hence Υ' is unimodular if and only if χ^* is injective. This completes the proof. $\square$

Changing the subject, we will show Theorem 4.1.

Proof of Theorem 4.1. We start by setting up notation. Let $M = H^1(\Sigma_g; \mathbb{Q})$. Let $E_\mathbf{z}$ be the total space of ξ as a closed 4-manifold with a C^∞ -map $\pi : E_\mathbf{z} \rightarrow S^2$, and let $E_\mathbf{z}^{\text{reg}}$ be the complimentary space $\pi^{-1}(Y) \subset E_\mathbf{z}$. Notice that, from the definition of Lefschetz fibrations, the restriction of $\pi : (E_\mathbf{z}^{\text{reg}}, \partial E_\mathbf{z}^{\text{reg}}) \rightarrow (Y, \partial Y)$ is a relative Σ_g -fibration.

Next, we will compute the rank of $\text{Ker}(\Gamma_\mathbf{z})$. Recalling the isomorphism $\mathcal{H}^1(Y, \partial Y; M) \cong \text{Ker}(\Gamma_\mathbf{z}) \oplus M$ in Proposition 3.10, we will focus on the Leary-Serre spectral sequence of π , which is written as

$$E_2^{p,q} \cong \mathcal{H}^p(Y, \partial Y; H^q(\Sigma_g; \mathbb{Q})) \implies E_\infty \cong H^*(E_\mathbf{z}^{\text{reg}}, \partial E_\mathbf{z}^{\text{reg}}; \mathbb{Q}).$$

Here we remark that if $q = 1$, the coefficients in the E_2 -term arise from the symplectic representation on $M = H^1(\Sigma_g; \mathbb{Q})$. Then, one can easily notice $E_2^{0,*} \cong 0$, because

$$\mathcal{H}^0(Y, \partial Y; H^q(\Sigma_g; \mathbb{Q})) \cong \mathcal{H}_2(Y; H^q(\Sigma_g; \mathbb{Q})) \cong 0$$

by duality. Therefore, the spectral sequence collapses at the E_2 -page, leading to isomorphisms

$$H^2(E_{\mathbf{z}}^{\text{reg}}, \partial E_{\mathbf{z}}^{\text{reg}}; \mathbb{Q}) \cong \mathcal{H}^1(Y, \partial Y; M) \oplus \mathbb{Q}, \quad (16)$$

$$H^3(E_{\mathbf{z}}^{\text{reg}}, \partial E_{\mathbf{z}}^{\text{reg}}; \mathbb{Q}) \cong \mathbb{Q}^{m-1} \oplus (M_{\pi_1(Y)}),$$

$$H^4(E_{\mathbf{z}}^{\text{reg}}, \partial E_{\mathbf{z}}^{\text{reg}}; \mathbb{Q}) \cong \mathcal{H}^2(Y, \partial Y; H^2(\Sigma_g; \mathbb{Q})) \cong \mathcal{H}_0(Y; H^2(\Sigma_g; \mathbb{Q})) \cong \mathbb{Q}.$$

Thus, for the attempt to compute $\mathcal{H}^1(Y, \partial Y; M)$, we shall study the rational cohomology $H^*(E_{\mathbf{z}}^{\text{reg}}, \partial E_{\mathbf{z}}^{\text{reg}}; \mathbb{Q})$. Noticing the excision axiom

$$H^*(E_{\mathbf{z}}^{\text{reg}}, \partial E_{\mathbf{z}}^{\text{reg}}; \mathbb{Q}) \cong H^*(E_{\mathbf{z}}, \pi^{-1}(D_1 \sqcup \cdots \sqcup D_m); \mathbb{Q}), \quad (17)$$

and, denoting the union $\bigsqcup_{i=1}^m \pi^{-1}(D_i)$ by W , we now consider the long exact sequence

$$0 \longrightarrow H^1(E_{\mathbf{z}}^{\text{reg}}) \longrightarrow H^1(W) \longrightarrow H^2(E_{\mathbf{z}}^{\text{reg}}, W) \longrightarrow \cdots \longrightarrow H^3(E_{\mathbf{z}}^{\text{reg}}) \longrightarrow H^3(W) \longrightarrow 0, \quad (18)$$

where the coefficients are trivial and rational. It is worth noticing the isomorphisms

$$H^1(W) \cong \mathbb{Q}^{2gm-m_{\text{ns}}}, \quad H^2(W) \cong \mathbb{Q}^{m-m_{\text{ns}}}$$

which immediately follow from Lemma 6.2 below. Therefore, with notation $H^i(E_{\mathbf{z}}) \cong \mathbb{Q}^{b_i}$, the exact sequence (17) reduces to

$$0 \rightarrow \mathbb{Q}^{b_1} \rightarrow \mathbb{Q}^{2gm-m_{\text{ns}}} \rightarrow H^2(E_{\mathbf{z}}^{\text{reg}}, \partial E_{\mathbf{z}}^{\text{reg}}) \rightarrow \mathbb{Q}^{b_2} \rightarrow \mathbb{Q}^{m-m_{\text{ns}}} \rightarrow \mathbb{Q}^{m-1} \oplus M_{\pi_1(Y)} \rightarrow \mathbb{Q}^{b_1} \rightarrow 0.$$

Recalling the Euler characteristic $\chi(E_{\mathbf{z}}) = m - 4g + 4$, we readily have $b_2 = m - 4g + 2 + 2b_1$. Moreover, noting $M_{\pi_1(Y)} \cong \mathbb{Q}^{b_1}$ by Lemma 6.2 again, the exact sequence therefore leads to

$$\dim(H^2(E_{\mathbf{z}}^{\text{reg}}, \partial E_{\mathbf{z}}^{\text{reg}})) = 2gm + b_1 + 1. \quad (19)$$

Hence, by combining the assumption $\mathcal{H}^1(Y, \partial Y; M) \cong \text{Ker}(\Gamma_{\mathbf{z}}) \oplus M$ with (16), the rank of $\text{Ker}(\Gamma_{\mathbf{z}})$ turns out to be $2g(m-1) + b_1$ as desired.

Next, we will compute the rank of the quotient free $\mathbb{Z}$ -module $\underline{M}_{\mathbf{z}}$ defined in (9). According to the proof of Theorem 3.13, this module is isomorphic to the cokernel $\text{Coker}(\delta_*)$. Notice $\text{rk}(\mathcal{H}^1(Y, \partial Y; M)) = 2gm + b_1$ from (19). Furthermore, noticing $H^0(\partial Y; M) \cong \bigoplus_i^m \text{Ker}(1 - e_{z_i})$ by definition, the rank is $2gm - m_{\text{ns}}$, since the matrix presentation of the symplectic representation implies that, if $z \in \mathcal{D}_g^{\text{ns}}$, the rank of $1 - e_z$ is 1; otherwise the rank of $1 - e_z$ is zero. Finally, noting $\text{rk} \mathcal{H}^0(Y; M) = \text{rk} H_1(\Sigma_g; \mathbb{Z})_{\pi_1(Y)} = b_1$ from Lemma 6.2, the sequence (13) implies the desired equality $\text{rk} \underline{M}_{\mathbf{z}} = \text{rk} \underline{M}_{\mathbf{z}} = m_{\text{ns}} - 2g + 2b_1$.

Finally, we will analyze the signature. Proposition 3.11 says that $\mathcal{Q}_{\psi}$ is symmetric, and that the signature of $\mathcal{Q}_{\psi}$ is equal to that of the cup product form on $\mathcal{H}^1(Y, \partial Y; M)$. Furthermore, regarding the symplectic form ω as the ring structure on $H^1(\Sigma_g)$, one can verify that the (relative) Leary-Serre spectral sequence of π preserves the cup product (see [McC, Section 5.3] or [Mey]). Hence the (relative) signature on $\mathcal{H}^1(Y, \partial Y; M)$ is

equal to that of $E_{\mathbf{z}}^{\text{reg}}$. Moreover $\sigma(E_{\mathbf{z}}^{\text{reg}}) = \sigma(E_{\mathbf{z}}) + m - m_{\text{ns}}$ is known as a result of Novikov-Wall additivity formula (see [Ozb, EN]). In summary, the signature of $\mathcal{Q}_{\psi}$ is equal to $\sigma(E_{\mathbf{z}}) + m - m_{\text{ns}}$ as claimed. $\square$

Lemma 6.2. *Let $D_i \subset S^2$ be the i -th open disk explained above. Then*

$$H_1(\pi^{-1}D_i; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{2g-1}, & \text{if } \gamma_i \in \mathcal{D}_g^{\text{ns}}, \\ \mathbb{Z}^{2g}, & \text{otherwise.} \end{cases} \quad H_2(\pi^{-1}D_i; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } \gamma_i \in \mathcal{D}_g^{\text{ns}}, \\ \mathbb{Z}^2, & \text{otherwise.} \end{cases}$$

Further, $H_n(\pi^{-1}D_i; \mathbb{Z})$ vanishes for $n \geq 3$.

In addition, the coinvariant $H_1(\Sigma_g; \mathbb{Z})_{\pi_1(Y)}$ is isomorphic to the integral first homology of $E_{\mathbf{z}}$.

Proof. The former claim follows from observing the 2-handle attaching associated with the singular value b_i (see [Ozb, Section 1] for details). In addition, it can be easily seen that a Mayer-Vietoris argument deduces the latter part on the homology $H_1(E_{\mathbf{z}}; \mathbb{Z})$. $\square$

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Appendix A. The fundamental quandle of the 2-sphere with m -points

We will show that the Hopf fibration $\mu : S^3 \rightarrow S^2$ yields a quandle isomorphism between some fundamental quandles (Proposition A.3). This discussion is analogous to the Dirac belt trick in physical understanding.

We begin by reviewing quandles and giving examples. A *quandle* [Joy] is a set, X , is a non-empty set with a binary operation $(x, y) \rightarrow x \triangleleft y$ such that, for any $x, y, z \in X$, the equalities $x \triangleleft x = x$ and $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ hold and there exists uniquely $w \in X$ satisfying $w \triangleleft y = x$.

Example A.1 (Conjugacy quandle). Every union of some conjugacy classes of a group G is a quandle with the conjugacy operation $x \triangleleft y := y^{-1}xy$ for any $x, y \in G$. For example, the subsets $\mathcal{D}_g, \mathcal{D}_g^{\text{ns}}$ of $\mathcal{M}_g$ in Example 3.4 are regarded as conjugacy quandles.

Example A.2 (Fundamental quandle [Joy]). Let M be a connected oriented C^∞ -manifold without boundaries, and $N \subset M$ a submanifold of codimension 2. For simplicity, assume the case where a tubular neighborhood of $N \subset M$ is diffeomorphic to $D^2 \times N$, where D^2 is the 2-disk. Fix two points $p_0 \in M \setminus N$ and $d_0 \in \partial D^2$. We define $Q(M, N)$ to be the set of homotopy classes of all pairs $(D^2 \times \{y\}, \alpha)$, where y runs over N and α is a path in $M \setminus N$ starting from p_0 and ending at d_0 (see, e.g., Figure 4). The set $Q(M, N)$ carries a quandle structure, called *fundamental quandle*, with an operation defined by

$$[(D_1 \times \{y_1\}, \alpha_1)] \triangleleft [(D_2 \times \{y_2\}, \alpha_2)] = [(D_1 \times \{y_1\}, \alpha_1 \cdot_{p_0} \alpha_2^{-1} \cdot_{d_0} \overline{\partial D_2} \cdot_{d_0} \alpha_2)].$$

Here the symbol $\overline{\partial D}$ is the path obtained from the boundary ∂D by cutting at d_0 , and the symbol $\cdot_y$ means the connection of two paths at a point y . See [Joy, Section 12–15] for details.

As examples, let $\overline{\mathbf{m}}$ be the set of m -points $\{b_1, \dots, b_m\}$ in S^2 with the same orientations. Then the Hopf fibration induces a quandle homomorphism $Q(S^3, T_{m,m}) \rightarrow Q(S^2, \overline{\mathbf{m}})$. Moreover

Proposition A.3. *The Hopf fibration μ induces a quandle isomorphism*

$$Q(S^3, T_{m,m}) \cong Q(S^2, \overline{\mathbf{m}}).$$

To show this, we consider the diagram of $T_{m,m}$ in the left hand side of Figure 3. Then, following from Wirtinger presentation for quandles (see [Joy, Section 15]), the fundamental quandle $Q(S^3, T_{m,m})$ is presented by

$$\text{generators :} \quad a_i \quad (i \in \mathbb{Z}),$$

$$\text{relations :} \quad (\cdots (a_i \triangleleft a_{i+1}) \triangleleft \cdots) \triangleleft a_{i+m-1} = a_i, \quad a_i = a_{m+i} \quad (i \in \mathbb{Z}). \quad (20)$$

Here, for $1 \leq i \leq m$, the generator a_i is represented by the path in the left of Figure 3. Then the image $c_i := \mu(a_i) \in Q(S^2, \overline{\mathbf{m}})$ is illustrated as the path shown in the right of Figure 3.

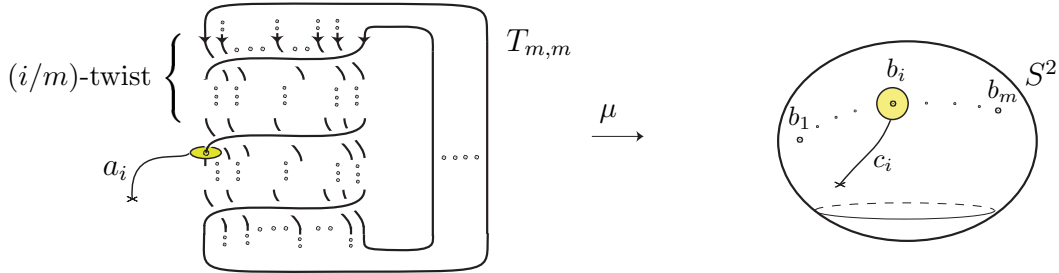
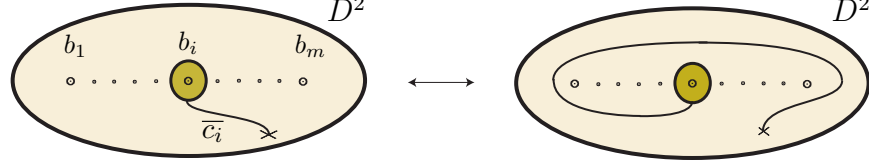


FIGURE 3. The quandle homomorphism induced from the Hopf fibration μ .

Proof of Proposition A.3. We will examine the image $Q(S^2, \overline{\mathbf{m}})$ for more details. Choose a disk D^2 in S^2 containing the points $\overline{\mathbf{m}}$. The inclusion $D^2 \subset S^2$ induces a quandle epimorphism $\kappa : Q(D^2, \overline{\mathbf{m}}) \rightarrow Q(S^2, \overline{\mathbf{m}})$ by definitions. Consider the element $\overline{c}_i \in Q(D^2, \overline{\mathbf{m}})$ drawn in the left of Figure 4. By van-Kampen theorem for the fundamental quandles (see [Joy, Theorem 13.1]), $Q(D^2, \overline{\mathbf{m}})$ is the free quandle generated by $\overline{c}_i$ (shown by induction on m), and the kernel of κ is generated by the equivalences in Figure 4. The kernel corresponds to the relation in (20); thus the presentations of $Q(S^3, T_{m,m})$ and $Q(S^2, \overline{\mathbf{m}})$ imply the desired bijectivity of μ . $\square$


 FIGURE 4. The relation in $Q(D^2, \mathbf{m})$.

Appendix B. Comparison with Meyer 2-cocycles and signature

This appendix compares Theorem 4.1 and some results of Meyer [Mey], in particular, a group 2-cocycle of the $\mathcal{M}_g$.

We start a brief review of some results of Meyer [Mey] (See also [EN, Appendix A]). Identifying the homology $M := H^1(\Sigma_g; \mathbb{R})$ with $\mathbb{R}^{2g}$, let $\mathcal{M}_g$ act on this $\mathbb{R}^{2g}$ as the symplectic representation. For arbitrary two elements $A, B \in \mathcal{M}_g$, we consider the subspace

$$V_{A,B} := \{ (x, y) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} \mid x \cdot (1 - A) + y \cdot (B - 1) = 0 \}$$

of the real vector space $\mathbb{R}^{2g} \times \mathbb{R}^{2g}$. Recalling the symplectic form $\omega : (\mathbb{R}^{2g})^2 \rightarrow \mathbb{R}$, let us define a bilinear form $\langle \cdot, \cdot \rangle_{A,B} : (V_{A,B})^2 \rightarrow \mathbb{R}$ by setting

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{A,B} := \omega(x_1 + y_1, y_2 \cdot (1 - B)).$$

We easily see this form $\langle \cdot, \cdot \rangle_{A,B}$ is symmetric; hence, we can define a map

$$c_{\mathcal{M}_g} : \mathcal{M}_g \times \mathcal{M}_g \longrightarrow \mathbb{Z}; \quad (A, B) \longmapsto \text{Sign}(V_{A,B}, \langle \cdot, \cdot \rangle_{A,B}).$$

Then Meyer showed that $c_{\mathcal{M}_g}$ is a normalized group 2-cocycle of $\mathcal{M}_g$.

We now explain a comparison with the bilinear form $\mathcal{Q}_\omega$ in Definition 3.1. To this aim,

Proposition B.1. *Let $G = Z = \mathcal{M}_g$, and let $M := H^1(\Sigma_g; \mathbb{R})$. For $A, B \in \mathcal{M}_g$, consider a 3-tuple $\mathbf{z} = (A, B, B^{-1}A^{-1}) \in Z^3$. Then the splitting surjection $M^3 \rightarrow M^2$ which sends (a, b, c) to $(a - b, b - c)$ gives rise to an isomorphism of bilinear forms*

$$(\text{Ker}(\Gamma_{\mathbf{z}}), \mathcal{Q}_\omega) \cong (V_{A,B}, \langle \cdot, \cdot \rangle_{A,B}) \oplus (M, 0).$$

Proof. Straightforward from the definitions. □

Here we give a geometric interpretation of the isomorphism. Let Y be the 2-sphere with three 2-disks removed, and consider the action $\pi_1(Y) \curvearrowright H^1(\Sigma_g; \mathbb{R}) = M$ similar to the proof of Theorem 4.1 (see Section 6). Meyer [Mey] showed, using the simplicial cohomology, that the space $V_{A,B}$ is isomorphic to $H^1(Y, \partial Y; M)$, and that the bilinear form $\langle \cdot, \cdot \rangle_{A,B}$ coincides with the cup product in the E_2 -term that we discussed in the proof with $m = 3$. Hence Proposition 3.10 readily ensures the desired isomorphism.

Furthermore, we will compare our formulation of signature (Theorem 4.1) with results [EN, Ozb]. According to Hopf theorem in group cohomology, Endo and Nagami showed

$$\text{Sign}(E) - m + m_{\text{ns}} = \sum_{i=2}^m c_{\mathcal{M}_g}(z_1 \cdots z_{i-1}, z_i),$$

for any Lefschetz fibration $E \rightarrow S^2$ associated with any tuple $(z_1, \dots, z_m)$; or see [Ozb, Theorem 3]. However, in general it is hard to compute signatures in m -times. Although there are some computations of the signatures, our formula in Theorem 4.1 has something applicable to computer program, in some sense. In computer experience, for $g \leq 9$, we can easily compute signatures of many Lefschetz fibrations. Actually, the signatures in Table 1 can be also computed in this way.

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DEPARTMENT OF MATHEMATICS, TOKYO INSTITUTE OF TECHNOLOGY, JAPAN
 Email address: nosaka@math.titech.ac.jp