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Mutation and recombination for hyperbolic 3-manifolds

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Abstract. We give a short topological proof for Ruberman's Theorem about mutation and volume, using the Maskit combination theorem and the homology of the linear group.

1. Introduction

Let $\rho : \Gamma \rightarrow SL(2, \mathbf{C})$ be a faithful representation with discrete, torsion-free image, and let $M = \Gamma \backslash \mathbf{H}^3$ be the associated orientable, hyperbolic 3-manifold. We assume that Γ has finite covolume, that is $vol(M) < \infty$.

Let $\Sigma \subset M$ be a properly embedded, incompressible, boundary-incompressible, 2-sided surface. Let $\tau : \Sigma \rightarrow \Sigma$ be a diffeomorphism. We will denote by M^τ the result of cutting M along Σ and regluing via τ.

Ruberman's Theorem ([9, Theorem 1.3]) states that for some specific pairs (Σ, τ), for example for the hyperelliptic involution of the genus 2 surface, one always has that M^τ is hyperbolic and $vol(M^\tau) = vol(M)$.

The pairs (Σ, τ) considered in [9] have in common that τ has finite order and that for each ρ as above the representation $\rho\mid_{\pi_1\Sigma} \circ \tau_* : \pi_1\Sigma \rightarrow SL(2, \mathbf{C})$ is conjugate in $SL(2, \mathbf{C})$ to $\rho\mid_{\pi_1\Sigma}$, see [9, Theorem 2.2].

As indicated in [9, Theorem 4.4] this is actually the only specific property of the pairs (Σ, τ) which is needed. That is, the conclusion of M^τ being hyperbolic and having the same volume as M does actually hold for all pairs (Σ, τ) such that $\rho\mid_{\pi_1\Sigma} \circ \tau_*$ is conjugate in $SL(2, \mathbf{C})$ to $\rho\mid_{\pi_1\Sigma}$ (and such that Σ is not a virtual fiber).

The proof in [9] uses hard results about minimal surfaces. As Ruberman points out in [9, Theorem 4.4], an alternative proof is possible using Bers' Theorem about the deformation space of quasifuchsian groups, yet another hard analytical theorem.

The aim of this paper is to give a purely topological proof of Ruberman's result which does not use any facts from analysis, but rather a simple version of the Maskit combination theorems and group-homological arguments. (The combination theorems have a purely topological proof in [7], although nowadays there exist also analytical proofs using minimal surfaces or harmonic maps.)

Key words and phrases. Mutation, volume, hyperbolic 3-manifold.

Theorem 1. *Let M be a compact, orientable, connected 3-manifold with (possibly empty) boundary. Let $\rho : \Gamma \to SL(2, \mathbf{C})$ be a representation that is a lift of a faithful representation $\Gamma \to Isom^+ \left(\mathbf{H}^3\right) = PSL(2, \mathbf{C})$ with discrete, torsion-free image. Assume that $\rho(\Gamma) \backslash \mathbf{H}^3$ has finite volume and is diffeomorphic to $int(M) = M - \partial M$.*

Let $\Sigma \subset M$ be a properly embedded, connected, incompressible, boundary-incompressible, 2-sided surface which is not a virtual fiber.

Let $\tau : (\Sigma, \partial\Sigma) \to (\Sigma, \partial\Sigma)$ be an orientation-preserving diffeomorphism of pairs such that $\tau^m = id$ for some $m \in \mathbf{N}$ and such that there exists some $A \in SL(2, \mathbf{C})$ with $\rho(\tau_* h) = A\rho(h) A^{-1}$ for all $h \in \pi_1\Sigma$.

Then $int(M^\tau)$ is hyperbolic and $vol(M^\tau) = vol(M)$.

The proof of $vol(M^\tau) = vol(M)$ will follow an argument analogous to the proof of [8, Theorem 2.13] but will use the constructions from [5, Section 4] to handle the case of manifolds with several cusps (where relative group homology does not directly apply).

2. Preliminaries

Let M be an orientable, complete, hyperbolic 3-manifold of finite volume and Σ a properly embedded, 2-sided surface. Let $\Gamma = \pi_1 M$. (We assume without further mention M and Σ to be connected and we omit basepoints from the notation.) The monodromy has image in $Isom^+ \left(\mathbf{H}^3\right) = PSL(2, \mathbf{C})$. By Culler's Theorem [4, Corollary 2.2] the monodromy comes from a representation $\rho : \Gamma \to SL(2, \mathbf{C})$ and we will henceforth assume such a representation ρ to be fixed.

Let $\tau : \Sigma \to \Sigma$ be a diffeomorphism. Throughout the paper we will work with the following assumption on (Σ, τ):

Assumption 1.1. *There exists some $A \in SL(2, \mathbf{C})$ with*

$$\rho(\tau_* h) = A\rho(h) A^{-1}$$

for all $h \in \pi_1\Sigma$.

An obvious example where this condition is satisfied, is the case that Σ is totally geodesic and τ is an isometry of the induced metric. In this case one can upon conjugation assume that $\rho(\pi_1\Sigma) \subset SL(2, \mathbf{R})$ and then A is an elliptic element in $SL(2, \mathbf{R}) \subset SL(2, \mathbf{C})$. Less obvious examples are provided by [9, Theorem 2.2] which asserts that Assumption 1.1 holds whenever (Σ, τ) is what [9] calls a symmetric surface, for example if Σ has genus 2 and τ is a hyperelliptic involution, or for certain symmetries of the 3- or 4-punctured sphere or the 1- or 2-punctured torus. For these symmetric surfaces (Σ, τ), Assumption 1.1 holds regardless how Σ is embedded (as an incompressible, boundary-incompressible surface) into a hyperbolic 3-manifold M.

Let M^τ be the result of cutting M along Σ and regluing via τ. Since Σ is a 2-sided, properly embedded surface, it has a neighborhood $N \simeq \Sigma \times [0, 1]$ in M, and a neighborhood $N^\tau \simeq \Sigma \times [0, 1]$ in M^τ. The complements $M - int(N)$ and $M^\tau - int(N^\tau)$ are diffeomorphic and we let X be the union of M and M^τ along this identification of

$M - int(N)$ and $M^\tau - int(N^\tau)$. The union of N and N^τ yields a copy of the mapping torus T^τ in X.

Let $\pi_1 M =< S \mid R >$ be a presentation of $\pi_1 M$. The Seifert-van Kampen Theorem implies

$$\pi_1 X =< S, t \mid R, tht^{-1} = \tau_* (h) \ \forall \ h \in \pi_1 \Sigma > .$$

If Assumption 1.1 holds, then we have a well-defined representation

$$\rho_X : \pi_1 X \to SL(2, \mathbb{C})$$

by $\rho_X (s) = \rho(s)$ for all $s \in S$ and $\rho_X (t) = A$.

Composition with the homomorphisms $\pi_1 M \to \pi_1 X$ and $\pi_1 M^\tau \to \pi_1 X$ induced by the inclusions yields the given representation ρ of $\pi_1 M$ and a representation ρ^τ of $\pi_1 M^\tau$. We will show in the Section 3 that under certain hypotheses (namely that Σ is incompressible, boundary-incompressible and not a virtual fiber) the representation ρ^τ will be discrete and faithful and therefore M^τ is a hyperbolic manifold (although X is not).

For use in Section 3 we mention that the Seifert-van Kampen Theorem also allows a description of $\pi_1 M^\tau$ (as a subgroup of $\pi_1 X$ with the above presentation), where we have to distinguish the cases whether the surface Σ is a separating surface in M or not.

If an incompressible, 2-sided surface Σ separates M into two submanifolds M_1 and M_2, then $\pi_1 M = \pi_1 M_1 *_{\pi_1 \Sigma} \pi_1 M_2$ is the amalgamated product of $\pi_1 M_1$ and $\pi_1 M_2$ over $\pi_1 \Sigma$ for two monomorphisms $\phi_1 : \pi_1 \Sigma \to \pi_1 M_1, \phi_2 : \pi_1 \Sigma \to \pi_1 M_2$. For the mutation M^τ we obtain that $\pi_1 M^\tau$ is the amalgamated product of $\pi_1 M_1$ and $t\pi_1 M_2 t^{-1}$ amalgamated over $\pi_1 \Sigma$ via the monomorphisms $\psi_1 : \pi_1 \Sigma \to \pi_1 M_1, \ \psi_2 : \pi_1 \Sigma \to \pi_1 M_2$ defined by

$$\psi_1 (h) = \phi_1 (\tau_* h), \ \psi_2 (h) = t\phi_2 (h) t^{-1}$$

for all $h \in \pi_1 \Sigma$.

If an incompressible, 2-sided surface Σ does not separate M, then there is a manifold N with $\partial N = \Sigma_1 \cup \Sigma_2$ such that M is the quotient of N under some diffeomorphism $f : \Sigma_1 \to \Sigma_2$. We assume the basepoint x_0 to belong to Σ_1, let u be some path in N from x_0 to $f(x_0)$, let $K_1 = \pi_1 \Sigma_1$ and let $K_2 \subset \pi_1 N$ the subgroup of all based homotopy classes of loops of the form $u * \sigma * \overline{u}$, where $\sigma : [0,1] \to \Sigma_2$ with $\sigma(0) = \sigma(1) = f(x_0)$. Then $\pi_1 M = \pi_1 N *_\alpha$ is the HNN-extension of $\pi_1 N$ for the two monomorphisms $\phi_1 : K_1 \to \pi_1 N, \ \phi_2 : K_2 \to \pi_1 N$ induced by the inclusions and the isomorphism $\alpha : K_1 \to K_2$ defined by $\alpha([\sigma]) = [u * f(\sigma) * \overline{u}]$, where $[\sigma]$ is the based homotopy class of $\sigma : [0,1] \to \Sigma_1$ with $\sigma(0) = \sigma(1) = x_0$. Let v be the extending element. For the mutation M^τ we obtain that $\pi_1 M^\tau = \pi_1 N *_{\alpha\tau_*}$ is the HNN-extension of $\pi_1 N$, with extending element vt, for the same monomorphisms ϕ_1, ϕ_2 and the isomorphism $\alpha\tau_* : K_1 \to K_2$.

We mention that X clearly is not a manifold, let alone a hyperbolic one. In fact, the representation ρ_X is not faithful and its image is not torsion-free, as the following elementary observation shows.

Observation 1.2. *Let $\rho : \Gamma \to SL(2, \mathbf{C})$ be a discrete, faithful representation of a torsion-free group, let $M = \rho(\Gamma) \backslash \mathbf{H}^3$ be the associated hyperbolic 3-manifold, and let $\Sigma \subset M$ be a properly embedded, incompressible, boundary-incompressible surface. Assume that Σ is not boundary-parallel.*

Let $\tau : \Sigma \to \Sigma$ be a diffeomorphism such that $\tau^m = id$ for some $m \in \mathbf{N}$ and such that there exists some $A \in SL(2, \mathbf{C})$ with $\rho(\tau_* h) = A\rho(h) A^{-1}$ for all $h \in \pi_1\Sigma$.

Then $A^m = \pm\mathbf{1}$.

Proof: We have $A^m \rho(h) A^{-m} = A^{m-1}\rho(\tau_* h) A^{-(m-1)} = \ldots = \rho(\tau_*^m h) = \rho(h)$ for all $h \in \pi_1\Sigma$, thus A^m conjugates $\rho(\pi_1\Sigma)$ to itself. If Σ is not boundary-parallel, then the image of $\pi_1\Sigma$ in $Isom^+(\mathbf{H}^3)$ contains at least two non-commuting hyperbolic elements. The corresponding matrices in $\rho(\pi_1\Sigma) \subset SL(2, \mathbf{C})$ are then diagonalisable with respect to two distinct bases.

A matrix that conjugates a diagonalizable 2-by-2-matrix (with 2 distinct eigenvalues) to itself must be diagonalizable with respect to the same basis. Thus the conjugating matrix $A^m \in SL(2, \mathbf{C})$ must be diagonal with respect to two distinct bases. This implies, by elementary linear algebra, that A^m is a multiple of the identity matrix. Together with $det(A^m) = 1$ this enforces $A^m = \pm\mathbf{1}$. *QED*

3. Discreteness of recombinations

Let M be an oriented (and connected) compact 3-manifold with boundary such that its interior $int(M) = M - \partial M$ is hyperbolic. The hyperbolic structure is given by the conjugacy class of a representation $\pi_1 M \to PSL(2, \mathbb{C})$ which by Culler's Theorem [4, Corollary 2.2] is the composition of a representation $\rho : \pi_1 M \to SL(2, \mathbb{C})$ with the canonical projection $SL(2, \mathbb{C}) \to PSL(2, \mathbb{C})$. For a diffeomorphism $\tau : \Sigma \to \Sigma$ we let $\rho^\tau : \pi_1 M^\tau \to SL(2, \mathbb{C})$ be the representation defined in Section 2.

Proposition 3.1. *If $\Sigma \subset M$ is a properly embedded, connected, incompressible, boundary-incompressible, 2-sided surface, which is not a virtual fiber, and if Assumption 1.1 holds for a diffeomeorphism $\tau : \Sigma \to \Sigma$, then $\rho^\tau(\pi_1 M^\tau)$ is a discrete subgroup of $SL(2, \mathbb{C})$. In particular, $int(M^\tau)$ is hyperbolic.*

Proof: Since Σ is incompressible, boundary-incompressible and not a virtual fiber, the subgroup $H := \rho(\pi_1\Sigma)$ is geometrically finite by the Thurston-Bonahon Theorem[1]. This implies that the limit set $W = \Lambda(H) \subset \partial_\infty \mathbf{H}^3$ is a quasi-circle, in particular W is a Jordan curve. The Schönflies theorem implies that W decomposes $\partial_\infty \mathbf{H}^3 \cong \mathbf{S}^2$ into two topological disks.

We distinguish the cases that Σ is separating in M (and thus in M^τ) or not. We wish to apply the Maskit combination theorems [7, Chapter VII] and have to check that their assumptions hold for $\rho^\tau(\pi_1 M^\tau)$.

[1]The Thurston-Bonahon Theorem is a combination of results in [6],[10],[1]. An explicit statement can be found in [3, Theorem 1.1] or [2, Corollary 8.3].

Case 1: Σ is separating. Then we obtain from Section 2 that $G = \rho(\pi_1 M)$ is an amalgamated product $G = G_1 *_H G_2$ with $G_1 = \rho(\pi_1 M_1)$ and $G_2 = \rho(\pi_1 M_2)$. The complement of the limit set $\partial_\infty \mathbf{H}^3 - W$ consists of two open topological disks B_1, B_2 such that B_i is precisely H-invariant in G_i for $i = 1, 2$, that is

$$h(B_i) \subset B_i \quad \forall\, h \in H$$

$$g(B_i) \subset B_{3-i} \quad \forall\, g \in G_i - H.$$

To apply the first Maskit combination theorem we have to check that the analogous condition is satisfied for the decomposition of $G^\tau := \rho^\tau(\pi_1 M^\tau)$ as amalgamated product of G_1 and AG_2A^{-1} over H.

We already know that $h(B_i) \subset B_i$ for $h \in H$ and $g(B_1) \subset B_2$ for $g \in G_1 - H$. We have to check that $AgA^{-1}(B_2) \subset B_1$ for $g \in G_2 - H$.

Assumption 1.1 implies that conjugation by $A \in SL(2, \mathbf{C})$ maps H to H, thus A and A^{-1} map the limit set $W = \Lambda(H)$ to itself. Since A is orientation-preserving it must then also map B_1 to B_1 and B_2 to B_2. Hence $g(B_2) \subset B_1$ implies $AgA^{-1}(B_2) \subset B_1$ for $g \in G_2 - H$.

The first combination theorem implies then that G^τ is discrete.

Case 2: Σ is non-separating. Then we obtain from Section 2 that $G = \rho(\pi_1 M)$ is an HNN-extension $G = G_0 *_\alpha$ of $G_0 := \rho(\pi_1 N)$ for some isomorphism $\alpha : H \to H_2$, where $H := \rho(K_1) = \rho(\pi_1 \Sigma_1), H_2 := \rho(K_2)$.

Let $f = \rho(v) \in G$ be the extending element of the HNN-extension $G = G_0 *_\alpha$ and let $W_2 = f(W) \subset \partial_\infty \mathbf{H}^3$. Then W and W_2 are disjoint simple closed curves and the complement $\partial_\infty \mathbf{H}^3 - (W \cup W_2)$ consists of two open topological disks B_1, B_2 and an open annulus R such that B_i is precisely H_i-invariant in G_0 for $i = 1, 2$ and, moreover,

$$f(R \cup B_2) = B_2.$$

To apply the second Maskit combination theorem we have to check that the analogous conditions are satisfied for the description of $G^\tau := \rho^\tau(\pi_1 M^\tau)$ as an HNN-extension $G_\tau = G_0 *_{\alpha\tau_*}$ for the isomorphism $\alpha\tau_* : H \to H_2$, with extending element $\rho(vt) = fA$.

This is clear for the first condition: we already know that B_i is precisely H_i-invariant in G_0.

Assumption 1.1 implies again that A and A^{-1} map the limit set $W = \Lambda(H)$ to itself. Since A is orientation-preserving it must then also map B_1 bijectively to B_1 and $R \cup B_2$ bijectively to $R \cup B_2$. In particular $f(R \cup B_2) = B_2$ implies

$$fA(R \cup B_2) = B_2.$$

The second combination theorem implies then that G^τ is discrete.

QED

One may wonder whether it is possible to apply the topological combination theorems ([7, Theorem VII.A.12.,C.13.]) directly to the actions on $\mathbf{H}^3$ rather than to the actions on $\partial_\infty \mathbf{H}^3$. Letting $\tilde{\phi} : \mathbf{H}^3 \to \mathbf{H}^3$ be the "lift" of τ defined as the lift of the isometry $\phi : \rho(\pi_1 \Sigma) \backslash \mathbf{H}^3 \to \rho(\pi_1 \Sigma) \backslash \mathbf{H}^3$ from [9, Lemma 2.5.] this would require to have a

$\tilde{\phi}$-invariant copy of $\widetilde{\Sigma}$ in $\mathbf{H}^3$. The main part of the argument in [9] actually consists in providing such a $\tilde{\phi}$-invariant surface, using heavy machinery from the theory of minimal surfaces. The advantage of using the combination theorem for the action on $\partial_\infty \mathbf{H}^3$ instead of the action on $\mathbf{H}^3$ is that one can avoid this machinery.

4. Volume of mutations

4.1. Recollections

We recollect some constructions from [5, Section 4] which will be needed in the proof of Theorem 1, in particular to handle the case of disconnected boundary, where relative group homology does not directly apply.

Let (X, Y) be a pair of topological spaces. Assume that X is path-connected and Y has path-components $Y_1, \ldots, Y_s$. Let $\Gamma = \pi_1(X, x)$. Using paths from y_i to x one can fix isomorphisms $l_i : \pi_1(Y_i, y_i) \to \Gamma_i$ to subgroups $\Gamma_i \subset \Gamma$ for $i = 1, \ldots, s$.

As in [5, Section 4.2.1], we will denote by $DCone\left(\cup_{i=1}^s Y_i \to X\right)$ the union along Y of X with the *disjoint* cones over $Y_1, \ldots, Y_s$. Let $\Psi_X : X \to \mid B\Gamma \mid$ be the classifying map for $\pi_1 X$. (We consider $B\Gamma$ as a simplicial set and denote $\mid B\Gamma \mid$ its geometric realization.)

As in [5, Definition 7] we denote $B\Gamma^{comp} = DCone\left(\cup_{i=1}^s B\Gamma_i \to B\Gamma\right)$, that is the quasi-simplicial set which is the union along $\cup_{i=1}^s B\Gamma_i$ of the simplicial set $B\Gamma$ and the cones $Cone\left(B\Gamma_i\right)$ over $B\Gamma_i$ with cone point c_i for $i = 1, \ldots, s$. Ψ_X extends to a continuous map $\Psi_X : DCone\left(\cup_{i=1}^s Y_i \to X\right) \to \mid B\Gamma^{comp} \mid$. If X and $Y_1, \ldots, Y_s$ are aspherical, then composition of $\left(\Psi_X\right)_*$ with the isomorphism $\iota : H_*\left(\mid B\Gamma^{comp} \mid\right) \to H_*^{simp}\left(B\Gamma^{comp}\right)$ yields the inverse of the Eilenberg-MacLane isomorphism:

$$\iota \circ \left(\Psi_X\right)_* = EM^{-1} : H_*\left(DCone\left(\cup_{i=1}^s Y_i \to X\right)\right) \to H_*^{simp}\left(B\Gamma^{comp}\right).$$

In particular, if M is a compact, connected, oriented, aspherical manifold with aspherical boundary $\partial M = \partial_1 M \cup \ldots \cup \partial_s M$, then there is for $* \geq 2$ a canonical isomorphism[2] $H_*\left(M, \partial M\right) \cong H_*\left(DCone\left(\cup_{i=1}^s \partial_i M \to M\right)\right)$ and we obtain the isomorphism ([5, Lemma 8])

$$EM^{-1} : H_*\left(M, \partial M\right) \to H_*^{simp}\left(B\Gamma^{comp}\right).$$

Now assume that $int\left(M\right) = M - \partial M$ is hyperbolic and $\rho : \Gamma \to SL\left(2, \mathbf{C}\right)$ is a representation defining the hyperbolic structure. As in [5, Definition 6] let

$$BSL\left(2, \mathbf{C}\right)^{comp} = DCone\left(\dot{\cup}_{c \in \partial_\infty \mathbf{H}^3} BSL\left(2, \mathbf{C}\right) \to BSL\left(2, \mathbf{C}\right)\right),$$

where each of the cone points of the disjoint copies of $Cone\left(BSL\left(2, \mathbf{C}\right)\right)$ is identified with a point $c \in \partial_\infty \mathbf{H}^3$.

Assume that $vol\left(M\right) < \infty$. Then for each $i = 1, \ldots, s$ there is a unique $c_i \in \partial_\infty \mathbf{H}^3$ with $\Gamma_i \subset Fix\left(c_i\right)$. Therefore $B\rho : B\Gamma \to BSL\left(2, \mathbf{C}\right)$ can be extended to

$$B\rho : B\Gamma^{comp} \to BSL\left(2, \mathbf{C}\right)^{comp}$$

[2]An explicit realisation of this isomorphism is as follows. Let $z \in Z_*\left(M, \partial M\right)$ be a relative cycle, then $\partial z \in Z_{*-1}\left(\partial M\right)$ and thus $z + Cone\left(\partial z\right) \in Z_*\left(DCone\left(\cup_{i=1}^s \partial_i M \to M\right)\right)$ is a cycle. This induces an isomorphism in homology.

by mapping the cone point of $Cone\left(B\Gamma_i\right)$ to the cone point $c_i \in \partial_\infty \mathbf{H}^3$ of $BSL\left(2,\mathbf{C}\right)^{comp}$ with $\Gamma_i \subset Fix\left(c_i\right)$.

In [5, Section 4.2.3] we defined a cocycle $\overline{cv_3} \in C^3_{simp}\left(BSL\left(2,\mathbf{C}\right)^{comp};\mathbf{R}\right)$ whose cohomology class $\left[\overline{cv_3}\right]$, by the proof of [5, Theorem 4], satisfies

$$< \left[\overline{cv_3}\right], \left(B\rho\right)_* EM^{-1}\left[M,\partial M\right] >= vol\left(M\right).$$

In particular, $vol\left(M\right)$ is determined by the element

$$\left(B\rho\right)_* EM^{-1}\left[M,\partial M\right] \in H_3^{simp}\left(BSL\left(2,\mathbf{C}\right)^{comp};\mathbf{Q}\right).$$

4.2. Proof of Theorem 1

Theorem 4.1. *Let M be a compact, orientable, connected 3-manifold with boundary ∂M. Let $\rho:\Gamma\rightarrow SL\left(2,\mathbf{C}\right)$ be a representation that is a lift of a faithful representation $\Gamma\rightarrow Isom^+\left(\mathbf{H}^3\right)=PSL\left(2,\mathbf{C}\right)$ with discrete, torsion-free image. Assume that $\rho\left(\Gamma\right)\backslash\mathbf{H}^3$ has finite volume and is diffeomorphic to $int\left(M\right)=M-\partial M$.*

Let $\Sigma\subset M$ be a properly embedded, connected, incompressible, boundary-incompressible, 2-sided surface which is not a virtual fiber.

Let $\tau:\left(\Sigma,\partial\Sigma\right)\rightarrow\left(\Sigma,\partial\Sigma\right)$ be an orientation-preserving diffeomorphism of pairs such that $\tau^m=id$ for some $m\in\mathbf{N}$ and such that there exists some $A\in SL\left(2,\mathbf{C}\right)$ with $\rho\left(\tau_* h\right)=A\rho\left(h\right)A^{-1}$ for all $h\in\pi_1\Sigma$.

Then $int\left(M^\tau\right)$ is hyperbolic and $vol\left(M^\tau\right)=vol\left(M\right)$.

Proof: We have proved in Proposition 2.1 that M^τ is hyperbolic, that is, the representation $\rho^\tau:\pi_1 M^\tau\rightarrow SL\left(2,\mathbf{C}\right)$ is faithful and has discrete image.

Let X be constructed as in Section 2 and let $\partial X = X\cap\left(\partial M\cup\partial M^\tau\right)$. Let $\Gamma^X:=\pi_1 X$ and let $\rho_X:\Gamma^X\rightarrow SL\left(2,\mathbf{C}\right)$ be the representation, defined in Section 1, with $\rho_X\mid_\Gamma=\rho$ and $\rho_X\left(t\right)=A$.

The construction of X implies that

$$i_{M*}\left[M,\partial M\right]-i_{M^\tau *}\left[M^\tau,\partial M^\tau\right]=i_{T^\tau *}\left[T^\tau,\partial T^\tau\right]\in H_3\left(X,\partial X;\mathbf{Q}\right), \tag{1}$$

where $i_M, i_{M^\tau}, i_{T^\tau}$ are the inclusions of M, M^τ and the mapping torus T^τ into X, and $\left[M,\partial M\right],\left[M^\tau,\partial M^\tau\right]$, and $\left[T^\tau,\partial T^\tau\right]$ are the fundamental classes in homology with $\mathbf{Q}$-coefficients.

We observe that there is a one-to-one correspondance between the path-components $\partial_1 M,\ldots,\partial_s M$ of ∂M with the path-components $\partial_1 M^\tau,\ldots,\partial_s M^\tau$ of ∂M^τ and with the path-components $\partial_1 X,\ldots,\partial_s X$ of ∂X. To each path-component $\partial_j\Sigma$ of $\partial\Sigma$ there is some path-component $\partial_{i_j}M$ of ∂M with $\partial_j\Sigma\subset\partial_{i_j}M$. The path-components $\partial_j\Sigma$ are in one-to-one correspondence with the path-components $\partial_j T^\tau$ of T^τ, and we have then $\partial_j T^\tau\subset\partial_{i_j}X$.

We choose the base point x of $\Gamma=\pi_1\left(M,x\right)$ to belong to $M\cap M^\tau\cap T^\tau$, for $i=1,\ldots,s$ the base point x_i of $\pi_1\left(\partial_i M,x_i\right)$ to belong to $\partial_i M\cap\partial_i M^\tau\cap\partial T^\tau$, and for all j the base point y_j of $\pi_1\left(\partial_j T^\tau,y_j\right)$ to belong to $\partial_j T^\tau\cap\partial_{i_j}M\cap\partial_{i_j}M^\tau$.

Let $i \in \{1, \ldots, s\}$. Recall that a path p_i from x_i to x yields an isomorphism $l_i : \pi_1(X, x_i) \to \pi_1(X, x)$. Let $\Gamma_i^X = l_i(\pi_1(\partial_i X, x_i)) \subset \Gamma^X := \pi_1(X, x)$, that is Γ_i^X is an isomorphic image of $\pi_1(\partial_i X, x_i)$ in Γ^X. We can choose the path p_i in $M \cap M^\tau$, thus if $i_{\partial_i M*} : \pi_1(\partial_i M, x_i) \to \pi_1(\partial_i X, x_i)$ is induced by the inclusion $i_{\partial_i M} : \partial_i M \to \partial_i X$, then $j_{\partial_i M} l_i \mid_{\pi_1(\partial_i M, x_i)} = l_i i_{\partial_i M*}$, where $j_{\partial_i M} : \Gamma_i \to \Gamma_i^X$ is the inclusion. An analogous fact holds by replacing M with M^τ.

For each $y_j \in \partial_j T^\tau \cap \partial_{i_j} M \cap \partial_{i_j} M^\tau$ we can choose a path from y_j to x_{i_j} in $\partial_{i_j} M \cap \partial_{i_j} M^\tau$. This yields an isomorphism l_j from $\pi_1(\partial_j T^\tau, y_j)$ to a subgroup of $\pi_1\left(\partial_{i_j} M, x_{i_j}\right)$. Composition of l_j with l_{i_j} yields an isomorphism from $\pi_1(\partial_j T^\tau, y_j)$ to a subgroup Γ_j^T of Γ. The same paths can be used to construct an isomorphism from $\pi_1(\partial_j T^\tau, y_j)$ to a subgroup of Γ^τ, and both images are then by construction the same subgroup of Γ^X.

By Section 4.1 $B\rho$ extends to a simplicial map $B\rho : B\Gamma^{comp} \to BSL(2, \mathbf{C})^{comp}$ where the cone point over $B\Gamma_i$ is mapped to the (unique) c_i with $\Gamma_i \subset Fix(c_i)$. However $B\rho_X : B\Gamma^X \to BSL(2, \mathbf{C})$ does not extend to $\left(B\Gamma^X\right)^{comp}$ in such a way that restriction to $B\Gamma^{comp}$ would give back $B\rho$. Indeed, in general there is no $c_i \in \partial_\infty \mathbf{H}^3$ with $\rho_X\left(\Gamma_i^X\right)$ in $Fix(c_i)$. (This is because $\rho_X(t) = A$ is not parabolic but elliptic.) Therefore we have to[3] go to a finite cover as follows.

Let $\pi_1 X =< S, t \mid R, tht^{-1} = \tau_*(h)\ \forall\, h \in \pi_1\Sigma >$ be the presentation from Section 2. Then

$$a(t) = 1,\ a(s) = 0 \quad \forall\ s \in S$$

yields a well-defined, surjective homomorphism

$$a : \pi_1 X \to \mathbf{Z}/2m\mathbf{Z}.$$

Let $\pi : \widehat{X} \to X$ be the $2m$-fold cyclic covering with $\Gamma^{\widehat{X}} := \pi_1\left(\widehat{X}, \hat{x}\right) \cong ker(a)$ for some $\hat{x} \in \pi^{-1}(x)$. For each $i \in \{1, \ldots, s\}$ we let $\partial_i \widehat{X} \subset \widehat{X}$ be the preimage of $\partial_i X$, we fix the preimage $\hat{p}_i$ of the path p_i ending at $\hat{x}$, and use $\hat{p}_i$ to define the isomorphism from $\pi_1 \partial_i \widehat{X}$ to a subgroup $\Gamma_i^{\widehat{X}} \subset \Gamma^{\widehat{X}}$. Then $\pi(\hat{p}_i) = p_i$ implies $\pi_*\left(\Gamma_i^{\widehat{X}}\right) \subset \Gamma_i^X$.

$\widehat{X}$ contains a copy of $\Sigma \times \mathbf{S}^1$ which is a $2m$-fold covering of T^τ. Let $\widehat{M} = \pi^{-1}(M)$, and $\widehat{M^\tau} = \pi^{-1}(M^\tau)$. Since $a \mid_{\pi_1 M}$ is trivial, $\widehat{M}$ consists of $2m$ copies of M. If Σ is separating, then $a \mid_{\pi_1 M^\tau}$ is trivial, thus also $\widehat{M^\tau}$ consists of $2m$ copies of M^τ. If Σ is non-separating, then $a \mid_{\pi_1 M^\tau}$ is surjective, thus $\widehat{M^\tau}$ is a connected $2m$-fold covering of M^τ.

Consider the transfer map $tr : H_*(X, \partial X; \mathbf{Q}) \to H_*\left(\widehat{X}, \partial\widehat{X}; \mathbf{Q}\right)$ of the finite covering $\widehat{X} \to X$. Application of the transfer map to Equation (1) yields

$$i_{\widehat{M}*}\left[\widehat{M}, \partial\widehat{M}\right] - i_{\widehat{M^\tau}*}\left[\widehat{M^\tau}, \partial\widehat{M^\tau}\right] = i_{\Sigma \times \mathbf{S}^1 *}\left[\Sigma \times \mathbf{S}^1, \partial\Sigma \times \mathbf{S}^1\right] \tag{2}$$

[3] If M is a closed manifold, then one can use $B\rho_X$ as in [8, Theorem 2.13] and does not need to go to a finite cover.

in $H_3\left(\widehat{X},\partial\widehat{X};\mathbf{Q}\right)$, where i_* denotes the respective inclusions into $\widehat{X}$.

Composition of $\pi_*:\pi_1\widehat{X}\rightarrow\pi_1 X$ with the representation $\rho_X:\pi_1 X\rightarrow SL\left(2,\mathbf{C}\right)$ yields a representation $\rho_{\widehat{X}}:\Gamma^{\widehat{X}}\rightarrow SL\left(2,\mathbf{C}\right)$. By Observation 1.2 we have $A^m=\pm\mathbf{1}$, thus $A^{2m}=\mathbf{1}$. Recall that $\rho_X\left(t\right)=A$, thus $\rho_{\widehat{X}}\left(t^{2m}\right)=A^{2m}=\mathbf{1}$.

The Seifert-van Kampen Theorem implies that $\pi_*\left(\Gamma_i^{\widehat{X}}\right)$ is generated by elements of Γ_i and t^{2m}. Since $\rho_X\left(t^{2m}\right)=A^{2m}=\mathbf{1}$, this implies $\rho_{\widehat{X}}\left(\Gamma_i^{\widehat{X}}\right)=\rho_X\left(\Gamma_i\right)\subset Fix\left(c_i\right)$. Thus we can extend

$$B\rho_{\widehat{X}}:\left(B\Gamma^{\widehat{X}}\right)^{comp}\rightarrow BSL\left(2,\mathbf{C}\right)^{comp},$$

by mapping the cone point over $B\Gamma_i^{\widehat{X}}$ to c_i, such that $B\rho_{\widehat{X}}Bj=B\rho$ for the homomorphism of pairs $j:\left(\Gamma,\Gamma_i\right)\rightarrow\left(\Gamma^{\widehat{X}},\Gamma_i^{\widehat{X}}\right)$ induced by the inclusion $i:\widehat{M}\rightarrow\widehat{X}$.

We do not know whether X and $\widehat{X}$ are aspherical or not, but we do have the classifying map $\Psi_{\widehat{X}}:\widehat{X}\rightarrow\mid B\Gamma^{\widehat{X}}\mid$, whose restriction to $\partial_i\widehat{X}$ is (upon a homotopy) the classifying map $\partial_i\widehat{X}\rightarrow\mid B\Gamma_i^{\widehat{X}}\mid$ and which therefore extends to

$$\Psi_{\widehat{X}}:DCone\left(\cup_{i=1}^s\partial_i\widehat{X}\rightarrow\widehat{X}\right)\rightarrow\mid\left(B\Gamma^{\widehat{X}}\right)^{comp}\mid.$$

Let $P_1:\Sigma\times\mathbf{S}^1\rightarrow\Sigma$ be the projection to the first factor, then the compositions $\mid B\rho_{\widehat{X}}\mid\Psi_{\widehat{X}}i_\Sigma P_1$ and $\mid B\rho_{\widehat{X}}\mid\Psi_{\widehat{X}}i_{\Sigma\times\mathbf{S}^1}$ from $DCone\left(\cup_j\partial_j\Sigma\times\mathbf{S}^1\rightarrow\Sigma\times\mathbf{S}^1\right)$ to $\mid BSL\left(2,\mathbf{C}\right)^{comp}\mid$ are homotopic since $\Sigma\times\mathbf{S}^1$ and $\partial_j\Sigma\times\mathbf{S}^1$ are aspherical and since the induced homomorphisms between fundamental groups agree because $\rho_{\widehat{X}}$ sends the generator t^{2m} of $\pi_*\left(\pi_1\mathbf{S}^1\right)\subset\pi_*\left(\pi_1\left(\Sigma\times\mathbf{S}^1\right)\right)$ to $A^{2m}=\mathbf{1}$.

Since the maps $\left(\mid B\rho_{\widehat{X}}\mid\Psi_{\widehat{X}}i_{\Sigma\times\mathbf{S}^1}\right)_*=\left(\mid B\rho_{\widehat{X}}\mid\Psi_{\widehat{X}}i_\Sigma P_1\right)_*$ from $H_3\left(\Sigma\times\mathbf{S}^1,\partial\Sigma\times\mathbf{S}^1\right)$ to $H_3\left(\mid BSL\left(2,\mathbf{C}\right)^{comp}\mid\right)$ factor over $H_3\left(\Sigma,\partial\Sigma\right)=0$ we obtain

$$\left(\mid B\rho_{\widehat{X}}\mid\Psi_{\widehat{X}}i_{\Sigma\times\mathbf{S}^1}\right)_*\left[\Sigma\times\mathbf{S}^1,\partial\Sigma\times\mathbf{S}^1\right]=0.$$

Thus Equation (2) implies that in $H_3\left(\mid BSL\left(2,\mathbf{C}\right)^{comp};\mathbf{Q}\mid\right)$ we get

$$\left(\mid B\rho_{\widehat{X}}\mid\Psi_{\widehat{X}}i_{\widehat{M}}\right)_*\left[\widehat{M},\partial\widehat{M}\right]=\left(\left(\mid B\rho_{\widehat{X}}\mid\Psi_{\widehat{X}}i_{\widehat{M}^\tau}\right)_*\left[\widehat{M}^\tau,\partial\widehat{M}^\tau\right]\right.\tag{3}$$

Let $\widehat{M}_1$ be the path-component of $\widehat{M}$ with $\hat{x}\in\widehat{M}_1$, let $i_{\widehat{M}_1}:\widehat{M}_1\rightarrow\widehat{X}$ be the inclusion and $I_1:M\rightarrow\widehat{M}_1$ the homeomorphism inverse to $\pi\mid_{\widehat{M}_1}$. Let $j_1=\left(i_{\widehat{M}_1}I_1\right)_*:\Gamma\rightarrow\Gamma^{\widehat{X}}$ be the induced homomorphism, then we have $\rho=\rho_{\widehat{X}}j_1$ and therefore

$$\begin{aligned}\left(\mid B\rho_{\widehat{X}}\mid\Psi_{\widehat{X}}i_{\widehat{M}_1}I_1\right)_* &= \left(\mid B\rho_{\widehat{X}}\mid\mid Bj_1\mid\Psi_M\right)_*=\left(\mid B\rho\mid\Psi_M\right)_*\\ &= \left(\mid B\rho\mid\right)_*\iota^{-1}EM^{-1}=\iota^{-1}\left(B\rho\right)_*EM^{-1}\end{aligned}$$

(where $\iota:H_*\left(\mid.\mid\right)\rightarrow H_*^{simp}\left(.\right)$ denotes the isomorphism between singular and simplicial homology).

If $\widehat{M_l}$ is another path-component of $\widehat{M}$, then we have a deck transformation σ_l of $\widehat{X}$ with $\sigma_l\left(\widehat{M_1}\right)=\widehat{M_l}$. We denote $I_l=\sigma_l I_1$ and obtain by the same computation $\left(\mid B\rho_{\widehat{X}}\mid \Psi_{\widehat{X}} i_{\widehat{M_l}} I_l\right)_*=\iota^{-1}\left(B\rho\right)_* EM^{-1}$. If Σ *is separating*, then the same argument applies to the path-components of $\widehat{M^\tau}$: let $\widehat{M_1^\tau}$ be the path-component of $\widehat{M^\tau}$ with $\hat{x}\in\widehat{M_1^\tau}$, let $I_1^\tau: M^\tau\to\widehat{M_1^\tau}$ be the homeomorphism inverse to $\pi\mid_{\widehat{M_1^\tau}}$ and $I_l^\tau=\sigma_l I_1^\tau$, then we obtain $\left(\mid B\rho_{\widehat{X}}\mid \Psi_{\widehat{X}} i_{\widehat{M_l^\tau}} I_l^\tau\right)_*=\iota^{-1}\left(B\rho^\tau\right)_* EM^{-1}$.

We have $\left[\widehat{M},\partial\widehat{M}\right]=\sum_{l=1}^{2m} I_{l*}\left[M,\partial M\right]$ and, if Σ is separating, we also have $\left[\widehat{M^\tau},\partial\widehat{M^\tau}\right]=\sum_{l=1}^{2m} I_{l*}^\tau\left[M^\tau,\partial M^\tau\right]$.

Thus Equation (3) implies

$$2m\left(B\rho\right)_* EM^{-1}\left[M,\partial M\right]=2m\left(B\rho^\tau\right)_* EM^{-1}\left[M^\tau,\partial M^\tau\right]$$

if Σ is separating, and

$$2m\left(B\rho\right)_* EM^{-1}\left[M,\partial M\right]=\left(B\hat{\rho}^\tau\right)_* EM^{-1}\left[\widehat{M^\tau},\partial\widehat{M^\tau}\right]$$

with $\hat{\rho}^\tau:=\rho_{\widehat{X}}\left(i_{\widehat{M^\tau}}\right)_*$ if Σ is non-separating, respectively. (In the latter case $\widehat{M^\tau}$ was connected.)

By Section 4.1 we have

$$<\left[\overline{c\nu_3}\right],\left(B\rho\right)_* EM^{-1}\left[M,\partial M\right]>=vol\left(M\right),$$

$$<\left[\overline{c\nu_3}\right],\left(B\rho^\tau\right)_* EM^{-1}\left[M^\tau,\partial M^\tau\right]>=vol\left(M^\tau\right),$$

and, if Σ is non-separating,

$$<\left[\overline{c\nu_3}\right],\left(B\hat{\rho}^\tau\right)_* EM^{-1}\left[\widehat{M^\tau},\partial\widehat{M^\tau}\right]>=vol\left(\widehat{M^\tau}\right)=2mvol\left(M^\tau\right).$$

Hence $vol\left(M\right)=vol\left(M^\tau\right)$.

QED

Remark: The assumption that Σ is not a virtual fiber is needed "only" for the proof that M^τ is hyperbolic, that is for the application of Proposition 2.1.

If Σ is a virtual fiber then M^τ may or may not be hyperbolic. For example, if $M=T^\alpha$ is a mapping torus of $\alpha:\Sigma\to\Sigma$, then M^τ is the mapping torus of $\alpha\circ\tau:\Sigma\to\Sigma$. By Thurston's hyperbolization theorem, the mapping torus T^α is hyperbolic if and only if α is pseudo-Anosov. However, if α is pseudo-Anosov and τ is of finite order, then $\alpha\circ\tau$ may or may not be pseudo-Anosov, so there is no general statement whether M^τ is hyperbolic or not.

Examples for both phenomena can be easily found in $SL\left(2,\mathbf{Z}\right)$, the mapping class group of the once-punctured torus. Here Anosov diffeomorphisms correspond to hyperbolic elements in $SL\left(2,\mathbf{Z}\right)$ and finite order diffeomorphisms correspond to elliptic elements in

$SL(2, \mathbf{Z})$. One easily finds hyperbolic elements A_1, A_2 and elliptic elements B_1, B_2 such that $A_1 B_1$ is hyperbolic while $A_2 B_2$ is not.

However, if M^τ happens to be hyperbolic, then the proof of Theorem 3.1 shows that $vol(M^\tau) = vol(M)$ even if Σ was a virtual fiber.

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