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Complex G_2 and Associative Grassmannian

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Abstract. We obtain defining equations of the smooth equivariant compactification of the Grassmannian of the complex associative 3-planes in $\mathbb{C}^7$, which is the parametrizing variety of all quaternionic subalgebras of the algebra of complex octonions $\mathbb{O} \cong \mathbb{C}^8$. By studying the torus fixed points, we compute the Poincaré polynomial of the compactification.

1. Introduction

The exceptional Lie group G_2, similar to any other Lie group, has different guises depending on the underlying field; it has two real forms and a complex form. All of these incarnations have descriptions as a stabilizer group. We denote these three forms by G_2^+, G_2^-, and by G_2, respectively, where first two are real forms. The first of these is compact, connected, simple, simply connected, of (real) dimension 14, the second group is non-compact, connected, simple, of (real) dimension 14. The complex form of G_2 is non-compact, connected, simple, simply connected, of (complex) dimension 14.

Let V denote either $\mathbb{R}^7$ or $\mathbb{C}^7$, and $e_1, \dots, e_7$ be its standard basis, and $x_1 = e_1^*, \dots, x_7 = e_7^*$ denote the dual basis. We write $GL_7(\mathbb{R})$ or $GL_7(\mathbb{C})$ instead of $GL(V)$ when there is no danger of confusion. If the underlying field does not play a role, then we write simply GL_7. Consider the fourth fundamental representation $\bigwedge^4 V$ of GL_7, which is irreducible. Note that $\bigwedge^3 V^*$ is naturally isomorphic, as a GL_7 representation, to $\bigwedge^4 V$, where V^* denotes the dual space. On the other hand, $\bigwedge^4 V$ and $\bigwedge^3 V$ are dual representations, hence $\bigwedge^4 V \simeq (\bigwedge^3 V)^* \simeq \bigwedge^3 V^*$. Assuming that i, j and k are distinct numbers from $\{1, \dots, 7\}$ let us use the shorthand e^{ijk} to denote the wedge product $x_i \wedge x_j \wedge x_k$. Following [9], we set:

$$\phi^+ := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$
$$\phi^- := e^{123} - e^{145} - e^{167} - e^{246} + e^{257} + e^{347} + e^{356}.$$

It turns out that G_2^- is the stabilizer of ϕ^- in $GL_7(\mathbb{R})$, G_2^+ is the stabilizer of ϕ^+ in $GL_7(\mathbb{R})$, and finally, G_2 is the stabilizer of $\phi := \phi^+$ in $GL_7(\mathbb{C})$. In fact, the orbit $GL_7 \cdot \phi$ is Zariski open in $\bigwedge^3 V^*$; over real numbers this orbit splits into two with stabilizers G_2^+ and G_2^-. A pleasant consequence of this openness is that $GL_7(\mathbb{C})$ has only finitely many orbits in $\bigwedge^3 V^*$.

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Let $\mathbb{O}_k$ denote the octonion algebra over a field k. To ease our notation, when $k = \mathbb{C}$ we write $\mathbb{O}$. Following [21], we view $\mathbb{O}_k$ as an 8-dimensional composition algebra; it is non-associative, unital (with unity $e \in \mathbb{O}_k$), and it is endowed with a norm $N : \mathbb{O}_k \to k$ such that $N(xy) = N(x)N(y)$ for all $x, y \in \mathbb{O}_k$.

Composition algebras exist only in dimensions 1,2,4 and 8. Moreover, they are uniquely determined (up to isotopy) by their quadratic form. When $k = \mathbb{C}$, there is a unique isomorphism class of quadratic forms and any member of this class is *isotropic*, that is to say the norm of the composition algebra vanishes on a nonzero element. When $k = \mathbb{R}$ there are essentially two isomorphism classes of quadratic forms, first of which gives isotropic composition algebras, and the second class gives composition algebras with positive-definite quadratic forms.

For any real octonion algebra $\mathbb{O}_\mathbb{R}$,the tensor product $\mathbb{C} \otimes_\mathbb{R} \mathbb{O}_\mathbb{R}$ is isomorphic to $\mathbb{O}$ (by the uniqueness of the octonion algebra over $\mathbb{C}$). In literature $\mathbb{C} \otimes_\mathbb{R} \mathbb{O}_\mathbb{R} = \mathbb{O}$ is known as the "complex bioctonion algebra". We denote by $G_\mathbb{R}$ the group of algebra automorphisms of $\mathbb{O}_\mathbb{R}$, and denote by G the group of algebra automorphisms of $\mathbb{O}$. By Proposition 2.4.6 of [21] we know that the group of $\mathbb{R}$-rational points of G is equal to $G_\mathbb{R}$. Of course(!), $G_\mathbb{R}$ is either G_2^+ or G_2^- depending on which $\mathbb{O}_k$ we start with. Meanwhile, G is equal to G_2. See Theorem 2.3.3, [21].

Let N_1 denote the restriction of the norm N to $e_0^\perp$, the (7 dimensional) orthogonal complement of the identity vector $e_0 \in \mathbb{O}_k$. For notational ease we are going to denote $e_0^\perp$ in $\mathbb{O}_k$ by $\mathbb{I}_k$ and denote $e_0^\perp$ in $\mathbb{O}$ simply by $\mathbb{I}$. The automorphism groups $G_\mathbb{R}$ and G preserve the norms on their respective octonion algebras, and obviously any automorphism maps identity to identity. Thus, we know that $G_\mathbb{R}$ is contained in $SO(\mathbb{I}_k)$ and G is contained in $SO(\mathbb{I})$. Here, $SO(W)$ denotes the group of orthogonal transformations of determinant 1 on a vector space W. If there is no danger of confusion, we write SO_n ($n = \dim W$) in place of $SO(W)$.

A *quaternion algebra*, $\mathbb{D}_k$ over a field k is a 4 dimensional composition algebra over k. As it is mentioned earlier, there are essentially (up to isomorphism) two quaternion algebras over $k = \mathbb{R}$ and there is a unique quaternion algebra over $k = \mathbb{C}$, which we denote simply by $\mathbb{D}$, and call it the split quaternion algebra. (More generally, any composition algebra over $\mathbb{C}$ is called split.) Any quaternion algebra over $\mathbb{C}$ is isomorphic to the algebra of 2×2 matrices over $\mathbb{C}$ with determinant as its norm.

The split octonion algebra $\mathbb{O}$ has a description which is built on $\mathbb{D}$ by the *Cayley-Dickson doubling process*: As a vector space, $\mathbb{O}$ is equal to $\mathbb{D} \oplus \mathbb{D}$ and its multiplicative structure is

$$(a, b)(c, d) = (ac + \bar{d}b, da + b\bar{c}), \quad \text{where } a, b, c, d \in \mathbb{D}, \tag{1}$$

and its norm is defined by $N((a, b)) = N(a) - N(b) = \det a - \det b$.

Let $\mathrm{Gr}_k(3, \mathbb{I}_k)$ denote the grassmannian of 3 dimensional subspaces in $\mathbb{I}_k$. Let $\mathbb{D}_k \subset \mathbb{O}_k$ be the quaternion subalgebra generated by the first four generators $e_1 = e, e_2, e_3, e_4$ of $\mathbb{O}_k$. As usual, if $k = \mathbb{C}$, then we set $\mathbb{D} = \mathbb{D}_k$. Let us denote by W_0 the intersection $\mathbb{D}_k \cap \mathbb{I}_k$, the span of e_2, e_3, e_4 in $\mathbb{I}_k$. By Corollary 2.2.4 [21], when $k = \mathbb{C}$, we know that G_2 acts transitively on the set of all quaternion subalgebras of $\mathbb{O}$. Since an algebra automorphism fixes the identity, under this action, W_0 is mapped to another 3-plane of the form $W' = D \cap \mathbb{I}$, for some other split quaternion subalgebra $D' \subset \mathbb{O}$. Thus, the G_2-action on $\mathrm{Gr}(3, \mathbb{I})$ has at least two orbits, one of which is $\mathrm{G}_2 \cdot W_0$ and there is at least one other orbit of the form $\mathrm{G}_2 \cdot W$ for some 3-plane W in $\mathbb{I}$. The goal of our paper is to obtain an understanding of the geometry of the Zariski closure of the orbit $\mathrm{G}_2 \cdot W_0$, which is the complexified version of the real associative Grassmannian $\mathrm{G}_2^+/\mathrm{SO}_4(\mathbb{R})$, where the deformation theory of [2] takes place. We achieve our goal by using techniques from calibrated geometries.

After we obtained some of our main results we learned from Michel Brion about the work of Alessandro Ruzzi [19, 20] on the classification of symmetric varieties of Picard number 1. Our work fits nicely with this classification scheme, so we will briefly mention the relevant results of Ruzzi.

Let G be a connected semisimple group defined over $\mathbb{C}$, θ be an involutory automorphism of G. We denote by G^θ the fixed locus of θ. Let H be any subgroup that is squeezed between $(G^\theta)^0$ and the normalizer subgroup $N_G(G^\theta)$. Here, the superscript 0 indicates the connected component of the identity element. The quotient varieties of the form G/H are called symmetric varieties. In his 2011 paper, Ruzzi classified all symmetric varieties of Picard number 1 and in [19, Theorem 2a)], he showed that the smooth equivariant completion with Picard number 1 of the symmetric variety $\mathrm{G}_2/\mathrm{SL}_2 \times \mathrm{SL}_2$ is the intersection of the grassmannian $\mathrm{Gr}(3, \mathbb{I})$ with a 27 dimensional G_2-stable linear space in $\mathbb{P}(\bigwedge^3 \mathbb{I})$. We will denote this equivariant completion by X_{min} and call it the (complex) *associative grassmannian*. Note that over $\mathbb{C}$, $\mathrm{SL}_2 \times \mathrm{SL}_2$ is identified with the special orthogonal group SO_4. Although Ruzzi has first showed that X_{min} is the unique smooth equivariant completion of $\mathrm{G}_2/\mathrm{SL}_2 \times \mathrm{SL}_2$ with Picard number 1, the question of finding its defining ideal as well as the computation of its Poincaré polynomial remained unanswered. In a sense our article finishes this program. More precisely, we prove the following results:

Theorem 1.1. As a subvariety of $\mathrm{Gr}(3, \mathbb{I})$, the compactification X_{min} of $\mathrm{G}_2/\mathrm{SO}_4$ is defined by the vanishing of the following seven linear forms in the Plücker coordinates of $\mathrm{Gr}(3, \mathbb{I})$:

(1) $p_{247} - p_{256} - p_{346} - p_{357}$,
(2) $p_{156} - p_{147} + p_{345} - p_{367}$,
(3) $-p_{245} + p_{267} + p_{146} + p_{157}$,
(4) $p_{567} + p_{127} - p_{136} + p_{235}$,
(5) $-p_{126} - p_{467} - p_{137} - p_{234}$,

(6) $p_{457} + p_{125} + p_{134} - p_{237}$,
(7) $p_{135} - p_{124} - p_{456} + p_{236}$.

Theorem 1.2. The Poincaré polynomial of X_{min} is

$$P_{X_{min}}(t^{1/2}) = 1 + t + 2t^2 + 2t^3 + 3t^4 + 2t^5 + 2t^6 + t^7 + t^8.$$

To prove these results we analyze the natural action of the maximal torus of G_2 on X_{min}. In particular, we determine the fixed points of the torus action and compute the Poincaré polynomial of X_{min} by using the Białynicki-Birula decomposition.

Let us emphasize once more that none of our results rely on Ruzzi's work but we use the techniques from calibrated geometries. In fact, over the field of real numbers, the analogous symmetric variety ASS $:= G_2^+/SO_4(\mathbb{R})$ is already compact, and its geometry is well understood; its defining equations are also given by certain linear equations arising from the calibration form. In this article, we essentially lifted these observations to the complex setting. Finally, let us mention that the complete description of the ring structure of the $H^*(\text{ASS}, \mathbb{Z})$ is described in [4].

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2. Grassmann of 3-planes

We call a 3-plane $W \in \text{Gr}(3, \mathbb{I}_k)$ associative if $W = D \cap \mathbb{I}_k$, where D is a quaternion subalgebra of $\mathbb{O}_k$. For a subset $S \subset \mathbb{O}_k$, we denote by $A(S)$ the subalgebra of $\mathbb{O}_k$ that is generated by S. Let $W \in \text{Gr}_k(3, \mathbb{I}_k)$ be a 3-plane and let $u_1, u_2, u_3 \in W$ be a basis. Thus, the vector space dimension of $A(W)$ is either $\dim A(W) = 4$ or $\dim A(W) = 8$. In latter case, obviously, $A(W) = \mathbb{O}_k$. In the former case, $A(W)$ is an associative subalgebra (as follows from Proposition 1.5.2 of [21]) but it does not need to be a composition subalgebra. Our orbit $G_2 \cdot W_0$ in $\text{Gr}_k(3, \mathbb{I}_k)$ contains the set of 3-planes W such that $A(W)$ is a 4 dimensional composition subalgebra, which we state in our next lemma:

Lemma 2.1. If $W \subset \mathbb{I}$ is a 3-plane that is in the G_2-orbit of W_0, then $A(W)$ is a quaternion subalgebra of $\mathbb{O}$. Moreover, the stabilizer subgroup of any such W is isomorphic to SO_4, the special orthogonal group of 4 by 4 matrices.

Proof. The subalgebra generated by W_0 is the quaternion algebra $\mathbb{D}$. Since G_2 acts transitively on the set of quaternion subalgebras (Corollary 2.2.4 [21]), the proof of our first assertion follows. To prove our second claim it is enough to prove it for W_0, the "origin of the orbit" since the stabilizer subgroups of other points are isomorphic to that of W_0 by conjugation.

Let g be an element from G_2 such that $g \cdot W_0 = W_0$. Then g acts on the orthogonal complement $\mathbb{D}^\perp$. Recall that G_2 is contained in $SO(\mathbb{I}_k)$. Therefore, on one hand we have an injection $\iota: \mathrm{Stab}_{\mathrm{G}_2}(W_0) \hookrightarrow SO(\mathbb{D}^\perp) \simeq \mathrm{SO}_4$. On the other hand, we know that the elements of $SO(\mathbb{D}^\perp)$ are completely determined by how they act on the part of the basis e_5, e_6, e_7, e_8 of $\mathbb{O}$. Indeed, we see this from Figure 1, which gives us the multiplicative structure of $\mathbb{O}$.

	e	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e	e	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_2	e_2	$-e$	e_4	$-e_3$	e_6	$-e_5$	$-e_8$	e_7
e_3	e_3	$-e_4$	$-e$	e_2	e_7	e_8	$-e_5$	$-e_6$
e_4	e_4	$-e_3$	$-e_2$	$-e$	e_8	$-e_7$	e_6	$-e_5$
e_5	e_5	$-e_6$	$-e_7$	$-e_8$	e	$-e_2$	$-e_3$	$-e_4$
e_6	e_6	e_5	$-e_8$	e_7	e_2	e	e_4	$-e_3$
e_7	e_7	e_8	e_5	$-e_6$	e_3	$-e_4$	e	e_2
e_8	e_8	$-e_7$	e_6	e_5	e_4	e_3	$-e_2$	e

FIGURE 1. Multiplication table for split octonions.

The multiplication table of e_5, e_6, e_7, e_8 includes e_2, e_3, e_4 and e, therefore, the action of g on W is uniquely determined by the action of g on e_5, e_6, e_7, e_8. It follows that ι is surjective as well, hence it is an isomorphism. $\square$

Remark 2.2. The element-wise stabilizer of $\mathbb{D}$ in G is isomorphic to SL_2. (See Proposition 2.2.1 [21]). Heuristically, this follows from the fact that $\mathbb{O} = \mathbb{D} \oplus \mathbb{D}$, and that $(\mathbb{D}, N) = (\mathrm{Mat}_2, \det)$.

Since $\mathbb{O} = \mathbb{D} \oplus \mathbb{D}$ and $\mathbb{D} = \mathrm{Mat}_2$, we take $\{(e_{ij}, 0) : i, j = 1, 2\} \cup \{(0, e_{ij}) : i, j = 1, 2\}$ as a basis for $\mathbb{O}$. Here, e_{ij} is the 2×2 matrix with 1 at the i, jth position and 0's everywhere else. Recall that $\mathbb{I}$ is the orthogonal complement of the identity $e = (e_{11} + e_{22}, 0)$ of $\mathbb{O}$. A straightforward computation shows that $(x, y) \in \mathbb{O}$ is in $\mathbb{I}$ if and only if the trace of x is 0. Thus, we write $\mathbb{I} = \mathfrak{sl}_2 \oplus \mathrm{Mat}_2$ (we are going to make use of Lie algebra structure on $\mathfrak{sl}_2$ in the sequel).

Let $W \in \mathrm{Gr}(3, \mathbb{I})$ be a 3-plane in $\mathbb{I}$ and let $\{u_1, u_2, u_3\}$ be a basis for W. The map $P : \mathrm{Gr}(3, \mathbb{I}) \to \mathbb{P}(\bigwedge^3 \mathbb{I})$ defined by $P(W) = [u_1 \wedge u_2 \wedge u_3]$ is the Plücker embedding of $\mathrm{Gr}(3, \mathbb{I})$ into the 34 dimensional projective space $\mathbb{P}(\bigwedge^3 \mathbb{I})$. Note that $\mathrm{GL}(\mathbb{I})$ acts on both of the varieties $\mathrm{Gr}(3, \mathbb{I})$ and $\bigwedge^3 \mathbb{I}$ via its natural action on $\mathbb{I}$. Note also that the Plücker embedding is equivariant with respect to these actions. In particular, it is equivariant with respect to the subgroup G_2.

We make the identifications

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{i} \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \mathbf{j} \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \mathbf{k} \leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{2}$$

and take $\{(\mathbf{i},0),(\mathbf{j},0),(\mathbf{k},0),(0,1),(0,\mathbf{i}),(0,\mathbf{j}),(0,\mathbf{k})\}$ as a basis for $\mathbb{I}$. Let W_0 denote the span of $\{(\mathbf{i},0),(\mathbf{j},0),(\mathbf{k},0)\}$ and let W_0^* denote the span of $\{(0,\mathbf{i}),(0,\mathbf{j}),(0,\mathbf{k})\}$. Thus,

$$\mathbb{I} = W_0 \oplus W_0^* \oplus \mathbb{C}. \tag{3}$$

Remark 2.3. It is noted earlier that a copy of $\mathfrak{sl}_2$ sits in $\mathbb{I}$:

$$\mathfrak{sl}_2 = \mathfrak{sl}_2 \oplus 0 \hookrightarrow \mathfrak{sl}_2 \oplus \mathrm{Mat}_2 = \mathbb{I}.$$

This copy of $\mathfrak{sl}_2$ is W_0 as a vector space.

Remark 2.4. A straightforward calculation shows that if $(0,v) \in W_0^*$, then for all $(x,0) \in \mathfrak{sl}_2$, $(x,0)(0,v) = (0,vx)$.

Next, we elaborate on a portion of the discussion from [14], §22.3 and analyze $\bigwedge^3 \mathbb{I}$ more closely. Let U denote $W_0 \oplus W_0^*$ so that we have

$$\bigwedge^3 (W_0 \oplus W_0^* \oplus \mathbb{C}) = \bigoplus_{n=0}^{3} \bigwedge^n U \otimes \bigwedge^{3-n} \mathbb{C} = \bigwedge^3 U \oplus \bigwedge^2 U.$$

Since $\dim W_0 = \dim W_0^* = 3$, we have canonical identifications $W_0 = \bigwedge^2 W_0^*$ and $W_0^* = \bigwedge^2 W_0$. It follows that

$$\begin{aligned} \bigwedge^3 U &= (\mathbb{C} \otimes \mathbb{C}) \oplus (W_0 \otimes \bigwedge^2 W_0^*) \oplus (\bigwedge^2 W_0 \otimes W_0^*) \oplus (\mathbb{C} \otimes \mathbb{C}) \\ &= \mathbb{C} \oplus (W_0 \otimes W_0) \oplus (W_0^* \otimes W_0^*) \oplus \mathbb{C} \end{aligned}$$

and that

$$\begin{aligned} \bigwedge^2 U &= \bigwedge^2 W_0 \otimes \mathbb{C} \oplus W_0 \otimes W_0^* \oplus \mathbb{C} \otimes \bigwedge^2 W_0^* \\ &= W_0^* \oplus (W_0 \otimes W_0^*) \oplus W_0. \end{aligned}$$

Putting all of the above together we see that

$$\begin{aligned} \bigwedge^3 \mathbb{I} &= \mathbb{C} \oplus (W_0 \otimes W_0) \oplus (W_0^* \otimes W_0^*) \oplus \mathbb{C} \oplus W_0^* \oplus (W_0 \otimes W_0^*) \oplus W_0 \\ &= \mathbb{I} \oplus (W_0 \otimes W_0) \oplus (W_0^* \otimes W_0^*) \oplus (W_0 \otimes W_0^*) \oplus \mathbb{C}. \end{aligned}$$

Next, we analyze $\mathrm{Sym}^2\mathbb{I}$ more closely;

$$\begin{aligned}
\mathrm{Sym}^2\mathbb{I} &= \mathrm{Sym}^2(U \oplus \mathbb{C}) \\
&= (\mathrm{Sym}^2 U \otimes \mathbb{C}) \oplus (\mathrm{Sym}^1 U \otimes \mathbb{C}) \oplus (\mathbb{C} \otimes \mathrm{Sym}^2\mathbb{C}) \\
&= \mathrm{Sym}^2 W_0 \oplus (W_0 \otimes W_0^*) \oplus \mathrm{Sym}^2 W_0^* \oplus (W_0 \oplus W_0^*) \oplus \mathbb{C} \\
&= (\mathrm{Sym}^2 W_0 \oplus W_0^*) \oplus (W_0 \otimes W_0^*) \oplus (\mathrm{Sym}^2 W_0^* \oplus W_0) \oplus \mathbb{C} \\
&= (\mathrm{Sym}^2 W_0 \oplus \bigwedge^2 W_0) \oplus (W_0 \otimes W_0^*) \oplus (\mathrm{Sym}^2 W_0^* \oplus \bigwedge^2 W_0^*) \oplus \mathbb{C} \\
&= \mathrm{End}(W_0) \oplus (W_0 \otimes W_0^*) \oplus \mathrm{End}(W_0^*) \oplus \mathbb{C} \\
&\simeq (W_0 \otimes W_0) \oplus (W_0 \otimes W_0^*) \oplus (W_0^* \otimes W_0^*) \oplus \mathbb{C}.
\end{aligned}$$

Remark 2.5. The last term is only an isomorphism since we are using non-canonical identification of W_0 with W_0^*.

Therefore, we see that

$$\bigwedge^3 \mathbb{I} \simeq \mathrm{Sym}^2\mathbb{I} \oplus \mathbb{I}. \tag{4}$$

Furthermore, it is true that $\mathrm{Sym}^2\mathbb{I} = \Gamma_{2,0} \oplus \mathbb{C}$, where

$$\begin{aligned}
\Gamma_{2,0} &= (W_0 \otimes W_0) \oplus (W_0^* \otimes W_0^*) \oplus (W_0 \otimes W_0^*) \\
&= (W_0 \otimes \bigwedge^2 W_0^*) \oplus (\bigwedge^2 W_0 \otimes W_0^*) \oplus (W_0 \otimes W_0^*).
\end{aligned} \tag{5}$$

is an irreducible representation of G_2 with highest weight $2\omega_1$, where ω_1 is the highest weight of the first fundamental representation $\mathbb{I}$ of G_2. (See [14], §22.3.) Once the root system $\Phi = \{\alpha_1, \ldots, \alpha_6, \beta_1, \ldots, \beta_6\}$ is chosen as in [14], §22.2 (pg. 347), we see that $2\omega_1 = \alpha_1 + \alpha_3 + \alpha_4$.

The structure of the representation of G_2 on $\mathbb{I}$ can be spelled out to a finer degree once we linearize the action. Let $\mathfrak{g}_2$ denote the Lie algebra of G_2. It is well known that $\mathfrak{g}_2$ contains a copy of $\mathfrak{g}_0 = \mathfrak{sl}_3$, and moreover, as a representation of $\mathfrak{g}_0$ it has the following decomposition:

$$\mathfrak{g}_2 = \mathfrak{g}_0 \oplus W \oplus W^*,$$

where W is isomorphic to the standard 3 dimensional representation $\mathbb{C}^3$ of $\mathfrak{sl}_3$ (See [14], §22.2.). Furthermore, the unique 7 dimensional irreducible representation V of $\mathfrak{g}_2$ can be identified with $V = W \oplus W^* \oplus \mathbb{C}$ as an $\mathfrak{sl}_3$-module. In our notation, we are going to take W as W_0. Before making this identification we choose a basis for W using the root system $\Phi = \{\alpha_1, \ldots, \beta_{12}\}$.

Let $V_i \subset V$ $(i = 1, \ldots, 6)$ denote the eigenspace (corresponding to the eigenvalue α_i) for the action of the maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}_2$ corresponding to Φ. Let Y_i be the root vector whose eigenvalue is $-\alpha_i = \beta_i$ for $i = 1, \dots, 6$. Arguing as in pg. 354 of [14], we have the basis $e_1 = v_4, e_2 = w_1, e_3 = w_3$ for W, where w_i's are found as follows:

$$\begin{aligned} v_3 &= Y_1(v_4), \\ v_1 &= -Y_2(v_3), \\ u &= Y_1(v_1), \\ w_1 &= \frac{1}{2}Y_1(u), \\ w_3 &= Y_2(w_1), \\ w_4 &= -Y_1(w_3). \end{aligned}$$

The corresponding dual basis elements e_1^*, e_2^*, e_3^* are given by w_4, v_1, v_3, respectively. The upshot of all of these is that we identify W_0 with W in such a way that the basis v_4, w_1, w_3 corresponds (in the given order) to $(\mathbf{i}, 0), (\mathbf{j}, 0), (\mathbf{k}, 0)$, and the basis w_4, v_1, v_3 for W^* corresponds to $(0, \mathbf{i}), (0, \mathbf{j}), (0, \mathbf{k})$. With respect to these identifications, we observe that the highest weight vector in $\mathrm{Sym}^2 V \subset \bigwedge^3 V$ of the highest weight $2\omega_1 = \alpha_1 + \alpha_3 + \alpha_4$ is given by the 3-form $v_1 \wedge v_3 \wedge v_4$, or by $(\mathbf{i}, 0) \wedge (0, \mathbf{j}) \wedge (0, \mathbf{k})$ in the case of $\mathrm{Sym}^2 \mathbb{I} \subset \bigwedge^3 \mathbb{I}$. It is clear that the 3-form $(\mathbf{i}, 0) \wedge (0, \mathbf{j}) \wedge (0, \mathbf{k})$ is actually an element of $W_0 \otimes \bigwedge^2 W_0^* \subset \Gamma_{2,0}$ by (5).

It is straightforward to verify that the octonions $(\mathbf{i}, 0), (0, \mathbf{j})$, and $(0, \mathbf{k})$ generate a quaternion algebra which we denote by $\mathbb{U}$. By Lemma 2.1 we see that the stabilizer subgroup of $\mathbb{U}$ is SO_4. It is well known that the highest weight vector in $\Gamma_{2,0} \subset \mathrm{Sym}^2(\mathbb{I})$ is the direction vector of the line that is stabilized by SO_4.

We view the projectivization $\mathbb{P}(\Gamma_{2,0})$ as a (closed) subvariety of $\mathbb{P}(\bigwedge^3 \mathbb{I})$. The image of $\mathrm{Gr}(3, \mathbb{I})$ intersects $\mathbb{P}(\Gamma_{2,0})$. In fact, by the above discussion we know that the image of the 3-plane $U_0 := \mathbb{U} \cap \mathbb{I} \in \mathrm{Gr}(3, \mathbb{I})$ under Plücker embedding is the SO_4-fixed point $[u_0] \in \mathbb{P}(\Gamma_{2,0})$. On one hand, since it is a G_2-equivariant isomorphism onto its image, the orbit $G_2 \cdot U_0$ in $\mathrm{Gr}(3, \mathbb{I})$ is mapped isomorphically onto $G_2 \cdot [u_0]$ in $\mathbb{P}(\Gamma_{2,0}) \subset \mathbb{P}(\bigwedge^3 \mathbb{I})$. On the other hand, as we are going to see in the sequel, the closure of the orbit $G_2 \cdot U_0$ in $\mathrm{Gr}(3, \mathbb{I})$ is smooth, however, the Zariski closure of the orbit $G_2 \cdot [u_0]$ in $\mathbb{P}(\bigwedge^3 \mathbb{I})$ is not. The latter closure is the smallest "degenerate" compactification of the symmetric variety G_2/SO_4, whereas the former compactification is the smallest, smooth G_2-equivariant compactification.

3. More on Octonions

In this section we collect and improve some known facts about alternating forms on (split) composition algebras.

The multiplicative structure of a quaternion algebra (over $\mathbb{R}$ or $\mathbb{C}$) is always associative (but not commutative). To measure how badly the associativity of multiplication fails in $\mathbb{O}$ one looks at the *associator*, defined by

$$[x, y, z] = (xy)z - x(yz) \text{ for all } x, y, z \in \mathbb{O}_k. \tag{6}$$

It is well known that the associator is an alternating 3-form (see Section 1.4 of [21]).

There are several other related multiplication laws on the imaginary part $\mathbb{I}_k$ of $\mathbb{O}_k$. For example, the "cross-product" is defined by

$$a \times b = \frac{1}{2}(ab - ba) \text{ for all } a, b \in \mathbb{I}_k.$$

Obviously, the cross-product is alternating. The "dot product" is defined by

$$a \cdot b = -\frac{1}{2}(ab + ba) \text{ for all } a, b \in \mathbb{I}_k.$$

It is also obvious that $ab = a \times b - a \cdot b$ for $a, b \in \mathbb{I}_k$. These products are easily extended to $\mathbb{O}_k$. Indeed, any element of $\mathbb{O}_k$ has the form $x = \alpha(1, 0) + a$, where $\alpha \in k$, $a \in \mathbb{I}_k$, and if $y = \beta(1, 0) + b$ is another element from $\mathbb{O}_k$, then

$$xy = (\alpha, a)(\beta, b) = (\alpha\beta - a \cdot b, \alpha a + \beta b + a \times b), \tag{7}$$

where we use the identification $\alpha(1, 0) + a = (\alpha, a)$. In particular, if $x = a$ and $y = b$ are from $\mathbb{I}_k$, then $xy = (-x \cdot y, x \times y)$, hence

$$N(xy) = (x \cdot y)^2 + N(x \times y). \tag{8}$$

Now we focus on $k = \mathbb{C}$ and extend some results from [16] to our setting. First, we re-label the basis $\{e, e_1, \ldots, e_7\}$ for $\mathbb{O}$ so that $\{e_1 = (\mathbf{i}, 0), e_2 = (\mathbf{j}, 0), e_3 = (\mathbf{k}, 0), e_4 = (0, 1), e_5 = (0, \mathbf{i}), e_6 = (0, \mathbf{j}), e_7 = (0, \mathbf{k})\}$ is the standard basis for $\mathbb{I}$. Consider the trilinear form

$$\varphi(x, y, z) = \langle x \times y, z\rangle, \qquad x, y, z \in \mathbb{I}.$$

Lemma 3.1. φ is an alternating 3-form on $\mathbb{I}$.

Proof. Since both cross-product and the inner product $\langle , \rangle$ are bilinear, we see it is enough to check the assertion on the basis $\{e_1, \ldots, e_7\}$. We verified this by using software called Maple. □

Remark 3.2. It follows from (7) that

$$\varphi(x, y, z) = \langle xy, z\rangle \text{ for } x, y, z \in \mathbb{I}. \tag{9}$$

It is not difficult to verify (by using Maple, or by hand) that φ is equal to the 3-form

$$\varphi = e^{123} - e^{145} + e^{167} - e^{246} - e^{257} - e^{347} + e^{356}, \tag{10}$$

where $e^{ijk} = de_i \wedge de_j \wedge de_k$ as before. In particular, we see that from (9) that $\mathrm{G}_2 = \mathrm{Aut}(\mathbb{O})$ stabilizes the form (10).

Definition 3.3. For a 3-plane $W \in \mathrm{Gr}(3, \mathbb{I})$ we define $\varphi(W)$ to be the evaluation of φ on any orthonormal basis $\{x, y, z\}$ of W.

Theorem 3.4. If a 3-plane $W \in \mathrm{Gr}(3, \mathbb{I})$ is associative, then $\varphi(x, y, z) \in \{-1, +1\}$ for any orthonormal basis $\{x, y, z\}$ of W.

Proof. Any two elements x, y of an orthonormal triplet (x, y, z) from $\mathbb{I}$ form a "special $(1,1)$-pair" in the sense of [21], Definition 1.7.4.[1] Since G_2 acts transitively on special $(1,1)$-pairs, and since $((\mathbf{i}, 0), (\mathbf{j}, 0))$ is such, there exists $g \in G_2$ such that $g(x) = (\mathbf{i}, 0)$, $g(y) = (\mathbf{j}, 0)$. In particular, it follows that $g(xy) = (\mathbf{k}, 0)$.

We claim that if x, y, z generates a quaternion algebra, then $g(z) = \pm(\mathbf{k}, 0)$. Indeed, unless xy is a scalar multiple of z, the span in $\mathbb{I}$ of e, x, y, z and xy is 5 dimensional, hence the composition algebra generated by x, y, z is not a quaternion subalgebra. It follows that, if $\{1, x, y, z\}$ is an orthonormal basis for a quaternion subalgebra, then z is a scalar multiple of xy. Since the norm of z is 1, we see that $z = \pm xy$, hence $g(z) = \pm(\mathbf{k}, 0)$. Then

$$\begin{aligned} \langle xy, z \rangle &= \langle g^{-1}((\mathbf{i}, 0)) g^{-1}((\mathbf{j}, 0)), g^{-1}(\pm(\mathbf{k}, 0)) \rangle \\ &= \langle g^{-1}((\mathbf{i}, 0)(\mathbf{j}, 0)), g^{-1}(\pm(\mathbf{k}, 0)) \rangle \\ &= \langle (\mathbf{i}, 0)(\mathbf{j}, 0), \pm(\mathbf{k}, 0) \rangle \\ &= \pm 1. \end{aligned}$$

□

We define the *triple-cross product* on $\mathbb{O}$ as follows

$$x \times y \times z = \frac{1}{2}(x(\bar{y}z) - z(\bar{y}x)) \quad \text{for all } x, y, z \in \mathbb{O}. \tag{11}$$

Lemma 3.5. The triple-cross product is trilinear and alternating. Moreover,

$$N(x \times y \times z) = N(x)N(y)N(z)$$

for all $x, y, z \in \mathbb{O}$ distinct from each other.

Proof. The trilinearity is obvious. To prove the second claim we check $x \times x \times z = 0$, $x \times y \times y = 0$, and $x \times y \times x = 0$. We use [21], Lemma 1.3.3 *i*) for the first two:

$$\begin{aligned} x \times x \times z &= \frac{1}{2}(x(\bar{x}z) - z(\bar{x}x)) = \frac{1}{2}(N(x)z - zN(x)) = 0, \\ x \times y \times y &= \frac{1}{2}(x(\bar{y}y) - y(\bar{y}x)) = \frac{1}{2}(xN(y) - N(y)x) = 0, \\ x \times y \times x &= \frac{1}{2}(x(\bar{y}x) - x(\bar{y}x)) = 0. \end{aligned}$$

Finally, to prove $N(x \times y \times z) = N(x)N(y)N(z)$ we expand $x \times y \times z$ in the orthonormal basis $e = (1, 0), e_1, \ldots, e_7$ of $\mathbb{O}$. In particular, this allows us to assume that x, y and z as scalar multiples of standard basis vectors. Now it is straightforward to verify (by Maple) that $x(\bar{y}z) = -z(\bar{y}x)$, hence $x \times y \times z = x(\bar{y}z)$ and our claim follows by taking norms. □

Next, we show that the "associator identity" of Harvey-Lawson (Theorem 1.6 [16]) holds for split octonion algebras.

[1] A pair (x, y) of elements from a composition algebra is called a special (λ, μ)-pair if $\langle x, e \rangle = \langle y, e \rangle = \langle x, y \rangle = 0$, $N(x) = \lambda$ and $N(y) = \mu$.

Theorem 3.6. For all $x, y, z \in \mathbb{I}$, the associator $[x, y, z]$ lies in $\mathbb{I}$, and

$$x \times y \times z = \langle xy, z\rangle e + [x, y, z], \tag{12}$$

where e is the identity element $(1, 0)$ of $\mathbb{O}$. Moreover,

$$|x \wedge y \wedge z| = \varphi(x, y, z)^2 + N([x, y, z]), \tag{13}$$

where $|x \wedge y \wedge z|$ is defined as $N(x)N(y)N(z)$.

Proof. By linearity and alternating property, it is enough to prove our first two assertions for (orthonormal) triplets (x, y, z) from the basis $\{e_1, \ldots, e_7\}$ and once again this is straightforward to verify by using Maple. To prove (13), we note as in the proof of Lemma 3.5 that $N(x\times y\times z) = N(x(\bar{y}z)) = N(x)N(y)N(z) = |x\wedge y\wedge z|$. Thus, our claim now follows from the simple fact that $N(\alpha e+u) = \alpha^2+N(u)$ whenever $u \in e^{\perp}, \alpha \in \mathbb{C}$. □

Corollary 3.7. Let $W \in \mathrm{Gr}(3, \mathbb{I})$ be a 3-plane and let x, y, z be an orthonormal basis for W (which always exists by Gram-Schmidt process). If W is associative, then $[x, y, z] = 0$.

Proof. By linearity and alternating property, it is enough to prove the statement "$\varphi(x, y, z) = \pm 1$ if and only if $[x, y, z] = 0$" on the orthonormal basis $\{e_1, \ldots, e_7\}$. We verified the cases by using Maple. The rest of the proof follows from Theorem 3.4. □

Recall that when x, y, and z are orthonormal vectors from $\mathbb{I}$, $x \times y \times z = x(\bar{y}z)$. We verified by using Maple that if the associator $[x, y, z]$ is nonzero for $\{x, y, z\} \subset \{e_1, \ldots, e_7\}$, then

$$x \times y \times z = [x, y, z] \quad \text{and} \quad \varphi(x, y, z) = 0, \tag{14}$$

and if $[x, y, z] = 0$, then

$$x \times y \times z = \varphi(x, y, z)e, \tag{15}$$

which conforms with (12). It is not difficult to show that the form $*\varphi(u, v, w, z)$ defined by

$$*\varphi(u, v, w, z) = \langle u \times v \times w, z\rangle$$

is an alternating 4-form, and it can be expressed as in

$$*\varphi = -e^{4567} + e^{2367} - e^{2345} + e^{1357} + e^{1346} + e^{1256} - e^{1247}. \tag{16}$$

This can be seen from the fact that the monomials of φ and $*\varphi$ are complementary in the sense that e^{ijkl} appears in $*\varphi$ if any only if there exists unique monomial e^{rst} in φ such that $\{r, s, t, i, j, k, l\} = \{1, 2, 3, 4, 5, 6, 7\}$. Also, it can be checked directly on the standard orthogonal basis. Now, we define a new 3-form $\chi(u, v, w)$ by the identity $\langle\chi(u, v, w), z\rangle = *\varphi(u, v, w, z)$.

An important consequence of these definitions and the above discussion (specifically, the equation (14)) is that if a 3-plane spanned by $x, y, z \in \mathbb{I}$ generates a quaternion subalgebra, then $[x, y, z] = 0$, which implies $\chi(x, y, z) = 0$. We record this in our next lemma.

Lemma 3.8. If the 3-plane W generated by $x, y, z \in \mathbb{I}$ is associative, then $\chi(x, y, z) = 0$.

The vanishing locus of χ on $\mathbb{P}(\bigwedge^3 \mathbb{I})$ can be made more precise since

$$\begin{aligned}\chi = &(e^{247} - e^{256} - e^{346} - e^{357})e_1\\ &+ (e^{156} - e^{147} + e^{345} - e^{367})e_2\\ &+ (-e^{245} + e^{267} + e^{146} + e^{157})e_3\\ &+ (e^{567} + e^{127} - e^{136} + e^{235})e_4\\ &+ (-e^{126} - e^{467} - e^{137} - e^{234})e_5\\ &+ (e^{457} + e^{125} + e^{134} - e^{237})e_6\\ &+ (e^{135} - e^{124} - e^{456} + e^{236})e_7,\end{aligned}$$

which follows from (16).

Remark 3.9. It appears that the idea of using these seven linear equations obtained from χ to study associative manifolds is first used in [3].

Remark 3.10. Let p_I (I is a d-element subset of $\{1, \ldots, n\}$) denote the (Plücker) coordinates on $\mathbb{P}(\bigwedge^d \mathbb{C}^n)$. The homogenous coordinate ring of the Grassmann variety of d dimensional subspaces in $\mathbb{C}^n$ is the quotient of the polynomial ring $\mathbb{C}[p_I : \ I \subset \{1, \ldots, n\}, \ |I| = d]$ by the ideal generated by the following quadratic polynomials: $\sum_{s=1}^{d+1} (-1)^s p_{i_1 i_2 \ldots i_{d-1} j_s} p_{j_1 j_2 \ldots \widehat{j_s} \ldots j_{d+1}}$, where $i_1, \ldots, i_{d-1}, j_1, \ldots, j_{d+1}$ are arbitrary numbers from $\{1, \ldots, n\}$. Here, the hatted entry $\widehat{j_s}$ is omitted from the sequence. Of course, the case of interest for us is when $d = 3$, $n = 7$, and (17) is a (at most) 4-term relation:

$$p_{i_1 i_2 j_1} p_{j_2 j_3 j_4} = p_{i_1 i_2 j_2} p_{j_1 j_3 j_4} - p_{i_1 i_2 j_3} p_{j_1 j_2 j_4} + p_{i_1 i_2 j_4} p_{j_1 j_2 j_3}. \tag{17}$$

In this notation, the vanishing locus of χ on $\mathbb{P}(\bigwedge^3 \mathbb{I})$ can be expressed in Plücker coordinates on $\mathbb{P}(\bigwedge^d \mathbb{C}^n)$ by the following 7 linear equations:

$$p_{247} - p_{256} - p_{346} - p_{357} = 0 \tag{18}$$
$$p_{156} - p_{147} + p_{345} - p_{367} = 0 \tag{19}$$
$$-p_{245} + p_{267} + p_{146} + p_{157} = 0 \tag{20}$$
$$p_{567} + p_{127} - p_{136} + p_{235} = 0 \tag{21}$$
$$-p_{126} - p_{467} - p_{137} - p_{234} = 0 \tag{22}$$
$$p_{457} + p_{125} + p_{134} - p_{237} = 0 \tag{23}$$
$$p_{135} - p_{124} - p_{456} + p_{236} = 0. \tag{24}$$

Moreover, it follows from Lemma 3.8 that the Zariski closure of the space of associative 3-planes in $\mathbb{P}(\bigwedge^3 \mathbb{I})$, namely the image of the associative grassmannian G_2/SO_4 under the Plücker embedding of $\mathrm{Gr}(3, \mathbb{I})$ lies in the intersection of these 7 hyperplanes with $\mathrm{Gr}(3, \mathbb{I})$.

Definition 3.11. We denote by X_{min} the intersection in $\mathbb{P}(\bigwedge^3 \mathbb{I})$ of the 7 hyperplanes (18)–(24) with the grassmannian $\mathrm{Gr}(3, \mathbb{I})$.

4. Two SL_2 actions

A 2-dimensional maximal torus T of G_2 is described by Springer and Veldkamp in Section 2.3 of [21] as the subgroup of automorphisms of $\mathbb{O}$ consisting of the following transformations:

$$t_{\lambda,\mu}: \ (x, y) \mapsto (c_\lambda x c_\lambda^{-1}, c_\mu y c_\lambda^{-1}),$$

where $(x, y) \in \mathbb{O}$, $\lambda, \mu \in k^*$ and c_λ, c_μ are the diagonal matrices $\mathrm{diag}(\lambda, \lambda^{-1})$, $\mathrm{diag}(\mu, \mu^{-1})$, respectively.

We look more closely at how T acts on the grassmannian, so we express the action in our coordinates. Let e_{ij} denote the elementary 2×2 matrix which has 1 at i, j'th position and 0's elsewhere. The set of pairs

$$\{(e_{11}, 0), (e_{12}, 0), (e_{21}, 0), (e_{22}, 0), (0, e_{11}), (0, e_{12}), (0, e_{21}), (0, e_{22})\}$$

forms a basis for $\mathbb{O}$. In this basis, $t_{\lambda,\mu}$ is the diagonal matrix

$$t_{\lambda,\mu} = \mathrm{diag}(1, \lambda^2, \lambda^{-2}, 1, \lambda^{-1}\mu, \lambda\mu, \lambda^{-1}\mu^{-1}, \lambda\mu^{-1}).$$

We are going to switch to the basis $\{e = (e_{11} + e_{22}, 0), e_1 = (\mathbf{i}, 0), e_2 = (\mathbf{j}, 0), e_3 = (\mathbf{k}, 0), e_4 = (0, e_{11} + e_{22}), e_5 = (0, \mathbf{i}), e_6 = (0, \mathbf{j}), e_7 = (0, \mathbf{k})\}$. The proof of the next lemma is straightforward so we skip it.

Lemma 4.1. The action of maximal torus $T = t_{\lambda,\mu}$ of G_2 on the basis $\{e_1, \ldots, e_7\}$ of $\mathbb{I}$ is given by

$$\begin{aligned}
t_{\lambda,\mu}(e) &= e \\
t_{\lambda,\mu}(e_1) &= e_1 \\
t_{\lambda,\mu}(e_2) &= \left(\frac{\lambda^2 + \lambda^{-2}}{2}\right) e_2 + i\left(\frac{-\lambda^2 + \lambda^{-2}}{2}\right) e_3 \\
t_{\lambda,\mu}(e_3) &= i\left(\frac{\lambda^2 - \lambda^{-2}}{2}\right) e_2 + \left(\frac{\lambda^2 + \lambda^{-2}}{2}\right) e_3 \\
t_{\lambda,\mu}(e_4) &= \left(\frac{\lambda^{-1}\mu + \lambda\mu^{-1}}{2}\right) e_4 + i\left(\frac{-\lambda^{-1}\mu + \lambda\mu^{-1}}{2}\right) e_5 \\
t_{\lambda,\mu}(e_5) &= i\left(\frac{\lambda^{-1}\mu - \lambda\mu^{-1}}{2}\right) e_4 + \left(\frac{\lambda^{-1}\mu + \lambda\mu^{-1}}{2}\right) e_5 \\
t_{\lambda,\mu}(e_6) &= \left(\frac{\lambda\mu + \lambda^{-1}\mu^{-1}}{2}\right) e_6 + i\left(\frac{-\lambda\mu + \lambda^{-1}\mu^{-1}}{2}\right) e_7 \\
t_{\lambda,\mu}(e_7) &= i\left(\frac{\lambda\mu - \lambda^{-1}\mu^{-1}}{2}\right) e_6 + \left(\frac{\lambda\mu + \lambda^{-1}\mu^{-1}}{2}\right) e_7.
\end{aligned}$$

There are two SL_2's naturally associated with the tori $t_{\lambda,\lambda}$ and $t_{\mathrm{id}_2,\mu}$.

Proposition 4.2. Let $x = (x_1, x_2)$ be an octonion from $\mathbb{I} = \mathfrak{sl}_2 \oplus \mathrm{Mat}_2$. The two SL_2 actions on $\mathbb{I}$ defined by

(1) $g \cdot x = (gx_1g^{-1}, gx_2g^{-1})$ and
(2) $g \cdot x = (x_1, gx_2)$

induce SL_2 actions on associative 3-planes.

Proof. Let $W \in \mathrm{Gr}(3, \mathbb{I})$ be an associative 3-plane spanned by the orthogonal basis $\{x, y, z\} \subset \mathbb{I} = \mathfrak{sl}_2 \oplus \mathrm{Mat}_2$. We know from Lemma 3.8 that W is associative if $[x, y, z] = 0$. Thus, it suffices to check the vanishing of the associator

$$[g \cdot x, g \cdot y, g \cdot z] = (g \cdot x\, g \cdot y)g \cdot z - g \cdot x(g \cdot y\, g \cdot z).$$

Note that $\overline{g} = g^{-1}$ for all $g \in \mathrm{SL}_2$. Note also that for any $x = (x_1, x_2), y = (y_1, y_2)$ from $\mathbb{I} = \mathfrak{sl}_2 \oplus \mathrm{Mat}_2$ we have

$$\begin{aligned}(g \cdot x)(g \cdot y) &= (gx_1y_1g^{-1} + \overline{gy_2g^{-1}}gx_2g^{-1}, gy_2x_1g^{-1} + gx_2g^{-1}\overline{gy_1g^{-1}})\\ &= (gx_1y_1g^{-1} + g\overline{y_2}g^{-1}gx_2g^{-1}, gy_2x_1g^{-1} + gx_2g^{-1}g\overline{y_1}g^{-1})\\ &= (gx_1y_1g^{-1} + g\overline{y_2}x_2g^{-1}, gy_2x_1g^{-1} + gx_2\overline{y_1}g^{-1})\\ &= g \cdot ((x_1, x_2)(y_1, y_2)).\end{aligned}$$

Therefore, if $[x, y, z] = 0$, then

$$\begin{aligned}[g \cdot x, g \cdot y, g \cdot z] = (g \cdot x\, g \cdot y)g \cdot z - g \cdot x(g \cdot y\, g \cdot z) &= (g \cdot (xy))g \cdot z - g \cdot x(g \cdot (yz))\\ &= g \cdot ((xy)z) - g \cdot (x(yz))\\ &= g \cdot ((xy)z - x(yz))\\ &= g \cdot [x, y, z]\\ &= 0.\end{aligned}$$

Next, we check our claim for the second action:

$$\begin{aligned}(g \cdot x)(g \cdot y) &= (x_1y_1 + \overline{gy_2}gx_2, gy_2x_1 + gx_2\overline{y_1})\\ &= (x_1y_1 + \overline{y_2}x_2, g(y_2x_1 + x_2\overline{y_1}))\\ &= g \cdot ((x_1, x_2)(y_1, y_2)).\end{aligned}$$

The rest follows as in the previous case. □

As a consequence of Proposition 4.2 we obtain two SL_2 actions on X_{min}.

Remark 4.3. We denote by U the following unipotent subgroup:

$$U = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} :\ u \in \mathbb{C} \right\} \subset \mathrm{SL}_2.$$

The matrices of the actions of a generic element $g_u := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in U$ on the ordered basis $e_1, \ldots, e_7$ of $\mathbb{I}$ are given by

$$(1)\ \ [g_u] = \begin{pmatrix} 1 & -iu & -u & 0 & 0 & 0 & 0 \\ iu & 1/2\,u^2+1 & -i/2u^2 & 0 & 0 & 0 & 0 \\ u & -i/2u^2 & 1-1/2\,u^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -iu & -u \\ 0 & 0 & 0 & 0 & iu & 1/2\,u^2+1 & -i/2u^2 \\ 0 & 0 & 0 & 0 & u & -i/2u^2 & 1-1/2\,u^2 \end{pmatrix}$$

$$(2)\ \ [g_u] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & u/2 & -i/2u \\ 0 & 0 & 0 & 0 & 1 & -i/2u & -u/2 \\ 0 & 0 & 0 & -u/2 & i/2u & 1 & 0 \\ 0 & 0 & 0 & i/2u & u/2 & 0 & 1 \end{pmatrix}.$$

For both of these actions of U on X_{min} the fixed point sets are positive dimensional. Indeed, the points $[e_{123}]$ and $[-e_{126}+ie_{127}+ie_{136}+e_{137}]$ of $\mathbb{P}(\bigwedge^3 \mathbb{I})$ lie on X_{min} (this can be verified by using eqs. (18)–(24)) and both of these points are fixed by the first action of U. By a result of Horrocks [15], we know that the fixed point set of a unipotent group acting on a connected complete variety is connected. Therefore, the fixed point set of U on X_{min} is positive dimensional for the first action. Similarly, the points $[e_{123}]$ and $[-e_{346}+ie_{347}-ie_{356}+e_{357}]$ of X_{min} are fixed by the second action of U, hence the fixed point set of this action is also positive dimensional.

5. Torus fixed points

Now we go back to analyzing fixed point set of the maximal torus $t_{\lambda,\mu}$ of G_2 on X_{min}. The action of $t_{\lambda,\mu}$ on the basis $\{e_1,\dots,e_7\}$ is computed in the previous section. The eigenvalues are $1, \frac{1}{\lambda^2}, \lambda^2, \frac{\lambda}{\mu}, \frac{\mu}{\lambda}, \frac{1}{\lambda\mu}, \lambda\mu$ and the respective eigenvectors are

$$\begin{aligned} \widetilde{e_1} &= e_1, \\ \widetilde{e_2} &= -ie_2+e_3, \\ \widetilde{e_3} &= ie_2+e_3, \\ \widetilde{e_4} &= -ie_4+e_5, \\ \widetilde{e_5} &= ie_4+e_5, \\ \widetilde{e_6} &= -ie_6+e_7, \\ \widetilde{e_7} &= ie_6+e_7. \end{aligned}$$

For i,j and k from $\{1,\dots,7\}$ we write $\widetilde{e}_{ijk}$ for $\widetilde{e}_i\wedge\widetilde{e}_j\wedge\widetilde{e}_k$. Accordingly, we write $\widetilde{p}_{ijk}$ for the transformed Plücker coordinate functions so that

$$\widetilde{p}_{ijk}(\widetilde{e}_{rst}) = \begin{cases} 1 & \text{if } i=r,\ j=s,\ k=t; \\ 0 & \text{otherwise.} \end{cases}$$

Our defining equations (18)–(24) become:

$$\widetilde{f}_1 := \widetilde{p}_{247} + \widetilde{p}_{356} = 0 \tag{25}$$

$$\widetilde{f}_2 := 2\widetilde{p}_{147} + 2\widetilde{p}_{156} + \widetilde{p}_{245} + \widetilde{p}_{345} - \widetilde{p}_{267} - \widetilde{p}_{367} = 0 \tag{26}$$

$$\widetilde{f}_3 := 2\widetilde{p}_{147} + 2\widetilde{p}_{156} + \widetilde{p}_{245} - \widetilde{p}_{345} - \widetilde{p}_{267} + \widetilde{p}_{367} = 0 \tag{27}$$

$$\widetilde{f}_4 := 2\widetilde{p}_{127} - 2\widetilde{p}_{136} + \widetilde{p}_{234} + \widetilde{p}_{235} + \widetilde{p}_{467} + \widetilde{p}_{567} = 0 \tag{28}$$

$$\widetilde{f}_5 := -2\widetilde{p}_{127} - 2\widetilde{p}_{136} + \widetilde{p}_{234} - \widetilde{p}_{235} + \widetilde{p}_{467} - \widetilde{p}_{567} = 0 \tag{29}$$

$$\widetilde{f}_6 := 2\widetilde{p}_{124} - 2\widetilde{p}_{135} - \widetilde{p}_{236} - \widetilde{p}_{237} + \widetilde{p}_{456} + \widetilde{p}_{457} = 0 \tag{30}$$

$$\widetilde{f}_7 := 2\widetilde{p}_{124} + 2\widetilde{p}_{135} - \widetilde{p}_{236} + \widetilde{p}_{237} + \widetilde{p}_{456} - \widetilde{p}_{457} = 0. \tag{31}$$

It is easily verified that the following 35 vectors are eigenvectors for the action of $t_{\lambda,\mu}$ on $\bigwedge^3 \mathbb{I}$ (together with the eigenvalues indicated on the left column):

$\frac{1}{\lambda^3\mu}$	$\widetilde{e}_{126}$
$\lambda^3\mu$	$\widetilde{e}_{137}$
μ^2	$\widetilde{e}_{157}$
μ^{-2}	$\widetilde{e}_{146}$
$\frac{\lambda^2}{\mu^2}$	$\widetilde{e}_{346}$
$\frac{\mu^2}{\lambda^2}$	$\widetilde{e}_{257}$
$\frac{\lambda^3}{\mu}$	$\widetilde{e}_{134}$
$\frac{\mu}{\lambda^3}$	$\widetilde{e}_{125}$
$\frac{1}{\lambda^2\mu^2}$	$\widetilde{e}_{246}$
$\lambda^2\mu^2$	$\widetilde{e}_{357}$
λ^4	$\widetilde{e}_{347}$
λ^{-4}	$\widetilde{e}_{256}$
$\frac{\mu}{\lambda}$	$\widetilde{e}_{567},\ \widetilde{e}_{235},\ \widetilde{e}_{127}$
$\frac{\lambda}{\mu}$	$\widetilde{e}_{467},\ \widetilde{e}_{234},\ \widetilde{e}_{136}$
$\frac{1}{\lambda\mu}$	$\widetilde{e}_{456},\ \widetilde{e}_{236},\ \widetilde{e}_{124}$
$\lambda\mu$	$\widetilde{e}_{457},\ \widetilde{e}_{237},\ \widetilde{e}_{135}$
$\frac{1}{\lambda^2}$	$\widetilde{e}_{267},\ \widetilde{e}_{245},\ \widetilde{e}_{156}$
λ^2	$\widetilde{e}_{367},\ \widetilde{e}_{345},\ \widetilde{e}_{147}$
1	$\widetilde{e}_{356},\ \widetilde{e}_{247},\ \widetilde{e}_{167},\ \widetilde{e}_{145},\ \widetilde{e}_{123}$

Theorem 5.1. Among the eigenvectors of $t_{\lambda,\mu}$ in $\bigwedge^3 \mathbb{I}$, only the images of the following vectors in $\mathbb{P}(\bigwedge^3 \mathbb{I})$ lie in X_{min}:

$\lambda^2\mu^2$	$\widetilde{e}_{357}$
$\frac{1}{\lambda^2\mu^2}$	$\widetilde{e}_{246}$
$\frac{1}{\lambda^3\mu}$	$\widetilde{e}_{126}$
$\lambda^3\mu$	$\widetilde{e}_{137}$
μ^2	$\widetilde{e}_{157}$
$\frac{1}{\mu^2}$	$\widetilde{e}_{146}$
$\frac{1}{\lambda^4}$	$\widetilde{e}_{256}$
λ^4	$\widetilde{e}_{347}$
$\frac{\mu}{\lambda^3}$	$\widetilde{e}_{125}$
$\frac{\lambda^3}{\mu}$	$\widetilde{e}_{134}$
$\frac{\mu^2}{\lambda^2}$	$\widetilde{e}_{257}$
$\frac{\lambda^2}{\mu^2}$	$\widetilde{e}_{346}$
1	$\widetilde{e}_{167},\ \widetilde{e}_{145},\ \widetilde{e}_{123}$

Proof. It is easily checked that the points that are given in the hypothesis of the theorem are all torus fixed and all of them lie in X_{min}. The only place we have to be careful is that X_{min} may intersect eigenspaces of dimension ≥ 2. Nevertheless, this potential problem does not occur; when we substitute a nontrivial linear combination of eigenvectors belonging to the same eigenvalue into equations (25)–(31), we get a contradiction. □

6. Smoothness

In this section, we will prove our main result. First, we have a remark on the dimensions.

Remark 6.1. The dimension of G_2/SO_4 is equal to $\dim G_2 - \dim SO_4 = 14 - 6 = 8$. We already pointed out in Remark 3.10 that G_2/SO_4 is an affine subvariety of X_{min}, therefore, the dimension of X_{min} is at least 8.

Next, we recall two standard facts.

(1) *Jacobian Criterion for Smoothness*: Let $I = (f_1, \ldots, f_m)$ be an ideal from $\mathbb{C}[x_1, \ldots, x_n]$ and let $x \in V(I)$ be a point from the vanishing locus of I in $\mathbb{C}^n$. Suppose $d = \dim V(I)$. If the rank of the Jacobian matrix $(\partial f_i/\partial x_j)_{i=1,\ldots,m,\ j=1,\ldots,n}$ at x is equal to $n - d$, then x is a smooth point of $V(I)$.

(2) *Open charts on the Grassmannian*: To see the complex manifold structure on $\mathrm{Gr}(d, \mathbb{C}^n)$, one looks at the intersections of $\mathrm{Gr}(d, \mathbb{C}^n)$ with the standard open charts in $\mathbb{P}(\wedge^d \mathbb{C}^n)$:

$$U_I := \mathrm{Gr}(d, \mathbb{C}^n) \cap \{x \in \mathbb{P}(\wedge^d \mathbb{C}^n) :\ p_I(x) \neq 0\}.$$

It is not difficult to show that the coordinate functions on U_I are given by p_J/p_I, where $J = j_1 \dots j_d$ is a sequence satisfying $|\{j_1, \dots, j_d\} \cap \{i_1, \dots, i_d\}| = d - 1$. Indeed, it is not difficult to verify (by using Plücker relations) that any other rational function of the form p_K/p_I is a polynomial in p_J/p_I's.

Theorem 6.2. The algebraic set X_{min} is a nonsingular projective variety of dimension 8.

Proof. Since X_{min} is a closed set (defined as the intersection of certain hyperplanes with the Grassmann variety) in a projective space, any irreducible component of X_{min} is a projective variety. Moreover, since X_{min} is stable under a torus action, each of these components is stable under the torus action as well. By Borel Fixed Point Theorem [5, Theorem 10.4], we know that any irreducible component of X_{min} contains at least one torus fixed point. In fact, there is a much stronger statement: Let V be a vector space and let $Y \subset \mathbb{P}(V)$ be a projective T-variety, where T is an algebraic torus. Finally, let Y^T denote the fixed point set of the torus action. In this case, Y^T contains at least $\dim Y + 1$ points. See [10, Lemma 2.4]. In Theorem 5.1, we showed that there are in total 15 torus fixed points in X_{min}. In the next few paragraphs we will show that each of these torus fixed points is smooth and its tangent space is 8 dimensional. Hence, each irreducible component of X_{min} is 8 dimensional, each component has at least 8+1=9 torus fixed points. A point in the intersection of two components is necessarily singular in X_{min}, hence, the Zariski tangent space at such a point would be at least 9 dimensional. In other words, the irreducible components of X_{min} do not intersect each other. But this implies that there is only one irreducible component, otherwise, in X_{min} there would at least be 18 torus fixed points. This finishes the proof of irreducibility.

We proceed to show that the torus fixed points are smooth. Note that the existence of a singular point in a T-variety implies the existence of a (possibly different) torus fixed singular point. Thus, it suffices to analyze neighborhoods of fixed points by using affine charts that are described earlier.

We start with the fixed point $m = [\widetilde{e}_{123}]$, which lies on the open chart $\widetilde{U}_{123}$ as its origin. Here, "tilde" indicates that we are using transformed Plücker coordinates. Recall that X_{min} is cut-out on $\widetilde{U}_{123}$ by the vanishing of the seven linear forms (25)–(31). A straightforward calculation shows that the Jacobian of these polynomials with respect to variables $\widetilde{q}_{124}, \widetilde{q}_{125}, \widetilde{q}_{126}, \widetilde{q}_{127}, \widetilde{q}_{134}, \widetilde{q}_{135}, \widetilde{q}_{136}, \widetilde{q}_{137}, \widetilde{q}_{234}, \widetilde{q}_{235}, \widetilde{q}_{236}, \widetilde{q}_{237}$ (in the written order) evaluated at the origin (which is $\widetilde{e}_{123}$), $\mathrm{Jac}(\widetilde{f}_1, \dots, \widetilde{f}_7)|_{\widetilde{q}_{ijk}=0}$, is equal to

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 0 & 0 & -1/2 & 0 & 1/4 & 1/4 & 0 & 0 \\
0 & 0 & 0 & -1/2 & 0 & 0 & -1/2 & 0 & 1/4 & -1/4 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & -1/4 & -1/4 \\
1/2 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & 0 & -1/4 & 1/4
\end{pmatrix}$$

which is obviously of rank 4. Hence, the dimension of the Zariski tangent space of X_{min} at m is $12 - 4 = 8$ dimensional. By Remark 6.1, we know that $\dim X_{min} \geq 8$, hence we have the equality, $\dim X_{min} = 8$. In particular, m is a smooth point of X_{min}.

We repeat this procedure for the other torus fixed points, which is tedious now. We verified this by using Maple. The outcome for each of the torus fixed points that are listed in Theorem 5.1 turns out to be the same. In summary, all of the 15 torus fixed points on X_{min} are nonsingular points, therefore, X_{min} is a smooth projective variety of dimension 8. □

Proof of Theorem 1.1. The G_2-orbit $\mathrm{G}_2/\mathrm{SO}_4 \subset X_{min}$ is irreducible and its dimension is equal to that of X_{min}. The proof follows from Theorem 6.2. □

7. Tangent space at $[\widetilde{e}_{123}]$

In this section we perform a sample calculation of the weights of a generic one-parameter $\gamma : \mathbb{C}^* \to t_{\lambda,\mu}$ subgroup on the tangent space of X_{min} at the $t_{\lambda,\mu}$-fixed point $m = [\widetilde{e}_{123}] \in X_{min}$. We use the term "generic" in the algebraic geometric sense, which is equivalent to the statement that the pairing between γ and any character $\alpha : t_{\lambda,\mu} \to \mathbb{C}^*$ is nonzero. In other words, we choose a regular one-parameter subgroup γ of $t_{\lambda,\mu}$ so that the fixed point set of γ on X_{min} is the same as that of $t_{\lambda,\mu}$. For example, $\gamma(s) := t_{s^{10},s}$, $s \in \mathbb{C}^*$ is regular.

Recall that the tangent space at $p = (p_1, \ldots, p_n)$ of an affine variety $V \subseteq \mathbb{C}^n$ defined by the vanishing of the polynomials $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$ is the intersection of the hyperplanes

$$\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(p)(x_i - p_i) = 0, \quad \text{for } j = 1, \ldots, r.$$

(Here we are abusing the notation. To be precise, x_i should be replaced by the vector field $\partial/\partial x_i$.) Equivalently, T_pV is the kernel of the Jacobian matrix of $f_1, \ldots, f_r$ (with respect to x_i's) evaluated at the point $p \in V$. In our case, p is $m = [\widetilde{e}_{123}]$ (the origin of the tangent space) and the Jacobian with respect to local coordinates

$$\widetilde{q}_{124}, \widetilde{q}_{125}, \widetilde{q}_{126}, \widetilde{q}_{127}, \widetilde{q}_{134}, \widetilde{q}_{135}, \widetilde{q}_{136}, \widetilde{q}_{137}, \widetilde{q}_{234}, \widetilde{q}_{235}, \widetilde{q}_{236}, \widetilde{q}_{237}$$

is as given in the proof of Theorem 6.2. It is straightforward to verify that

$$\left\{ -\frac{1}{2}x_{135} + x_{237}, \frac{1}{2}x_{124} + x_{236}, -\frac{1}{2}x_{127} + x_{235}, \frac{1}{2}x_{136} + x_{234}, x_{137}, x_{134}, x_{126}, x_{125} \right\} \quad (32)$$

is a basis for the kernel of the Jacobian matrix computed in the proof of Theorem 6.2. Here, x_{ijk} stands for the tangent vector $\frac{\partial}{\partial \widetilde{q}_{ijk}}$.

Recall from Section 4 that $t_{\lambda,\mu}$ acts on $\mathbb{I}$ according to

$$\begin{aligned}
t_{\lambda,\mu}(\widetilde{e}_1) &= \widetilde{e}_1\\
t_{\lambda,\mu}(\widetilde{e}_2) &= \frac{1}{\lambda^2}\ \widetilde{e}_2\\
t_{\lambda,\mu}(\widetilde{e}_3) &= \lambda^2\ \widetilde{e}_3\\
t_{\lambda,\mu}(\widetilde{e}_4) &= \frac{\lambda}{\mu}\ \widetilde{e}_4\\
t_{\lambda,\mu}(\widetilde{e}_5) &= \frac{\mu}{\lambda}\ \widetilde{e}_5\\
t_{\lambda,\mu}(\widetilde{e}_6) &= \frac{1}{\lambda\mu}\ \widetilde{e}_6\\
t_{\lambda,\mu}(\widetilde{e}_7) &= \lambda\mu\ \widetilde{e}_7.
\end{aligned}$$

Let us denote by $w_{ijk}(\lambda,\mu)$ the weight (the eigenvalue) of the action $t_{\lambda,\mu}\cdot\widetilde{e}_{ijk}$.

The action of $t_{\lambda,\mu}$ on a Plücker coordinate $\widetilde{p}_{ijk}$ is given by

$$t_{\lambda,\mu}\cdot\widetilde{p}_{ijk}(x) = \widetilde{p}_{ijk}(t_{\lambda^{-1},\mu^{-1}}\cdot x) = w_{ijk}(\lambda^{-1},\mu^{-1})\widetilde{p}_{ijk},$$

and therefore, its action on a local Plücker coordinate function $\widetilde{q}_{rst}$ on $\widetilde{U}_{ijk}$ is given by

$$t_{\lambda,\mu}\cdot\widetilde{q}_{rst}(x) = \frac{\widetilde{p}_{rst}(t_{\lambda^{-1},\mu^{-1}}\cdot x)}{\widetilde{p}_{ijk}(t_{\lambda^{-1},\mu^{-1}}\cdot x)} = \frac{w_{rst}(\lambda^{-1},\mu^{-1})}{w_{ijk}(\lambda^{-1},\mu^{-1})}\widetilde{q}_{rst}.$$

Consequently, if $v=\sum_{r,s,t} a_{rst}\frac{\partial}{\partial\widetilde{q}_{rst}}$ is a tangent vector at $\widetilde{e}_{ijk}\in\widetilde{U}_{ijk}$, then the action of the one-parameter subgroup $\gamma(\lambda)=t_{\lambda^{10},\lambda}$, $\lambda\in\mathbb{C}^*$ on v is given by

$$\gamma\cdot v = \sum_{r,s,t}\lim_{\lambda\to 1}\left(\frac{w_{rst}(\lambda^{10},\lambda)}{w_{ijk}(\lambda^{10},\lambda)}\right)\frac{\partial}{\partial\widetilde{q}_{rst}}. \tag{33}$$

For example, the action of γ on the basis vectors (32), which we denote by $v_1,\dots,v_8$ in the written order, is given by

$$\begin{aligned}
\gamma\cdot v_1 &= 11v_1\\
\gamma\cdot v_2 &= -11v_2\\
\gamma\cdot v_3 &= -9v_3\\
\gamma\cdot v_4 &= 9v_4\\
\gamma\cdot v_5 &= 31v_5\\
\gamma\cdot v_6 &= 29v_6\\
\gamma\cdot v_7 &= -31v_7\\
\gamma\cdot v_8 &= -29v_8.
\end{aligned}$$

8. Białynicki-Birula Decomposition

Let X be a smooth projective variety over $\mathbb{C}$ on which an algebraic torus T acts with finitely many fixed points. Let T' be a 1 dimensional subtorus with $X^{T'} = X^T$. For $p \in X^{T'}$, define the sets

$$C_p^+ = \{y \in X : \lim_{s \to 0} s \cdot y = p,\ s \in T'\}$$

and

$$C_p^- = \{y \in X : \lim_{s \to \infty} s \cdot y = p,\ s \in T'\},$$

called the plus and minus cells of p, respectively.

The following result is customarily called the Białynicki-Birula decomposition theorem in the literature.

Theorem 8.1 ([7])**.** If X, T and T' are as in the above paragraph, then

(1) both of the sets C_p^+ and C_p^- are locally closed subvarieties in X, furthermore they are isomorphic to an affine space;

(2) if T_pX is the tangent space of X at p, then C_p^+ (resp., C_p^-) is T'-equivariantly isomorphic to the subspace T_p^+X (resp., T_p^-X) of T_pX spanned by the positive (resp., negative) weight spaces of the action of T' on T_pX.

As a consequence of the BB-decomposition, there exists a filtration

$$X^{T'} = V_0 \subset V_1 \subset \cdots \subset V_n = X, \qquad n = \dim X,$$

of closed subsets such that for each $i = 1, \dots, n$, $V_i - V_{i-1}$ is the disjoint union of the plus (resp., minus) cells in X of (complex) dimension i. It follows that the odd-dimensional integral cohomology groups of X vanish, the even-dimensional integral cohomology groups of X are free, and the Poincaré polynomial $P_X(t) := \sum_{i=0}^{2n} \dim H^i(X; \mathbb{C}) t^i$ of X is given by

$$P_X(t) = \sum_{p \in X^{T'}} t^{2 \dim C_p^+} = \sum_{p \in X^{T'}} t^{2 \dim C_p^-}.$$

Now, let T' denote the 1 dimensional subtorus of $T = t_{\lambda,\mu}$ that is given by the image of the regular one-parameter subgroup $\gamma(\lambda) = t_{\lambda^{10},\lambda}$, $\lambda \in \mathbb{C}^*$. In the rest of this section, we are going to compute the weights of T' on the tangent spaces at the torus fixed points. We have already made a sample calculation of this sort in Section 7.

(1) $p = [\widetilde{e}_{246}]$. An eigenbasis for tangent space at p is given by

$$\{-x_{234} + x_{467}, -1/2x_{124} + x_{456}, x_{346}, x_{245} + x_{267}, x_{256}, 1/2x_{124} + x_{236}, x_{146}, x_{126}\}$$

The weights (in the order of the eigenvectors) are $31, 11, 40, 2, -18, 11, 20, -9$.

(2) $p = [\widetilde{e}_{157}]$. An eigenbasis for tangent space at p is given by

$$\{x_{125}, -1/2\,x_{127} + x_{567}, 1/2\,x_{135} + x_{457}, x_{357}, x_{257}, x_{167}, x_{145}, x_{137}\}$$

The weights (in the order of the eigenvectors) are $-31, -11, 9, 20, -20, -2, -2, 29$.

(3) $p = [\widetilde{e}_{256}]$. An eigenbasis for tangent space at p is given by

$$\{-x_{235} + x_{567}, x_{236} + x_{456}, 1/2\, x_{156} + e_{267}, x_{257}, x_{246}, -1/2\, x_{156} + x_{245}, x_{126}, x_{125}\}$$

The weights (in the order of the eigenvectors) are $31, 29, 10, 22, 18, 10, 9, 11$.

(4) $p = [\widetilde{e}_{126}]$. An eigenbasis for tangent space at p is given by

$$\{1/2\, x_{156} + x_{267}, x_{256}, x_{246}, 1/2\, x_{124} + x_{236}, x_{167}, x_{146}, x_{125}, x_{123}\}$$

The weights (in the order of the eigenvectors) are $11, -9, 9, 10, 31, 29, 2, 31$.

(5) $p = [\widetilde{e}_{167}]$. An eigenbasis for tangent space at p is given by

$$\{-1/2\, x_{127} + x_{567}, 1/2\, x_{136} + x_{467}, -1/2\, x_{147} + x_{367}, 1/2\, x_{156} + x_{267}, x_{157}, x_{146}, x_{137}, x_{126}\}$$

The weights (in the order of the eigenvectors) are $-9, 9, 10, -10, 2, -2, 31, -31$.

(6) $p = [\widetilde{e}_{145}]$. An eigenbasis for tangent space at p is given by

$$\{1/2\, x_{135} + x_{457}, -1/2\, x_{124} + x_{456}, 1/2\, x_{147} + x_{345}, -1/2\, x_{156} + x_{245}, x_{157}, x_{146}, x_{134}, x_{125}\}$$

The weights (in the order of the eigenvectors) are $11, -11, 10, -10, 2, -2, 29, -29$.

(7) $p = [\widetilde{e}_{123}]$. An eigenbasis for tangent space at p is given by

$$\{-1/2\, x_{135} + x_{237}, 1/2\, x_{124} + x_{236}, -1/2\, x_{127} + x_{235}, 1/2\, x_{136} + x_{234}, x_{137}, x_{134}, x_{126}, x_{125}\}$$

The weights (in the order of the eigenvectors) are $11, -11, -9, 9, 31, 29, -31, -29$.

(8) $p = [\widetilde{e}_{137}]$. An eigenbasis for tangent space at p is given by

$$\{-1/2\, x_{147} + x_{367}, x_{357}, x_{347}, -1/2\, x_{135} + x_{237}, x_{167}, x_{157}, x_{134}, x_{123}\}$$

The weights (in the order of the eigenvectors) are $-11, -9, 9, -10, -31, -29, -2, -31$.

(9) $p = [\widetilde{e}_{125}]$. An eigenbasis for tangent space at p is given by

$$\{x_{257}, x_{256}, -1/2\, x_{156} + x_{245}, -1/2\, x_{127} + x_{235}, x_{157}, x_{145}, x_{126}, x_{123}\}$$

The weights (in the order of the eigenvectors) are $11, -11, 9, 10, 31, 29, -2, 29$.

(10) $p = [\widetilde{e}_{257}]$. An eigenbasis for tangent space at p is given by

$$\{x_{157}, x_{125}, -1/2\, x_{127} + x_{567}, x_{237} + x_{457}, x_{357}, x_{245} + x_{267}, x_{256}, -1/2\, x_{127} + x_{235}\}$$

The weights (in the order of the eigenvectors) are $20, -11, 9, 29, 40, -2, -22, 9$.

(11) $p = [\widetilde{e}_{357}]$. An eigenbasis for tangent space at p is given by

$$\{[x_{137}, -x_{235} + x_{567}, 1/2\, x_{135} + x_{457}, x_{345} + x_{367}, x_{347}, x_{257}, -1/2\, x_{135} + x_{237}, x_{157}\}$$

The weights (in the order of the eigenvectors) are $9, 31, -31, -11, -2, 18, -40, -11, -20$.

(12) $p = [\widetilde{e}_{146}]$. An eigenbasis for tangent space at p is given by

$$\{x_{346}, x_{246}, x_{167}, x_{145}, x_{134}, x_{126}, 1/2\, x_{136} + x_{467}, -1/2\, x_{124} + x_{456}\}$$

The weights (in the order of the eigenvectors) are $20, -20, 2, 2, 31, -29, 11, -9$.

(13) $p = [\widetilde{e}_{347}]$. An eigenbasis for tangent space at p is given by

$$\{-x_{234} + x_{467}, x_{237} + x_{457}, -1/2\, x_{147} + x_{367}, x_{357}, x_{346}, 1/2\, x_{147} + x_{345}, x_{137}, x_{134}\}$$

The weights (in the order of the eigenvectors) are $-31, -29, -10, -18, -22, -10, -9, -11$

(14) $p = [\widetilde{e}_{134}]$. An eigenbasis for tangent space at p is given by

$$\{x_{146}, x_{145}, x_{137}, x_{123}, x_{347}, x_{346}, 1/2\, x_{147} + x_{345}, 1/2\, x_{136} + x_{234}\}$$

The weights (in the order of the eigenvectors) are $-31, -29, 2, -29, 11, -11, -9, -10$.

(15) $p = [\widetilde{e}_{346}]$. An eigenbasis for tangent space at p is given by

$$\{1/2\, x_{136} + x_{234}, x_{146}, x_{134}, 1/2\, x_{136} + x_{467}, x_{236} + x_{456}, x_{345} + x_{367}, x_{347}, x_{246}\}$$

The weights (in the order of the eigenvectors) are $-9, -20, 11, -9, -29, 2, 22, -40$.

Theorem 8.2. The Poincaré polynomial of X_{min} is

$$P_X(t^{1/2}) = 1 + t + 2t^2 + 2t^3 + 3t^4 + 2t^5 + 2t^6 + t^7 + t^8.$$

Proof. The proof follows from the discussion at the beginning of this section and the computations made above. □

Corollary 8.3. The Picard number of X_{min} is 1.

Proof. For a nonsingular projective variety X over $\mathbb{C}$, the Picard number $\rho(X)$ of X satisfies $1 \leq \rho(X) \leq b_2$, where b_2 is the second Betti number of X. In the light of this fact, the proof follows from Theorem 8.2. □

Remark 8.4. In [19, Theorem 2], Ruzzi showed that there exists unique smooth equivariant completion of G_2/SO_4 with Picard number 1. It follows from our Corollary 8.3 that X_{min} is the completion that Ruzzi found.

We finish our paper with a general remark.

Remark 8.5. The theory of equivariant embeddings of symmetric varieties is a very active and fascinating branch of algebraic geometry (see [8, 22]). There are many G_2-equivariant compactifications of G_2/SO_4. For example, there is the well known “wonderful compactification” due to DeConcini and Procesi, [12]. The basic invariants of this compactification are determined in [13]. More general than the wonderful compactification are the “special embeddings” of symmetric varieties (see [11] and [23]). For of G_2/SO_4, there are many special embeddings whose posets of G_2-orbits are isomorphic to that of X_{min}. However, these special embeddings (except the wonderful compactification) of G_2/SO_4 are not smooth.

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