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On the number of solutions to the asymptotic Plateau problem

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Abstract. By using a simple topological argument, we show that the space of closed, orientable, codimension-1 submanifolds of $S^{n-1}_\infty(\mathbf{H}^n)$ which bound a unique absolutely area minimizing hypersurface in $\mathbf{H}^n$ is dense in the space of closed, orientable, codimension-1 submanifolds of $S^{n-1}_\infty(\mathbf{H}^n)$. In particular, in dimension 3, we prove that the set of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding a unique absolutely area minimizing surface in $\mathbf{H}^3$ is not only dense, but also a countable intersection of open dense subsets of the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ with C^0 topology. We also show that the same is true for least area planes in $\mathbf{H}^3$. Moreover, we give some non-uniqueness results in dimension 3.

1. Introduction

The asymptotic Plateau problem in hyperbolic space asks for the existence of an absolutely area minimizing hypersurface $\Sigma \subset \mathbf{H}^n$ asymptotic to a given closed codimension-1 submanifold Γ in $S^{n-1}_\infty(\mathbf{H}^n)$. This problem has been solved by Michael Anderson in his seminal paper [4]. He proved the existence of a solution for any given closed submanifold in the sphere at infinity. Then, Hardt and Lin studied the asymptotic regularity of these solutions in [14], [17]. Lang generalized Anderson's methods to solve this problem in Gromov-Hadamard spaces in [16].

On the other hand, there are only a few results so far on the number of the absolutely area minimizing hypersurfaces for a given asymptotic boundary. In [4], Anderson showed that if the given asymptotic boundary Γ bounds a convex domain in $S^{n-1}_\infty(\mathbf{H}^n)$, then there exists a unique absolutely area minimizing hypersurface in $\mathbf{H}^n$. Then, Hardt and Lin generalized this result to the closed codimension-1 submanifolds bounding star shaped domains in $S^{n-1}_\infty(\mathbf{H}^n)$ in [14]. Recently, the author showed a generic uniqueness result in dimension 3 for least area planes in [7], [8]. For other results on asymptotic Plateau problem, see the survey article [11].

Key words and phrases. Asymptotic Plateau problem, uniqueness, hyperbolic space, least area plane, absolutely area minimizing surface.

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In this paper, we give some uniqueness results for the solutions to the asymptotic Plateau problem. We first show that the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding a unique least area plane in $\mathbf{H}^3$ is a countable intersection of open dense subsets of the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ by using simple topological arguments.

Theorem 3.3. Let A be the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ and let $A' \subset A$ be the subspace containing the simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding a unique least area plane in $\mathbf{H}^3$. Then, A' is dense in A with respect to the C^0-topology. Indeed, A' is a countable intersection of open dense subsets of A with respect to the C^0-topology.

We show that the same is true for absolutely area minimizing hypersurfaces in $\mathbf{H}^3$ as well.

Corollary 4.5. Let A be the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ and let $A' \subset A$ be the subspace containing the simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding a unique absolutely area minimizing surface in $\mathbf{H}^3$. Then, A' is dense in A with respect to the C^0-topology. Indeed, A' is a countable intersection of open dense subsets of A with respect to the C^0-topology.

For higher dimensions, we have the following density result.

Theorem 4.4. Let B be the space of connected, closed, orientable codimension-1 submanifolds of $S^{n-1}_\infty(\mathbf{H}^n)$, and let $B' \subset B$ be the subspace containing the submanifolds of $S^{n-1}_\infty(\mathbf{H}^n)$ bounding a unique absolutely area minimizing hypersurface in $\mathbf{H}^n$. Then B' is dense in B with respect to the C^0-topology.

A short outline of the technique to prove the main result is the following: For simplicity, we will focus on the case of the least area planes in $\mathbf{H}^3$ (Theorem 3.3). Let Γ_0 be a simple closed curve in $S^2_\infty(\mathbf{H}^3)$. First, we will show that either there exists a unique least area plane Σ_0 in $\mathbf{H}^3$ with $\partial_\infty\Sigma_0 = \Gamma_0$, or there exist two *disjoint* least area planes Σ_0^+, Σ_0^- in $\mathbf{H}^3$ with $\partial_\infty\Sigma_0^\pm = \Gamma_0$. Now, take a small neighborhood $N(\Gamma_0) \subset S^2_\infty(\mathbf{H}^3)$ which is an annulus. Then foliate $N(\Gamma_0)$ by simple closed curves $\{\Gamma_t\}$ where $t \in (-\epsilon, \epsilon)$, i.e., $N(\Gamma_0) \simeq \Gamma \times (-\epsilon, \epsilon)$. By the above fact, for any Γ_t either there exists a unique least area plane Σ_t, or there are two least area planes $\Sigma_t^\pm$ disjoint from each other. Also, since these are least area planes, if they have disjoint asymptotic boundary, then they are disjoint by Meeks-Yau exchange roundoff trick. This means, if $t_1 < t_2$, then Σ_{t_1} is disjoint and *below* from Σ_{t_2} in $\mathbf{H}^3$. Consider this collection of least area planes. Note that for curves Γ_t bounding more than one least area plane, we have a canonical region N_t in $\mathbf{H}^3$ between the disjoint least area planes $\Sigma_t^\pm$, see Figure 1.

Now, $N(\Gamma)$ separates $S^2_\infty(\mathbf{H}^3)$ into two parts. Take a geodesic $\beta \subset \mathbf{H}^3$ which is asymptotic to two points that belong to these two different parts. This geodesic is transverse to the collection of these least area planes asymptotic to the curves in $\{\Gamma_t\}$. Also, a finite segment of this geodesic intersects the entire collection. Let the length of this finite segment be C. Now, consider the *thickness* of the neighborhoods N_t assigned to the asymptotic curves $\{\Gamma_t\}$. Let s_t be the length of the segment I_t of β between Σ_t^+ and Σ_t^-, which is the *width* of N_t assigned to Γ_t. Then, the curves Γ_t bounding more than one least area plane have positive width, and contribute to total thickness of the collection, and the curves bounding unique least area planes have 0 width and do not contribute to the total thickness. Since $\sum_{t\in(-\epsilon,\epsilon)} s_t < C$, the total thickness is finite. This implies for only countably many $t \in (-\epsilon, \epsilon)$, $s_t > 0$, i.e., Γ_t bounds more than one least area plane. For the remaining uncountably many $t \in (-\epsilon, \epsilon)$, $s_t = 0$, and there exists a unique least area plane for those t. This proves the space of simple closed curves of uniqueness is dense in the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$. Then, we will show this space is not only dense, but also a countable intersection of open dense subsets.

We should note that this technique is quite general and it can be applied to many different settings of the Plateau problem (see Concluding remarks).

On the other hand, after the above uniqueness results, it is a reasonable question whether all simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bound a unique absolutely area minimizing surface or a unique least area plane. Up till now, it has not been known whether all simple closed curves in $S^2_\infty(\mathbf{H}^3)$ have a unique solution to the asymptotic Plateau problem or not. The only known results about nonuniqueness also come from Anderson in [5]. He constructs examples of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding more than one complete *minimal surface* in $\mathbf{H}^3$. These examples are also area minimizing in their topological class. However, none of them are absolutely area minimizing, i.e., a solution to the asymptotic Plateau problem.

In this paper, we show the existence of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ with nonunique solution to the asymptotic Plateau problem.

Theorem 5.2. There exists a simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$ such that Γ bounds more than one absolutely area minimizing surface $\{\Sigma_i\}$ in $\mathbf{H}^3$ with $\partial_\infty \Sigma_i = \Gamma$.

By using similar ideas, we also show the existence of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding more than one least area planes in $\mathbf{H}^3$ (Theorem 5.3).

The organization of the paper is as follows: In the next section we will cover some basic results which will be used in the following sections. In section 3, we will show the uniqueness results for least area planes in $\mathbf{H}^3$. Then in section 4, we will show the results on absolutely area minimizing hypersurfaces in $\mathbf{H}^n$. In Section 5, we will prove the nonuniqueness results. Finally in section 6, we will have some concluding remarks.

2. Preliminaries

In this section, we will overview the basic results which we will use in the following sections. For details on the notions and results in this section, see the survey article [11].

First, we will give the definitions of area minimizing surfaces. First set of the definitions are about compact surfaces and hypersurfaces. The second set of the definitions are their generalizations to the noncompact surfaces and hypersurfaces.

Definition 2.1. (Compact Case) A *least area disk* (area minimizing disk) is a disk which has the smallest area among the disks with the same boundary. An *absolutely area minimizing surface* is a surface which has the smallest area among all the surfaces (with no topological restriction) with the same boundary. An *absolutely area minimizing hypersurface* is a hypersurface which has the smallest volume among all hypersurfaces with the same boundary.

Definition 2.2. (Noncompact Case) A *least area plane* is a plane such that any compact subdisk in the plane is a least area disk. We will also call a complete noncompact surface as *absolutely area minimizing surface* if any compact subsurface is an absolutely area minimizing surface. Similarly, we will call a complete noncompact hypersurface as *absolutely area minimizing hypersurface*, if any compact part (codimension-0 submanifold with boundary) of the hypersurface is an absolutely area minimizing hypersurface.

Now, we will quote the basic results on asymptotic Plateau problem.

Lemma 2.3. [4] *Let Γ be a codimension-1 closed submanifold of $S^{n-1}_\infty(\mathbf{H}^n)$. Then there exists a complete, absolutely area minimizing $n-1$-rectifiable current Σ in $\mathbf{H}^n$ with $\partial_\infty \Sigma = \Gamma$.*

Note that the rectifiable current here is indeed a smooth hypersurface of $\mathbf{H}^n$ except for a singular set of Hausdorff dimension at most $n-8$ by the regularity result stated below. For convenience, we will call area minimizing codimension-1 rectifiable currents as area minimizing hypersurfaces throughout the paper.

Lemma 2.4. [5] *Let Γ be a simple closed curve in $S^2_\infty(\mathbf{H}^3)$. Then, there exists a complete, least area plane Σ in $\mathbf{H}^3$ asymptotic to Γ.*

The following fact about interior regularity theory of geometric measure theory is well-known.

Lemma 2.5. [12] *Let Σ be a $(n-1)$-dimensional area minimizing rectifiable current. Then Σ is a smooth, embedded manifold in the interior except for a singular set of Hausdorff dimension at most $n-8$.*

Finally, we will state a theorem about limits of sequences of least area planes. Here, the limit is the pointwise limit of the planes, and for each limit point, there is a disk containing the point in the limit set such that it is the limit of a sequence of subdisks in the planes [13, Lemma 3.3].

Lemma 2.6. [13] *Let $\{\Sigma_i\}$ be a sequence of least area planes in $\mathbf{H}^3$ with $\partial_\infty \Sigma_i = \Gamma_i$ simple closed curves in $S^2_\infty(\mathbf{H}^3)$ for any i. If $\Gamma_i \to \Gamma$, then there exists a subsequence $\{\Sigma_{i_j}\}$ of $\{\Sigma_i\}$ such that $\Sigma_{i_j} \to \widehat{\Sigma}$, where $\widehat{\Sigma}$ is a collection of least area planes whose asymptotic boundaries are Γ.*

The next theorem is a similar limit theorem about absolutely area minimizing hypersurfaces in $\mathbf{H}^n$.

Lemma 2.7. *Let $\{\Gamma_i\}$ be a sequence of connected closed codimension-1 submanifolds in $S^{n-1}_\infty(\mathbf{H}^n)$ which are pairwise disjoint. Let $\{\Sigma_i\}$ be a sequence of complete absolutely area minimizing hypersurfaces in $\mathbf{H}^n$ with $\partial_\infty(\Sigma_i) = \Gamma_i$. If Γ_i converges to a closed codimension-1 submanifold Γ in $S^{n-1}_\infty(\mathbf{H}^n)$, then there exists a subsequence of $\{\Sigma_i\}$ which converges to a complete absolutely area minimizing hypersurface Σ in $\mathbf{H}^n$ with $\partial_\infty\Sigma = \Gamma$.*

*Proof:*To prove the Lemma 2.3, Anderson constructs a sequence of absolutely area minimizing surfaces $\{T_i\}$ with $\partial T_i \to \Gamma$, and for any compact set K, gives lower and upper bounds for the area of the surfaces in the sequence when restricted to K. Then using isoperimetric inequality, he shows that the sequence satisfies the requirements for the compactness theorem of geometric measure theory [12]. By taking K as enlarging closed balls $B_i(x_0)$, and by using a diagonal sequence argument, he proves the existence of complete, absolutely area minimizing rectifiable $n-1$-current Σ with $\partial_\infty\Sigma = \Gamma$.

By using Anderson's method, all we need to show is that we can induce a suitable sequence of compact absolutely area minimizing hypersurfaces $\{S_i\}$ from $\{\Sigma_i\}$ where $\gamma_i = \partial S_i$ converges to Γ in $S^2_\infty(\mathbf{H}^3)$.

Let $K > 0$ be sufficiently large so that $\Sigma_i \cap N_K(CH(\Gamma)) \neq \emptyset$ where $N_K(CH(\Gamma))$ is the K neighborhood of the convex hull of Γ. Then, let $S_i = N_K(CH(\Gamma)) \cap \Sigma_i$. Notice that $\partial S_i \subset \partial N_K(CH(\Gamma))$. Since $K > 0$ is fixed, $\partial N_K(CH(\Gamma))$ would be asymptotic to Γ. Since $\Sigma_i \subset CH(\Gamma_i)$ and $\Gamma_i \to \Gamma$, then ∂S_i must converge to Γ asymptotically.

In order to get the upper and lower bounds needed in the Anderson's proof, since all the surfaces in the sequence are in $N_K(CH(\Gamma))$, while the surfaces in the sequence in Anderson's case are in $CH(\Gamma)$, the same arguments would work with slight modification for the uniform mass bounds coming from $N_K(CH(\Gamma))$ as $K > 0$ is fixed. Hence, again by compactness theorem [12], we get convergent subsequences for enlarging closed balls $B_i(x_0)$. Then, again by using a diagonal sequence argument, we get a subsequence $\{S_{i_j}\}$ which converges to a complete absolutely area minimizing hypersurface Σ in $\mathbf{H}^n$ with $\partial_\infty\Sigma = \Gamma$.

□

Convention: All the surfaces and hypersurfaces in the paper are assumed to be orientable.

3. Least area planes in $\mathbf{H}^3$

In this section, we will prove that the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding a unique least area plane in $\mathbf{H}^3$ is dense in the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$.

First, we will show that if two least area planes have disjoint asymptotic boundaries, then they are disjoint.

Lemma 3.1. *Let Γ_1 and Γ_2 be two disjoint simple closed curves in $S^2_\infty(\mathbf{H}^3)$. If Σ_1 and Σ_2 are least area planes in $\mathbf{H}^3$ with $\partial_\infty\Sigma_i = \Gamma_i$, then Σ_1 and Σ_2 are disjoint.*

Proof: Assume that $\Sigma_1 \cap \Sigma_2 \neq \emptyset$. Since asymptotic boundaries Γ_1 and Γ_2 are disjoint, the intersection cannot contain an infinite line. So, the intersection between Σ_1 and Σ_2 must contain a simple closed curve γ. Since Σ_1 and Σ_2 are also minimal, and the tangential intersection of minimal surfaces must be isolated by maximum principle, the intersection must be transverse on a subarc τ of γ.

Now, γ bounds two area minimizing disks D_1 and D_2 in $\mathbf{H}^3$, with $D_i \subset \Sigma_i$. Now, take a larger subdisk E_1 of Σ_1 containing D_1, i.e., $D_1 \subset E_1 \subset \Sigma_1$. By definition, E_1 is also an area minimizing disk. Now, modify E_1 by swaping the disks D_1 and D_2. Then, we get a new disk $E_1' = \{E_1 - D_1\} \cup D_2$. Now, E_1 and E_1' have same area, but E_1' has a folding curve along γ. Since Σ_1 and Σ_2 is transverse along τ, by smoothing out E_1' near τ as in [19], we get a disk with smaller area, which contradicts to E_1 being area minimizing. Note that this technique is known as Meeks-Yau exchange roundoff trick.

$\square$

The following lemma is very essential for our technique. Mainly, the lemma says that for any given simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$, either there exists a unique least area plane Σ in $\mathbf{H}^3$ asymptotic to Γ, or there exist two least area planes $\Sigma^\pm$ in $\mathbf{H}^3$ which are asymptotic to Γ and disjoint from each other. Even though this lemma is also proven in [10], because of its importance for the technique, and to set the notation for the main result, we give a proof here. Note that Brian White proved a similar version of this lemma by using geometric measure theory methods in [22].

Lemma 3.2. *Let Γ be a simple closed curve in $S^2_\infty(\mathbf{H}^3)$. Then either there exists a unique least area plane Σ in $\mathbf{H}^3$ with $\partial_\infty \Sigma = \Gamma$, or there are two canonical disjoint extremal least area planes Σ^+ and Σ^- in $\mathbf{H}^3$ with $\partial_\infty \Sigma^\pm = \Gamma$. Moreover, any least area plane Σ' with $\partial_\infty \Sigma' = \Gamma$ is disjoint from $\Sigma^\pm$, and it is captured in the region bounded by Σ^+ and Σ^- in $\mathbf{H}^3$.*

Proof: Let Γ be a simple closed curve in $S^2_\infty(\mathbf{H}^3)$. Γ separates $S^2_\infty(\mathbf{H}^3)$ into two parts, say D^+ and D^-. Define sequences of pairwise disjoint simple closed curves $\{\Gamma_i^+\}$ and $\{\Gamma_i^-\}$ such that $\Gamma_i^+ \subset D^+$, and $\Gamma_i^- \subset D^-$ for any i, and $\Gamma_i^+ \to \Gamma$, and $\Gamma_i^- \to \Gamma$.

By Lemma 2.4, for any $\Gamma_i^+ \subset S^2_\infty(\mathbf{H}^3)$, there exists a least area plane Σ_i^+ in $\mathbf{H}^3$ asymptotic to Γ_i^+. This defines a sequence of least area planes $\{\Sigma_i^+\}$. Now, by using Lemma 2.6, we take the limit of a convergent subsequence. In the limit we get a collection of least area planes $\widehat{\Sigma}^+$ with $\partial_\infty \widehat{\Sigma}^+ = \Gamma$, as $\partial_\infty \Sigma_i^+ = \Gamma_i^+ \to \Gamma$.

Now, we claim that the collection $\widehat{\Sigma}^+$ consists of only one least area plane. Assume that there are two least area planes Σ_a^+ and Σ_b^+ in the collection $\widehat{\Sigma}^+$. Since we have $\partial_\infty \Sigma_a^+ = \partial_\infty \Sigma_b^+ = \Gamma$, Σ_a^+ and Σ_b^+ might not be disjoint, but they are disjoint from least area planes in the sequence, i.e., $\Sigma_i^+ \cap \Sigma_{a,b}^+ = \emptyset$ for any i, by Lemma 3.1.

If Σ_a^+ and Σ_b^+ are disjoint, say Σ_a^+ is *above* Σ_b^+. By Lemma 3.1, we know that for any i, Σ_i^+ is *above* both Σ_a^+ and Σ_b^+. However this means that Σ_a^+ is a barrier between the sequence $\{\Sigma_i^+\}$ and Σ_b^+, and so, Σ_b^+ cannot be limit of this sequence, which is a contradiction.

If Σ_a^+ and Σ_b^+ are not disjoint, then they intersect each other, and in some region, Σ_b^+ is *above* Σ_a^+. However since Σ_a^+ is the limit of the sequence $\{\Sigma_i^+\}$, this would imply Σ_b^+ must intersect planes Σ_i^+ for sufficiently large i. However, this contradicts the fact that Σ_b^+ is disjoint from Σ_i^+ for any i, as they have disjoint asymptotic boundary. So, there exists a unique least area plane Σ^+ in the collection $\widehat{\Sigma}^+$. Similarly, $\widehat{\Sigma}^- = \Sigma^-$. By using similar arguments, one can conclude that these least area planes Σ^+, and Σ^- are canonical, i.e., independent of the choice of the sequence $\{\Gamma_i^\pm\}$ and $\{\Sigma_i^\pm\}$.

Now, let Σ' be any least area plane with $\partial_\infty \Sigma' = \Gamma$. If $\Sigma' \cap \Sigma^+ \neq \emptyset$, then some part of Σ' must be *above* Σ^+. Since $\Sigma^+ = \lim \Sigma_i^+$, for sufficiently large i, $\Sigma' \cap \Sigma_i^+ \neq \emptyset$. However, $\partial_\infty \Sigma_i^+ = \Gamma_i^+$ is disjoint from $\Gamma = \partial_\infty \Sigma'$. Then, by Lemma 3.1, Σ' must be disjoint from Σ_i^+. This is a contradiction.

Similarly, this is true for Σ^-, too. Moreover, let $N \subset \mathbf{H}^3$ be the region between Σ^+ and Σ^-, i.e., $\partial N = \Sigma^+ \cup \Sigma^-$. Then by construction, N is also a canonical region for Γ, and for any least area plane Σ' with $\partial_\infty \Sigma' = \Gamma$, Σ' is contained in the region N, i.e., $\Sigma' \subset N$. This shows that if $\Sigma^+ = \Sigma^-$, there exists a unique least area plane asymptotic to Γ. If $\Sigma^+ \neq \Sigma^-$, then they must be disjoint. $\square$

Now, we are going to prove the main theorem of the section.

Theorem 3.3. *Let A be the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ and let $A' \subset A$ be the subspace containing the simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding a unique least area plane in $\mathbf{H}^3$. Then, A' is dense in A with respect to the C^0-topology. Indeed, A' is a countable intersection of open dense subsets of A with respect to the C^0-topology.*

Proof: We will prove this theorem in 2 steps.

Claim 1: A' is dense in A with C^0-topology.

Proof: Let $\Gamma_0 \in A$ be a simple closed curve in $S^2_\infty(\mathbf{H}^3)$. Since Γ_0 is simple, there exists a small neighborhood $N(\Gamma_0)$ of Γ_0 which is an open annulus in $S^2_\infty(\mathbf{H}^3)$. Then, we can find a homeomorphism $\phi : S^1 \times (-\epsilon, \epsilon) \to N(\Gamma_0)$ such that $\phi(S^1 \times \{0\}) = \Gamma_0$. Then, let $\phi(S^1 \times t) = \Gamma_t$. Since ϕ is a homeomorphism, $\{\Gamma_t\}$ foliates $N(\Gamma_0)$ with simple closed curves Γ_t. In other words, $\{\Gamma_t\}$ are pairwise disjoint simple closed curves, and $N(\Gamma_0) = \bigcup_{t \in (-\epsilon, \epsilon)} \Gamma_t$.

Now, $N(\Gamma_0)$ separates $S^2_\infty(\mathbf{H}^3)$ into two parts, say D^+ and D^-, i.e., $N(\Gamma_0) \cup D^+ \cup D^-$ gives $S^2_\infty(\mathbf{H}^3)$. Let p^+ be a point in D^+ and let p^- be a point in D^- such that for a small δ, $B_\delta(p^\pm)$ are in the interior of $D^\pm$. Let β be the geodesic in $\mathbf{H}^3$ asymptotic to p^+ and to p^-.

By Lemma 3.2, for any Γ_t either there exists a unique least area plane Σ_t in $\mathbf{H}^3$, or there is a canonical region N_t in $\mathbf{H}^3$ between the canonical least area planes Σ_t^+ and Σ_t^- (In Figure 1, Γ_t and Γ_s bound more than one least area plane in $\mathbf{H}^3$, whereas Γ_0 bounds a unique least area plane Σ_0 in $\mathbf{H}^3$). With abuse of notation, if Γ_t bounds a unique least area plane Σ_t in $\mathbf{H}^3$, define $N_t = \Sigma_t$ as a degenerate canonical neighborhood for Γ_t (In Figure 1, N_t and N_s represent nondegenerate canonical neighborhoods, and $N_0 = \Sigma_0$ represents degenerate canonical neighborhood). Then, let $\widehat{N} = \{N_t\}$ be the collection of these degenerate and nondegenerate canonical neighborhoods for $t \in (-\epsilon, \epsilon)$. Clearly, degenerate neighborhood N_t means Γ_t bounds unique least area plane, and nondegenerate neighborhood N_s means that Γ_s bounds more than one least area plane. Note that by Lemma 3.1, all canonical neighborhoods in the collection are pairwise disjoint. On the other hand, by construction the geodesic β intersects all the canonical neighborhoods in the collection $\widehat{N}$.

We claim that the part of β which intersects $\widehat{N}$ is a finite line segment. Let P^+ be the geodesic plane asymptotic to the round circle $\partial B_\delta(p^+)$ in D^+. Similarly, define P^-. By Lemma 3.1, $P^\pm$ are disjoint from the collection of canonical regions $\widehat{N}$. Let $\beta \cap P^\pm = \{q^\pm\}$. Then the part of β which intersects $\widehat{N}$ is the line segment $l \subset \beta$ with endpoints q^+ and q^-. Let C be the length of this line segment l.

Now, for each $t \in (-\epsilon, \epsilon)$, we will assign a real number $s_t \geq 0$. If there exists a unique least area plane Σ_t in $\mathbf{H}^3$ for Γ_t (N_t is degenerate), then let s_t be 0. If not, let $I_t = \beta \cap N_t$, and s_t be the length of I_t. Clearly if Γ_t bounds more than one least area plane (N_t is nondegenerate), then $s_t > 0$. Also, it is clear that for any t, $I_t \subset l$ and $I_t \cap I_s = \emptyset$ for any $t \neq s$. Then, $\sum_{t\in(-\epsilon,\epsilon)} s_t < C$ where C is the length of l. This means for only countably

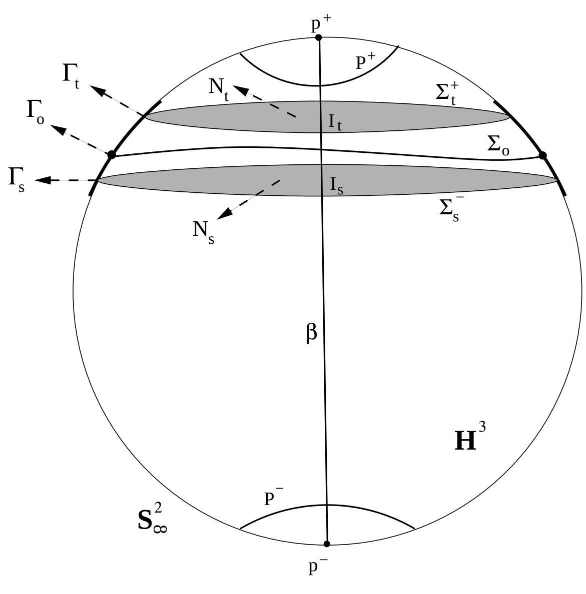

FIGURE 1. A finite segment of geodesic γ intersects the collection of least area planes Σ_t in $\mathbf{H}^n$ asymptotic to Γ_t in $S^{n-1}_\infty(\mathbf{H}^n)$.

many $t \in (-\epsilon, \epsilon)$, $s_t > 0$. So, there are only countably many nondegenerate N_t for $t \in (-\epsilon, \epsilon)$. Hence, for all other t, N_t is degenerate. This means there exist uncountably many $t \in (-\epsilon, \epsilon)$, where Γ_t bounds a unique least area plane. Since Γ_0 is arbitrary, this proves A' is dense in A.

$\square$

Claim 2: A' is a countable intersection of open dense subsets in A with C^0-topology.

Proof: We will define a sequence of open dense subsets $U^i \subset A$ such that their intersection will give us A', i.e., $A' = \bigcap U^i$.

Let $\Gamma \in A$ be a simple closed curve in $S^2_\infty(\mathbf{H}^3)$, as in the Claim 1. Let $N(\Gamma) \subset S^2_\infty(\mathbf{H}^3)$ be a neighborhood of Γ in $S^2_\infty(\mathbf{H}^3)$, which is an open annulus. Then, define an open neighborhood U_Γ of Γ in A, such that $U_\Gamma = \{\alpha \in A \mid \alpha \subset N(\Gamma),\ \alpha \text{ is homotopic to } \Gamma\}$. Clearly, $A = \bigcup_{\Gamma \in A} U_\Gamma$. Now, define a geodesic β_Γ as in Claim 1, which intersects all the least area planes asymptotic to curves in U_Γ.

Now, for any $\alpha \in U_\Gamma$, by Lemma 3.2, there exists a canonical region N_α in $\mathbf{H}^3$ (which can be degenerate if α bounds a unique least area plane). Let $I_{\alpha,\Gamma} = N_\alpha \cap \beta_\Gamma$. Then let $s_{\alpha,\Gamma}$ be the length of $I_{\alpha,\Gamma}$ ($s_{\alpha,\Gamma}$ is 0 if N_α degenerate). Hence, for every element α in U_Γ, we assign a real number $s_{\alpha,\Gamma} \geq 0$.

Define the sequence of open dense subsets in U_Γ by $U^i_\Gamma = \{\alpha \in U_\Gamma \mid s_{\alpha,\Gamma} < 1/i\ \}$. We claim that U^i_Γ is an open subset of U_Γ and A. Let $\alpha \in U^i_\Gamma$, and let $s_{\alpha,\Gamma} = \lambda < 1/i$. So, the interval $I_{\alpha,\Gamma} \subset \beta_\Gamma$ has length λ. Let $I' \subset \beta_\Gamma$ be an interval containing $I_{\alpha,\Gamma}$ in its interior, and has length less than $1/i$. By the proof of Claim 1, we can find two simple closed curves $\alpha^+, \alpha^- \in U_\Gamma$ with the following properties:

- $\alpha^\pm$ are disjoint from α,
- $\alpha^\pm$ are lying in opposite sides of α in $S^2_\infty(\mathbf{H}^3)$,
- $\alpha^\pm$ bounds unique least area planes $\Sigma_{\alpha^\pm}$,
- $\Sigma_{\alpha^\pm} \cap \beta_\Gamma \subset I'$.

The existence of such curves is clear from the proof Claim 1, since if one takes any foliation $\{\alpha_t\}$ of a small neighborhood of α in $S^2_\infty(\mathbf{H}^3)$, there are uncountably many curves in the family bounding a unique least area plane, and one can choose sufficiently close pair of curves to α, to ensure the conditions above.

After finding $\alpha^\pm$, consider the open annulus F_α in $S^2_\infty(\mathbf{H}^3)$ bounded by α^+ and α^-. Let $V_\alpha = \{\gamma \in U_\Gamma \mid \gamma \subset F_\alpha,\ \gamma \text{ is homotopic to } \alpha\}$. Clearly, V_α is an open subset of U_Γ. If we can show $V_\alpha \subset U^i_\Gamma$, then this proves U^i_Γ is open for any i and any $\Gamma \in A$.

Let $\gamma \in V_\alpha$ be any curve, and N_γ be its canonical neighborhood given by Lemma 3.2. Since $\gamma \subset F_\alpha$, α^+ and α^- lie in opposite sides of γ in $S^2_\infty(\mathbf{H}^3)$. This means Σ_{α^+} and Σ_{α^-} lie in opposite sides of N_γ. By choice of $\alpha^\pm$, this implies $N_\gamma \cap \beta_\Gamma = I_{\gamma,\Gamma} \subset I'$. So, the length $s_{\gamma,\Gamma}$ is less than $1/i$. This implies $\gamma \in U^i_\Gamma$, and so $V_\alpha \subset U^i_\Gamma$. Hence, U^i_Γ is open in U_Γ and A.

Now, we can define the sequence of open dense subsets. Let $U^i = \bigcup_{\Gamma \in A} U^i_\Gamma$ be an open subset of A. Since the elements in A' represent the curves bounding a unique least area plane, $s_{\alpha,\Gamma} = 0$ for any $\alpha \in A'$, and for any $\Gamma \in A$. This means $A' \subset U^i$ for any i. As A' is dense in A by Claim 1, U^i is not only open, but also dense in A for any $i > 0$. On the other hand, $A' = \bigcap_{i>0} U^i$, as $s_{\alpha,\Gamma} = 0$ for any $\alpha \in A'$, and for any $\Gamma \in A$. Then, A' is a countable intersection of open dense subsets $\{U^i\}$ of A. This implies A' is a countable intersection of open dense subsets in A with C^0-topology. $\square$

$\square$

Remark 3.1. Notice that we did not use the term *generic* for the countable intersection of open dense subsets in the theorem. This is because even though C^0-topology is complete, the space A of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ with C^0-topology is not complete as they have non-simple (non-embedded) closed curves in the limit. Hence, even though A' is countable intersection of open dense subsets of A, this does not mean that A' is generic in A in Baire sense as A is not a complete metric space.

Remark 3.2. This result is similar to the uniqueness results in [8]. In [8], we used a heavy machinery of analysis to prove that there exists an open dense subset in the space of $C^{3,\mu}$-smooth embeddings of circle into sphere, where any simple closed curve in this space bounds a unique least area plane in $\mathbf{H}^3$. In the above result, the argument is fairly simple, and does not use the analytical machinery.

Remark 3.3. Note that by using similar techniques, the same result can be obtained by using Hausdorff topology instead of C^0 topology on the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$.

4. Area minimizing hypersurfaces in $\mathbf{H}^n$

In this section, we will show that the space of codimension-1 closed submanifolds of $S^{n-1}_\infty(\mathbf{H}^n)$ bounding a unique absolutely area minimizing hypersurface in $\mathbf{H}^n$ is dense in the space of all codimension-1 closed submanifolds of $S^{n-1}_\infty(\mathbf{H}^n)$. The idea is similar to the previous section.

First, we need to show a simple topological lemma.

Lemma 4.1. *Any codimension-*1 *closed, orientable submanifold* Γ *of* S^{n-1} *is separating in* S^{n-1}. *Moreover, if* Σ *is a hypersurface with boundary in the closed unit ball* B^n*, where the boundary* $\partial\Sigma \subset \partial B^n$*, then* Σ *is also separating, too.*

Proof: If a codimension-1 closed, orientable submanifold Γ is non-separating in S^{n-1}, then it does not bound any codimension-0 submanifold in S^{n-1}. This implies Γ generates a nontrivial homology in $n-2$ level. However, since $H_{n-2}(S^{n-1})$ is trivial, this is a contradiction.

Let Σ be as in the assumption. Take the double of B^n, then $B^n \sqcup \widehat{B^n} = S^n$, and $\Sigma \sqcup \widehat{\Sigma}$ is a codimension-1 closed submanifold of S^n. By above, $\Sigma \sqcup \widehat{\Sigma}$ is separating in S^n. Hence, Σ is separating in B^n. $\square$

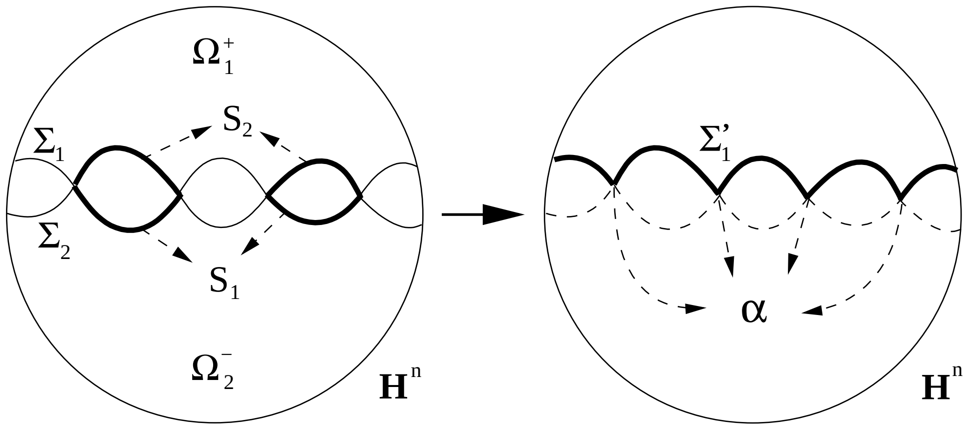

FIGURE 2. S_1 is the part of Σ_1 lying below Σ_2, and S_2 is the part of Σ_2 lying above Σ_1. After swaping S_1 and S_2, we get a new area minimizing hypersurface Σ_1' with singularity along $\alpha = \Sigma_1 \cap \Sigma_2$.

Now, we will prove a disjointness lemma analogous to Lemma 3.1. This lemma roughly says that if asymptotic boundaries of two absolutely area minimizing hypersurfaces in $\mathbf{H}^n$ are disjoint in $S^{n-1}_\infty(\mathbf{H}^n)$, then they are disjoint in $\mathbf{H}^n$.

Lemma 4.2. *Let Γ_1 and Γ_2 be two disjoint connected, closed, orientable codimension-1 submanifolds in $S^{n-1}_\infty(\mathbf{H}^n)$. If Σ_1 and Σ_2 are absolutely area minimizing hypersurfaces in $\mathbf{H}^n$ with $\partial_\infty \Sigma_i = \Gamma_i$, then Σ_1 and Σ_2 are disjoint, too.*

Proof: Assume that the absolutely area minimizing hypersurfaces are not disjoint, i.e., $\Sigma_1 \cap \Sigma_2 \neq \emptyset$. By Lemma 4.1, Σ_1, and Σ_2 separates $\mathbf{H}^n$ into two parts. So we can write $\mathbf{H}^n - \Sigma_i = \Omega_i^+ \cup \Omega_i^-$.

Now, consider the intersection of hypersurfaces $\alpha = \Sigma_1 \cap \Sigma_2$. We claim that the intersection set α is in the compact part of $\mathbf{H}^n$. Clearly, as $\Sigma_i \subset CH(\Gamma_i)$, then α is in $CH(\Gamma_1) \cap CH(\Gamma_2)$. Consider the compactification of $\mathbf{H}^n$ in the Poincare ball model. Then, $\overline{\mathbf{H}^n} = \mathbf{H}^n \cup S^{n-1}_\infty(\mathbf{H}^n)$ would be a closed n-ball topologically. Since $\partial_\infty CH(\Gamma_i) = \Gamma_i$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, then $CH(\Gamma_1) \cap CH(\Gamma_2) \cap S^{n-1}_\infty(\mathbf{H}^n) = \emptyset$. As $CH(\Gamma_1)$ and $CH(\Gamma_2)$ are compact subsets of $\overline{\mathbf{H}^n}$ and $CH(\Gamma_1) \cap CH(\Gamma_2) \cap S^{n-1}_\infty(\mathbf{H}^n) = \emptyset$, then $CH(\Gamma_1) \cap CH(\Gamma_2)$ would be compact in $\mathbf{H}^n$, too. This shows that α is in a compact subset of $\mathbf{H}^n$. Moreover, by maximum principle [20], the intersection cannot have isolated tangential intersections.

Now, without loss of generality, we assume that Σ_1 is *above* Σ_2 (the noncompact part of Σ_1 lies in Ω_2^+). Now define the compact subhypersurfaces S_i in Σ_i as $S_1 = \Sigma_1 \cap \Omega_2^-$, and $S_2 = \Sigma_2 \cap \Omega_1^+$. In other words, S_1 is the part of Σ_1 lying *below* Σ_2, and S_2 is the part of Σ_2 lying *above* Σ_1. Then, $\partial S_1 = \partial S_2 = \alpha$. See Figure 2. Note that if $S_1 = \emptyset$ or $S_2 = \emptyset$, then this implies the intersection is tangential and Σ_1 lies in one side of Σ_2, but this contradicts to maximum principle for area minimizing hypersurfaces ([20], Corollary 1). Hence, we can assume $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$.

On the other hand, since Σ_1 and Σ_2 are absolutely area minimizing, then by definition, so are S_1 and S_2, too. Then by swaping the surfaces, we can get new absolutely area minimizing hypersurfaces. In other words, $\Sigma_1' = \{\Sigma_1 - S_1\} \cup S_2$, and $\Sigma_2' = \{\Sigma_2 - S_2\} \cup S_1$ are also absolutely area minimizing hypersurfaces. Note that by construction, Σ_1' lies in one side of Σ_2' and vice versa.

Now, if the intersection of Σ_1 and Σ_2 is transverse in some part of α, we will have a codimension-1 singularity set in Σ_1', which contradicts to regularity theorem for absolutely area minimizing hypersurfaces, i.e., Lemma 2.5. If the intersection along α is completely tangential, since Σ_1' lies in one side of Σ_2', this contradicts to maximum principle for area minimizing hypersurfaces ([20], Corollary 1). The proof follows. $\square$

Lemma 4.3. *Let Γ be a connected, closed, orientable codimension-1 submanifold of $S^{n-1}_\infty(\mathbf{H}^n)$. Then either there exists a unique absolutely area minimizing hypersurface Σ in $\mathbf{H}^n$ asymptotic to Γ, or there are two canonical disjoint extremal absolutely area minimizing hypersurfaces Σ^+ and Σ^- in $\mathbf{H}^n$ asymptotic to Γ.*

Proof: Let Γ be a connected, closed, orientable codimension-1 submanifold of $S^{n-1}_\infty(\mathbf{H}^n)$. Then by Lemma 4.1, Γ separates $S^{n-1}_\infty(\mathbf{H}^n)$ into two parts, say Ω^+ and Ω^-. Define sequences of pairwise disjoint closed submanifolds of the same topological type $\{\Gamma_i^+\}$ and $\{\Gamma_i^-\}$ in $S^{n-1}_\infty(\mathbf{H}^n)$ such that $\Gamma_i^+ \subset \Omega^+$, and $\Gamma_i^- \subset \Omega^-$ for any i, and $\Gamma_i^+ \to \Gamma$, and $\Gamma_i^- \to \Gamma$ in Hausdorff metric. In other words, $\{\Gamma_i^+\}$ and $\{\Gamma_i^-\}$ converges to Γ from opposite sides.

By Lemma 2.3, for any $\Gamma_i^+ \subset S^2_\infty(\mathbf{H}^3)$, there exists an absolutely area minimizing hypersurface Σ_i^+ in $\mathbf{H}^n$. This defines a sequence of absolutely area minimizing hypersurfaces $\{\Sigma_i^+\}$. By Lemma 2.7, we get a convergent subsequence $\Sigma_{i_j}^+ \to \Sigma^+$. Hence, we get the absolutely area minimizing hypersurface Σ^+ in $\mathbf{H}^n$ asymptotic to Γ. Similarly, we get the absolutely area minimizing hypersurface Σ^- in $\mathbf{H}^n$ asymptotic to Γ. Similar arguments show that these absolutely area minimizing hypersurfaces $\Sigma^\pm$ are canonical by their construction, i.e., independent of the choice of the sequence $\{\Gamma_i^\pm\}$ and $\{\Sigma_i^\pm\}$.

Assume that $\Sigma^+ \neq \Sigma^-$, and they are not disjoint. Since these are absolutely area minimizing hypersurfaces, nontrivial intersection implies some part of Σ^- lies *above* Σ^+. Since $\Sigma^+ = \lim \Sigma_{i_j}^+$, Σ^- must also intersect some $\Sigma_{i_j}^+$ for sufficiently large i_j. However by Lemma 4.2, $\Sigma_{i_j}^+$ is disjoint from Σ^- as $\partial_\infty \Sigma_{i_j}^+ = \Gamma_{i_j}^+$ is disjoint from $\partial_\infty \Sigma^- = \Gamma$. This is a contradiction. This shows Σ^+ and Σ^- are disjoint.

Similar arguments show that $\Sigma^\pm$ are disjoint from any absolutely area minimizing hypersurface Σ' asymptotic to Γ. As the sequences of Σ_i^+ and Σ_i^- form a barrier for other absolutely area minimizing hypersurfaces asymptotic to Γ, any such absolutely area minimizing hypersurface must lie in the region bounded by Σ^+ and Σ^- in $\mathbf{H}^n$. This shows that if $\Sigma^+ = \Sigma^-$, then there exists a unique absolutely area minimizing hypersurface asymptotic to Γ. $\square$

Remark 4.1. By above theorem and its proof, if Γ bounds more than one absolutely area minimizing hypersurface, then there exists a canonical region N_Γ in $\mathbf{H}^n$ asymptotic to Γ such that N_Γ is the region between the canonical absolutely area minimizing hypersurfaces Σ^+ and Σ^-. Moreover, by using similar ideas to the proof of Lemma 3.2, one can show that any absolutely area minimizing hypersurface in $\mathbf{H}^n$ asymptotic to Γ is in the region N_Γ.

Now, we can prove the main result of the paper.

Theorem 4.4. *Let B be the space of connected, closed, orientable codimension-1 submanifolds of $S^{n-1}_\infty(\mathbf{H}^n)$, and let $B' \subset B$ be the subspace containing the submanifolds of $S^{n-1}_\infty(\mathbf{H}^n)$ bounding a unique absolutely area minimizing hypersurface in $\mathbf{H}^n$. Then B' is dense in B with C^0-topology.*

Proof: Let B be the space of connected, closed, orientable codimension-1 submanifolds of $S^{n-1}_\infty(\mathbf{H}^n)$ with C^0-topology. Let $\Gamma_0 \in B$ be a locally flat (tame), closed submanifold in $S^{n-1}_\infty(\mathbf{H}^n)$. Since Γ_0 is a locally flat, closed, orientable submanifold, there exists a small tubular neighborhood $N(\Gamma_0)$ of Γ_0 in $S^{n-1}_\infty(\mathbf{H}^n)$, which is homeomorphic to $Y \times I$ where Y is closed $n-1$-dimensional manifold homeomorphic to Γ_0 [18]. Then, we can find a homeomorphism $\phi : Y \times (-\epsilon, \epsilon) \to N(\Gamma_0)$ such that $\phi(Y \times \{0\}) = \Gamma_0$. Then, let $\phi(Y \times t) = \Gamma_t$. Since ϕ is a homeomorphism, $\{\Gamma_t\}$ foliates $N(\Gamma_0)$ with closed hypersurfaces Γ_t. In other words, $\{\Gamma_t\}$ are pairwise disjoint closed hypersurfaces in $S^{n-1}_\infty(\mathbf{H}^n)$, and $N(\Gamma_0) = \bigcup_{t\in(-\epsilon,\epsilon)} \Gamma_t$.

By Lemma 4.1, $N(\Gamma_0)$ separates $S^{n-1}_\infty(\mathbf{H}^n)$ into two parts, say Ω^+ and Ω^-, i.e., $S^{n-1}_\infty(\mathbf{H}^n) = N(\Gamma_0) \cup \Omega^+ \cup \Omega^-$. Let p^+ be a point in Ω^+ and let p^- be a point in Ω^- such that for a small δ, $B_\delta(p^\pm)$ are in the interior of $\Omega^\pm$. Let β be the geodesic in $\mathbf{H}^n$ asymptotic to p^+ and p^-.

By Lemma 4.3 and Remark 4.1, for any Γ_t either there exists a unique absolutely area minimizing hypersurface Σ_t in $\mathbf{H}^n$, or there is a canonical region N_t in $\mathbf{H}^n$ asymptotic to Γ_t, namely the region between the canonical absolutely area minimizing hypersurfaces Σ^+_t and Σ^-_t. With abuse of notation, if Γ_t bounds a unique absolutely area minimizing hypersurface Σ_t in $\mathbf{H}^n$, define $N_t = \Sigma_t$ as a degenerate canonical neighborhood for Γ_t. Then, let $\widehat{N} = \{N_t\}$ be the collection of these degenerate and nondegenerate canonical neighborhoods for $t \in (-\epsilon, \epsilon)$. Clearly, degenerate neighborhood N_t means Γ_t bounds a unique absolutely area minimizing hypersurface, and nondegenerate neighborhood N_s means that Γ_s bounds more than one absolutely area minimizing hypersurface. Note that by Lemma 4.2, all canonical neighborhoods in the collection are pairwise disjoint. On the other hand, by Lemma 4.1, the geodesic β intersects all the canonical neighborhoods in the collection $\widehat{N}$.

We claim that the part of β which intersects $\widehat{N}$ is a finite line segment. Let P^+ be the geodesic hyperplane asymptotic to the round sphere $\partial B_\delta(p^+)$ in Ω^+. Similarly, define P^-. By Lemma 4.2, $P^\pm$ are disjoint from the collection of canonical regions $\widehat{N}$. Let $\beta \cap P^\pm = \{q^\pm\}$. Then the part of β which intersects $\widehat{N}$ is the line segment $l \subset \beta$ with endpoints q^+ and q^-. Let C be the length of this line segment l.

Now, for each $t \in (-\epsilon, \epsilon)$, we will assign a real number $s_t \geq 0$. If there exists a unique absolutely area minimizing hypersurface Σ_t for Γ_t (N_t is degenerate), then let s_t be 0. If not, let $I_t = \beta \cap N_t$, and s_t be the length of I_t. Clearly if Γ_t bounds more than one least area plane (N_t is nondegenerate), then $s_t > 0$. Also, it is clear that for any t, $I_t \subset l$ and $I_t \cap I_s = \emptyset$ for any $t \neq s$. Then, $\sum_{t \in (-\epsilon, \epsilon)} s_t < C$ where C is the length of l. This means for only countably many $t \in (-\epsilon, \epsilon)$, $s_t > 0$. So, there are only countably many nondegenerate N_t for $t \in (-\epsilon, \epsilon)$. Hence, for all other t, N_t is degenerate. This means there exist uncountably many $t \in (-\epsilon, \epsilon)$, where Γ_t bounds a unique absolutely area minimizing hypersurface. Since Γ_0 is locally flat, and locally flat embeddings are dense in B, [3] (for $n \geq 5$), [2] (for $n = 4$), [6] (for $n = 3$), this proves B' is dense in B. $\square$

On the other hand, in dimension 3, the unique curves for absolutely area minimizing surfaces are not only dense, but also a countable intersection of open dense subsets just like the least area planes case.

Corollary 4.5. *Let A be the space of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ and let $A' \subset A$ be the subspace containing the simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding a unique absolutely area minimizing surface in $\mathbf{H}^3$. Then, A' is dense in A with respect to the C^0-topology. Indeed, A' is a countable intersection of open dense subsets of A with respect to the C^0-topology.*

Proof: By Theorem 4.4, we know that A' is dense in A. Then by using the proof of Claim 2 in Theorem 3.3 in this setting, it is clear that A' is a countable intersection of open dense subsets in A with C^0-topology. $\square$

5. Nonuniqueness results

In this section, we will show that there exists a simple closed curve in $S^2_\infty(\mathbf{H}^3)$ which is the asymptotic boundary of more than one absolutely area minimizing surface in $\mathbf{H}^3$. First, we need a lemma about the limits of absolutely area minimizing surfaces.

A short outline of the method is the following. We first construct a simple closed curve Γ_0 in $S^2_\infty(\mathbf{H}^3)$ which is the asymptotic boundary of more than one *minimal surface* in $\mathbf{H}^3$. Then, we foliate $S^2_\infty(\mathbf{H}^3)$ with simple closed curves $\{\Gamma_t\}$ where Γ_0 is a leaf in the foliation. We show that if each Γ_t bounds a unique absolutely area minimizing surface Σ_t in $\mathbf{H}^3$, then the family of surfaces $\{\Sigma_t\}$ must foliate the whole $\mathbf{H}^3$. However, since we chose Γ_0 to bound more than one minimal surface in $\mathbf{H}^3$, one of the surfaces must have a tangential intersection with one of the leaves in the foliation. This contradicts to the maximum principle for minimal surfaces.

Now, we quote a result on the existence of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ which are the asymptotic boundaries of more than one minimal surface in $\mathbf{H}^3$.

Lemma 5.1. [5] *There is a set Δ of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ such that for any $\Gamma \in \Delta$, there exist infinitely many complete, smoothly embedded minimal surfaces in $\mathbf{H}^3$ asymptotic to Γ.*

Remark 5.1. Alternatively, one can construct simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding more than one minimal surface in $\mathbf{H}^3$ as follows. By using the technique in [Ha], one can construct a simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$ such that the absolutely area minimizing surface Σ asymptotic to Γ has positive genus (see Figure 3). Then, Σ separates $\mathbf{H}^3$ into two parts Ω^+ and Ω^- which are both mean convex domains. Then by using Meeks-Yau's results in [19], one can get sequences of least area disks $\{D_i^\pm\}$ in $\Omega^\pm$ whose boundaries converge to Γ in $S^2_\infty(\mathbf{H}^3)$. By taking the limit, one can get two least area planes P^+ and P^- with $\partial_\infty P^\pm = \Gamma$. We used Σ as a barrier to get distinct limits from the sequences $\{D_i^+\}$ and $\{D_i^-\}$. Note that the planes $P^\pm$ are least area just in $\Omega^\pm$. However, $P^\pm$ may not be least area in $\mathbf{H}^3$, but of course, they are still minimal planes.

Theorem 5.2. *There exists a simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$ such that Γ bounds more than one absolutely area minimizing surface $\{\Sigma_i\}$ in $\mathbf{H}^3$, i.e., $\partial_\infty \Sigma_i = \Gamma$.*

Proof: Assume that for any simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$, there exists a unique complete absolutely area minimizing surface Σ in $\mathbf{H}^3$ with $\partial_\infty \Sigma = \Gamma$. Let Γ_0 be a simple closed curve in $S^2_\infty(\mathbf{H}^3)$ such that M_1 and M_2 are two distinct minimal surfaces in $\mathbf{H}^3$ asymptotic to Γ as in Lemma 5.1.

Now, foliate $S^2_\infty(\mathbf{H}^3)$ by simple closed curves $\{\Gamma_t\}_{-1\leq t\leq 1}$ where Γ_0 is a leaf in the foliation. Note that, there are only two singular leaves $\{\Gamma_{-1}, \Gamma_1\}$ in the foliation which are points, and all other leaves are embedded simple closed curves in $S^2_\infty(\mathbf{H}^3)$. By assumption, for any Γ_t, there exists a unique absolutely area minimizing surface Σ_t in $\mathbf{H}^3$.

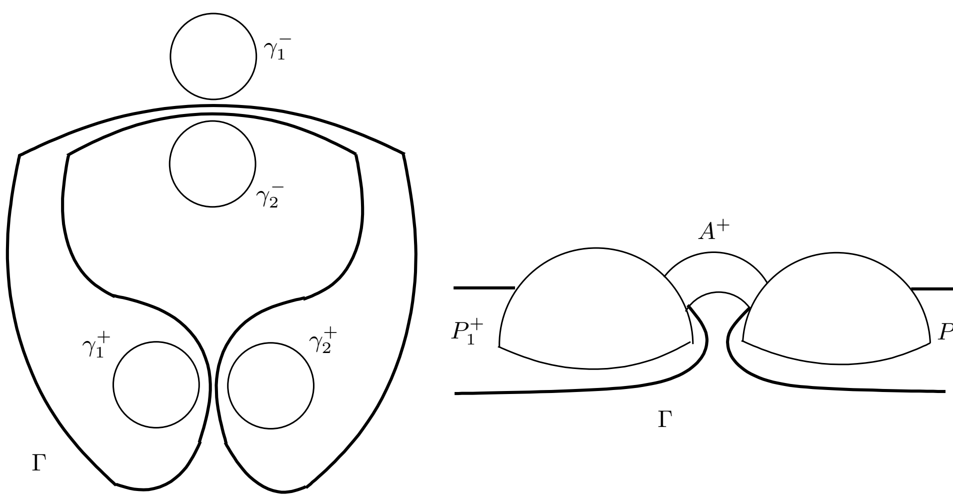

FIGURE 3. Γ is a simple closed curve in $S^2_\infty(\mathbf{H}^3)$. γ_i^+ and γ_i^- are round circles in $S^2_\infty(\mathbf{H}^3)$ bounding the geodesic planes P_i^+ and P_i^- in $\mathbf{H}^3$.

We claim that $\{\Sigma_t\}$ will be a foliation of $\mathbf{H}^3$. By Lemma 4.2, $\Sigma_t \cap \Sigma_s = \emptyset$ for any $s \neq t$. Hence, the only way to fail to be a foliation for $\{\Sigma_t\}$ is to have a gap between two leaves.

Now, assume that there is a gap between the leaves $\{\Sigma_t \mid t > s\}$ and $\{\Sigma_t \mid t < s\}$. This implies if we have sequences $\{\Gamma_{t_i^+}\}$ and $\{\Gamma_{t_i^-}\}$, where $t_i^+ \to s$ from positive side, and $t_i^- \to s$ from negative side, then one of the sequences $\{\Sigma_{t_i^+}\}$ and $\{\Sigma_{t_i^-}\}$ has no subsequences converging to Σ_s, because of the existence of the gap between the leaves (Recall that we assume there is a unique absolutely area minimizing surface Σ_t with $\partial_\infty \Sigma_t = \Gamma_t$). However, this contradicts to Lemma 2.7. So, $\{\Sigma_t\}$ will be a foliation of $\mathbf{H}^3$.

Now, by assumption, there are two distinct minimal surfaces M_1 and M_2 which are asymptotic to Γ_0. This implies at least one of these surfaces is not a leaf of the foliation, say M_2, and must intersect the leaves in the foliation nontrivially. Since $\{\Sigma_t\}$ foliates whole $\mathbf{H}^3$, this means M_2 must intersect tangentially (lying in one side) one of the leaves in the foliation. However, this contradicts to the maximum principle. So, one of the simple closed curves in $\{\Gamma_t\}$ must bound more than one absolutely area minimizing surface in $\mathbf{H}^3$. The proof follows. $\square$

By using similar ideas, one can prove an analogous theorem for least area planes in $\mathbf{H}^3$.

Theorem 5.3. *There exists a simple closed curve Γ in $S^2_\infty(\mathbf{H}^3)$ such that Γ bounds more than one least area plane $\{P_i\}$ in $\mathbf{H}^3$, i.e., $\partial_\infty P_i = \Gamma$.*

Proof: The proof is completely analogous to the proof of the previous theorem. Again, we start with the same foliation of $S^2_\infty(\mathbf{H}^3)$ with simple closed curves $\{\Gamma_t\}$ containing Γ_0 which bounds more than one complete minimal surface in $\mathbf{H}^3$.

Assume that P_t is the unique least area plane in $\mathbf{H}^3$ with $\partial_\infty P_t = \Gamma_t$. Again, we claim that $\{P_t\}$ is a foliation of $\mathbf{H}^3$. By Lemma 3.1, $P_t \cap P_s = \emptyset$ for any $s \neq t$. Hence, the only way to fail to be a foliation for $\{P_t\}$ is to have a gap between two leaves.

Now, assume that there is a gap between the leaves $\{P_t \mid t > s\}$ and $\{P_t \mid t < s\}$. This implies if we have sequences $\{P_{t_i^+}\}$ and $\{P_{t_i^-}\}$, where $t_i^+ \to s$ from positive side, and $t_i^- \to s$ from negative side, then one of the sequences $\{P_{t_i^+}\}$ and $\{P_{t_i^-}\}$ has no subsequences converging to Σ_s, because of the existence of the gap between the leaves (Recall that we assume there is a unique absolutely area minimizing surface Σ_t with $\partial_\infty \Sigma_t = \Gamma_t$). However, one can construct a sequence of least area disks $\{D_{t_i^+}\}$ where $D_{t_i^+} \subset P_{t_i^+}$ with $\partial D_{t_i^+} \to \Gamma_s$. By using Lemma 2.6, we get a subsequence with $D_{t_{i_j}^+} \to P_s^+$ where P_s^+ is a least area plane with $\partial_\infty P_s^+ = \Gamma_s$. Similarly, one can get a least area plane P_s^- with $\partial_\infty P_s^- = \Gamma_s$. However, since there is a gap between the leaves, $P_s^+ \neq P_s^-$. This contradicts to the uniqueness assumption as Γ_s bounds unique least area plane. Also, $\{P_t\}$ fills $\mathbf{H}^3$ by the construction of $\{\Gamma_t\}$. Hence, this shows that $\{P_t\}$ is a foliation of $\mathbf{H}^3$.

However, like in the proof of previous theorem, Γ_0 bounds more than one complete minimal surface, and at least one of them is not a leaf of the foliation, say M_2. Hence, M_2 must intersect tangentially (lying in one side) one of the leaves in the foliation. Again, this contradicts to the maximum principle for minimal surfaces. The proof follows. $\square$

Remark 5.2. The same proof may not work for area minimizing surfaces in a specified topological class. The problem is that Lemma 2.7 may not be true for this case as the limiting surface might not be in the same topological class (the genus might drop in the limit).

Remark 5.3. As the introduction suggests, there is no known example of a simple closed curve of $S^2_\infty(\mathbf{H}^3)$ with nonunique solution to the asymptotic Plateau problem. Unfortunately, the results above show the existence of such an example, but they do not give one. The main problem to find such an example is the noncompactness of the objects. In compact case, the quantitative data enables you to find such examples like baseball curve on a sphere, but in the asymptotic case the techniques do not work because of the lack of the quantitative data. In other words, the explicit examples in compact case, like baseball curve on a sphere, are indeed area minimizing surfaces, and being able to compare the areas of surfaces and symmetry enables one to show the baseball curve bounds two different area minimizing disks in the ball. Even though the baseball curve γ in $S^2_\infty(\mathbf{H}^3)$ might be good candidate for a curve bounding two different least area planes in $\mathbf{H}^3$, it is not easy to show that the least area plane bounding γ in $S^2_\infty(\mathbf{H}^3)$ is fixed or not by the involution of $\overline{BH^3}$ which is fixing γ without using the area tool like in the compact case. To overcome this problem, it might be possible to employ the renormalized area defined by Alexakis-Mazzeo in [1] in order to construct an explicit simple closed curve in $S^2_\infty(\mathbf{H}^3)$ bounding more than one absolutely area minimizing surface.

6. Concluding remarks

In this paper, we showed that the space of closed, codimension-1 submanifolds of $S^{n-1}_\infty(\mathbf{H}^n)$ has a dense subspace of closed, codimension-1 submanifolds of $S^{n-1}_\infty(\mathbf{H}^n)$ bounding a unique absolutely area minimizing hypersurface in $\mathbf{H}^n$. As we discussed in the introduction, Anderson showed this result for closed submanifolds bounding convex domains in $S^{n-1}_\infty(\mathbf{H}^n)$ in [4]. Then, Hardt and Lin generalized this result to closed submanifolds bounding star shaped domains in $S^{n-1}_\infty(\mathbf{H}^n)$ in [14]. These were the only cases known so far. Hence, our results show that they are indeed very abundant.

The technique which we employ here is very general, and it applies to many different settings of Plateau problem. In particular, it can naturally be generalized to the Gromov-Hadamard spaces which is studied by Lang in [16], and it can be generalized to the mean convex domains with spherical boundary which is studied by Lin in [17]. Generalizing this technique in the context of constant mean curvature hypersurfaces in hyperbolic space also gives similar results. On the other hand, they can also be applied in Gromov hyperbolic 3-spaces with cocompact metric where the author solved the asymptotic Plateau problem [9].

On the other hand, it was not known whether all closed codimension-1 submanifolds in $S^{n-1}_\infty(\mathbf{H}^n)$ have a unique solution to the asymptotic Plateau problem or not. The only known results about nonuniqueness also come from Anderson in [5]. He constructs examples of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ bounding more than one complete *minimal* surface in $\mathbf{H}^3$. These examples are also area minimizing in their topological class. However, none of them are absolutely area minimizing, i.e., a solution to the asymptotic Plateau problem. In Section 5, we prove the existence of simple closed curves in $S^2_\infty(\mathbf{H}^3)$ with nonunique solution to asymptotic Plateau problem, and hence, give an answer for dimension 3. However, there is no result in higher dimensions yet. In other words, it is not known whether there exist closed codimension-1 submanifolds in $S^{n-1}_\infty(\mathbf{H}^n)$ bounding more than one absolutely area minimizing hypersurfaces or not for $n > 3$.

It might be possible to extend the techniques in this paper to address the nonuniqueness question in higher dimensions. It is possible to use the technique in Remark 5.1 (Figure 3), to get a codimension-1 sphere Γ_0 in $S^{n-1}_\infty(\mathbf{H}^n)$ bounding an absolutely area minimizing hypersurface Σ_0 in $\mathbf{H}^n$ which is not a hyperplane. Like in the proof of Theorem 5.2, by foliating $S^{n-1}_\infty(\mathbf{H}^n)$ by closed, codimension-1 submanifolds $\{\Gamma_t\}$, and by assuming uniqueness of absolutely area minimizing hypersurfaces, one can get a foliation of $\mathbf{H}^n$ by the absolutely area minimizing hypersurfaces $\{\Sigma_t\}$ with $\partial_\infty \Sigma_t = \Gamma_t$. Let γ be a $n-2$-sphere in Σ_0 which is "close" to Γ_0. Then by using [23], one can get a compact area minimizing hyperplane M whose boundary is γ. Hence by construction, M cannot be in a leaf in the foliation $\{\Sigma_t\}$. Like in the proof of Theorem 5.2, it might be possible to get a contradiction by studying the intersection of M with the foliation $\{\Sigma_t\}$.

Unfortunately, the maximum principle [20] does not work here since even though Σ_t is absolutely area minimizing, M is not. Also, the maximum principles due to Solomon-White [21] and Ilmanen [15] which are for stationary varifolds are not enough to get a contradiction since the minimizing hyperplane M given by [23] might have codimension-1 singularities, while [21] and [15] works up to codimension-2 singularities. So, to get a contradiction here, one needs a stronger maximum principle, or a more regular area minimizing hyperplane M.

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References

[1] S. Alexakis and R. Mazzeo, *Renormalized area and properly embedded minimal surfaces in hyperbolic 3-manifolds*, Comm. Math. Phys. **297** (2010) 621–651.

[2] F. D. Ancel, *Resolving wild embeddings of codimension-one manifolds in manifolds of dimensions greater than 3*, Special volume in honor of R. H. Bing (1914-1986). Topology Appl. **24** (1986) no. 1-3, 13–40.

[3] F. D. Ancel, J. W. Cannon, *The locally flat approximation of cell-like embedding relations*, Ann. of Math. (2) **109** (1979), no. 1, 61–86.

[4] M. Anderson, *Complete minimal varieties in hyp. space*, Invent. Math. **69** (1982) 477–494.

[5] M. Anderson, *Complete minimal hypersurfaces in hyperbolic n-manifolds*, Comment. Math. Helv. **58** (1983) 264–290.

[6] R. H. Bing, *A decomposition of E^3 into points and tame arcs such that the decomposition space is topologically different from E^3*, Ann. of Math. (2) **65** (1957) 484–500.

[7] B. Coskunuzer, *Minimal planes in hyp. space*, Comm. Anal. Geom. **12** (2004) 821–836.

[8] B. Coskunuzer, *Generic uniqueness of least area planes in hyperbolic space*, Geom. & Topology **10** (2006) 401–412.

[9] B. Coskunuzer, *Uniform 1-cochains and genuine lamin.*, Topology **45** (2006) 751–784.

[10] B. Coskunuzer, *Mean convex hulls and least area disks spanning extreme curves*, Math Z. **252** (2006) 811–824.

[11] B. Coskunuzer, *Asymptotic Plateau problem*, preprint; arXiv:0907.0552

[12] H. Federer, *Geometric measure theory*, Springer-Verlag, New York 1969.

[13] D. Gabai, *On the geometric and topological rigidity of hyperbolic* 3*-manifolds*, J. Amer. Math. Soc. **10** (1997) 37–74.

[14] R. Hardt and F.H. Lin, *Regularity at infinity for absolutely area minimizing hypersurfaces in hyperbolic space*, Invent. Math. **88** (1987) 217–224.

[15] T. Ilmanen, *A strong maximum principle for singular minimal hypersurfaces*, Calc. Var. Partial Differential Equations **4** (1996) 443–467.

[16] U. Lang, *Asymptotic Plateau problem in Gromov hyperbolic manifolds*, Calc. Var. Partial Differential Equations **16** (2003) 31–46.

[17] F.H. Lin, *Plateau's problem for H-convex curves*, Manuscripta Math. **58** (1987) 497–511.

[18] B. Morton, *Locally flat imbeddings of topological manifolds*, Ann. of Math. (2) **75** (1962) 331–341.

[19] W. Meeks and S.T. Yau, *The classical Plateau problem and the topology of three manifolds*, Topology **21** (1982) 409–442.

[20] L. Simon, *A strict maximum principle for area minimizing hypersurfaces*, J. Differential Geom. **26** (1987) 327–335.

[21] B. Solomon and B. White, *A strong maximum principle for varifolds that are stationary with respect to even parametric elliptic functionals*, Indiana Univ. Math. J. **38** (1989) 683–691.

[22] B. White, *On the topological type of minimal submanifolds*, Topology **31** (1992) 445–448.

[23] B. White, *Mappings that minimize area in their homotopy classes*, J. Differential Geom. **20** (1984) 433–446.

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