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Hurwitz complete sets of factorizations in the modular group and the classification of Lefschetz elliptic fibrations over the disk

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Abstract. Given any matrix B in $SL(2,\mathbb{Z})$, we describe an algorithm that provides at least one relatively minimal Lefschetz elliptic fibration over the disk within each topological equivalence class, whose total monodromy is in the conjugacy class of B.

1. Introduction

Locally holomorphic fibrations (or maps) have received a great deal of attention due to the close relationship between the existence of a symplectic structure on a smooth 4-dimensional manifold M and the existence of a locally holomorphic fibration on M, i.e. a smooth map f from M into some 2-dimensional oriented manifold Σ, that is locally modeled by holomorphic maps (see [6]). The elliptic case, i.e. when the general fiber is a torus, has been extensively studied by several authors. In [10], Moishezon i) shows how the study of elliptic locally holomorphic fibrations essentially reduces, up to deformation, to the study of relatively minimal Lefschetz ones, and ii) solves the topological classification problem for the latter, under the assumption that $\Sigma = S^2$. In [9] Matsumoto generalizes ii) by allowing Σ to be a closed connected 2-manifold of arbitrary genus. Following Moishezon, we will refer to elliptic locally holomorphic fibrations as *topological elliptic fibrations*.

It can be verified that a smooth surjective map $f : M \to \Sigma$ between compact connected 2-dimensional manifolds is locally holomorphic for an appropriate choice of orientarion for M and Σ, if and and only if it is a branched cover. Furthermore, it can be seen that the Lefschetz condition (see Definition 2.3) for a locally holomorphic map whose domain and target are 4-dimensional and 2-dimensional, respectively, corresponds to the *simplicity* condition for a branched cover: a branched cover $f : M \to \Sigma$ between 2-dimensional manifolds is *simple* if the preimage of any point of Σ has at least $deg(f) - 1$ points. Simplicity is also equivalent to the demand that local monodromies around singular values are transpositions. Results analogous to the ones mentioned above have been obtained in the context of branched covers. In [11], Natanzon classifies branched covers over a

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disk, up to topological equivalence, using geometric topology methods. Khovanskii and Zdravkovska obtain the same result by combinatorial means (see [7]).

The problem of classification of relatively minimal Lefschetz topological elliptic fibrations over a disk (see Definition 2.3), up to topological equivalence, is equivalent to the problem of studying the set

$$\left\{ (g_1, \ldots, g_n) : n \geq 0 \text{ and } g_i \in SL(2,\mathbb{Z}) \text{ is a conjugate of } U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\},$$

modulo the equivalence relation $\sim_{H+C}$ that declares that two n-tuples are equivalent if one can be obtained from the other by a finite sequence of Hurwitz moves followed by a global conjugation (see [1] , and Definition 2.5). A satisfactory answer to this problem would comprise:

(1) An algorithm that for any $B \in SL(2,\mathbb{Z})$ would produce a "simple" collection of factorizations of B in terms of conjugates of U, such that any other such factorization of B can be transformed into at least one of the members of the collection.
(2) An algorithm that would decide whether two factorizations of B in conjugates of U are Hurwitz equivalent.

In this article we construct an algorithm that solves part (1). More precisely, this algorithm outputs a finite list of factorizations of B such that the *infinite* collection obtained by concatenating the members of this list with appropriate powers of a particular product $(s_0 s_1)$ contains at least one representative of each Hurwitz equivalence class, as explained in Section 5.

Part (2) seems to be a difficult problem. In fact, there are examples of groups within which the Hurwitz equivalence problem is undecidable, even for very special types of factorizations (see [12]). Our results do not rule out the possibility that two factorizations of an element of $SL(2,\mathbb{Z})$ in terms of conjugates of U having the same number of factors are always Hurwitz equivalent. This was pointed out to us by Prof. Auroux in a private communication. Soon after Prof. Auroux wrote an article [2] where he exhibits matrices in $SL(2,\mathbb{Z})$ admiting Hurwitz inequivalent factorizations in terms of conjugates of U and having the same number of factors.

The article is organized as follows: in Section 2 we introduce the basic notions concerning topological elliptic fibrations over the unit disk and the classification of relatively minimal Lefschetz ones. The central result there is Theorem 2.8 which relates the problem of classifying all of *special* elliptic fibrations over the disk to the problem of classifying their monodromy representations, up to conjugation and Hurwitz equivalence, in the modular group. Section 3 is devoted to the study of the relationship between (u-)*special* factorizations in $PSL(2,\mathbb{Z})$ and their liftings to $SL(2,\mathbb{Z})$. Section 4 deals with a combinatorial study of Hurwitz equivalence of (s_1-)special factorization in the modular group.

Section 5 presents an algorithm for generating a relatively simple set that is H-complete in the set of (s_1-)special factorizations of any given element in the modular group.

2. Special elliptic fibrations over the disk and Hurwitz equivalence

Definition 2.1. Let Σ be a compact, connected and oriented smooth two dimensional manifold (with or without boundary). A *topological elliptic fibration over* Σ is a smooth function $f : M \to \Sigma$ such that

1. (1) M is a compact, connected and oriented four dimensional smooth manifold (with or without boundary).
2. (2) f is surjective.
3. (3) $f(\text{int}(M)) = \text{int}(\Sigma)$ and $f(\partial M) = \partial\Sigma$.
4. (4) f has a finite number (possibly zero) of critical values $q_1, \ldots, q_n$ all contained in $\text{int}(\Sigma)$.
5. (5) f is locally holomorphic, that is, for each $p \in \text{int}(M)$ there exist orientation preserving charts from neighborhoods of p and $f(p)$, to open sets of $\mathbb{C}^2$ and $\mathbb{C}$ (endowed with their standard orientations), respectively, relative to which f is holomorphic.
6. (6) The preimage of each regular value is a smooth two dimensional manifold that is closed and connected, and has genus one.

Next we define when two topological elliptic fibrations are considered as topologically the same.

Definition 2.2. Two topological elliptic fibrations $f_1 : M_1 \to \Sigma_1$ and $f_2 : M_2 \to \Sigma_2$ are *topologically equivalent*, written as $f_1 \sim_{\text{Top}} f_2$, if there exist orientation preserving diffeomorphisms $H : M_1 \to M_2$ and $h : \Sigma_1 \to \Sigma_2$, such that $h \circ f_1 = f_2 \circ H$.

Definition 2.3. A topological elliptic fibration $f : M \to \Sigma$ will be called

- (i) *Relatively minimal* if none of its fibers contains an embedded sphere with selfintersection -1.
- (ii) *Lefschetz* if for each critical point p (necessarily contained in $\text{int}(M)$) of f there exist charts as in condition (5) above relative to which f takes the form $(z_1, z_2) \to z_1^2 + z_2^2$, and f is injective when restricted to the set of critical points.

For the sake of brevity, topological elliptic fibrations will be referred to as *elliptic fibrations*, and being relatively minimal and Lefschetz will be abreviated by the adjective *special*. Hence, instead of saying relatively minimal Lefschetz topological elliptic fibrations, we will say *special elliptic fibrations*.

We notice that being *special* is preserved by topological equivalence.

In what follows we will only consider special elliptic fibrations over the closed unit disk, $D = \{z \in C : |z| \leq 1\}$*, endowed with its standard orientation.*

Definition 2.4. Let G be a group.

(i) Any n-tuple of elements of G, $\alpha = (g_1, \ldots, g_n)$, $n \geq 0$, will be called a *factorization*. The only 0-tuple (the empty tuple) will be denoted by ( ).

(ii) The element $g_1 \cdots g_n$ will be called the *product* of the factorization, and will be denoted by $\mathrm{prod}(\alpha)$. When α is empty, we define its product as the identity element of G. Given any g in G, we will say that *α is a factorization of g* if its product is equal to g.

Definition 2.5. Let G be a group.

(i) Let $n \geq 2$. For any integer $1 \leq i \leq n-1$, a *Hurwitz right move at position i*, is the function $H_i : G^n \to G^n$ defined as

$$H_i(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \ldots, g_n)\,.$$

The inverse function is called a *Hurwitz left move at position i*, which is given by

$$H_i^{-1}(g_1, \ldots, g_i, g_{i+1}, \ldots, g_n) = (g_1, \ldots, g_{i-1}, g_i g_{i+1} g_i^{-1}, g_i, g_{i+2}, \ldots, g_n)\,.$$

When $H_i(g_1, \ldots, g_n) = (g_1', \ldots, g_n')$ (resp. $H_i^{-1}(g_1, \ldots, g_n) = (g_1', \ldots, g_n')$) we will say that $(g_1', \ldots, g_n')$ *is obtained from* $(g_1, \ldots, g_n)$ *by a Hurwitz right move* (respectively, *by a Hurwitz left move*) *at position i.*

(ii) Let $\alpha = (g_1, \ldots, g_n)$ and $\alpha' = (g_1', \ldots, g_m')$ be factorizations in G. If α' is obtained from $\alpha = (g_1, \ldots, g_n)$ by successive applications of Hurwitz moves, we will say that *α and α' are H-equivalent*, which we denote by $\alpha \sim_H \alpha'$. In this case, it follows immediately that $n = m$ and $prod(\alpha) = prod(\alpha')$. On the other hand, if there exists an element h such that $\alpha \sim_H (h^{-1} g_1' h, \ldots, h^{-1} g_n' h)$, we will say that *$\alpha$ and α' are $H+C$-equivalent.* This will be denoted by $\alpha \sim_{H+C} \alpha'$. In this case, it follows immediately that $prod(\alpha)$ and $prod(\alpha')$ are conjugate with each other.

(iii) Given a set $\mathcal{S}$ of factorizations, a set $\mathcal{T} \subset \mathcal{S}$ will be said to be *H-complete in $\mathcal{S}$* (*$H + C$-complete in $\mathcal{S}$*) if for each $\alpha \in \mathcal{S}$ there is at least one $\alpha' \in \mathcal{T}$ such that $\alpha \sim_H \alpha'$ ($\alpha \sim_{H+C} \alpha'$, respectively).

It is clear that being $H + C$-equivalent is weaker than being H-equivalent.

Definition 2.6. Factorizations $(G_1, \ldots, G_n)$ in $SL(2, \mathbb{Z})$ such that each G_i is a conjugate of U will be called *special factorizations in $SL(2, \mathbb{Z})$* or simply *special factorizations.*

Let $f : M \to D$ be any special elliptic fibration over the disk. Let us denote by q_0 the point $(1, 0)$, and by C the boundary of the disk with its standard counterclockwise orientation. As usual,

$$\rho : \pi_1(D - \{q_1, \ldots, q_n\}, q_0) \to SL(2, \mathbb{Z})$$

will stand for the *monodromy representation* where we have identified the mapping class group of T^2, a fixed model of the regular fiber, with $SL(2, \mathbb{Z})$. The mapping ρ is an anti-homomorphism determined by its action on any basis of the rank n free group $\pi_1(D - \{q_1, \ldots, q_n\}, q_0)$. We may take $\{[\gamma_1], \ldots, [\gamma_n]\}$ the standard basis consisting of the classes of clockwise oriented, pairwise disjoint arcs where each γ_i surrounds exclusively the critical value q_i, $i = 1, \dots, n$. We may choose the γ_i's in such a way that (for an appropriate numbering of the q_i's) the product $[\gamma_1] \cdots [\gamma_n]$ equals the class of C. The conjugacy class in $SL(2, \mathbb{Z})$ of $\rho([C])$ is called the *total monodromy* of the fibration. It can be readily seen that this is a well defined notion.

Remark 2.7. If $f : M \to D$ is any special elliptic fibration over the disk, since each singular fiber has a single ordinary double point (of type I_1, in Kodaira's classification [8]) the monodromy around any of these fibers is in the conjugacy class of $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $SL(2, \mathbb{Z})$.

Special elliptic fibrations over D can be classified up to conjugation and Hurwitz moves. More precisely:

Theorem 2.8. *Let $f_1 : M_1 \to D$ and $f_2 : M_2 \to D$ be two special elliptic fibrations. Let us fix monodromy representations ρ and ρ', and bases $\{[\gamma_1], \dots, [\gamma_n]\}$, and $\{[\gamma'_1], \dots, [\gamma'_n]\}$, for f_1 and f_2, respectively. Let $g_i = \rho([\gamma_i])$, and $g'_i = \rho'([\gamma'_i])$. Then, f_1 and f_2 are topologically equivalent if and only if $\alpha = (g_1, \dots, g_n)$ and $\alpha' = (g'_1, \dots, g'_n)$ are equivalent under the equivalence relation $\sim_{H+C}$.*

For a proof see [5].

Hence, special factorizations modulo $\sim_{H+C}$ are in bijective correspondence with special elliptic fibrations over the disk modulo topological equivalence. Therefore, in order to classify these fibrations, it suffices to describe special factorizations modulo $\sim_{H+C}$. In this article *we present an algorithm that for any given matrix B in $SL(2, \mathbb{Z})$ produces a set that is H-complete in the set of special factorizations of B.* In general, this set could be redundant in the sense that it might contain more than one representative in some equivalence classes. Since $H+C$-equivalence is weaker than H-equivalence, it is clear that this set is also $H+C$-complete. *Therefore, for any given B, this algorithm will provide at least one special elliptic fibration over the disk within each topological equivalence class, whose total monodromy is in the conjugacy class of B.*

3. Special factorizations in the modular group

Even though it is well known that $SL(2, \mathbb{Z})$ is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

it is important for our purposes that a decomposition of any matrix in $SL(2, \mathbb{Z})$ as product of powers of S and U (or equivalently, as a product of powers of S and $R = SU$) can be achieved algorithmically. This is the content of the next proposition.

Proposition 3.1. *Every matrix in $SL(2, \mathbb{Z})$ can be written as a product of powers of S and U. Moreover, there is an algorithm that given any matrix B in $SL(2, \mathbb{Z})$ yields one of such factorizations.*

Proof. For any matrix A, U^nA is the matrix obtained from A by performing the row operation corresponding to adding n times the second row to the first, while SA is the matrix obtained from A by performing the row operation corresponding to interchanging the first and second row, and multiplying the first row by -1.

For any matrix $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, since $\det(B) = 1$, the entries a and c must be relatively prime. If $|c| < |a|$, by the Euclidean algorithm, if $a = cn + r$, then by premultiplying by U^{-n} we obtain a matrix of the form $U^{-n}B = \begin{pmatrix} r & b' \\ c & d \end{pmatrix}$ with $b' = b - nd$. In case where $|a| < |c|$, we may first multiply by S to interchange the rows. Thus, in any case, premultiplying by U^{-n}, or by $U^{-n}S$, has the effect of putting B in the form $\begin{pmatrix} r & b' \\ c & d \end{pmatrix}$, where $\gcd(a, c) = \gcd(c, r)$ (gcd denotes the greatest common divisor). By successively premultiplying by S, and suitable powers of U, we may transform B into a matrix of the form $B' = \begin{pmatrix} \pm 1 & m \\ 0 & k \end{pmatrix}$. That is, $B' = PB$, where P is a product of S and powers of U. Since $B'$is in $SL(2, \mathbb{Z})$, k must be equal to ± 1. Therefore, $B' = \pm I_2 U^{\pm m}$. Since $S^2 = -I_2$, then $B = P^{-1}(\pm I_2)U^{\pm m}$. □

The modular group, $SL(2, \mathbb{Z})/\{\pm I_2\}$, is usually denoted by $PSL(2, \mathbb{Z})$. For the sake of brevity, we will denote this group by $\mathcal{M}$. The classes of S, U and R will be denoted by ω, u and b, respectively. Note that $b = \omega u$. It is a well known fact

$$\mathcal{M} = \left\langle \omega, b \,\middle|\, \omega^2 = b^3 = 1 \right\rangle .$$

The following corollary is an immediate consequence of the previous proposition.

Corollary 3.2. *There is an algorithm that expresses any element in $\mathcal{M}$ as a product of positive powers of ω and b.*

Let $\pi : SL(2, \mathbb{Z}) \to \mathcal{M}$ denote the canonical homomorphism to the quotient.

Definition 3.3. A factorization $\alpha = (g_1, \ldots, g_n)$ in $\mathcal{M}$ will be called *u-special* if each g_i is a conjugate of u.

A special factorization $(A_1, \ldots, A_n)$ in $SL(2, \mathbb{Z})$ will be called a *lift of* α, if $\pi(A_i) = g_i$ for each i.

We observe that each u-special factorization $\alpha = (g_1, \ldots, g_n)$ in $\mathcal{M}$ has exactly one lift. Indeed, if $g_i = a_i u a_i^{-1}$, then its preimages are $\pm A_i U A_i^{-1}$, where A_i is any preimage of a_i. But only $A_i U A_i^{-1}$ is a conjugate of U, since the trace$(-A_i U A_i^{-1}) = -2$, and every conjugate of U has trace 2. The lift of α will be denoted by lift(α).

Now, in $\mathcal{M}$, if α' is obtained from α by performing a Hurwitz move, then lift(α') can be obtained from lift(α) by the corresponding move. Reciprocally, Hurwitz moves in $SL(2, \mathbb{Z})$ can be transformed into Hurwitz moves in $\mathcal{M}$ via π. Therefore, $\alpha \sim_H \alpha'$ if and only if lift$(\alpha) \sim_H$ lift(α'). From this, it follows that H-complete sets in the set of special factorizations of a matrix B in $SL(2,\mathbb{Z})$, can be obtained from H-complete sets in the set of u-special factorizations of $\pi(B)$. More precisely:

Proposition 3.4. *Let B be an element of $SL(2,\mathbb{Z})$. If $\mathcal{S}$ is H-complete in the set of u-special factorizations of $\pi(B)$, then the collection $\mathcal{R} = \{lift(\alpha) : \alpha \in \mathcal{S}\}$ is H-complete in the set of special factorizations of B.*

4. H-complete sets in the sets of special factorizations of elements of $\mathcal{M}$

In order to make this article more leisurely readable, we present in this section a series of propositions that are basically a reformulation of results that appear in [4], [9], and [10]. For instance, Proposition 4.7 and its consequences are a restatement of the addendum of Theorem 4.25 of [4], using Lemma 2.5 in [9].

In this section, we identify $\mathcal{M}$ with the free product

$$\left\langle \omega, b \,\middle|\, \omega^2 = b^3 = 1 \right\rangle .$$

There is a unique automorphism ϕ of $\mathcal{M}$ that sends ω into itself and b into b^2. Let us denote by $c_b : \mathcal{M} \to \mathcal{M}$ conjugation by b, i.e., $c_b(z) = bzb^{-1}$, and by h the composition $h = c_b \circ \phi$. The problem of finding sets which are H-complete in the sets of factorizations, in terms of conjugates of $u = \omega b$, of given elements of $\mathcal{M}$, is equivalent, via h, to the problem of finding sets which are H-complete in the sets of factorizations, in terms of conjugates of $h(u) = b\omega b$, of given elements of $\mathcal{M}$.

It is important to have a symbol for the empty word: We will denote it by 1.

It is a standard fact that each element a in $\mathcal{M}$ can be written uniquely as a product $a = t_k \cdots t_1$, where each t_i is either ω, b, or b^2 and no consecutive pair $t_i t_{i-1}$ is formed either by two powers of b or two copies of ω. We call the product $t_k \cdots t_1$ the *reduced expression of a*, and we call k the *length* of a, denoted by $l(a)$. Let s_1 denote the element $b\omega b$. The shortest conjugates of s_1 in $\mathcal{M}$ are precisely (see [4]) $s_0 = b^2(b\omega b)b = \omega b^2$ and $s_2 = b(b\omega b)b^2 = b^2\omega$. The element s_1 is trivially a conjugate of itself of length 3. It can be easily seen that if g is a conjugate of greater length, its reduced expression is of the form $Q^{-1}s_1Q$, where Q is a reduced word that begins with ω (see [4]), and $l(g) = 2l(Q) + 3$. We will call a conjugate of s_1 (*conjugate* will always mean conjugate of s_1 in $\mathcal{M}$) *short* if $g \in \{s_0, s_1, s_2\}$, otherwise it will be called *long*.

The following notion is the key ingredient for understanding the reduced expression of a product of conjugates of s_1.

Definition 4.1. We will say that two conjugates g and h of s_1 *join well* if we have $l(gh) \geq \max(l(g), l(h))$. Otherwise, we say they *join badly*.

Definition 4.2. A factorization $\alpha = (g_1, \ldots, g_n)$ in $\mathcal{M}$ is called s_1-*special* if each g_i is a conjugate of s_1. We say α is *well joined* if each pair of elements g_i, g_{i+1} join well. Otherwise, we say that α is *badly joined*.

The empty factorization will be regarded as being s_1-special, and well joined. s_1-Special factorizations with just one element will also be regarded as well joined.

Remark 4.3. We notice that the following identities hold:
$s_2s_2 = b^2\omega b^2\omega$, $s_1s_1 = b\omega b^2\omega b$,
$s_0s_0 = \omega b^2\omega b^2$, $s_2s_1 = b^2\omega b\omega b$,
and, $s_1s_0 = b\omega b\omega b^2$ and $s_0s_2 = \omega b\omega$.
Hence, the corresponding factorizations in each case are well joined. On the other hand, since $s_0s_1 = s_1s_2 = s_2s_0 = b$, the corresponding factorizations are badly joined.

The following propositions will be useful for the proof of one of the main results used for the construction of sets H-complete in the sets of s_1-special factorizations of given elements of $\mathcal{M}$.

Proposition 4.4. *Every s_1-special factorization $\alpha = (g_1, \ldots, g_n)$ can be transformed by Hurwitz moves into a factorization $\beta = (h_1, \ldots, h_n)$(necessarily s_1-special, and with the same number of factors), satisfying:*
(i) Each h_i is short, or
(ii) β is well joined and at least one of the $h_i's$ is long.

Proof. See [4]. □

Proposition 4.5. *Every s_1-special factorization (g_1, g_2, g_3) in which each g_i is short, and where g_1, g_2 join badly, is H-equivalent to a factorization (g'_1, g'_2, g'_3), where each g'_i is short, and g'_2, g'_3 join badly.*

Proof. The only pairs of short conjugates that do not join well are $(s_0, s_1), (s_1, s_2), (s_2, s_0)$. It follows that for each s_i there exists an s_j such that (s_j, s_i) does not join well. We also notice that any two of these pairs are H-equivalent. Hence, for g_3, there is s_j such that (s_j, g_3) does not join well. Therefore, after a Hurwitz move performed on the pair (g_1, g_2), transforming it into (g'_1, g'_2), with $g'_2 = s_j$, then, the factorization (g'_1, g'_2, g'_3) with $g'_3 = g_3$, is Hurwitz equivalent to (g_1, g_2, g_3), and (g'_2, g'_3) does not join well. □

Proposition 4.6. *(i) Every factorization $(g_1, \ldots, g_n)$ where each g_i is a short conjugate, and where not all pairs of elements g_i, g_{i+1} join well, is H-equivalent to a factorization $(g'_1, \ldots, g'_n)$, where (g'_{n-1}, g'_n) join badly.*
(ii) Every factorization $(g_1, \ldots, g_n)$ in short conjugates where not all pairs of elements g_i, g_{i+1} join well is H-equivalent to a factorization $(g'_1, \ldots, g'_n)$ in which we have $(g'_{n-1}, g'_n) = (s_0, s_1)$.

Proof. For each factorization $\alpha = (g_1, \ldots, g_n)$ $(n \geq 2)$ in short conjugates where not all pairs of elements g_i, g_{i+1} join well we associate the integer

$$k(\alpha) = n - \max\{r : (g_r, g_{r+1}) \text{ does not join well}\}.$$

The proof proceeds by induction on $k = k(\alpha)$. If $k = 1$, then (g_{n-1}, g_n) join badly, and the result follows. For $k_0 \geq 1$, let us suppose that the result is true for all α such that $k(\alpha) \leq k_0$. Let $\beta = (g_1, \ldots, g_n)$ be a factorization with $k(\beta) = k_0 + 1$. This implies that (g_{n-k_0-1}, g_{n-k_0}) does not join well. Applying Proposition 4.5 we infer that $(g_{n-k_0-1}, g_{n-k_0}, g_{n-k_0+1})$ is H-equivalent to a factorization $(g'_{n-k_0-1}, g'_{n-k_0}, g'_{n-k_0+1})$ in short conjugates, such that $(g'_{n-k_0}, g'_{n-k_0+1})$ join badly. Summarizing, the original factorization β is H-equivalent to a factorization in short conjugates $\beta' = (g'_1, \ldots, g'_n)$ in which $(g'_{n-k_0}, g'_{n-k_0+1})$ join badly. Clearly $k(\beta') < k(\beta)$, thus the proposition holds for β', i.e., β' is H-equivalent to another factorization in short conjugates $\beta'' = (g''_1, \ldots, g''_n)$ in which (g''_{n-1}, g''_n) join badly. We conclude that the result also holds for β, since β is Hurwitz equivalent to β''. This proves the first statement. The second assertion easily follows from the fact that all pairs of short conjugates that join badly are H-equivalent to (s_0, s_1). □

Proposition 4.7. *Every factorization $(g_1, \ldots, g_n)$ in short conjugates is H-equivalent to another factorization in short conjugates, of the form $(g'_1, \ldots, g'_m, s_0, s_1, \ldots, s_0, s_1)$, $0 \leq m \leq n$, where there are $(n-m)/2$ pairs of s_0, s_1, and $(g'_1, \ldots, g'_m)$ is well joined.*

Proof. Let $\alpha = (g_1, \ldots, g_n)$ be a factorization in short conjugates. Each factorization $\beta = (h_1, \ldots, h_n)$ in short conjugates that is H-equivalent to α can be written uniquely as $(h_1, \ldots, h_m, s_0, s_1, \ldots, s_0, s_1)$ where there are $r \geq 0$ pairs s_0, s_1, and where $m \geq 0$ and $(h_{m-1}, h_m) \neq (s_0, s_1)$, if $m \geq 2$. The integer r will be denoted by $r(\beta)$ to indicate its dependence on β. Let $\gamma = (g'_1, \ldots, g'_m, s_0, s_1, \ldots, s_0, s_1)$ be a factorization in short conjugates, H-equivalent to α, such that $r(\gamma) \geq r(\beta)$ for any other factorization in short conjugates β, H-equivalent to α. Let us verify that $(g'_1, \ldots, g'_m)$ is well joined. If $(g'_1, \ldots, g'_m)$ is badly joined, and $m \geq 2$, by the second part of Proposition 4.6 there would be another factorization in short conjugates $(g''_1, \ldots, g''_m)$ H-equivalent to $(g'_1, \ldots, g'_m)$, and such that $(g''_{m-1}, g''_m) = (s_0, s_1)$. Hence, γ would also be (and, therefore α), H-equivalent to a factorization in short conjugates $(g''_1, \ldots, g''_{m-2}, s_0, s_1, \ldots, s_0, s_1)$ with $r(\gamma) + 1$ pairs s_0, s_1, in contradiction with the maximality of γ. Thus, $(g'_1, \ldots, g'_m)$ is well joined. □

Proposition 4.8. *Each s_1-special factorization $(g_1, \ldots, g_n)$ is H-equivalent to a factorization of the form $(g'_1, \ldots, g'_m, s_0, s_1, \ldots, s_0, s_1)$, where there are $r \geq 0$ pairs s_0, s_1, and where $(g'_1, \ldots, g'_m)$ is well joined. Moreover, $(g'_1, \ldots, g'_m)$ is a factorization in short conjugates, whenever $r > 0$.*

Proof. By Proposition 4.4, $(g_1, \ldots, g_n)$ is H-equivalent to a factorization $\beta = (g'_1, \ldots, g'_n)$ that either is well joined and at least one of the $g'_i s$ is a long conjugate, or it is badly joined and all $g'_i s$ are short conjugates. In the first case, β already has the desired form, since the fact that the factors join well implies that $(g'_{n-1}, g'_n) \neq (s_0, s_1)$, and consequently $r = 0$. Now, in case β consists of short conjugates that join well, then it also has already the desired form for the same reason. Hence, let us suppose that β is a factorization in short conjugates that is badly joined. By Proposition 4.7, this factorization is H-equivalent to another one in short conjugates, of the form $(g''_1, \ldots, g''_m, s_0, s_1, \ldots, s_0, s_1)$, with $(n-m)/2$ pairs s_0, s_1, and where $(g''_1, \ldots, g''_m)$ is well joined. □

An immediate consequence is the following theorem.

Theorem 4.9. *For each $g \in \mathcal{M}$, the set of all s_1-special factorizations of g having either of the following two forms is H-complete in the set of s_1-special factorizations of g:*

(i) $(g_1, \ldots, g_m, s_0, s_1, \ldots, s_0, s_1)$, *where there are $r > 0$ pairs s_0, s_1, $(g_1, \ldots, g_m)$ is well joined, and each g_i is short.*

(ii) $(g_1, \ldots, g_p)$, *where this factorization is well joined.*

5. An algorithm to produce H-complete sets

For h in the modular group, let us denote by $WJ(h)$ the set formed by all s_1-*special factorizations* of h that are well joined, and by $WJS(h)$ the set consisting of those factorizations in $WJ(h)$ that are in short conjugates. Remember that we regard the empty factorization ( ) as a *well joined s_1-special factorization of the identity* 1, *in short conjugates.*

Since $s_0 s_1 = b$ and $b^3 = 1$, we have that $(s_0 s_1)^{3k+l}$ equals 1 if $l = 0$, b if $l = 1$ and b^2 if $l = 2$. According to Theorem 4.9, for any fixed element g, the union of the following four sets of factorizations of g is H-complete in the set of all s_1-special factorizations of g:

(i) $A = \{\alpha : \alpha$ is a *well joined s_1-special* factorization of $g\}$.

(ii) $B = \{(g_1, \ldots, g_m, s_0, s_1, \ldots, s_0, s_1) : (g_1, \ldots, g_m)$ is a *well joined s_1-special* factorization of g in *short* conjugates and the number of pairs s_0, s_1 is of the form $3k$, with $k \geq 1\}$.

(iii) $C = \{(g_1, \ldots, g_m, s_0, s_1, \ldots, s_0, s_1) : (g_1, \ldots, g_m)$ is a *well joined s_1-special* factorization of gb^2 in *short* conjugates and the number of pairs s_0, s_1 is of the form $3k+1$, with $k \geq 0\}$.

(iv) $D = \{(g_1, \ldots, g_m, s_0, s_1, \ldots, s_0, s_1) : (g_1, \ldots, g_m)$ is a *well joined s_1-special* factorization of gb in *short* conjugates and the number of pairs s_0, s_1 is of the form $3k+2$, with $k \geq 0\}$.

Remark 5.1. As a consequence, in order to find a set that is H-complete in the set of s_1-special factorizations of an element g we need i) an algorithm that takes an element h in the modular group, and produces the set $WJ(h)$, and ii) an algorithm that extracts the subset $WJS(h)$.

This second task is trivial, but the first one is less so. The key ingredient to formulate the algorithm in i) is discussed next.

Definition 5.2. We define the *left part* of short conjugates of $b\omega b$ as follows:

$$\begin{aligned} left(s_0) &= left(\omega b^2) = \omega \\ left(s_1) &= left(b\omega b) = b\omega \\ left(s_2) &= left(b^2\omega) = b^2\omega. \end{aligned}$$

For long conjugates, $left(P^{-1} b\omega b P) = P^{-1} b\omega$, where P is an element of the modular group that begins with ω.

The following result is Lemma 2.4 in [9].

Proposition 5.3. *If* $(h_1, \ldots, h_n)$ *with* $n \geq 1$ *is a* s_1*-special factorization that is well joined, the reduced expression of its product* $h_1 \cdots h_n$ *begins with* $left(h_1)$*.*

Proof. See [9]. □

If $(h_1, \ldots, h_n)$ is a well joined s_1-special factorization of h, then there is only a finite number of possible values for h_1, at most one for each factor ω in the reduced expression of h. The first possibility for h_1 is either ωb^2, $b\omega b$, or $b^2\omega$, depending on whether the reduced expression of h begins with ω, $b\omega$ or $b^2\omega$, respectively. Every other factor ω yields a possible value for h_1 if its preceding factor is b: In this case the prefix of the reduced expression of h which ends at the mentioned factor ω has the form $P^{-1}b\omega$, where P begins with ω, and the possible value of h_1 is $P^{-1}b\omega bP$.

In particular, the element 1 has only one well joined s_1-special factorization, namely the empty factorization. Also, b and b^2 admit no well joined s_1-special factorizations.

Now we give an algorithm, that we will call *FirstFactor*, which takes as input any element h in the modular group, with h not in the set $\{1, b, b^2\}$, and produces all possible candidates to be first factors in any well joined s_1-special factorization of h. This algorithm outputs a set that contains:

- **a:** ωb^2, if h begins with ω,
- **b:** $b\omega b$, if h begins with $b\omega$,
- **c:** $b^2\omega$, if h begins with $b^2\omega$
- **d:** For each occurrence of $b\omega$ that is not at the beginning of h, the element $P^{-1}b\omega bP$, where P^{-1} is the initial section of h ending right before the occurrence of $b\omega$ starts.

Then we define another algorithm that we will call *Sibling*. This algorithm receives as input an ordered pair $((g_1, \ldots, g_n), z)$, where $(g_1, \ldots, g_n)$ is a s_1-special factorization and z is any element in the modular group. Then, *Sibling* takes the following actions:

(1) If $z = 1$, then *Sibling* outputs the set $\{((g_1, \ldots, g_n), z)\}$.
(2) If z is b or b^2, then *Sibling* outputs the empty set $\{\ \}$.
(3) If z is different from 1, b and b^2, and $(g_1, \ldots, g_n)$ is the empty factorization, then *Sibling* computes the (necessarily nonempty) set $F = \mathrm{FirstFactor}(z)$, and then outputs the set $\{((g), g^{-1}z) : g \in F\}$.
(4) If z is different from 1, b and b^2, and $(g_1, \ldots, g_n)$ is not the empty factorization, *Sibling* computes the (necessarily nonempty) set $F = \mathrm{FirstFactor}(z)$ and then outputs the set $\{((g_1, \ldots, g_n, g), g^{-1}z) : g \in F \text{ and } (g_n, g) \text{ join well}\}$.

We make the following elementary but important observations:

(1) For each pair $((h_1, \ldots, h_n), z)$ formed by a factorization and any element z in the modular group, we call $h_1 \cdots h_n z$ *the product of the pair.* Then *Sibling* preserves products, i.e., each pair in *Sibling*$(((g_1, \ldots, g_n), z))$ has the same product as the pair $((g_1, \ldots, g_n), z)$. Notice that this statement is true even if z is b or b^2.
(2) If $((g_1, \ldots, g_n), z)$, where $(g_1, \ldots, g_n)$ is a well joined s_1-special factorization with $n \geq 0$, then the first component of each element of *Sibling*$(((g_1, \ldots, g_n), z))$ is a

s_1-special factorization that is also well joined. Notice that this is true even if z is b or b^2.

(3) (a) *Sibling*$(((g_1,\ldots,g_n),\omega)) = \{((g_1,\ldots,g_n,\omega b^2),b)\}$ for any s_1-special factorization $(g_1,\ldots,g_n)$ with $n \geq 0$. Therefore

$$\textit{Sibling}(\textit{Sibling}(((g_1,\ldots,g_n),\omega))) = \{\ \}$$

for any s_1-special factorization $(g_1,\ldots,g_n)$ with $n \geq 0$.

(b) *Sibling*$(((g_1,\ldots,g_n),z)) = \{\ \}$, if $z = b, b^2$ and $(g_1,\ldots,g_n)$ is a s_1-special factorization with $n \geq 0$.

(c) Sibling$(((g_1,\ldots,g_n),z)) = \{((g_1,\ldots,g_n),z)\}$, if z is 1 and $(g_1,\ldots,g_n)$ is a s_1-special factorization with $n \geq 0$.

(d) Let $z \notin \{1,\omega,b,b^2\}$ and let $g \in \text{FirstFactor}(z)$. Let us see that $l(g^{-1}z) < l(z)$. Let z begin with ω and $g = \omega b^2$. Then z will be of the form $\omega b^\delta Q$, where $\delta = 1,2$ and Q is a reduced word that is 1 or begins with ω. We have

$$g^{-1}z = (\omega b^2)^{-1}(\omega b^\delta\ Q) = (b\omega)(\omega b^\delta)Q = b^\gamma Q,$$

where γ is 0 or 2, and clearly $l(b^\gamma Q) < l(\omega b^\delta Q)$. Let z begin with $b\omega$ and $g = b\omega b$. In this case z is of the form $b\omega Q$, where Q is a reduced word that is either 1 or begins with b or b^2. We have $g^{-1}z = (b^2\omega b^2)(b\omega Q) = b^2 Q$, and clearly $l(b^2Q) < l(b\omega Q)$. Let z begin with $b^2\omega$ and let $g = b^2\omega$. It is clear that $l(g^{-1}z) < l(z)$ in this case. Let z begin with $P^{-1}b\omega$, where P is a reduced word that begins with ω, and let $g = P^{-1}b\omega bP$. Then z is of the form $P^{-1}b\omega Q$, where Q is a reduced word that is either 1 or begins with b^δ with $\delta = 1,2$. Then $g^{-1}z = (P^{-1}b^2\omega b^2 P)(P^{-1}b\omega Q) = P^{-1}b^2Q$. Clearly, $l(P^{-1}b^2Q) < l(P^{-1}b\omega Q)$.

Now we define another routine, that we will call *SiblingSets* that takes as input a set S whose elements are ordered pairs of the form $((g_1,\ldots,g_n),z)$, and outputs the set $\cup_{s\in S}\text{Sibling}(s)$. Notice that *SiblingSets* applied to the empty set gives the empty set. Finally, we define a routine, that we call *Welljoined* that takes an element h in the modular group as input, then calculates the result of applying $l(h)+1$ times *SiblingSets* to the set $\{((\ ),h)\}$, i.e., calculates $T = \textit{SiblingSets}^{l(h)+1}(\{((\ ),h)\})$, and then outputs the set formed by the first components of the ordered pairs in T.

By all the observations above, the algorithm *Welljoined* finds all possible well joined s_1-special factorizations of any element h. By Remark 5.1 this is all we needed in order to find a set that is H-complete in the set of s_1-special factorizations of an element g.

A close look at the routine *SiblingSets* reveals that an application of this routine $l(h)+1$ times gives an upper bound for the number of elements of $WJ(h)$, namely $|WJ(h)| \leq l(h)!$

As properly suggested by the referee, an alternative approach based on results in [4] gives a much better upper bound:

$$|WJ(h)| \leq 2^{\text{number of } \omega\text{'s in } h} \leq 2^{\lceil \frac{l(h)}{2} \rceil}$$

In order to show this is true, it suffices to notice that if $(g_1, \ldots, g_n)$ is a well joined factorization of h, "the central ω" of each factor remains in the reduced expression of h (the first factor g_1 is determined by the position of the leftmost central ω in each iteration).

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