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Isotropy subgroup of the Hurwitz action of the 4-braid group on braid systems

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Abstract. We study the Hurwitz action of the 4-braid group B_4 on the 4-fold direct product B_5^4 of the 5-braid group B_5 and determine explicitly a system of generators of the isotropy subgroup of the Hurwitz action of B_4 at the standard generators of B_5. As a corollary 125 representatives of its coset space are also presented.

1. Introduction

Let B_n denote the n-braid group, which has the following presentation [1, 2]:

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & (|i-j| = 1), \\ \sigma_i\sigma_j = \sigma_j\sigma_i & (|i-j| > 1) \end{array} \right\rangle,$$

where σ_i is the i-th standard generator.

Definition 1.1. The *Hurwitz action* of the n-braid group B_n on the n-fold Cartesian product G^n of a group G is the right action defined by

$$(b_1, \dots, b_{i-1}, b_i, b_{i+1}, b_{i+2}, \dots, b_n) \cdot \sigma_i = (b_1, \dots, b_{i-1}, b_{i+1}, {b_{i+1}}^{-1} b_i b_{i+1}, b_{i+2}, \dots, b_n),$$

where $\sigma_1, \dots, \sigma_{n-1}$ are the standard generators of B_n.

The Hurwitz action by the standard generators σ_i and their inverses σ_i^{-1} are called *elementary transformations*.

When G is the mapping class group of a Riemann surface Σ_g of genus g and when each b_j ($j = 1, \dots, n$) is a positive Dehn twist along a simple closed curve on Σ_g, the orbit of an n-tuple $(b_1, \dots, b_n)$ by the Hurwitz action of B_n corresponds to an equivalence class of a genus g Lefschetz fibration $f : W \to D^2$ over a 2-disk with n singular fibers (cf. [3]). When G is the m-braid group B_m and when each b_j ($j = 1, \dots, n$) is a conjugate of σ_1, the orbit of an n-tuple $(b_1, \dots, b_n)$ corresponds to an equivalence class of an "algebraic" braided surface in a bi-disk $D^2 \times D^2$ with n branch points [4, 6, 7]. Here we say that two braided surfaces S and S' are *equivalent* if there is a fiber-preserving diffeomorphism $f : D^2 \times D^2 \to D^2 \times D^2$ (that is, $f(D^2 \times \{x\}) = D^2 \times \{g(x)\}$ for some diffeomorphism $g : D^2 \to D^2$) carrying S to S' rel $D^2 \times \partial D^2$. Moreover if the fiber-preserving diffeomorphism $f : D^2 \times D^2 \to D^2 \times D^2$ keeps every fiber $D^2 \times \{x\}$ setwise for

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all $x \in D^2$ (i.e. $g : D^2 \to D^2$ is the identity map), we say that S and S' are *isomorphic* (cf. [4]). By definition, isomorphic braided surfaces are equivalent and have the same branch point set. It is natural to ask if two given braided surfaces are equivalent or not, and if given equivalent braided surfaces with the same branch point set are isomorphic or not. The former question is equivalent to asking if two given n-tuples $(b_1, \ldots, b_n)$ and $(b'_1, \ldots, b'_n)$ determined by their monodromies are in the same Hurwitz orbit or not. This is a very hard problem and no algorithm to solve it is known. The second question is quite easy to solve. Two braided surfaces S and S' with the same branch point set are isomorphic if and only if their monodromies determine the same n-tuples $(b_1, \ldots, b_n)$ with respect to the same generating system of $D^2 \setminus \{\text{branch points}\}$. So an interesting question is to ask how many braided surfaces are there which are equivalent to S and have the same branch point set with S but are not isomorphic to each other. This number is exactly equal to the cardinal number of the orbit of an n-tuple $(b_1, \ldots, b_n)$ of S [8]. Therefore it is important to study the orbit of an n-tuple $(b_1, \ldots, b_n) \in B_m^n$ under the Hurwitz action. We shall consider a special case of $(\sigma_1, \ldots, \sigma_n)$ in B_{n+1}^n, since it corresponds to the "standard" braided surface of degree $n+1$ with n branch points, and it is related to an unknotted surface braid (cf. [4, 7]).

An element of the n-fold direct product B_m^n of B_m is called a *braid system of degree m and of length n* (cf. [4]).

Let $(b_1, \ldots, b_n)$ be a braid system of degree m and of length n. We denote the isotropy subgroup of the Hurwitz action of B_n at the braid system $(b_1, \ldots, b_n)$ by $H(b_1, \ldots, b_n)$. Namely, $H(b_1, \ldots, b_n) = \{\beta \in B_n \mid (b_1, \ldots, b_n) \cdot \beta = (b_1, \ldots, b_n)\}$. The elements of the orbit of $(b_1, \ldots, b_n)$ under the Hurwitz action correspond to the cosets of B_n by the isotropy subgroup $H(b_1, \ldots, b_n)$. In particular, the cardinal number of the orbit is that of the coset space $H(b_1, \ldots, b_n)\backslash B_n$.

In [8] we described the isotropy subgroup of Hurwitz action of B_3 at the standard generators of B_4 and its coset space which consists of 16 elements.

In this paper, we study the isotropy subgroup of Hurwitz action of B_4 at those of B_5 and its cosets. To state the main theorem, we need some preparations.

For integers i $(0 \le i \le 3)$ and s $(0 \le s \le 4)$, let a_i^s, b_i^s, c_i^s, d_i^s, e_0^s, $e_1^s, \ldots, e_8^s$ be elements of B_4 defined by

$$\begin{array}{lllllll}
a_i^s & = & (\sigma_1\sigma_2\sigma_3)^s(\sigma_2\sigma_3)^i, & \quad & b_i^s & = & (\sigma_1\sigma_2\sigma_3)^s(\sigma_2\sigma_3)^i\sigma_2, \\
c_i^s & = & (\sigma_1\sigma_2\sigma_3)^s(\sigma_2\sigma_3)^i\sigma_3, & & d_i^s & = & (\sigma_1\sigma_2\sigma_3)^s(\sigma_2\sigma_3)^i{\sigma_2}^2, \\
e_0^s & = & (\sigma_1\sigma_2\sigma_3)^s{\sigma_1}^2, & & e_1^s & = & (\sigma_1\sigma_2\sigma_3)^s{\sigma_1}^2\sigma_2, \\
e_2^s & = & (\sigma_1\sigma_2\sigma_3)^s{\sigma_1}^2\sigma_2\sigma_3, & & e_3^s & = & (\sigma_1\sigma_2\sigma_3)^s{\sigma_1}^2\sigma_2\sigma_3\sigma_2, \\
e_4^s & = & (\sigma_1\sigma_2\sigma_3)^s{\sigma_1}^2(\sigma_2\sigma_3)^2, & & e_5^s & = & (\sigma_1\sigma_2\sigma_3)^s{\sigma_1}^2(\sigma_2\sigma_3)^2\sigma_2, \\
e_6^s & = & (\sigma_1\sigma_2\sigma_3)^s{\sigma_1}^2(\sigma_2\sigma_3)^3, & & e_7^s & = & (\sigma_1\sigma_2\sigma_3)^s{\sigma_1}^2(\sigma_2\sigma_3)^3\sigma_2, \\
e_8^s & = & (\sigma_1\sigma_2\sigma_3)^s{\sigma_1}^2(\sigma_2\sigma_3)^4. & & & &
\end{array}$$

Now we define two subsets of B_4 denoted by $\mathcal{X}$ and $\mathcal{S}$.

Let $\mathcal{X} = (\bigcup_{i=0}^{3} \mathcal{A}_i) \cup (\bigcup_{i=0}^{3} \mathcal{B}_i) \cup (\bigcup_{i=0}^{3} \mathcal{C}_i) \cup (\bigcup_{i=0}^{3} \mathcal{D}_i) \cup (\bigcup_{i=0}^{8} \mathcal{E}_i)$ be the subset of B_4 defined by

$$\begin{array}{rcll} \mathcal{A}_i & = & \{a_i^s \mid 0 \le s \le 4\} & (i = 0, \ldots, 3), \\ \mathcal{B}_i & = & \{b_i^s \mid 0 \le s \le 4\} & (i = 0, \ldots, 3), \\ \mathcal{C}_i & = & \{c_i^s \mid 0 \le s \le 4\} & (i = 0, \ldots, 3), \\ \mathcal{D}_i & = & \{d_i^s \mid 0 \le s \le 4\} & (i = 0, \ldots, 3), \\ \mathcal{E}_i & = & \{e_i^s \mid 0 \le s \le 4\} & (i = 0, \ldots, 8). \end{array}$$

Moreover, let $\mathcal{Y}_j$ $(j = 1, 2, 3)$ be the subset of $\mathcal{X}$ defined by

$$\begin{array}{rcl} \mathcal{Y}_1 & = & \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{D}_1 \cup \mathcal{D}_3 \cup \mathcal{E}_3 \cup \mathcal{E}_4, \\ \mathcal{Y}_2 & = & \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_4 \cup \mathcal{E}_5 \cup \mathcal{E}_6 \cup \mathcal{E}_7, \\ \mathcal{Y}_3 & = & \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3 \cup \mathcal{E}_4 \cup \mathcal{E}_7 \cup \mathcal{E}_8 \end{array}$$

and let $\mathcal{Z}_j$ be the complement of $\mathcal{Y}_j$ in $\mathcal{X}$ for $j = 1, 2, 3$.

Finally a subset $\mathcal{S}$ of B_4 is defined by

$$\mathcal{S} \;=\; \{\beta\sigma_j^2\beta^{-1} \mid \beta \in \mathcal{Y}_j,\ j = 1, 2, 3\} \;\cup\; \{\beta\sigma_j^3\beta^{-1} \mid \beta \in \mathcal{Z}_j,\ j = 1, 2, 3\}.$$

The following theorem is the main result of this paper.

Theorem 1.2. *Let s_1, s_2, s_3, s_4 be the standard generators of B_5. Then, we have*

(1) *$H(s_1, s_2, s_3, s_4)$ is generated by $\mathcal{S}$.*
(2) *$\#(H(s_1, s_2, s_3, s_4)\backslash B_4) = 125$. Moreover, all the representatives are given by $\mathcal{X}$.*

Remark 1.3. For any integer $n \ge 2$, the equality $\#(H(s_1, \ldots, s_n)\backslash B_n) = (n+1)^{n-1}$ has been proven in [5]. In particular, the first half of Theorem 1.2(2) was known. So, the purpose of this paper is to give explicitly a system of generators $\mathcal{S}$ of the isotropy subgroup and 125 representatives of its cosets. In fact, we give a direct proof of Theorem 1.2 without using the above mentioned result of [5].

Remark 1.4. In [9], for any integer $n \ge 2$ and any element φ of the symmetric group S_n, the author shows the equality $\#(H(s_{\varphi(1)}, \ldots, s_{\varphi(n)})\backslash B_n) = (n+1)^{n-1}$.

2. Proof of Theorem 1.2

The notation $\beta_1 \equiv \beta_2 \pmod{H}$ stands for $H\beta_1 = H\beta_2$ (or equivalently $\beta_1\beta_2^{-1} \in H$), where H is a subgroup of B_4 and $\beta_1, \beta_2 \in B_4$. In the middle column of Table 2.1, $b_1 * b_2$ stands for ${b_2}^{-1}b_1b_2$ for $b_1, b_2 \in B_5$ and "1, 2, 3, 4" are abbreviations of "s_1, s_2, s_3, s_4" of B_5, respectively. In the right column of Table 2.1, (ij) is the transposition of i and j for $1 \le i < j \le 5$, which is an element of the symmetric group S_5. For each $\beta \in \mathcal{X}$, we write the result of $(1, 2, 3, 4) \cdot \beta$ in the middle column of Table 2.1 and that of $(\pi(1), \pi(2), \pi(3), \pi(4)) \cdot \beta$ in the right column of Table 2.1, where π is the natural homomorphism from B_5 to S_5 which sends $i(= s_i)$ to the transposition $(i\, i{+}1)$. For any distinct elements β_1 and β_2 of $\mathcal{X}$, we see that $(\pi(1), \pi(2), \pi(3), \pi(4)) \cdot \beta_1 \ne (\pi(1), \pi(2), \pi(3), \pi(4)) \cdot \beta_2$. Thus, $(s_1, s_2, s_3, s_4) \cdot \beta_1 \ne (s_1, s_2, s_3, s_4) \cdot \beta_2$, and we obtain the following lemma.

Lemma 2.1. *For any distinct elements β_1 and β_2 of $\mathcal{X}$, $\beta_1 \not\equiv \beta_2 \pmod{H(s_1, s_2, s_3, s_4)}$.*

The following proposition is given in [8] and it is useful to prove Lemma 2.3.

Proposition 2.2. *Let (b_1, b_2) be a braid system of degree m and of length 2.*

(1) $\sigma_1^2 \in H(b_1, b_2)$ *if and only if* $b_1 b_2 = b_2 b_1$.
(2) $\sigma_1^3 \in H(b_1, b_2)$ *if and only if* $b_1 b_2 b_1 = b_2 b_1 b_2$.

Lemma 2.3.

(1) *If* $\beta \in \mathcal{Y}_j$ *for* $j = 1, 2$ *or* 3, *then* $\beta\sigma_j^2 \equiv \beta \pmod{H(s_1, s_2, s_3, s_4)}$.
(2) *If* $\beta \in \mathcal{Z}_j$ *for* $j = 1, 2$ *or* 3, *then* $\beta\sigma_j^3 \equiv \beta \pmod{H(s_1, s_2, s_3, s_4)}$.

	β	$(1,2,3,4)\cdot\beta$	$(\pi(1),\pi(2),\pi(3),\pi(4))\cdot\beta$
$\mathcal{A}_0$	a_0^0	$(1,2,3,4)$	$((12),(23),(34),(45))$
	a_0^1	$(2,3,4,1*(234))$	$((23),(34),(45),(15))$
	a_0^2	$(3,4,1*(234),1)$	$((34),(45),(15),(12))$
	a_0^3	$(4,1*(234),1,2)$	$((45),(15),(12),(23))$
	a_0^4	$(1*(234),1,2,3)$	$((15),(12),(23),(34))$
$\mathcal{A}_1$	a_1^0	$(1,3,4,2*(34))$	$((12),(34),(45),(25))$
	a_1^1	$(2,4,1*(234),1*2)$	$((23),(45),(15),(13))$
	a_1^2	$(3,1*(234),1,2*3)$	$((34),(15),(12),(24))$
	a_1^3	$(4,1,2,3*4)$	$((45),(12),(23),(35))$
	a_1^4	$(1*(234),2,3,1*(23))$	$((15),(23),(34),(14))$
$\mathcal{A}_2$	a_2^0	$(1,4,2*(34),2)$	$((12),(45),(25),(23))$
	a_2^1	$(2,1*(234),1*2,3)$	$((23),(15),(13),(34))$
	a_2^2	$(3,1,2*3,4)$	$((34),(12),(24),(45))$
	a_2^3	$(4,2,3*4,1*(234))$	$((45),(23),(35),(15))$
	a_2^4	$(1*(234),3,1*(23),1)$	$((15),(34),(14),(12))$
$\mathcal{A}_3$	a_3^0	$(1,2*(34),2,3)$	$((12),(25),(23),(34))$
	a_3^1	$(2,1*2,3,4)$	$((23),(13),(34),(45))$
	a_3^2	$(3,2*3,4,1*(234))$	$((34),(24),(45),(15))$
	a_3^3	$(4,3*4,1*(234),1)$	$((45),(35),(15),(12))$
	a_3^4	$(1*(234),1*(23),1,2)$	$((15),(14),(12),(23))$
$\mathcal{B}_0$	b_0^0	$(1,3,2*3,4)$	$((12),(34),(24),(45))$
	b_0^1	$(2,4,3*4,1*(234))$	$((23),(45),(35),(15))$
	b_0^2	$(3,1*(234),1*(23),1)$	$((34),(15),(14),(12))$
	b_0^3	$(4,1,2*(34),2)$	$((45),(12),(25),(23))$
	b_0^4	$(1*(234),2,1*2,3)$	$((15),(23),(13),(34))$
$\mathcal{B}_1$	b_1^0	$(1,4,3*4,2*(34))$	$((12),(45),(35),(25))$
	b_1^1	$(2,1*(234),1*(23),1*2)$	$((23),(15),(14),(13))$
	b_1^2	$(3,1,2*(34),2*3)$	$((34),(12),(25),(24))$
	b_1^3	$(4,2,1*2,3*4)$	$((45),(23),(13),(35))$
	b_1^4	$(1*(234),3,2*3,1*(23))$	$((15),(34),(24),(14))$

Table 2.1(a)

	β	$(1,2,3,4)\cdot\beta$	$(\pi(1),\pi(2),\pi(3),\pi(4))\cdot\beta$
$\mathcal{B}_2$	b_2^0	$(1,2*(34),2*3,2)$	$((12),(25),(24),(23))$
	b_2^1	$(2,1*2,3*4,3)$	$((23),(13),(35),(34))$
	b_2^2	$(3,2*3,1*(23),4)$	$((34),(24),(14),(45))$
	b_2^3	$(4,3*4,2*(34),1*(234))$	$((45),(35),(25),(15))$
	b_2^4	$(1*(234),1*(23),1*2,1)$	$((15),(14),(13),(12))$
$\mathcal{B}_3$	b_3^0	$(1,2,3*4,3)$	$((12),(23),(35),(34))$
	b_3^1	$(2,3,1*(23),4)$	$((23),(34),(14),(45))$
	b_3^2	$(3,4,2*(34),1*(234))$	$((34),(45),(25),(15))$
	b_3^3	$(4,1*(234),1*2,1)$	$((45),(15),(13),(12))$
	b_3^4	$(1*(234),1,2*3,2)$	$((15),(12),(24),(23))$
$\mathcal{C}_0$	c_0^0	$(1,2,4,3*4)$	$((12),(23),(45),(35))$
	c_0^1	$(2,3,1*(234),1*(23))$	$((23),(34),(15),(14))$
	c_0^2	$(3,4,1,2*(34))$	$((34),(45),(12),(25))$
	c_0^3	$(4,1*(234),2,1*2)$	$((45),(15),(23),(13))$
	c_0^4	$(1*(234),1,3,2*3)$	$((15),(12),(34),(24))$
$\mathcal{C}_1$	c_1^0	$(1,3,2*(34),2*3)$	$((12),(34),(25),(24))$
	c_1^1	$(2,4,1*2,3*4)$	$((23),(45),(13),(35))$
	c_1^2	$(3,1*(234),2*3,1*(23))$	$((34),(15),(24),(14))$
	c_1^3	$(4,1,3*4,2*(34))$	$((45),(12),(35),(25))$
	c_1^4	$(1*(234),2,1*(23),1*2)$	$((15),(23),(14),(13))$
$\mathcal{C}_2$	c_2^0	$(1,4,2,3*4)$	$((12),(45),(23),(35))$
	c_2^1	$(2,1*(234),3,1*(23))$	$((23),(15),(34),(14))$
	c_2^2	$(3,1,4,2*(34))$	$((34),(12),(45),(25))$
	c_2^3	$(4,2,1*(234),1*2)$	$((45),(23),(15),(13))$
	c_2^4	$(1*(234),3,1,2*3)$	$((15),(34),(12),(24))$
$\mathcal{C}_3$	c_3^0	$(1,2*(34),3,2*3)$	$((12),(25),(34),(24))$
	c_3^1	$(2,1*2,4,3*4)$	$((23),(13),(45),(35))$
	c_3^2	$(3,2*3,1*(234),1*(23))$	$((34),(24),(15),(14))$
	c_3^3	$(4,3*4,1,2*(34))$	$((45),(35),(12),(25))$
	c_3^4	$(1*(234),1*(23),2,1*2)$	$((15),(14),(23),(13))$
$\mathcal{D}_0$	d_0^0	$(1,2*3,2,4)$	$((12),(24),(23),(45))$
	d_0^1	$(2,3*4,3,1*(234))$	$((23),(35),(34),(15))$
	d_0^2	$(3,1*(23),4,1)$	$((34),(14),(45),(12))$
	d_0^3	$(4,2*(34),1*(234),2)$	$((45),(25),(15),(23))$
	d_0^4	$(1*(234),1*2,1,3)$	$((15),(13),(12),(34))$
$\mathcal{D}_1$	d_1^0	$(1,3*4,3,2*(34))$	$((12),(35),(34),(25))$
	d_1^1	$(2,1*(23),4,1*2)$	$((23),(14),(45),(13))$
	d_1^2	$(3,2*(34),1*(234),2*3)$	$((34),(25),(15),(24))$
	d_1^3	$(4,1*2,1,3*4)$	$((45),(13),(12),(35))$
	d_1^4	$(1*(234),2*3,2,1*(23))$	$((15),(24),(23),(14))$

Table 2.1(b)

	β	$(1,2,3,4)\cdot\beta$	$(\pi(1),\pi(2),\pi(3),\pi(4))\cdot\beta$
$\mathcal{D}_2$	d_2^0	$(1,2*3,4,2)$	$((12),(24),(45),(23))$
	d_2^1	$(2,3*4,1*(234),3)$	$((23),(35),(15),(34))$
	d_2^2	$(3,1*(23),1,4)$	$((34),(14),(12),(45))$
	d_2^3	$(4,2*(34),2,1*(234))$	$((45),(25),(23),(15))$
	d_2^4	$(1*(234),1*2,3,1)$	$((15),(13),(34),(12))$
$\mathcal{D}_3$	d_3^0	$(1,3*4,2*(34),3)$	$((12),(35),(25),(34))$
	d_3^1	$(2,1*(23),1*2,4)$	$((23),(14),(13),(45))$
	d_3^2	$(3,2*(34),2*3,1*(234))$	$((34),(25),(24),(15))$
	d_3^3	$(4,1*2,3*4,1)$	$((45),(13),(35),(12))$
	d_3^4	$(1*(234),2*3,1*(23),2)$	$((15),(24),(14),(23))$
$\mathcal{E}_0$	e_0^0	$(1*2,1,3,4)$	$((13),(12),(34),(45))$
	e_0^1	$(2*3,2,4,1*(234))$	$((24),(23),(45),(15))$
	e_0^2	$(3*4,3,1*(234),1)$	$((35),(34),(15),(12))$
	e_0^3	$(1*(23),4,1,2)$	$((14),(45),(12),(23))$
	e_0^4	$(2*(34),1*(234),2,3)$	$((25),(15),(23),(34))$
$\mathcal{E}_1$	e_1^0	$(1*2,3,1,4)$	$((13),(34),(12),(45))$
	e_1^1	$(2*3,4,2,1*(234))$	$((24),(45),(23),(15))$
	e_1^2	$(3*4,1*(234),3,1)$	$((35),(15),(34),(12))$
	e_1^3	$(1*(23),1,4,2)$	$((14),(12),(45),(23))$
	e_1^4	$(2*(34),2,1*(234),3)$	$((25),(23),(15),(34))$
$\mathcal{E}_2$	e_2^0	$(1*2,3,4,1)$	$((13),(34),(45),(12))$
	e_2^1	$(2*3,4,1*(234),2)$	$((24),(45),(15),(23))$
	e_2^2	$(3*4,1*(234),1,3)$	$((35),(15),(12),(34))$
	e_2^3	$(1*(23),1,2,4)$	$((14),(12),(23),(45))$
	e_2^4	$(2*(34),2,3,1*(234))$	$((25),(23),(34),(15))$
$\mathcal{E}_3$	e_3^0	$(1*2,4,3*4,1)$	$((13),(45),(35),(12))$
	e_3^1	$(2*3,1*(234),1*(23),2)$	$((24),(15),(14),(23))$
	e_3^2	$(3*4,1,2*(34),3)$	$((35),(12),(25),(34))$
	e_3^3	$(1*(23),2,1*2,4)$	$((14),(23),(13),(45))$
	e_3^4	$(2*(34),3,2*3,1*(234))$	$((25),(34),(24),(15))$
$\mathcal{E}_4$	e_4^0	$(1*2,4,1,3*4)$	$((13),(45),(12),(35))$
	e_4^1	$(2*3,1*(234),2,1*(23))$	$((24),(15),(23),(14))$
	e_4^2	$(3*4,1,3,2*(34))$	$((35),(12),(34),(25))$
	e_4^3	$(1*(23),2,4,1*2)$	$((14),(23),(45),(13))$
	e_4^4	$(2*(34),3,1*(234),2*3)$	$((25),(34),(15),(24))$
$\mathcal{E}_5$	e_5^0	$(1*2,1,4,3*4)$	$((13),(12),(45),(35))$
	e_5^1	$(2*3,2,1*(234),1*(23))$	$((24),(23),(15),(14))$
	e_5^2	$(3*4,3,1,2*(34))$	$((35),(34),(12),(25))$
	e_5^3	$(1*(23),4,2,1*2)$	$((14),(45),(23),(13))$
	e_5^4	$(2*(34),1*(234),3,2*3)$	$((25),(15),(34),(24))$

Table 2.1(c)

	β	$(1,2,3,4)\cdot\beta$	$(\pi(1),\pi(2),\pi(3),\pi(4))\cdot\beta$
$\mathcal{E}_6$	e_6^0	$(1*2,1,3*4,3)$	$((13),(12),(35),(34))$
	e_6^1	$(2*3,2,1*(23),4)$	$((24),(23),(14),(45))$
	e_6^2	$(3*4,3,2*(34),1*(234))$	$((35),(34),(25),(15))$
	e_6^3	$(1*(23),4,1*2,1)$	$((14),(45),(13),(12))$
	e_6^4	$(2*(34),1*(234),2*3,2)$	$((25),(15),(24),(23))$
$\mathcal{E}_7$	e_7^0	$(1*2,3*4,1,3)$	$((13),(35),(12),(34))$
	e_7^1	$(2*3,1*(23),2,4)$	$((24),(14),(23),(45))$
	e_7^2	$(3*4,2*(34),3,1*(234))$	$((35),(25),(34),(15))$
	e_7^3	$(1*(23),1*2,4,1)$	$((14),(13),(45),(12))$
	e_7^4	$(2*(34),2*3,1*(234),2)$	$((25),(24),(15),(23))$
$\mathcal{E}_8$	e_8^0	$(1*2,3*4,3,1)$	$((13),(35),(34),(12))$
	e_8^1	$(2*3,1*(23),4,2)$	$((24),(14),(45),(23))$
	e_8^2	$(3*4,2*(34),1*(234),3)$	$((35),(25),(15),(34))$
	e_8^3	$(1*(23),1*2,1,4)$	$((14),(13),(12),(45))$
	e_8^4	$(2*(34),2*3,2,1*(234))$	$((25),(24),(23),(15))$

Table 2.1(d)

Proof of Lemma 2.3. The result of $(s_1,s_2,s_3,s_4)\cdot\beta$ $(=\ (1,2,3,4)\cdot\beta)$ for each $\beta\in\mathcal{X}$ is written in the middle column of the table above. In this proof, we denote it by (b_1,b_2,b_3,b_4) for each $\beta\in\mathcal{X}$.

(1) If $\beta\in\mathcal{Y}_j$ for $j=1,2$ or 3, we see that $b_jb_{j+1}=b_{j+1}b_j$ by direct calculations. So, by Proposition 2.2(1), we have $(s_1,s_2,s_3,s_4)\cdot\beta\sigma_j^2=(s_1,s_2,s_3,s_4)\cdot\beta$.

(2) If $\beta\in\mathcal{Z}_j$ for $j=1,2$ or 3, we see that $b_jb_{j+1}b_j=b_{j+1}b_jb_{j+1}$ by direct calculations. So, by Proposition 2.2(2), we have $(s_1,s_2,s_3,s_4)\cdot\beta\sigma_j^3=(s_1,s_2,s_3,s_4)\cdot\beta$. □

By Lemma 2.3 $\beta\sigma_j^2\beta^{-1}\in H(s_1,s_2,s_3,s_4)$ if $\beta\in\mathcal{Y}_j$ and $\beta\sigma_j^3\beta^{-1}\in H(s_1,s_2,s_3,s_4)$ if $\beta\in\mathcal{Z}_j$ hold for $j=1,2,3$. Thus, we see that any element of $\mathcal{S}$ belongs to $H(s_1,s_2,s_3,s_4)$ and hence $\langle\mathcal{S}\rangle\subset H(s_1,s_2,s_3,s_4)$.

Now we consider the following commutative diagram:

$$\begin{tikzcd} & \mathcal{X} \arrow[dl, "i'"'] \arrow[dr, "i"] & \\ \langle\mathcal{S}\rangle\backslash B_4 \arrow[rr, "j"'] & & H(s_1,s_2,s_3,s_4)\backslash B_4 \end{tikzcd}$$

where i and i' are the restrictions $p|_{\mathcal{X}}$ and $p'|_{\mathcal{X}}$ of the natural projections

$p : B_4 \to H(s_1, s_2, s_3, s_4)\backslash B_4$ and $p' : B_4 \to \langle \mathcal{S} \rangle \backslash B_4$, and j is the natural surjection. By Lemma 2.1 we see that i is injective and hence i' is also injective.

We will show that i' is surjective in Lemma 2.9 by preparing Lemmas 2.4, 2.5, 2.6, 2.7 and 2.8. Hereafter we will use the abbreviation $\alpha \equiv \beta$ instead of $\alpha \equiv \beta \pmod{\langle \mathcal{S} \rangle}$ unless otherwise stated.

Lemma 2.4. *For $1 \leq j \leq 3$, we have the following formulas.*

(1) *If $\beta \in \mathcal{Y}_j$, then $\beta\sigma_j^2 \equiv \beta$.*
(2) *If $\beta \in \mathcal{Z}_j$, then $\beta\sigma_j^3 \equiv \beta$.*

Proof. This is obtained immediately from the definition of $\mathcal{S}$. □

Lemma 2.5. *For $0 \leq s \leq 4$, we have the following formulas.*

(1) $a_0^s(\sigma_2\sigma_3)^4 \equiv a_0^s$,
(2) $e_2^s(\sigma_2\sigma_3)^3\sigma_2 \equiv e_2^s$ *and*
(3) $e_0^s(\sigma_2\sigma_3)^5\sigma_2 \equiv e_0^s$.

Proof. (1) In the braid group B_4, we have $a_0^s(\sigma_2\sigma_3)^4 = a_0^s\sigma_2^2\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3^2$. Since a_0^s is in $\mathcal{A}_0 \subset \mathcal{Z}_2$, by Lemma 2.4(2) we have $a_0^s\sigma_2^2\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3^2 \equiv a_0^s\sigma_2^{-1}\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3^2$. In B_4, we have $a_0^s\sigma_2^{-1}\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3^2 = d_1^s\sigma_3\sigma_2^{-1}\sigma_3$. Since $d_1^s \in \mathcal{D}_1 \subset \mathcal{Y}_3$, by Lemma 2.4(1) we have $d_1^s\sigma_3\sigma_2^{-1}\sigma_3 \equiv d_1^s\sigma_3^{-1}\sigma_2^{-1}\sigma_3 = a_1^s\sigma_2^2\sigma_3^{-1}\sigma_2^{-1}\sigma_3$. Since $a_1^s \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $a_1^s\sigma_2^2\sigma_3^{-1}\sigma_2^{-1}\sigma_3 \equiv a_1^s\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3 = a_0^s$. Thus, we obtain $a_0^s(\sigma_2\sigma_3)^4 \equiv a_0^s$.

(2) Since $e_2^s(\sigma_2\sigma_3)^3\sigma_2 = e_5^s\sigma_3^2\sigma_2\sigma_3$ and $e_5^s \in \mathcal{Z}_3$, using Lemma 2.4(2) we have $e_5^s\sigma_3^2\sigma_2\sigma_3 \equiv e_5^s\sigma_3^{-1}\sigma_2\sigma_3 = e_4^s\sigma_2^2\sigma_3\sigma_2^{-1}$. Since $e_4^s \in \mathcal{Y}_2$, using Lemma 2.4(1) we have $e_4^s\sigma_2^2\sigma_3\sigma_2^{-1} \equiv e_4^s\sigma_3\sigma_2^{-1} = e_3^s\sigma_3^2\sigma_2^{-1}$. Since e_3^s is in $\mathcal{Y}_3$, using Lemma 2.4(1) we have $e_3^s\sigma_3^2\sigma_2^{-1} \equiv e_3^s\sigma_2^{-1} = e_2^s$.

(3) Since $e_0^s(\sigma_2\sigma_3)^5\sigma_2 = e_7^s\sigma_3^2\sigma_2\sigma_3$ and $e_7^s \in \mathcal{Y}_3$, by Lemma 2.4(1) we have $e_7^s\sigma_3^2\sigma_2\sigma_3 \equiv e_7^s\sigma_2\sigma_3 = e_6^s\sigma_2^2\sigma_3$. Since $e_6^s \in \mathcal{Y}_2$, by Lemma 2.4(1) we have $e_6^s\sigma_2^2\sigma_3 \equiv e_6^s\sigma_3 = e_5^s\sigma_3^2$. Since $e_5^s \in \mathcal{Z}_3$, by Lemma 2.4(2) we have $e_5^s\sigma_3^2 \equiv e_5^s\sigma_3^{-1} = e_1^s\sigma_3^2\sigma_2$. Since $e_1^s \in \mathcal{Y}_3$, by Lemma 2.4(1) we have $e_1^s\sigma_3^2\sigma_2 \equiv e_1^s\sigma_2 = e_0^s\sigma_2^2$. Since $e_0^s \in \mathcal{Y}_2$, by Lemma 2.4(1) we have $e_0^s\sigma_2^2 \equiv e_0^s$. □

Lemma 2.6. $(\sigma_1\sigma_2\sigma_3)^5 \equiv id$.

Proof. In B_4, we have $(\sigma_1\sigma_2\sigma_3)^5 = c_0^2\sigma_2\sigma_1\sigma_2\sigma_3{}^2\sigma_2\sigma_3{}^2$. Since $c_0^2 \in \mathcal{Y}_2$, by Lemma 2.4(1) $c_0^2\sigma_2\sigma_1\sigma_2\sigma_3{}^2\sigma_2\sigma_3{}^2 \equiv c_0^2\sigma_2^{-1}\sigma_1\sigma_2\sigma_3{}^2\sigma_2\sigma_3{}^2 = a_0^2\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_3{}^2\sigma_2\sigma_3{}^2$. Since $a_0^s \in \mathcal{Z}_3$, by Lemma 2.4(2) we have $a_0^2\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_3{}^2\sigma_2\sigma_3{}^2 \equiv a_0^2\sigma_3^{-2}\sigma_2^{-1}\sigma_1\sigma_2\sigma_3{}^2\sigma_2\sigma_3{}^2$. In B_4, we have $a_0^2\sigma_3^{-2}\sigma_2^{-1}\sigma_1\sigma_2\sigma_3{}^2\sigma_2\sigma_3{}^2 = b_0^0\sigma_1^2\sigma_3\sigma_2\sigma_3^2\sigma_2\sigma_3^2$. Since $b_0^0 \in \mathcal{Y}_1$, by Lemma 2.4(1) we have $b_0^0\sigma_1^2\sigma_3\sigma_2\sigma_3^2\sigma_2\sigma_3^2 \equiv b_0^0\sigma_3\sigma_2\sigma_3^2\sigma_2\sigma_3^2 = (\sigma_2\sigma_3)^4$. Using Lemma 2.5(1), we have $(\sigma_2\sigma_3)^4 = a_0^0(\sigma_2\sigma_3)^4 \equiv a_0^0 = id$. □

Lemma 2.7. (1) *The following equations hold.*

$$
\begin{array}{rlclcll}
\text{(i)} & a_0^s{\sigma_1}^2 & \equiv & a_3^t\sigma_1 & \equiv & e_0^s & \textit{for } s,t \in \{0,1,2,3,4\} \textit{ with } t \equiv s+1 \pmod 5, \\
\text{(ii)} & a_1^s\sigma_1 & \equiv & c_2^t & & & \textit{for } s,t \in \{0,1,2,3,4\} \textit{ with } t \equiv s+2 \pmod 5, \\
\text{(iii)} & a_2^s\sigma_1 & \equiv & b_0^t & & & \textit{for } s,t \in \{0,1,2,3,4\} \textit{ with } t \equiv s+3 \pmod 5, \\
\text{(iv)} & b_1^s\sigma_1 & \equiv & c_1^t & & & \textit{for } s,t \in \{0,1,2,3,4\} \textit{ with } t \equiv s+3 \pmod 5, \\
\text{(v)} & b_2^s\sigma_1^2 & \equiv & e_6^t\sigma_1 & \equiv & b_3^t & \textit{for } s,t \in \{0,1,2,3,4\} \textit{ with } t \equiv s+4 \pmod 5, \\
\text{(vi)} & c_0^s\sigma_1^2 & \equiv & c_3^t\sigma_1 & \equiv & e_5^s & \textit{for } s,t \in \{0,1,2,3,4\} \textit{ with } t \equiv s+1 \pmod 5, \\
\text{(vii)} & d_0^s\sigma_1^2 & \equiv & e_7^t\sigma_1 & \equiv & e_2^u & \textit{for } s,t,u \in \{0,1,2,3,4\} \\
 & & & & & & \qquad\qquad \textit{with } u \equiv t+2 \equiv s+3 \pmod 5, \\
\text{(viii)} & d_1^s\sigma_1 & \equiv & e_4^t & & & \textit{for } s,t \in \{0,1,2,3,4\} \textit{ with } t \equiv s+2 \pmod 5, \\
\text{(ix)} & d_2^s\sigma_1^2 & \equiv & e_8^t\sigma_1 & \equiv & e_1^u & \textit{for } s,t,u \in \{0,1,2,3,4\} \\
 & & & & & & \qquad\qquad \textit{with } u \equiv t+2 \equiv s+3 \pmod 5, \\
\text{(x)} & d_3^s\sigma_1 & \equiv & e_3^t & & & \textit{for } s,t \in \{0,1,2,3,4\} \textit{ with } t \equiv s+2 \pmod 5.
\end{array}
$$

(2) *For each* $s \in \{0,1,2,3,4\}$, *the following equations hold.*

$$
\begin{array}{rlclcl}
\text{(i)} & a_0^s{\sigma_2}^2 & \equiv & b_0^s\sigma_2 & \equiv & d_0^s, \\
\text{(ii)} & a_1^s\sigma_2^2 & \equiv & b_1^s\sigma_2 & \equiv & d_1^s, \\
\text{(iii)} & a_2^s\sigma_2^2 & \equiv & b_2^s\sigma_2 & \equiv & d_2^s, \\
\text{(iv)} & a_3^s\sigma_2^2 & \equiv & b_3^s\sigma_2 & \equiv & d_3^s, \\
\text{(v)} & c_0^s\sigma_2 & \equiv & c_2^s, & & \\
\text{(vi)} & c_1^s\sigma_2 & \equiv & c_3^s, & & \\
\text{(vii)} & e_0^s\sigma_2 & \equiv & e_1^s, & & \\
\text{(viii)} & e_2^s\sigma_2^2 & \equiv & e_3^s\sigma_2 & \equiv & e_8^s, \\
\text{(ix)} & e_4^s\sigma_2 & \equiv & e_5^s, & & \\
\text{(x)} & e_6^s\sigma_2 & \equiv & e_7^s. & &
\end{array}
$$

(3) *For each* $s \in \{0,1,2,3,4\}$, *the following equations hold.*

$$
\begin{array}{rlclcl}
\text{(i)} & a_0^s{\sigma_3}^2 & \equiv & c_0^s\sigma_3 & \equiv & b_3^s, \\
\text{(ii)} & a_1^s\sigma_3^2 & \equiv & c_1^s\sigma_3 & \equiv & b_0^s, \\
\text{(iii)} & a_2^s\sigma_3^2 & \equiv & c_2^s\sigma_3 & \equiv & b_1^s, \\
\text{(iv)} & a_3^s\sigma_3^2 & \equiv & c_3^s\sigma_3 & \equiv & b_2^s, \\
\text{(v)} & d_0^s\sigma_3 & \equiv & d_2^s, & & \\
\text{(vi)} & d_1^s\sigma_3 & \equiv & d_3^s, & & \\
\text{(vii)} & e_0^s\sigma_3^2 & \equiv & e_5^s\sigma_3 & \equiv & e_6^s, \\
\text{(viii)} & e_1^s\sigma_3 & \equiv & e_2^s, & & \\
\text{(ix)} & e_3^s\sigma_3 & \equiv & e_4^s, & & \\
\text{(x)} & e_7^s\sigma_3 & \equiv & e_8^s. & &
\end{array}
$$

Proof. First, we prove (2).

Since $d_i^s = b_i^s\sigma_2 = a_i^s\sigma_2^2$ for $0 \leq i \leq 3$, we have (i), (ii), (iii) and (iv).

Let i be 0 or 1. Then, we have $c_i^s\sigma_2 = b_{i+1}^s\sigma_3^{-1}$ in B_4. Since $b_{i+1}^s \in \mathcal{Z}_3$, by Lemma 2.4(2) we have $b_{i+1}^s\sigma_3^{-1} \equiv b_{i+1}^s\sigma_3^2 = c_{i+2}^s$. Hence, (v) and (vi) hold.

Let $i \in \{0, 2, 4, 6\}$. Then, $e_i^s\sigma_2 = e_{i+1}^s$. Hence, (vii), (ix), (x) and $e_2^s{\sigma_2}^2 \equiv e_3^s\sigma_2$ of (viii) hold.

We show $e_3^s\sigma_2 \equiv e_8^s$ of (viii). Note that $e_3^s\sigma_2 = e_2^s\sigma_2^2$. By Lemma 2.5(2), we have $e_2^s\sigma_2^2 \equiv e_2^s(\sigma_2\sigma_3)^3\sigma_2^3 = e_8^s\sigma_2^3$. Since $e_8^s \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $e_8^s\sigma_2^3 \equiv e_8^s$.

Next, we prove (3).

Let $i \in \{0, 1, 2, 3\}$. By Lemma 2.5(1), $a_i^s = a_0^s(\sigma_2\sigma_3)^i \equiv a_0^s(\sigma_2\sigma_3)^{i+4}$. Then, $a_i^s\sigma_3^2 = a_0^s(\sigma_2\sigma_3)^{i+3}\sigma_2\sigma_3^3$. When $i = 0$, we see that $a_0^s(\sigma_2\sigma_3)^{i+3}\sigma_2\sigma_3^3 = b_3^s\sigma_3^3$. Since $b_3^s \in \mathcal{Z}_3$, by Lemma 2.4(2) we have $b_3^s\sigma_3^3 \equiv b_3^s$. When $i \in \{1, 2, 3\}$, by Lemma 2.5(1) we have $a_0^s(\sigma_2\sigma_3)^{i+3}\sigma_2\sigma_3^3 \equiv a_0^s(\sigma_2\sigma_3)^{i-1}\sigma_2\sigma_3^3 = b_{i-1}^s\sigma_3^3$. Since $b_{i-1}^s \in \mathcal{Z}_3$, by Lemma 2.4(2) we have $b_{i-1}^s\sigma_3^3 \equiv b_{i-1}^s$. Since $c_i^s = a_i^s\sigma_3$, we have (i), (ii), (iii) and (iv).

Let i be 0 or 1. Then, $d_i^s\sigma_3 = a_{i+2}^s\sigma_2^{-1}$ in B_4. Since $a_{i+2}^s \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $a_{i+2}^s\sigma_2^{-1} \equiv a_{i+2}^s\sigma_2^2 = d_{i+2}^s$. Thus, we have (v) and (vi).

Let $i \in \{1, 3, 5, 7\}$. Then, $e_i^s\sigma_3 = e_{i+1}^s$. Hence, (viii), (ix), (x) and $e_5^s\sigma_3 \equiv e_6^s$ of (vii) hold.

We show $e_0^s\sigma_3 \equiv e_5^s$ of (vii). Since $e_0^s\sigma_3 = e_3^s{\sigma_3}^{-1}\sigma_2^{-1}$ in B_4 and $e_3^s \in \mathcal{Y}_3$, by Lemma 2.4(1) we have $e_3^s{\sigma_3}^{-1}\sigma_2^{-1} \equiv e_3^s\sigma_3\sigma_2^{-1} = e_4^s\sigma_2^{-1}$. Since $e_4^s \in \mathcal{Y}_2$, by Lemma 2.4(1) we have $e_4^s\sigma_2^{-1} \equiv e_4^s\sigma_2 = e_5^s$.

Finally, we prove (1).

We show (i). It is obvious that $a_0^s\sigma_1^2 = e_0^s$. We show $a_0^s\sigma_1 \equiv a_3^t$ for $s, t \in \{0, 1, 2, 3, 4\}$ with $t \equiv s + 1 \pmod 5$. Note that $a_0^s\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{s+1}(\sigma_2\sigma_3)^{-1}$. When $s \neq 4$, $(\sigma_1\sigma_2\sigma_3)^{s+1}(\sigma_2\sigma_3)^{-1} = a_0^{s+1}(\sigma_2\sigma_3)^{-1}$. By Lemma 2.5(1), we have $a_0^{s+1}(\sigma_2\sigma_3)^{-1} \equiv a_0^{s+1}(\sigma_2\sigma_3)^3 = a_3^{s+1} = a_3^t$. When $s = 4$, by Lemma 2.6 $(\sigma_1\sigma_2\sigma_3)^{s+1}(\sigma_2\sigma_3)^{-1} = (\sigma_1\sigma_2\sigma_3)^5(\sigma_2\sigma_3)^{-1} \equiv (\sigma_2\sigma_3)^{-1}$. By Lemma 2.5(1), we have $(\sigma_2\sigma_3)^{-1} = a_0^0(\sigma_2\sigma_3)^{-1} \equiv a_0^0(\sigma_2\sigma_3)^3 = a_3^0$.

We show (iii) before (ii). Since $a_2^s \in \mathcal{Y}_1$, it suffices to prove $a_2^s\sigma_1^{-1} \equiv b_0^t$ with $t \equiv s + 3 \pmod 5$. Since $b_0^t\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{t+2}(\sigma_2\sigma_3)^{-2}$, we will prove $(\sigma_1\sigma_2\sigma_3)^{t+2}(\sigma_2\sigma_3)^{-2} \equiv a_2^s$, where $t \equiv s+3 \pmod 5$. When $t = 0, 1, 2$, we have $(\sigma_1\sigma_2\sigma_3)^{t+2}(\sigma_2\sigma_3)^{-2} = a_0^{t+2}(\sigma_2\sigma_3)^{-2}$. By Lemma 2.5(1), we have $a_0^{t+2}(\sigma_2\sigma_3)^{-2} \equiv a_0^{t+2}(\sigma_2\sigma_3)^2 = a_2^{t+2} = a_2^s$. When $t = 3, 4$, by Lemma 2.6 $(\sigma_1\sigma_2\sigma_3)^{t+2}(\sigma_2\sigma_3)^{-2} \equiv (\sigma_1\sigma_2\sigma_3)^{t-3}(\sigma_2\sigma_3)^{-2} = a_0^{t-3}(\sigma_2\sigma_3)^{-2}$. By Lemma 2.5(1), $a_0^{t-3}(\sigma_2\sigma_3)^{-2} \equiv a_0^{t-3}(\sigma_2\sigma_3)^2 = a_2^{t-3} = a_2^s$.

We show (ii). Note that $a_1^s\sigma_1 = b_0^s\sigma_1\sigma_3$. Since $b_0^s \in \mathcal{Y}_1$, by Lemma 2.4(1) we have $b_0^s\sigma_1\sigma_3 \equiv b_0^s\sigma_1^{-1}\sigma_3$. By (1)(iii), $b_0^s\sigma_1^{-1} \equiv a_2^t$ for $t \in \{0, 1, 2, 3, 4\}$ with $t \equiv s + 2 \pmod 5$. Thus, we have $b_0^s\sigma_1^{-1}\sigma_3 \equiv a_2^t\sigma_3 = c_2^t$.

We show (iv). In B_4, $b_1^s\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{s+3}{\sigma_2}^{-1}{\sigma_3}^{-2}\sigma_2^{-1}\sigma_3^{-1}$. When s is 0 or 1, it is $a_0^{s+3}{\sigma_2}^{-1}{\sigma_3}^{-2}\sigma_2^{-1}\sigma_3^{-1}$. Since $a_0^{s+3} \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $a_0^{s+3}{\sigma_2}^{-1}{\sigma_3}^{-2}\sigma_2^{-1}\sigma_3^{-1}$ $\equiv a_0^{s+3}{\sigma_2}^2{\sigma_3}^{-2}\sigma_2^{-1}\sigma_3^{-1} = d_0^{s+3}{\sigma_3}^{-2}\sigma_2^{-1}\sigma_3^{-1}$. Since $d_0^{s+3} \in \mathcal{Y}_3$, by Lemma 2.4(1) we have $d_0^{s+3}{\sigma_3}^{-2}\sigma_2^{-1}\sigma_3^{-1} \equiv d_0^{s+3}\sigma_2^{-1}\sigma_3^{-1} = b_0^{s+3}\sigma_3^{-1}$. Since $b_0^{s+3} \in \mathcal{Z}_3$, by Lemma 2.4(2) we have $b_0^{s+3}\sigma_3^{-1} \equiv b_0^{s+3}\sigma_3^2 = c_1^{s+3} = c_1^t$. When $s \in \{2, 3, 4\}$, by Lemma 2.6 we get

$b_1^s\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_2{}^{-1}\sigma_3{}^{-2}\sigma_2^{-1}\sigma_3^{-1} \equiv a_0^{s-2}\sigma_2{}^{-1}\sigma_3{}^{-2}\sigma_2^{-1}\sigma_3^{-1}$. By the same argument as in the case where $s \in \{0,1\}$, we see that $a_0^{s-2}\sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_3^{-1} \equiv c_1^{s-2} = c_1^t$.

We show (v). First, we show $b_2^s\sigma_1 \equiv e_6^t$ with $t \equiv s+4 \pmod 5$. Note that $b_2^s\sigma_1 = b_2^s\sigma_3\sigma_3{}^{-1}\sigma_1 = a_0^s(\sigma_2\sigma_3)^3\sigma_3{}^{-1}\sigma_1$. By Lemma 2.5(1), we have $a_0^s(\sigma_2\sigma_3)^3\sigma_3{}^{-1}\sigma_1 \equiv a_0^s(\sigma_2\sigma_3)^{-1}\sigma_3^{-1}\sigma_1$. When $s \neq 0$, we have $a_0^s(\sigma_2\sigma_3)^{-1}\sigma_3^{-1}\sigma_1 = e_2^{s-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}$. Since $e_2^{s-1} \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $e_2^{s-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1} \equiv e_2^{s-1}\sigma_2^2\sigma_3^{-1}\sigma_2^{-1} = e_3^{s-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3$. Since $e_3^{s-1} \in \mathcal{Y}_3$, by Lemma 2.4(1) we have $e_3^{s-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_3 \equiv e_3^{s-1}\sigma_3\sigma_2^{-1}\sigma_3 = e_4^{s-1}\sigma_2^{-1}\sigma_3$. Since $e_4^{s-1} \in \mathcal{Y}_2$, by Lemma 2.4(1) we have $e_4^{s-1}\sigma_2^{-1}\sigma_3 \equiv e_4^{s-1}\sigma_2\sigma_3 = e_6^{s-1} = e_6^t$. When $s = 0$, by Lemma 2.6 we have $a_0^s(\sigma_2\sigma_3)^{-1}\sigma_3^{-1}\sigma_1 = (\sigma_2\sigma_3)^{-1}\sigma_3^{-1}\sigma_1 \equiv (\sigma_1\sigma_2\sigma_3)^5(\sigma_2\sigma_3)^{-1}\sigma_3^{-1}\sigma_1 = e_2^4\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}$. By the same argument as in the case where $s \neq 0$, we see that $e_2^4\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1} \equiv e_6^4 = e_6^t$.

Next we show $b_2^s\sigma_1^2 \equiv b_3^t$ with $t \equiv s+4 \pmod 5$. Note that $b_2^s\sigma_1^2 = b_2^s\sigma_3\sigma_3{}^{-1}\sigma_1^2 = a_0^s(\sigma_2\sigma_3)^3\sigma_3{}^{-1}\sigma_1^2$. By Lemma 2.5(1), we have $a_0^s(\sigma_2\sigma_3)^3\sigma_3{}^{-1}\sigma_1^2 \equiv a_0^s(\sigma_2\sigma_3)^{-1}\sigma_3^{-1}\sigma_1^2$. When $s \neq 0$, $a_0^s(\sigma_2\sigma_3)^{-1}\sigma_3^{-1}\sigma_1^2 = a_0^{s-1}\sigma_1\sigma_3^{-1}\sigma_1^2 = a_0^{s-1}\sigma_1^3\sigma_3^{-1}$. Since $a_0^{s-1} \in \mathcal{Z}_1$, by Lemma 2.4(2) we have $a_0^{s-1}\sigma_1^3\sigma_3^{-1} \equiv a_0^{s-1}\sigma_3^{-1}$. By Lemma 2.5(1), we have $a_0^{s-1}\sigma_3^{-1} \equiv a_0^{s-1}(\sigma_2\sigma_3)^4\sigma_3^{-1} = b_3^{s-1} = b_3^t$. When $s = 0$, by Lemma 2.6, we have $a_0^s(\sigma_2\sigma_3)^{-1}\sigma_3^{-1}\sigma_1^2 \equiv (\sigma_1\sigma_2\sigma_3)^5(\sigma_2\sigma_3)^{-1}\sigma_3^{-1}\sigma_1^2 = a_0^4\sigma_1^3\sigma_3^{-1}$. By the same argument as in the case where $s \neq 0$, we see that $a_0^4\sigma_1^3\sigma_3^{-1} \equiv b_3^4 = b_3^t$.

We show (vi). Note that $c_0^s\sigma_1^2 = a_0^s\sigma_3\sigma_1^2 = a_0^s\sigma_1^2\sigma_3$. By (1)(i), we have $a_0^s\sigma_1^2\sigma_3 \equiv a_3^t\sigma_1\sigma_3 = a_3^t\sigma_3\sigma_1$ with $t \equiv s+1 \pmod 5$. By (3)(iv), we have $a_3^t\sigma_3\sigma_1 \equiv c_3^t\sigma_1$. Thus, we have $c_0^s\sigma_1^2 \equiv c_3^t\sigma_1$ with $t \equiv s+1 \pmod 5$. On the other hand, by (1)(i) we have $a_3^t\sigma_1\sigma_3 \equiv e_0^s\sigma_3$. By (3)(vii), we see that $e_0^s\sigma_3 \equiv e_5^s$. Thus, we have $c_0^s\sigma_1^2 \equiv e_5^s$.

We show (vii). Since $d_0^s \in \mathcal{Z}_1$, by Lemma 2.4(2) we have $d_0^s\sigma_1^2 \equiv d_0^s\sigma_1^{-1}$. Note that $d_0^s\sigma_1^{-1} = a_0^s\sigma_2^2\sigma_1^{-1}$. Since $a_0^s \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $a_0^s\sigma_2^2\sigma_1^{-1} \equiv a_0^s\sigma_2^{-1}\sigma_1^{-1}$. In B_4, we have $a_0^s\sigma_2^{-1}\sigma_1^{-1} = (\sigma_1\sigma_2\sigma_3)^{s-2}\sigma_1^2\sigma_2\sigma_3$. When $s = 2,3,4$, we see $(\sigma_1\sigma_2\sigma_3)^{s-2}\sigma_1^2\sigma_2\sigma_3 = e_2^{s-2}$. When $s = 0,1$, by Lemma 2.6 we have $(\sigma_1\sigma_2\sigma_3)^{s-2}\sigma_1^2\sigma_2\sigma_3 \equiv (\sigma_1\sigma_2\sigma_3)^{s+3}\sigma_1^2\sigma_2\sigma_3 = e_2^{s+3}$. Thus, we have $d_0^s\sigma_1^2 \equiv e_2^u$ with $u \equiv s+3 \pmod 5$. We show that $d_0^s\sigma_1 \equiv e_7^t$ for $t \in \{0,1,2,3,4\}$ with $t = s+1 \pmod 5$. In B_4, $d_0^s\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{s+1}\sigma_1^2\sigma_3^{-1}\sigma_2^{-1}$. When $s \neq 4$, we have $(\sigma_1\sigma_2\sigma_3)^{s+1}\sigma_1^2\sigma_3^{-1}\sigma_2^{-1} = e_0^{s+1}\sigma_3^{-1}\sigma_2^{-1}$. By Lemma 2.5(3), we have $e_0^{s+1}\sigma_3^{-1}\sigma_2^{-1} \equiv e_0^{s+1}(\sigma_2\sigma_3)^5\sigma_2\sigma_3{}^{-1}\sigma_2^{-1} = e_7^{s+1}\sigma_3^2$. Since $e_7^{s+1} \in \mathcal{Y}_3$, by Lemma 2.4(1) we have $e_7^{s+1}\sigma_3^2 \equiv e_7^{s+1}$. Thus, $d_0^s\sigma_1 \equiv e_7^{s+1} = e_7^t$. When $s = 4$, by Lemma 2.6 we have $(\sigma_1\sigma_2\sigma_3)^{s+1}\sigma_1^2\sigma_3^{-1}\sigma_2^{-1} \equiv \sigma_1^2\sigma_3^{-1}\sigma_2^{-1} = e_0^0\sigma_3^{-1}\sigma_2^{-1}$. By the same argument as in the case where $s \neq 4$, we see that $e_0^0\sigma_3^{-1}\sigma_2^{-1} \equiv e_7^0 = e_7^t$.

We show (viii). In B_4, $d_1^s\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{s+1}\sigma_2^2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_3$. When $s \in \{0,1,2\}$, we have $(\sigma_1\sigma_2\sigma_3)^{s+1}\sigma_2^2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_3 = a_0^{s+1}\sigma_2^2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_3$. Since $a_0^{s+1} \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $a_0^{s+1}\sigma_2^2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_3 \equiv a_0^{s+1}\sigma_2^{-1}\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_3$. In B_4, it is $a_0^{s+2}\sigma_1^{-1}\sigma_3^{-2}\sigma_2^{-1}$. Since $a_0^{s+2} \in \mathcal{Z}_1$ and $e_0^{s+2} \in \mathcal{Z}_3$, by Lemma 2.4(2) we have $a_0^{s+2}\sigma_1^{-1}\sigma_3^{-2}\sigma_2^{-1} \equiv a_0^{s+2}\sigma_1^2\sigma_3^{-2}\sigma_2^{-1} = e_0^{s+2}\sigma_3^{-2}\sigma_2^{-1} \equiv e_0^{s+2}\sigma_3\sigma_2^{-1}$. Since $e_0^{s+2} \in \mathcal{Y}_2$, by Lemma 2.4(1) we have $e_0^{s+2}\sigma_3\sigma_2^{-1} \equiv e_0^{s+2}\sigma_2^2\sigma_3\sigma_2^{-1} = e_1^{s+2}\sigma_3^{-1}\sigma_2\sigma_3$. Since $e_1^{s+2} \in \mathcal{Y}_3$, we have $e_1^{s+2}\sigma_3^{-1}\sigma_2\sigma_3 \equiv e_1^{s+2}\sigma_3\sigma_2\sigma_3 = e_4^{s+2} = e_4^t$. When $s \in \{3,4\}$, by Lemma 2.6, we have $(\sigma_1\sigma_2\sigma_3)^{s+1}\sigma_2^2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_3 \equiv$

$(\sigma_1\sigma_2\sigma_3)^{s-4}\sigma_2^2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_3$. By the same argument as in the case where $s \in \{0,1,2\}$, we see that $(\sigma_1\sigma_2\sigma_3)^{s-4}\sigma_2^2\sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_3 \equiv e_4^{s-3} = e_4^t$.

We show (ix). First, we show $d_2^s\sigma_1 \equiv e_8^t$ with $t+2 \equiv s+3 \pmod 5$. Note that $d_2^s\sigma_1 = a_2^s\sigma_2^2\sigma_1$. Since $a_2^s \in \mathcal{Z}_2$ and $a_0^s \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $a_2^s\sigma_2^2\sigma_1 \equiv a_2^s\sigma_2^{-1}\sigma_1 = a_0^s\sigma_2^2\sigma_3\sigma_1 \equiv a_0^s\sigma_2^{-1}\sigma_3\sigma_1$. In B_4, we have $a_0^s\sigma_2^{-1}\sigma_3\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{s+1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1}$. When $s \neq 4$, we see that $(\sigma_1\sigma_2\sigma_3)^{s+1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1} = a_0^{s+1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1}$. Since $a_0^{s+1} \in \mathcal{Z}_1$, by Lemma 2.4(2) we have $a_0^{s+1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2^{-1} \equiv a_0^{s+1}\sigma_1^2\sigma_2\sigma_3^{-1}\sigma_2^{-1} = e_1^{s+1}\sigma_3^{-1}\sigma_2^{-1}$. Since $e_1^{s+1} \in \mathcal{Y}_3$, by Lemma 2.4(1) we have $e_1^{s+1}\sigma_3^{-1}\sigma_2^{-1} \equiv e_1^{s+1}\sigma_3\sigma_2^{-1} = e_2^{s+1}\sigma_2^{-1}$. Since $e_2^{s+1} \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $e_2^{s+1}\sigma_2^{-1} \equiv e_2^{s+1}\sigma_2^2$. By Lemma 2.5(2), we have $e_2^{s+1}\sigma_2^2 \equiv e_2^{s+1}(\sigma_2\sigma_3)^3\sigma_2^3 = e_8^{s+1}\sigma_2^3$. Since $e_8^{s+1} \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $e_8^{s+1}\sigma_2^3 \equiv e_8^{s+1} = e_8^t$. When $s = 4$, by Lemma 2.6, $(\sigma_1\sigma_2\sigma_3)^{s+1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_2^{-1} \equiv (\sigma_1\sigma_2\sigma_3)^{s-4}\sigma_1^{-1}\sigma_2\sigma_3\sigma_2^{-1} = a_0^0\sigma_1^{-1}\sigma_2\sigma_3\sigma_2^{-1}$. By the same argument as in the case where $s \neq 4$, we see $a_0^0\sigma_1^{-1}\sigma_2\sigma_3\sigma_2^{-1} \equiv e_8^0 = e_8^t$.

Next, we show $e_8^t\sigma_1 \equiv e_1^u$ with $u \equiv t+2 \pmod 5$. Note that $e_8^t\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{t+2}\sigma_1{}^2\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_3$. When $t \in \{0,1,2\}$, it is $e_0^{t+2}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_3$. Since $e_0^{t+2} \in \mathcal{Y}_2$, by Lemma 2.4(1) we have $e_0^{t+2}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_3 \equiv e_0^{t+2}\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_3 = e_3^{t+2}\sigma_1^2\sigma_2^{-1}\sigma_3$. Since $e_3^{t+2} \in \mathcal{Y}_1$ and $e_1^{t+2} \in \mathcal{Y}_3$, by Lemma 2.4(1) we have that $e_3^{t+2}\sigma_1^2\sigma_2^{-1}\sigma_3 \equiv e_3^{t+2}\sigma_2^{-1}\sigma_3 = e_1^{t+2}\sigma_3^2 \equiv e_1^{t+2} = e_1^u$. When $t \in \{3,4\}$, by Lemma 2.6 we have $(\sigma_1\sigma_2\sigma_3)^{t+2}\sigma_1{}^2\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_3 \equiv (\sigma_1\sigma_2\sigma_3)^{t-3}\sigma_1{}^2\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_3 = e_0^{t-3}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_3$. By the same argument as in the case where $t \in \{0,1,2\}$, we see that $e_0^{t-3}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_3 \equiv e_1^{t-3} = e_1^u$.

We show (x). In B_4, $d_3^s\sigma_1 = a_3^s\sigma_2^2\sigma_1$. Since $a_3^s \in \mathcal{Z}_2$, by Lemma 2.4(2) we have $a_3^s\sigma_2^2\sigma_1 \equiv a_3^s\sigma_2^{-1}\sigma_1 = a_0^s(\sigma_2\sigma_3)^3\sigma_2^{-1}\sigma_1$. By Lemma 2.5(1), we have $a_0^s(\sigma_2\sigma_3)^3\sigma_2^{-1}\sigma_1 \equiv a_0^s(\sigma_2\sigma_3)^{-1}\sigma_2^{-1}\sigma_1 = (\sigma_1\sigma_2\sigma_3)^{s-1}\sigma_1\sigma_2^{-1}\sigma_1$. In B_4, it is $(\sigma_1\sigma_2\sigma_3)^{s-3}\sigma_1^2\sigma_2\sigma_3\sigma_2\sigma_1^2$. When s is 3 or 4, we see that $(\sigma_1\sigma_2\sigma_3)^{s-3}\sigma_1^2\sigma_2\sigma_3\sigma_2\sigma_1^2 = e_3^{s-3}\sigma_1^2$. Since $e_3^{s-3} \in \mathcal{Y}_1$, by Lemma 2.4(1) we have $e_3^{s-3}\sigma_1^2 \equiv e_3^{s-3} = e_3^t$. When $s = 0,1,2$, by Lemma 2.6 we have the indentity $(\sigma_1\sigma_2\sigma_3)^{s-3}\sigma_1^2\sigma_2\sigma_3\sigma_2\sigma_1^2 \equiv (\sigma_1\sigma_2\sigma_3)^{s+2}\sigma_1^2\sigma_2\sigma_3\sigma_2\sigma_1^2 = e_3^{s+2}\sigma_1^2$. By the same argument as in the case where $s \in \{3,4\}$, we have $e_3^{s+2}\sigma_1^2 \equiv e_3^{s+2} = e_3^t$. $\square$

By Lemmas 2.4 and 2.7, we shall obtain Lemma 2.8.

Lemma 2.8. *For any element $\alpha \in \mathcal{X}$ and for any power σ_j^l of each standard generator σ_j $(j = 1,2,3)$ of B_4, there is an element β of $\mathcal{X}$ such that $\alpha\sigma_j^l \equiv \beta$.*

Proof. Since $\mathcal{X} = \mathcal{Y}_j \coprod \mathcal{Z}_j$, α belongs to either $\mathcal{Y}_j$ or $\mathcal{Z}_j$. Suppose that $\alpha \in \mathcal{Y}_j$. Then, by Lemma 2.4(1) we have $\alpha\sigma_j^l \equiv \alpha$ for an even number l or $\alpha\sigma_j^l \equiv \alpha\sigma_j$ for an odd number l. We will show that for $\alpha \in \mathcal{Y}_j$ there is an element β of $\mathcal{X}$ such that $\alpha\sigma_j \equiv \beta$. By Lemma 2.7(1)(ii) and Lemma 2.4(1), we have $a_1^s\sigma_1^2 \equiv c_2^t\sigma_1 \equiv a_1^s$ (*i.e.*, $a_1^s\sigma_1 \equiv c_2^t$ and $c_2^t\sigma_1 \equiv a_1^s$) for $s,t \in \{0,1,2,3,4\}$ with $t \equiv s+2 \pmod 5$. Similarly, by Lemma 2.4(1) and Lemma 2.7(1)(iii), (1)(iv), (1)(viii), (1)(x); (2)(v), (2)(vi), (2)(vii), (2)(ix), (2)(x); (3)(v), (3)(vi), (3)(viii), (3)(ix) and (3)(x) we have $a_2^s\sigma_1^2 \equiv b_0^t\sigma_1 \equiv a_2^s$ for $s,t \in \{0,1,2,3,4\}$ with $t \equiv s+3 \pmod 5$, $b_1^s\sigma_1^2 \equiv c_1^t\sigma_1 \equiv b_1^s$ for $s,t \in \{0,1,2,3,4\}$ with $t \equiv s+3 \pmod 5$, $d_1^s\sigma_1^2 \equiv e_4^t\sigma_1 \equiv$

d_1^s for $s,t \in \{0,1,2,3,4\}$ with $t \equiv s+2 \pmod 5$, $d_3^s\sigma_1^2 \equiv e_3^t\sigma_1 \equiv d_3^s$ for $s,t \in \{0,1,2,3,4\}$ with $t \equiv s+2 \pmod 5$; $c_0^s\sigma_2 \equiv c_2^s$, $c_2^s\sigma_2 \equiv c_0^s$, $c_1^s\sigma_2 \equiv c_3^s$, $c_3^s\sigma_2 \equiv c_1^s$, $e_0^s\sigma_2 \equiv e_1^s$, $e_1^s\sigma_2 \equiv e_0^s$, $e_4^s\sigma_2 \equiv e_5^s$, $e_5^s\sigma_2 \equiv e_4^s$, $e_6^s\sigma_2 \equiv e_7^s$, $e_7^s\sigma_2 \equiv e_6^s$; $d_0^s\sigma_3 \equiv d_2^s$, $d_2^s\sigma_3 \equiv d_0^s$, $d_1^s\sigma_3 \equiv d_3^s$, $d_3^s\sigma_3 \equiv d_1^s$, $e_1^s\sigma_3 \equiv e_2^s$, $e_2^s\sigma_3 \equiv e_1^s$, $e_3^s\sigma_3 \equiv e_4^s$, $e_4^s\sigma_3 \equiv e_3^s$, $e_7^s\sigma_3 \equiv e_8^s$ and $e_8^s\sigma_3 \equiv e_7^s$ for $s \in \{0,1,2,3,4\}$. Hence, we have the result when $\alpha \in \mathcal{Y}_j$.

We consider the case where $\alpha \in \mathcal{Z}_j$. By Lemma 2.4(2) we have $\alpha\sigma_j^l \equiv \alpha$ for $l \in 3\mathbb{Z}$, $\alpha\sigma_j^l \equiv \alpha\sigma_j$ for $l \in 3\mathbb{Z}+1$ or $\alpha\sigma_j^l \equiv \alpha\sigma_j^2$ for $l \in 3\mathbb{Z}+2$. We will show that for $\alpha \in \mathcal{Z}_j$, there are elements β and γ of $\mathcal{X}$ such that $\alpha\sigma_j \equiv \beta$ and $\alpha\sigma_j^2 \equiv \gamma$. By Lemma 2.7(1)(i) and Lemma 2.4(2), we have $a_0^s\sigma_1^3 \equiv a_3^t\sigma_1^2 \equiv e_0^s\sigma_1 \equiv a_0^s$ (*i.e.*, $a_0^s\sigma_1 \equiv a_3^t$, $a_3^t\sigma_1 \equiv e_0^s$ and $e_0^s\sigma_1 \equiv a_0^s$) for $s,t \in \{0,1,2,3,4\}$ with $t \equiv s+1 \pmod 5$. Similarly, by Lemma 2.4(2) and Lemma 2.7(1)(v), (1)(vi), (1)(vii), (1)(ix); (2)(i), (2)(ii), (2)(iii), (2)(iv), (2)(viii); (3)(i), (3)(ii), (3)(iii), (3)(iv) and (3)(vii) we have $b_2^s\sigma_1 \equiv e_6^t$, $e_6^t\sigma_1 \equiv b_3^t$ and $b_3^t\sigma_1 \equiv b_2^s$ for $s,t \in \{0,1,2,3,4\}$ with $t \equiv s+4 \pmod 5$, $c_0^s\sigma_1^2 \equiv c_3^t\sigma_1 \equiv e_5^s$, $c_3^t\sigma_1^2 \equiv e_5^s\sigma_1 \equiv c_0^s$ and $e_5^s\sigma_1^2 \equiv c_0^s\sigma_1 \equiv c_3^t$ for $s,t \in \{0,1,2,3,4\}$ with $t \equiv s+1 \pmod 5$, $d_0^s\sigma_1 \equiv e_7^t$, $e_7^t\sigma_1 \equiv e_2^u$ and $e_2^u\sigma_1 \equiv d_0^s$ for $s,t,u \in \{0,1,2,3,4\}$ with $u \equiv t+2 \equiv s+3 \pmod 5$, $d_2^s\sigma_1 \equiv e_8^t$, $e_8^t\sigma_1 \equiv e_1^u$ and $e_1^u\sigma_1 \equiv d_2^s$ for $s,t,u \in \{0,1,2,3,4\}$ with $u \equiv t+2 \equiv s+3 \pmod 5$; $a_0^s\sigma_2 \equiv b_0^s$, $b_0^s\sigma_2 \equiv d_0^s$, $d_0^s\sigma_2 \equiv a_0^s$, $a_1^s\sigma_2 \equiv b_1^s$, $b_1^s\sigma_2 \equiv d_1^s$, $d_1^s\sigma_2 \equiv a_1^s$, $a_2^s\sigma_2 \equiv b_2^s$, $b_2^s\sigma_2 \equiv d_2^s$, $d_2^s\sigma_2 \equiv a_2^s$, $a_3^s\sigma_2 \equiv b_3^s$, $b_3^s\sigma_2 \equiv d_3^s$, $d_3^s\sigma_2 \equiv a_3^s$, $e_2^s\sigma_2 \equiv e_3^s$, $e_3^s\sigma_2 \equiv e_8^s$, $e_8^s\sigma_2 \equiv e_2^s$; $a_0^s\sigma_3 \equiv c_0^s$, $c_0^s\sigma_3 \equiv b_3^s$, $b_3^s\sigma_3 \equiv a_0^s$, $a_1^s\sigma_3 \equiv c_1^s$, $c_1^s\sigma_3 \equiv b_0^s$, $b_0^s\sigma_3 \equiv a_1^s$, $a_2^s\sigma_3 \equiv c_2^s$, $c_2^s\sigma_3 \equiv b_1^s$, $b_1^s\sigma_3 \equiv a_2^s$, $a_3^s\sigma_3 \equiv c_3^s$, $c_3^s\sigma_3 \equiv b_2^s$, $b_2^s\sigma_3 \equiv a_3^s$, $e_0^s\sigma_3 \equiv e_5^s$, $e_5^s\sigma_3 \equiv e_6^s$ and $e_6^s\sigma_3 \equiv e_0^s$ for $s \in \{0,1,2,3,4\}$. Hence, we have the result when $\alpha \in \mathcal{Z}_j$. $\square$

By Lemma 2.8, we have the following lemma;

Lemma 2.9. *i' is surjective.*

Proof. Let ω be an element in B_4. Then we may assume $\omega = \sigma_{i_1}{}^{j_1} \cdots \sigma_{i_k}{}^{j_k}$, where $i_l \in \{1,2,3\}$ and j_l is a non-zero integer for $l = 1,\dots,k$ and with $i_{l_1} \neq i_{l_2}$ for $|l_1 - l_2| = 1$. Since $id = a_0^0 \in \mathcal{X}$, we see by Lemma 2.8 that there exists an element $\beta_1 \in \mathcal{X}$ such that $\omega = id \cdot \sigma_{i_1}{}^{j_1} \cdots \sigma_{i_k}{}^{j_k} \equiv \beta_1\sigma_{i_2}{}^{j_2} \cdots \sigma_{i_k}{}^{j_k}$. Applying Lemma 2.8 repeatedly, we can inductively find elements $\beta_1, \beta_2, \dots, \beta_l$ in $\mathcal{X}$ such that $\omega \equiv \beta_l\sigma_{i_{l+1}}{}^{j_{l+1}} \cdots \sigma_{i_k}{}^{j_k}$ for $l = 1,\dots,k$. Hence, we get $\omega \equiv \beta_k \in \mathcal{X}$ at the end. $\square$

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Recall that i and i' are injective and j is surjective. Since i' is surjective by Lemma 2.9, the maps i', $i = j \circ i'$ and j are bijective. The bijectivity of i implies that $\mathcal{X}$ is a complete system of representatives of $H(s_1,s_2,s_3,s_4)\backslash B_4$. Since $\mathcal{X}$ consists of 125 elements, we have $\#(H(s_1,s_2,s_3,s_4)\backslash B_4) = 125$. This completes the proof of Theorem 1.2(2). Moreover, the bijectivity of j implies $H(s_1,s_2,s_3,s_4) = \langle \mathcal{S} \rangle$. This completes the proof of Theorem 1.2 (1). $\square$

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