GGT Journal of Gökova Geometry Topology
Volume 5 (2011) 86 – 102

The Catanese-Ciliberto-Mendes Lopes surface

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Abstract. We draw a handlebody picture of the complex surface defined by Catanese-Ciliberto-Mendes Lopes. This is a surface obtained by taking the quotient of the product of surfaces $\Sigma_2 \times \Sigma_3$ of genus 2 and 3, under the product of involutions $\tau_2 \times \tau_3$, where τ_2 is the elliptic involution of Σ_2, and τ_3 is a free involution on Σ_3.

1. Introduction

Catanese-Ciliberto-Mendes Lopes surface (CaCiMe or CCM surface in short) M is a complex surface constructed in [CCM] (and discussed in [HP] and [P]), which is topologically a genus 2 surface bundle over a surface of genus 2. Recently in [AP] this surface is used in interesting smooth manifold constructions. While inspecting [AP] we felt that this interesting complex surface M deserves a careful topological study of its own.

Generally, drawing handlebody pictures of S^1 bundles over 3-manifolds, or 3-manifold bundles over circles is relatively easy compared to surface bundles of surfaces (e.g., [AK] and [A1]). Here we take this opportunity to introduce a new technique to draw a surface bundle over a surface, which avoids the "turning handles upside down" process. This manifold M is a good test case to understand many of the general difficulties one encounters in drawing surface bundles over surfaces, as well as taking their fiber sum. We draw the handlebody picture of M in such a way that all the tori used in the constructions of [AP] are clearly visible. This combined with the log transform picture (e.g., [AY]) will allow one to see the "Luttinger surgery" constructions of [AP] in a concrete geometric way.

2. Construction

Let Σ_g denote the surface of genus g. Let $\tau_2 : \Sigma_2 \to \Sigma_2$ be the hyperelliptic involution and $\tau_3 : \Sigma_3 \to \Sigma_3$ be the free involution induced by 180^o rotation, as indicated in Figure 1. The CaCiMe surface M is the complex surface obtained by taking the quotient of $\Sigma_2 \times \Sigma_3$ by the product involution:

$$M = (\Sigma_2 \times \Sigma_3)/(\tau_2 \times \tau_3)$$

1991 *Mathematics Subject Classification.* 58D27, 58A05, 57R65.

The author is partially supported by NSF grant DMS 0905917.

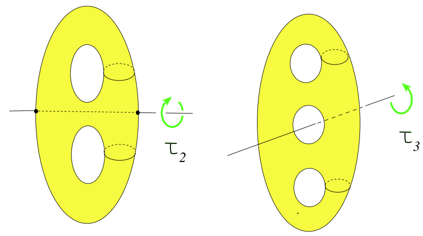

FIGURE 1. Involutions τ_2 and τ_3

By projecting to the second factor we can describe M as a Σ_2-bundle over $\Sigma_2 = \Sigma_3/\tau_3$. Let A denote the twice punctured 2-torus $A = T^2 - \left(D^2_- \sqcup D^2_+\right)$. Then clearly M is obtained by identifying the two boundary components of $\Sigma_2 \times A$ by the involution induced by τ_2 (notice that A is the interior of the fundamental domain of the action τ_3). By deforming A as in Figure 2, we see that $M = E \natural E'$ is obtained by fiber summing two Σ_2 bundles over T^2, where E is the trivial bundle $\Sigma_2 \times T^2 \to T^2$, and

$$E' = \Sigma_2 \times S^1 \times [0,1]/(x,y,0) \sim (\tau_2(x), y, 1)$$

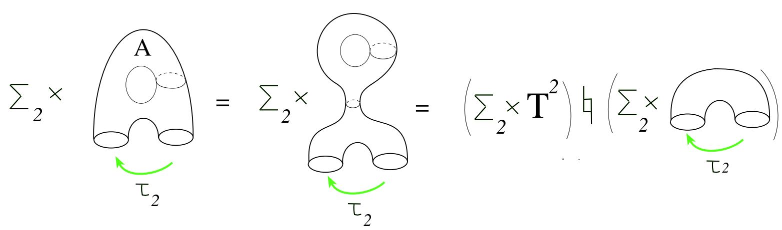

FIGURE 2. Decomposing $M = E \natural E'$

We will build a handlebody of M by a step by step process drawing the following handlebodies in the given order, also we will concretely identify the indicated diffeomorphisms:

(a) $E_0 = E - \Sigma_2 \times D^2 = \Sigma_2 \times (T^2 - D^2)$
(b) $E_0' = E' - \Sigma_2 \times D^2$
(c) $f_1 : \partial E_0 \xrightarrow{\approx} \Sigma_2 \times S^1$
(d) $f_2 : \partial E_0' \xrightarrow{\approx} \Sigma_2 \times S^1$
(e) $M = -E_0 \smile_{f_2^{-1} \circ f_1} E_0'$

One way to to perform the gluing operation (e) is to turn the handlebody E_0 upside down and attach its dual handlebody to top of E_0', getting $M = -E_0 \smile E_0'$ (e.g., the technique used in [A1]). In this paper we choose another way which amounts to identifying the boundaries of the two handlebodies $-E_0$ and E_0' by a cylinder $\partial E_0 \times I$

$$M = -E_0 \smile_{f_1} (\Sigma_2 \times S^1) \times I \smile_{f_2^{-1}} E_0'$$

Though this seems like a trivial distinction, it makes a big difference in constructing the handlebodies. One advantage of this technique is that we see the imbedded tori used in the construction of [AP] clearly.

3. Constructing E_0

Figure 3 (a disk with two pairs of 1-handles and a 2-handle, where only the attaching arcs of the 1-handles are drawn) describes a handlebody for Σ_2, and Figure 4 is $\Sigma_2 \times [0,1]$. Hence Figure 5 is a handlebody of $\Sigma_2 \times S^1$ (compare [AK]). Figure 6 is the same as Figure 5, except it is drawn as a Heegaard diagram. So Figure 7 describes a handlebody picture of of $\Sigma_2 \times S^1 \times S^1 = \Sigma_2 \times T^2$. A close inspection shows that removing the 2-handle, denoted by c, from Figure 7 we get the handlebody of E_0 (c is the disk boundary in $T^2 - D^2$, as the attaching circle of the 2-handle corresponding to D^2; more precisely it is the upside down 2-handle of the missing $\Sigma_2 \times D^2$ which is removed from E_0).

Next, in Figures 8 through 11 we gradually convert the "pair of balls" notation of the 1-handles of Figure 7 to the "circle-with-dot" notation of [A2] (i.e., carving). Figure 11 is the same as Figure 7, except that all of its 1-handles are drawn in circle-with-dot notation. For the benefit of the reader we did this transition in several steps: First in Figure 8 we converted a pair of 1-handles of Figure 7 to the circle-with-dot notation, then in Figure 11 we converted the remaining 1-handles. Figure 9 shows how to perform local isotopies near the attaching balls of 1-handles to go to the intermediate picture Figure 10 where the attaching balls are drawn as flat arcs. We then converted the flat arcs to the circles-with-dots. In Figure 11 all the 2-handles are attached with 0-framing.

4. Diffeomorphism $f_1 : \partial E_0 \to \Sigma_2 \times S^1$

Next we construct a difeomorphism $f_1 : \partial E_0 \xrightarrow{\approx} \Sigma_2 \times S^1$. First by an isotopy we go from Figure 11 to Figure 12, then by replacing the circles-with-dots with 0-framed circles, and by performing the handle slides to Figure 12 as indicated by the arrows, we obtain the first picture of Figure 13, and by further handle slides and cancellations we obtain the second picture of Figure 13, which is $\Sigma_2 \times D^2$. In Figure 13 we also indicate where this diffeomorphism throws the linking loops a_1, b_1, a_2, b_2, c. Finally in Figure 14 we describe the diffeomprphism we constructed $f_1 : \partial E_0 \to \Sigma_2 \times S^1$ in a much more concrete way by indicating the images of the arcs shown in the figure. Though going from Figure 11 to Figure 14 is locally a routine process, finding the correct handle sliding moves and locating and keeping the track of those arcs is the most time consuming part of this work.

5. Constructing E_0' and $f_2 : \partial E_0' \to \Sigma_2 \times S^1$

Figures 15 and 16 shows that the diffoemorphism $\tau_2 : \Sigma_2 \to \Sigma_2$ is induced from 180^0 rotation of the disk with four 1-handles. Having noted this, we proceed exactly as in Figure 7 through Figure 11, except that we replace Figure 7 by Figure 17 (due to the twisting by $\tau_2 : \Sigma_2 \to \Sigma_2$). So Figure 20 is the handlebody of E_0' (without the curve denoted by c'), and Figure 22 describes a diffeomorphism $f_2 : \partial E_0' \to \Sigma_2 \times S^1$.

6. Constructing $M = -E_0 \smile E_0'$

To construct $M = -E_0 \smile E_0'$ we draw the handlebodies $-E_0$ and E_0' side by side, and glue their boundaries to the two boundary components of the cylinder $\Sigma_2 \times S^1 \times I$. This gluing is done by identifying the 1-handle circles $\{a_1, b_1, a_2, b_2, c\}$ (Figure 13) of $\partial E_0 \approx \Sigma_2 \times S^1$ and of $\partial E_0' \approx \Sigma_2 \times S^1$ by 2-handles. Now Figure 14 and Figure 22 gives us exactly the information needed to draw the CaCiMe surface $M = -E_0 \smile_{f_2^{-1} \circ f_1} E_0'$ as shown in Figure 23 (all the circles are 0-framed 2-handles).

7. Epilogue

The CaCiMe surface has its own place in the classification scheme of complex surfaces, as stated in the following theorem:

Theorem 7.1. *(*[HP], [P]*) If X is a smooth minimal complex projective surface of general type with $p_g(X) = q(X) = 3$, then one of following hold:*

(a) $K_X^2 = 6$ *and* $X = Sym^2(\Sigma_3)$
(b) $K_X^2 = 8$ *and* $X=$ *CaCiMe surface.*

Recall the formulas $b_1(X) = 2q(X)$, $K_X = c_1(X)$ and $3\sigma(X) = c_1^2(X) - 2\chi(X)$, and the Noether formula: $1 - q(X) + p_g(X) = \frac{1}{12}[\ c_1^2(X) + c_2(X)\ ]$. So if (b) holds then $b_2(X) = 14$ and $\sigma(X) = 0$. Hence the CaCiMe surface X is homology equivalent to $\#7(S^2 \times S^2)\#6(S^1 \times S^3)$, and its fundamental group presumably can be calculated from its fibration structure provided its monodromies are determined, But now we can easily calculate the fundamental group as well as the other topological invariants from its handlebody picture in Figure 23.

Remark 7.2. One of the reason we decided to study the handlebody structure of the Cacime surface is that, it appears to be the starting point of many other interesting manifolds, for example the construction techniques used in [A3], [A4] and [A5] are all driven from the construction of the handlebody of the Cacime surface. In particular the reader is encouraged to look at [A3], where a simpler version of the construction of Section 3 is used to build handlebodies for T^4 and $T^2 \times (T^2 - D^2)$.

Acknowledgements: We would like to thank Anar Akhmedov for introducing us to the Catanese-Ciliberto-Mendes Lopes complex surface, and explaining [AP]. We also thank IMBM (Istanbul mathematical sciences research institute) for providing us an inspiring environment where a large part of this work was done.

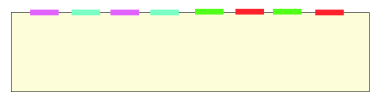

FIGURE 3. Σ_2

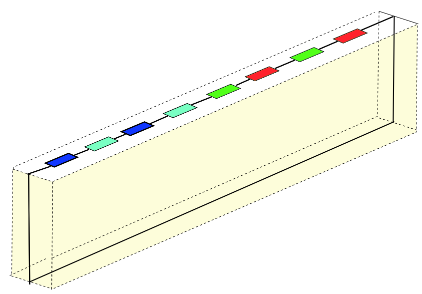

FIGURE 4. $\Sigma_2 \times [0,1]$

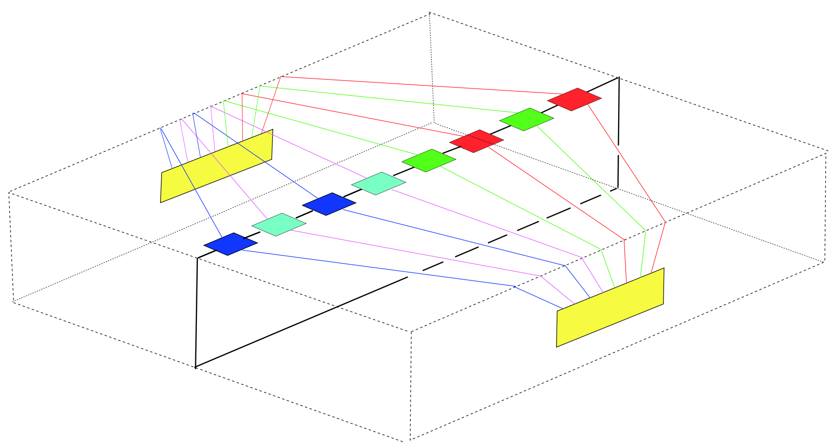

FIGURE 5. $\Sigma_2 \times S^1$

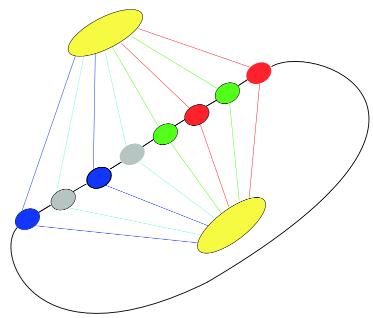

FIGURE 6. $\Sigma_2 \times S^1$

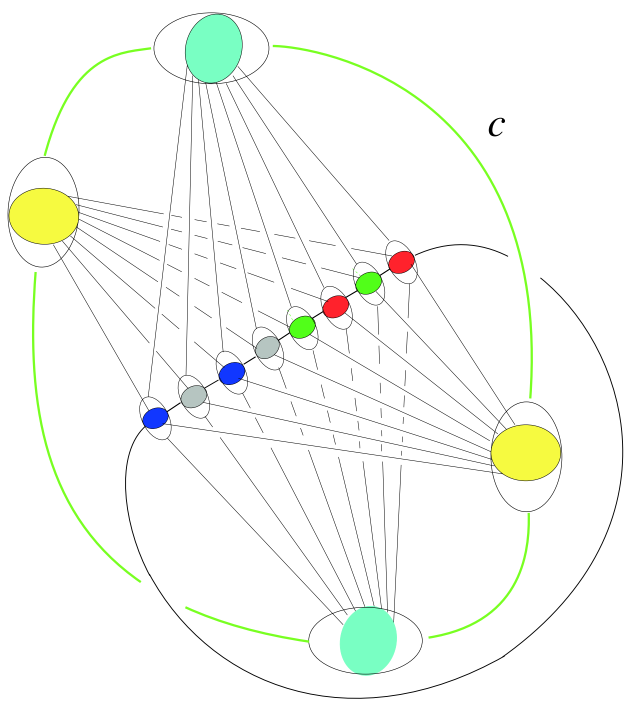

FIGURE 7. $\Sigma_2 \times T^2$

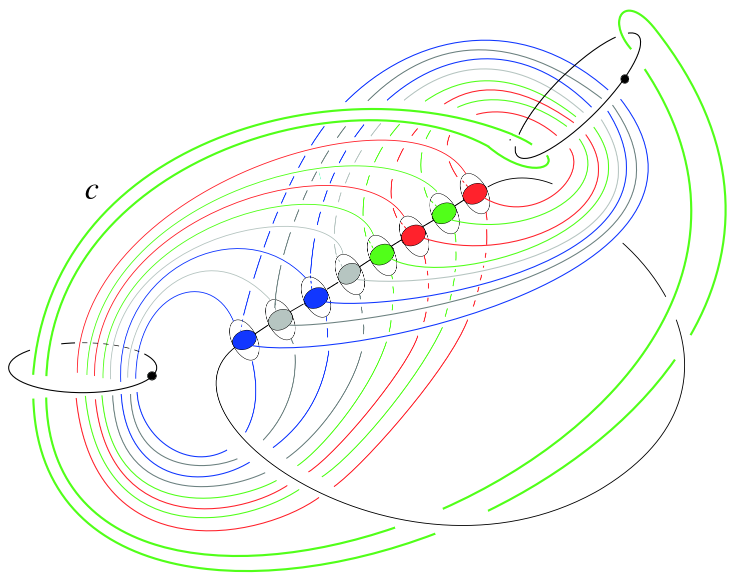

FIGURE 8. E_0

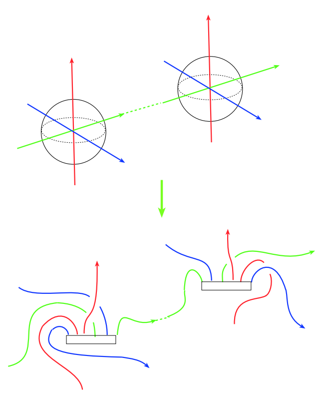

FIGURE 9. Local isotopies used to flatten 1-handle attaching balls

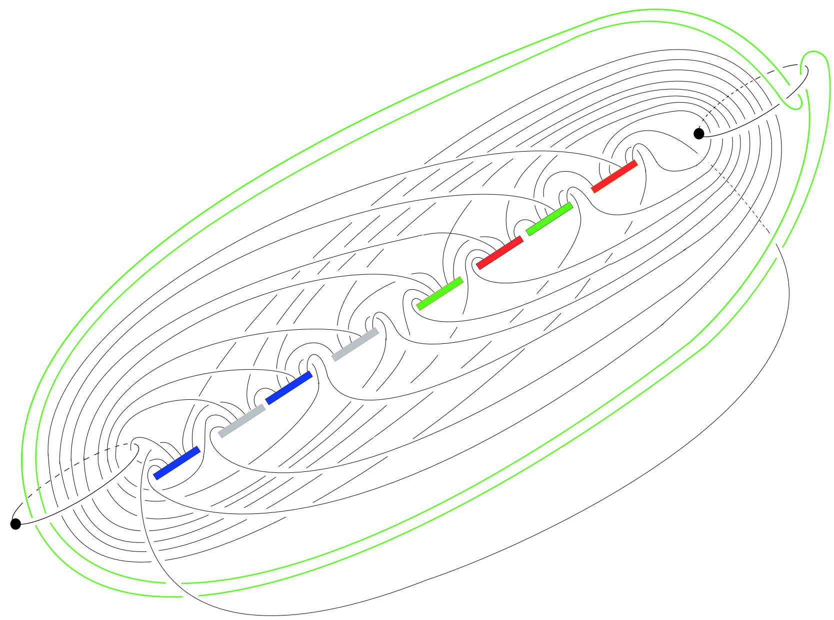

FIGURE 10. Converting 1-handle notation

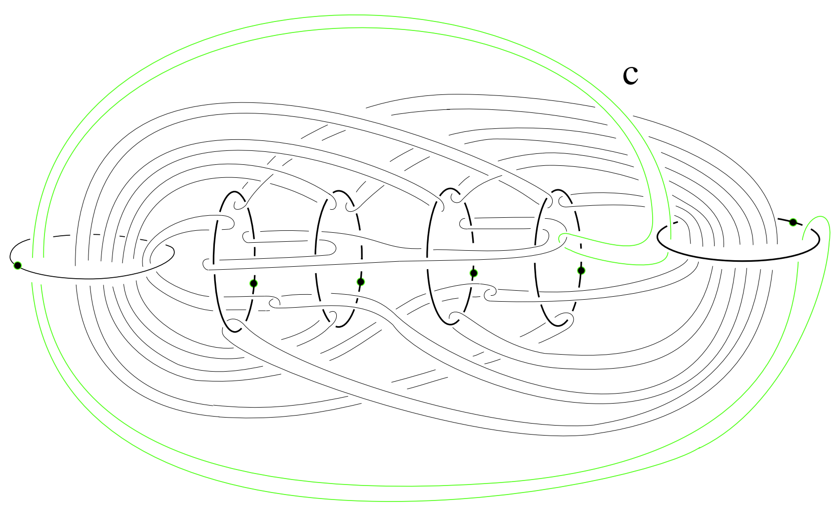

FIGURE 11. E_0

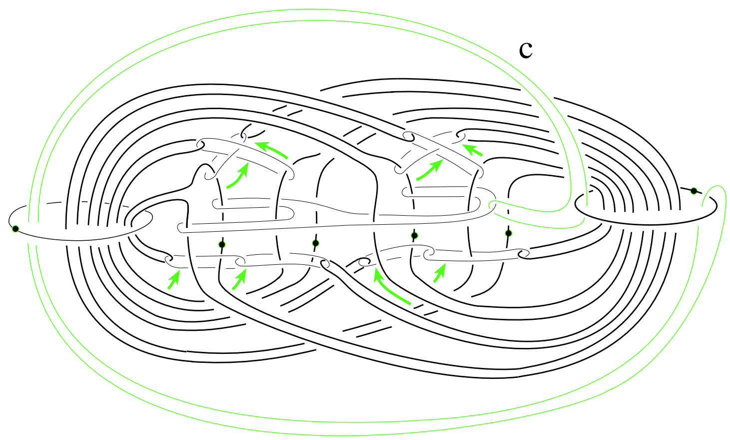

FIGURE 12. Surgering inside of E_0

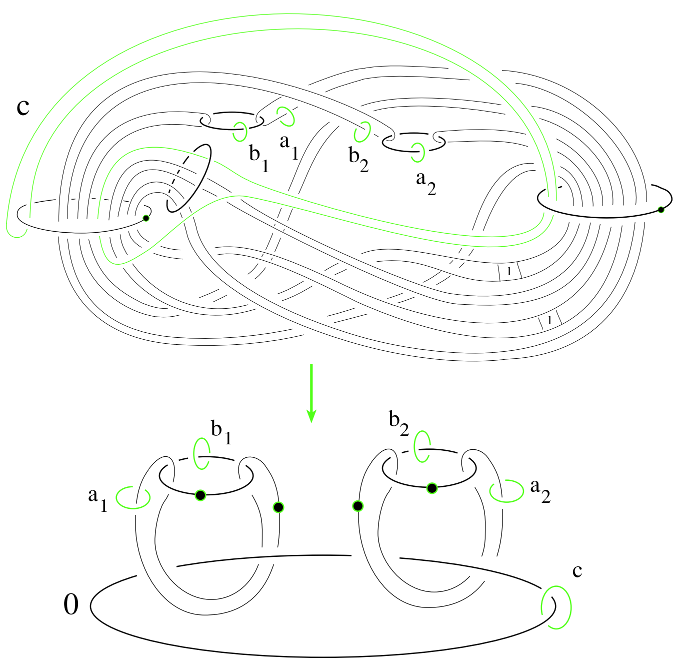

FIGURE 13. Checking $\partial E_0 \approx \Sigma_2 \times S^1$

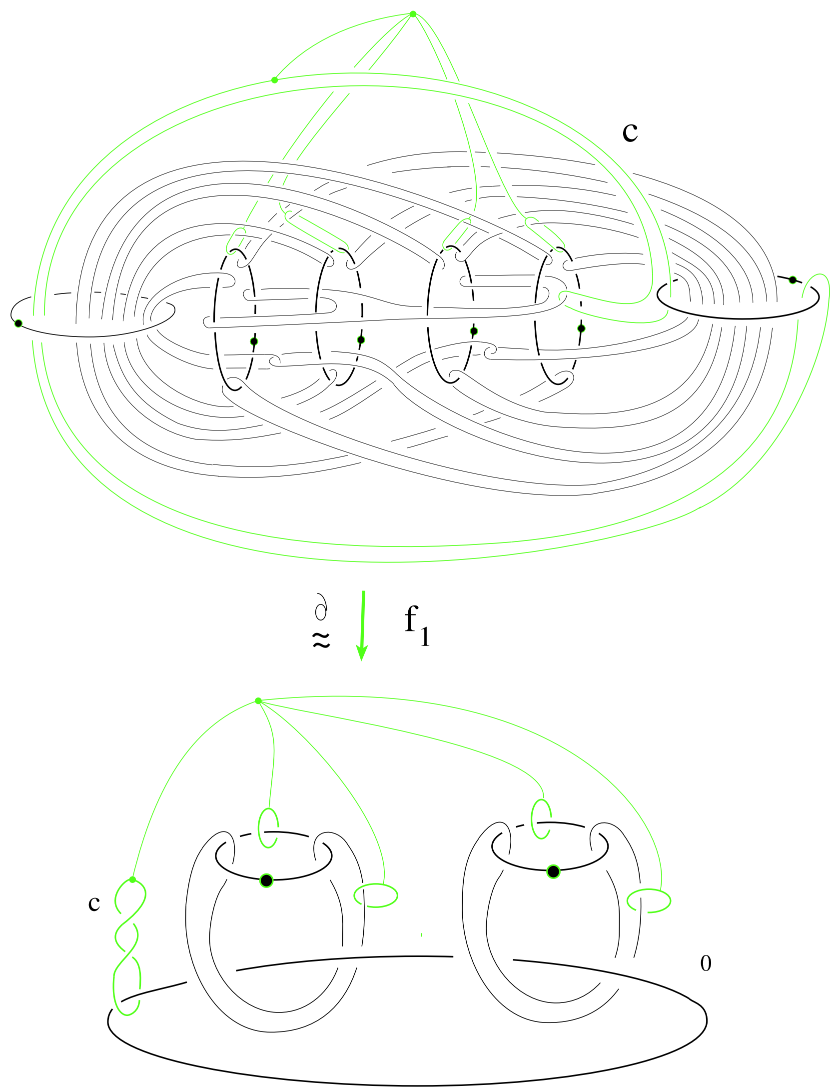

FIGURE 14. Diffeomorphism $\partial E_0 \approx \Sigma_2 \times S^1$ made concrete

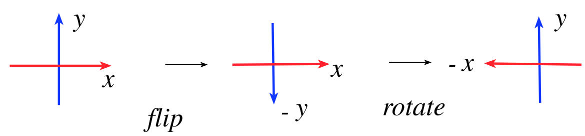

FIGURE 15. Action of τ_2 on the zero handle of Σ_2

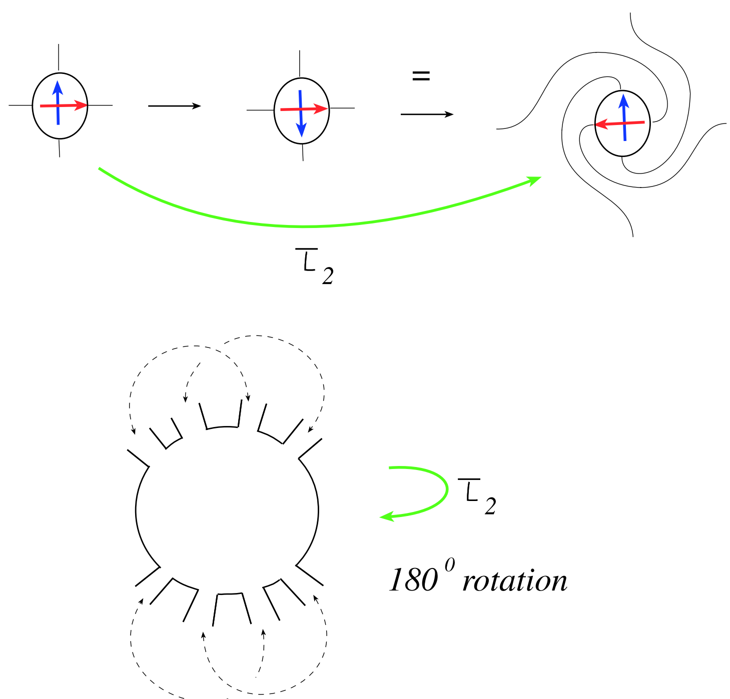

FIGURE 16. Describing $\tau_2 : \Sigma_2 \to \Sigma_2$

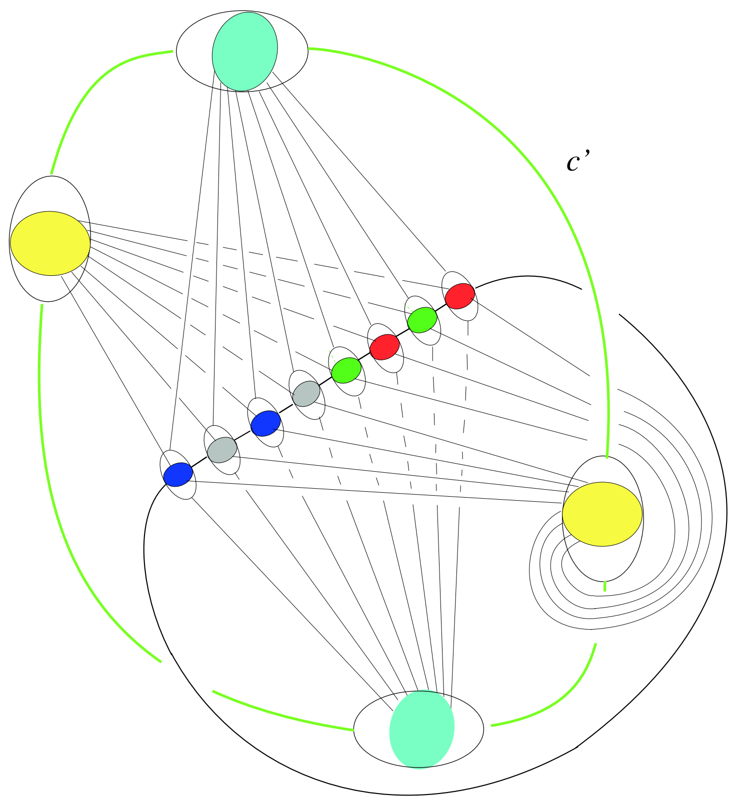

FIGURE 17. E_0'

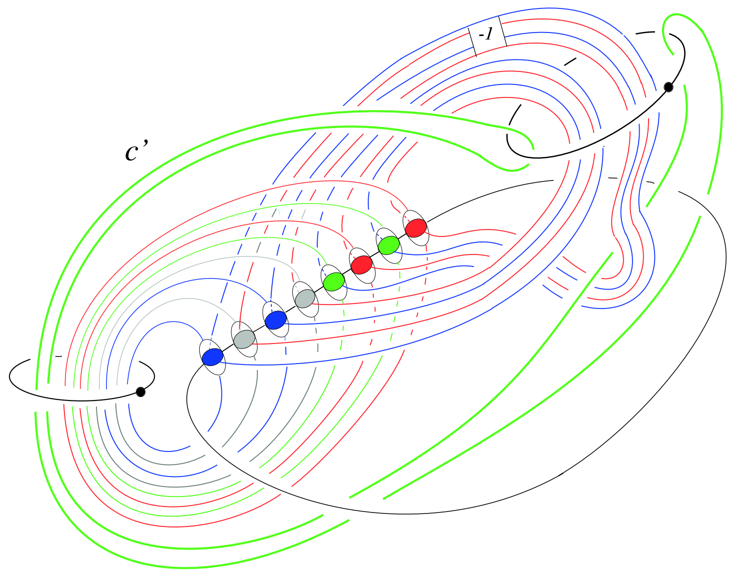

FIGURE 18. E'_0

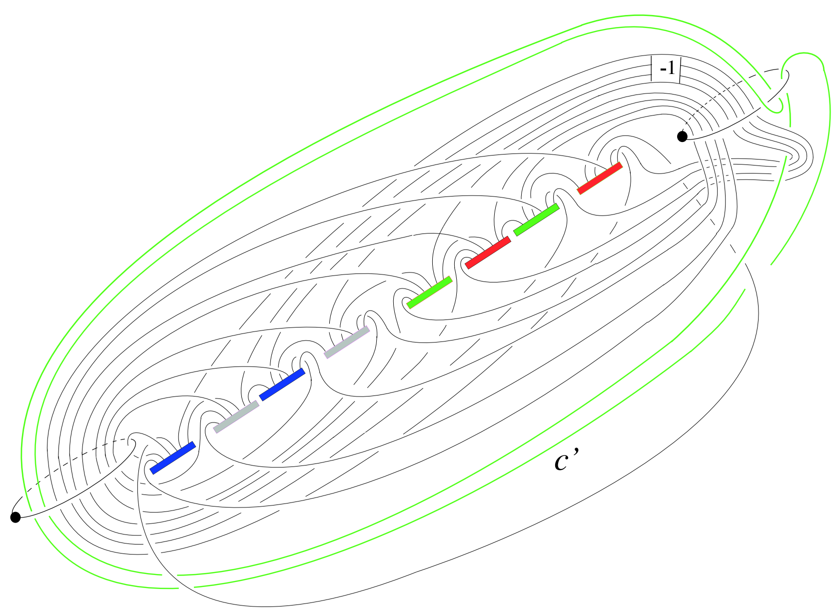

FIGURE 19. Converting 1-handle notation

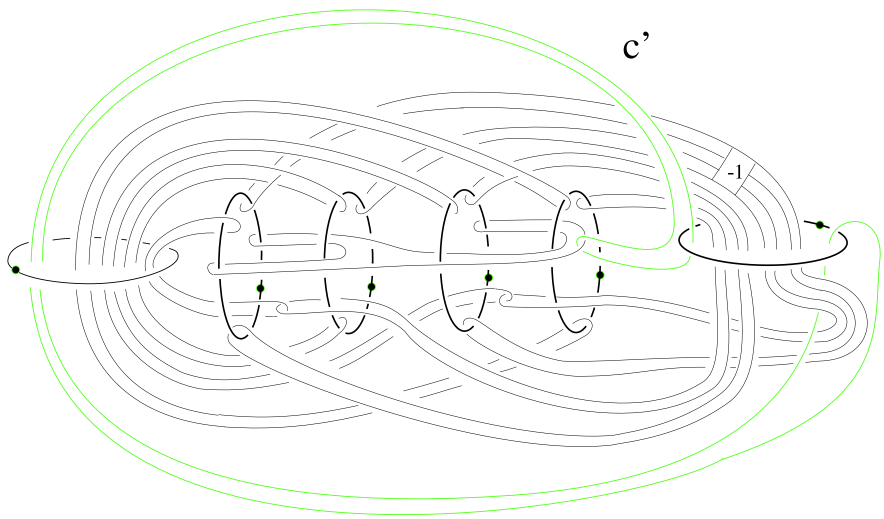

FIGURE 20. E_0'

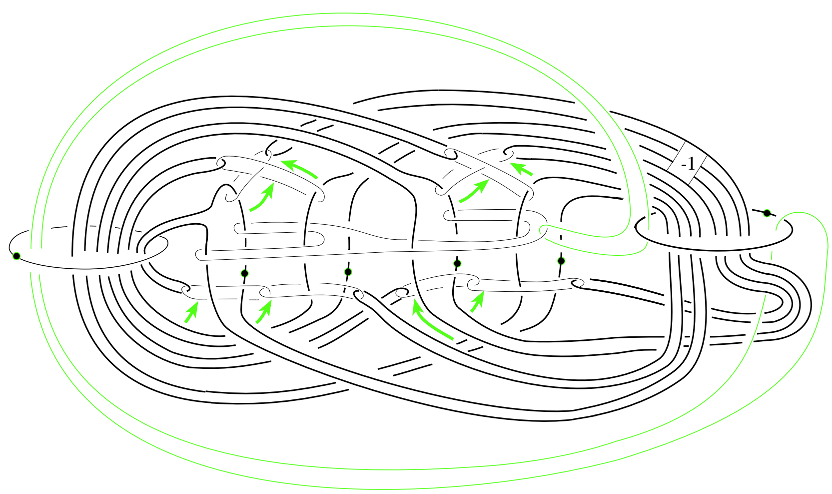

FIGURE 21. Surgering inside of E_0'

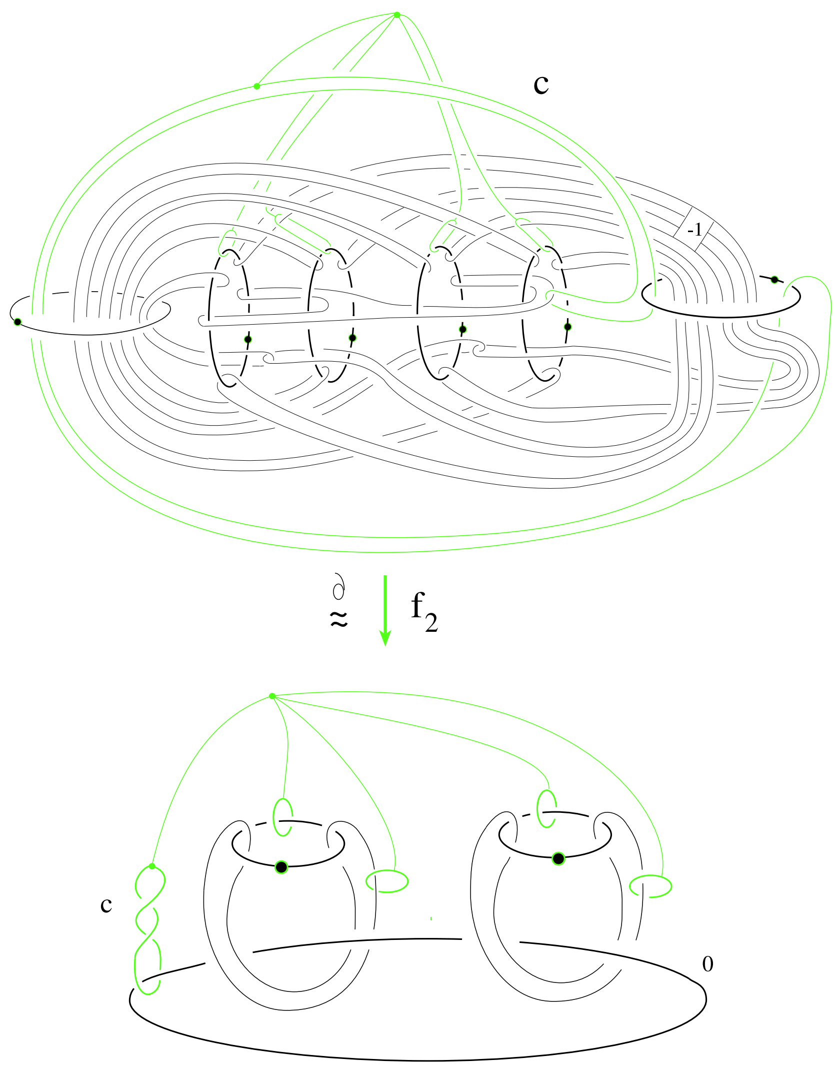

FIGURE 22. Diffeomorphism $\partial E_0' \approx \Sigma_2 \times S^1$ made concrete

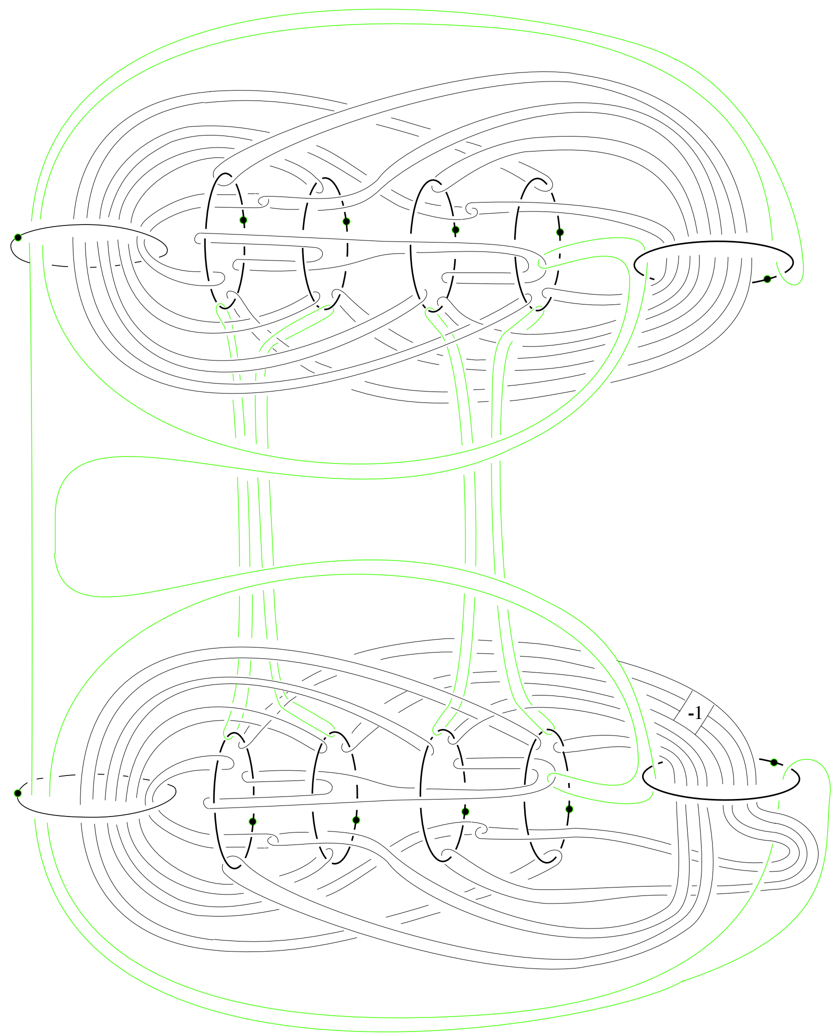

Figure 23. The CaCiMe surface M

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