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Spin symplectic 4-manifolds near Bogomolov-Miyaoka-Yau line

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Abstract. We construct the first known family of simply connected spin symplectic 4-manifolds M_k satisfying $c_1^2(M_k)/\chi_h(M_k) \to 9$ as $k \to \infty$. Using M_k, we also construct many other new spin symplectic 4-manifolds with nonnegative signature.

1. Introduction

Throughout, let M be a closed oriented smooth 4-manifold. Let $e(M)$ and $\sigma(M)$ denote the Euler characteristic and the signature of M, respectively. We define

$$\chi_h(M) = \frac{e(M) + \sigma(M)}{4} \quad \text{and} \quad c_1^2(M) = 2e(M) + 3\sigma(M). \tag{1}$$

If M is a complex surface, then $\chi_h(M)$ and $c_1^2(M)$ are the holomorphic Euler characteristic and the square of the first Chern class $c_1(M)$, respectively. If M admits a symplectic structure, then $\chi_h(M) \in \mathbb{Z}$.

Given an ordered pair $(\chi, c) \in \mathbb{Z}\times\mathbb{Z}$, the "symplectic geography problem" asks whether there exists a minimal symplectic 4-manifold M with $(\chi_h(M), c_1^2(M)) = (\chi, c)$. This problem was first posed by McCarthy and Wolfson in [15], and much of the foundational work was done by Gompf in [8]. It is the symplectic analogue of the geography problem for complex surfaces of general type which was posed by Persson in [22].

To make the problem more manageable, we require our minimal symplectic 4-manifold M to be simply connected, which implies that $\chi_h(M) \geq 1$ and $c_1^2(M) \geq 0$ (cf. [25], [13]). We also require M to be spin, in which case

$$c_1^2(M) - 8\chi_h(M) = \sigma(M) \equiv 0 \pmod{16}$$

by the work of Rohlin (cf. [23]). A spin 4-manifold cannot contain any surface with odd self-intersection and hence a spin symplectic 4-manifold is automatically minimal.

In this paper we study the geography problem for simply connected spin symplectic 4-manifolds: Given an ordered pair of integers (χ, c) satisfying

$$\chi \geq 1, \quad c \geq 0, \quad c - 8\chi \equiv 0 \pmod{16},$$

is there a simply connected spin symplectic 4-manifold M with $(\chi_h(M), c_1^2(M)) = (\chi, c)$?

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Our problem has been answered affirmatively when $c < 8\chi$, i.e., when the signature is negative (cf. [20]). We also know that the answer is yes for most of the points in the region $8\chi \leq c \leq 8.76\chi$ (cf. [21], [1], [8]). As far as we know, nothing was known about the existence of simply connected spin symplectic 4-manifold in the region $c > 8.76\chi$ until now. The following definition will be useful when stating some of our results.

Definition 1.1. We say that a 4-manifold M has ∞^2-*property* if there exist infinitely many pairwise nondiffeomorphic irreducible symplectic 4-manifolds and infinitely many pairwise nondiffeomorphic irreducible nonsymplectic 4-manifolds, all of which are homeomorphic to M. We also say that a symmetric bilinear form has ∞^2-*property* if it is the intersection form of infinitely many pairwise nondiffeomorphic simply connected irreducible symplectic 4-manifolds and infinitely many pairwise nondiffeomorphic simply connected irreducible nonsymplectic 4-manifolds.

Let $\mathbb{N}$ denote the set of positive integers. Our first result is the following.

Theorem 1.2. *There exists an infinite family of closed simply connected spin symplectic 4-manifolds* $\{M_k \mid k \in \mathbb{N}\}$ *satisfying* $8.92 < c_1^2(M_k)/\chi_h(M_k) < 9$ *for every* $k \geq 2$, *and*

$$\lim_{k\to\infty} \frac{c_1^2(M_k)}{\chi_h(M_k)} = 9.$$

Moreover, M_k *has* ∞^2*-property for every* $k \geq 1$.

The proof of Theorem 1.2 will be given in Section 3. Corollary 5.3 in Section 5 gives a similar result when the fundamental group is any finitely presented group. The line $c = 9\chi$ is commonly called the Bogomolov-Miyaoka-Yau line because these mathematicians proved that $c_1^2(M) \leq 9\chi_h(M)$ when M is a complex surface of general type (cf. [2], [17], [28]). We recall that infinite families of simply connected *nonspin* symplectic 4-manifolds approaching the BMY line from below were constructed in [24] and [19]. The construction of M_k's in Theorem 1.2 is similar to Niepel's construction in [19], but delicate care must be taken in order to ensure that M_k's are *spin*.

There is no known example of a smooth complex projective surface M of general type with $\pi_1(M) = 1$ and slope $c_1^2(M)/\chi_h(M)$ in the interval $[852/97, 9)$ (cf. [26]). Note that by the work of Yau and Miyaoka (cf. [28], [18]), any smooth complex projective surface M of general type satisfying $c_1^2(M) = 9\chi_h(M)$ has as universal cover the complex ball, and so it cannot be simply connected. Currently the only known simply connected symplectic 4-manifold lying on the BMY line is the complex projective plane $\mathbb{CP}^2$. It also remains an open problem whether there exists any symplectic 4-manifold in the region $c > 9\chi$ strictly above the BMY line.

Let $b_i(M) = \dim_{\mathbb{R}} H_i(M; \mathbb{R})$ be the i-th Betti number of M. The famous 11/8 conjecture (cf. [14]) speculates that

$$b_2(M) \geq \frac{11}{8}|\sigma(M)|, \tag{2}$$

whenever M is a closed oriented smooth spin 4-manifold. When $\sigma(M) \geq 0$, (2) is equivalent to

$$c_1^2(M) \leq \frac{1}{19}(184\chi_h(M) + 16b_1(M) - 16) \approx 9.68\chi_h(M) + 0.84(b_1(M) - 1).$$

The 11/8 conjecture remains open, but there are some partial results. In [7], Furuta proved that either $b_2(M) = \sigma(M) = 0$ or

$$b_2(M) \geq \frac{10}{8}|\sigma(M)| + 2, \tag{3}$$

whenever M is a closed oriented smooth spin 4-manifold. When $\sigma(M) \geq 0$, (3) is equivalent to

$$c_1^2(M) \leq \frac{1}{9}(88\chi_h(M) + 8b_1(M) - 16) \approx 9.78\chi_h(M) + 0.89b_1(M) - 1.78.$$

Hence a closed spin symplectic 4-manifold cannot lie very far above the BMY line.

From now on, let M be a closed simply connected spin symplectic 4-manifold with $\sigma(M) \geq 0$. For such M, the pair $(\chi_h(M), c_1^2(M))$ determines the intersection form of M, and vice versa. According to the classification of symmetric bilinear integral forms that are indefinite and unimodular (cf. [16]), the intersection form of M is isomorphic to $pE_8 \oplus qH$, where

$$E_8 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

Moreover, p is a nonnegative even integer and q is a positive odd integer since

$$p = \frac{\sigma(M)}{8} \quad \text{and} \quad q = b_2^-(M) \equiv b_2^+(M) \pmod 2.$$

Here, $b_2^+(M)$ and $b_2^-(M)$ denote the dimensions of the maximal positive definite and the maximal negative definite subspaces of $H_2(M;\mathbb{Z})$, respectively, under the intersection form. Very often it is easier to express things in (p, q) coordinates rather than in (χ, c) coordinates. For example, (2) is equivalent to $q \geq \frac{3}{2}p$, while (3) is equivalent to $q \geq p+1$.

Given a nonnegative even integer p, it is now well known (cf. [1], [21]) that $pE_8 \oplus qH$ has ∞^2-property when the odd integer q is greater than some constant that depends only on p.

Definition 1.3. For an even integer $p \geq 0$, let Λ_p denote the smallest positive odd integer such that $pE_8 \oplus qH$ has ∞^2-property for every odd integer $q \geq \Lambda_p$.

In light of the 11/8 conjecture, it may not be too optimistic to make the following conjecture.

Conjecture 1.4. *Λ_p is the least positive odd integer that is greater than or equal to $\frac{3}{2}p$.*

Unfortunately, Conjecture 1.4, as well as the 11/8 conjecture, seems quite difficult to tackle with the existing techniques. Our next result gives a new asymptotic upper bound on Λ_p.

Corollary 1.5. *As $p \to \infty$, we have $\Lambda_p \leq 8p + O(p^{6/7})$.*

Corollary 1.5 follows immediately from Theorem 4.3 in Section 4. It is an improvement over the asymptotic upper bound $\Lambda_p \leq 20p + O(p^{2/3})$ that was obtained in [1].

2. Spin complex surfaces near BMY line

In this section, we study certain complex surfaces that we will use as building blocks in the construction of M_k in Theorem 1.2. For each integer $n \geq 2$, let X_n denote the minimal complex surface of general type constructed by Hirzebruch in [11] satisfying

$$e(X_n) = n^7 \quad \text{and} \quad \sigma(X_n) = \frac{1}{3}(n^7 - 4n^5).$$

In what follows, we briefly summarize the topological properties of X_n that we will need. We will use the notation in Section 1 of [11] whenever possible. If Σ is a collection of 2-dimensional surfaces in a 4-manifold, we let $[\Sigma]$ denote the 2-dimensional cohomology class that is Poincaré dual to the homology class of Σ.

Let $\zeta = e^{2\pi i/6}$ and let T be the elliptic curve $\mathbb{C}/\langle \mathbb{Z} \cdot 1 + \mathbb{Z}\zeta \rangle$. Consider the abelian surface $T \times T$. Denote the points in $T \times T$ by (z, w) and define four elliptic curves in $T \times T$:

$$T_0 = \{(z,w) \mid w = 0\}, \quad T_\infty = \{(z,w) \mid z = 0\},$$
$$T_1 = \{(z,w) \mid w = z\}, \quad T_\zeta = \{(z,w) \mid w = \zeta z\}.$$

These 2-tori intersect at $(0,0)$ and they do not intersect anywhere else. The abelian group $U_n = (\mathbb{Z}/n\mathbb{Z})^4$ acts on $T \times T \cong \mathbb{T}^4$. Let $Y_n = \mathbb{T}^4 \# n^4 \overline{\mathbb{CP}}^2$ be the blow-up of $T \times T$ at n^4 points in $U_n(0,0)$, the image of $(0,0)$ under the action of U_n. Let $I = \{0, \infty, 1, \zeta\}$, and for each $i \in I$, let $D_i = U_n(T_i)$, and let $\widetilde{D}_i$ be the proper transform of D_i under the blow-ups. Each $\widetilde{D}_i$ is the disjoint union of n^2 2-tori of self-intersection $-n^2$.

There exists a degree n^3 branched covering map $\pi : X_n \to Y_n$, where the branch locus of π is the disjoint union $\bigcup_{i \in I} \widetilde{D}_i$. Inside $H^2(Y_n; \mathbb{Z})$, we have

$$\sum_{i \in I} [\widetilde{D}_i] = n^2 \sum_{i \in I} [T_i'] - 4 \sum_{j \in U_n} [L_j],$$

where L_j denotes the exceptional curve of the j-th blow-up and T_i' denotes the total transform of T_i under the blow-ups.

Lemma 2.1. *If $n \equiv 4 \pmod 8$, then X_n is spin.*

Proof. It was shown in [11] that the canonical class $\bar{K}$ of X_n is given by

$$\bar{K} = \sum_{j\in U_n}[\bar{L}_j] + (n-1)\sum_{i\in I}[\bar{D}_i], \tag{5}$$

where $[\bar{L}_j] = \pi^*[L_j]$ and $[\bar{D}_i] = \frac{1}{n}\pi^*[\widetilde{D}_i]$. If $n = 4m$, we can rewrite (5) as

$$\begin{aligned}\bar{K} &= (3m-1)\sum_{i\in I}[\bar{D}_i] + \sum_{j\in U_n}[\bar{L}_j] + m\sum_{i\in I}[\bar{D}_i] \\ &= (3m-1)\sum_{i\in I}[\bar{D}_i] + \frac{1}{4}\left(4\sum_{j\in U_n}[\bar{L}_j] + n\sum_{i\in I}[\bar{D}_i]\right) \\ &= (3m-1)\sum_{i\in I}[\bar{D}_i] + \frac{1}{4}\pi^*\left(4\sum_{j\in U_n}[L_j] + \sum_{i\in I}[\widetilde{D}_i]\right) \\ &= (3m-1)\sum_{i\in I}[\bar{D}_i] + \frac{1}{4}\pi^*\left(n^2\sum_{i\in I}[T_i']\right).\end{aligned}$$

If $n = 4m$ with an odd integer $m > 0$, then $w_2(X_n) \equiv \bar{K} \equiv 0 \pmod 2$. □

The following useful lemma was proved by Niepel in [19].

Lemma 2.2. (Niepel) *Each X_n contains an embedded symplectic surface F_n of genus $g(F_n) = 3n^5 - 3n^4 + n^3 + 1$ and self-intersection $2n^3$. Moreover, the inclusion induced homomorphism $\pi_1(F_n) \to \pi_1(X_n)$ is surjective.* □

Remark 2.3. Let $\delta : Y_n \to T \times T$ be the blow-down map, and let z_0 and w_0 be generic points in T. According to [19], we can take F_n to be the surface obtained from the union of preimages

$$(\delta\circ\pi)^{-1}(\{z_0\}\times T) \quad \text{and} \quad (\delta\circ\pi)^{-1}(T\times\{w_0\})$$

by symplectically resolving the n^3 intersection points between them. Since X_n is a projective algebraic surface, we can choose a Kähler form on X_n. F_n is a symplectic submanifold of X_n with respect to the chosen Kähler form. We note that each X_n is also smoothly minimal, i.e., X_n does not contain any smoothly embedded 2-sphere with self-intersection -1 (cf. [6]).

3. Construction of M_k

Let $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}}^2$, a rational elliptic surface that is the complex projective plane blown up nine times. For a positive integer r, let $E(r)$ denote the fiber sum of r copies of $E(1)$. Then $E(r)$ is a simply connected elliptic surface without multiple fibers such that $\chi_h(E(r)) = r$ and $c_1^2(E(r)) = 0$. Let F be a smooth torus fiber of $E(r)$.

Let K be a knot of genus $g(K)$ in S^3. Let $E(r)_K$ denote the homotopy elliptic surface that was constructed by Fintushel and Stern in [3]. Recall that $E(r)_K$ is built by performing a knot surgery on $E(r)$:

$$E(r)_K = \big[E(r)\setminus\nu(F)\big] \cup \big[S^1\times(S^3\setminus\nu(K))\big], \tag{6}$$

where the ν's denote tubular neighborhoods. In (6), we glue the 3-torus boundaries in such a way that the meridians of F get identified with the longitudes of K. Note that $E(r)_K$ is homeomorphic to $E(r)$ so we have $\pi_1(E(r)_K) = 1$,

$$e(E(r)_K) = e(E(r)) = 12r \quad \text{and} \quad \sigma(E(r)_K) = \sigma(E(r)) = -8r.$$

If r is even, then $E(r)$ and $E(r)_K$ are all spin. If K is a fibered knot, then $E(r)_K$ admits a symplectic structure. A sphere section of $E(r)$ and a Seifert surface of K give rise to a symplectic submanifold Σ_K of genus $g(K)$ and self-intersection $-r$ inside $E(r)_K$.

Lemma 3.1. *Let $\nu(\Sigma_K)$ be a tubular neighborhood of Σ_K inside $E(r)_K$. Then the complement $E(r)_K \setminus \nu(\Sigma_K)$ is simply connected.*

Proof. The surface Σ_K transversely intersects once a topological sphere in $E(r)_K$ coming from a cusp fiber of $E(r)$. Thus any meridian of Σ_K is nullhomotopic in $E(r)_K \setminus \nu(\Sigma_K)$. Hence we conclude that $\pi_1(E(r)_K \setminus \nu(\Sigma_K)) = \pi_1(E(r)_K) = 1$. □

From now on, we assume that $r = 2n^3$ with $n \geq 2$. Let K be a fibered knot of genus $g(K) = 3n^5 - 3n^4 + n^3 + 1$. Consider the symplectic pairs (X_n, F_n) and $(E(2n^3)_K, \Sigma_K)$. Note that the genus of F_n and Σ_K are both equal to $3n^5 - 3n^4 + n^3 + 1$. Also, we have $[F_n]^2 = -[\Sigma_K]^2 = 2n^3$. We define

$$Z_n = X_n \#_{F_n = \Sigma_K} E(2n^3)_K = \left[X_n \setminus \nu(F_n)\right] \cup \left[E(2n^3)_K \setminus \nu(\Sigma_K)\right]$$

to be a symplectic normal sum (cf. [8, 15]) of X_n and $E(2n^3)_K$ along symplectic submanifolds F_n and Σ_K.

Theorem 3.2. *For every integer $n \geq 2$, the symplectic 4-manifold Z_n is simply connected, irreducible, and satisfies*

$$\chi_h(Z_n) = \frac{1}{3}(n^7 + 8n^5) - 3n^4 + 3n^3,$$
$$c_1^2(Z_n) = 3n^7 + 20n^5 - 24n^4 + 8n^3.$$

Z_n has ∞^2-property (cf. Definition 1.1) for every $n \geq 2$, and Z_n is spin if $n \equiv 4 \pmod 8$. For every $n \geq 2$, Z_n contains $2n^3$ disjoint symplectic torus submanifolds T_j ($j = 1, \ldots, 2n^3$) of self-intersection 0 that represent $2n^3$ linearly independent homology classes and satisfy $\pi_1(Z_n \setminus \nu(\bigcup_{j=1}^{2n^3} T_j)) = 1$.

Proof. Since $\pi_1(E(2n^3)_K \setminus \nu(\Sigma_K)) = 1$ by Lemma 3.1, Seifert-Van Kampen theorem implies that

$$\pi_1(Z_n) = \frac{\pi_1(X_n \setminus \nu(F_n))}{\langle \pi_1(\partial\nu(F_n)) \rangle}, \tag{7}$$

where $\partial\nu(F_n)$ is the boundary of the tubular neighborhood $\nu(F_n)$ and $\langle \pi_1(\partial\nu(F_n)) \rangle$ is the normal subgroup of $\pi_1(X_n \setminus \nu(F_n))$ generated by the image of $\pi_1(\partial\nu(F_n))$ under the inclusion induced homomorphism. Note that $\partial\nu(F_n)$ is a circle bundle over F_n with Euler number $2n^3$. It is well known (cf. Proposition 10.4 in [4]) that

$$\pi_1(\partial\nu(F_n)) = \langle \alpha_i, \beta_i, \mu \mid \textstyle\prod_{i=1}^{g(F_n)} [\alpha_i, \beta_i] = \mu^{2n^3},\ \alpha_i \mu \alpha_i^{-1} = \mu,\ \beta_i \mu \beta_i^{-1} = \mu \rangle,$$

where the index i ranges over $1, \ldots, g(F_n)$. Here, μ is represented by a fiber circle which is a meridian of F_n, and α_i, β_i are the parallel push-offs of the standard generators of $\pi_1(F_n)$. In (7), we have $\mu = 1$ in the quotient group since $\mu \in \langle \pi_1(\partial\nu(F_n))\rangle$. Thus we can write

$$\pi_1(Z_n) = \frac{\pi_1(X_n)}{\langle \pi_1(F_n)\rangle}.$$

Since $\pi_1(F_n) \to \pi_1(X_n)$ is surjective by Lemma 2.2, we conclude that $\pi_1(Z_n) = 1$.

Next we compute that

$$e(Z_n) = e(X_n) + e(E(2n^3)_K) - 2e(F_n) = n^7 + 12n^5 - 12n^4 + 28n^3,$$

$$\sigma(Z_n) = \sigma(X_n) + \sigma(E(2n^3)_K) = \frac{1}{3}(n^7 - 4n^5) - 16n^3.$$

From (1), we can now compute $\chi_h(Z_n)$ and $c_1^2(Z_n)$. The symplectic normal sum Z_n is minimal by Usher's theorem in [27]. Any simply connected minimal symplectic 4-manifold is irreducible (cf. [12], [10]). If $n \equiv 4 \pmod 8$, then X_n and $E(2n^3)_K$ are both spin, and hence Z_n can also be given a spin structure according to Proposition 1.2 in [8].

Now recall from [9] that $E(2)$ contains three disjoint copies of Gompf nucleus. Since $E(2n^3)$ can be viewed as the fiber sum of n^3 copies of $E(2)$, the homotopy elliptic surface $E(2n^3)_K$ contains $2n^3$ nuclei of $E(2)$ that are all disjoint from $\nu(\Sigma_K)$. Let N_j ($j = 1, \ldots, 2n^3$) denote these nuclei that are contained in $[E(2n^3)_K \setminus \nu(\Sigma_K)] \subset Z_n$, and let T_j be a smooth torus fiber in N_j. By changing the symplectic form on the corresponding $E(2)$ part if necessary, we can always arrange every T_j to be a symplectic submanifold of Z_n. Since T_j transversely intersects once a sphere section of N_j with self-intersection -2, every meridian of T_j is nullhomotopic. It follows that $\pi_1(Z_n \setminus \nu(\bigcup_{j=1}^{2n^3} T_j)) = \pi_1(Z_n) = 1$.

To obtain infinite families of pairwise nondiffeomorphic 4-manifolds that are homeomorphic to Z_n, we apply Fintushel-Stern knot surgery to Z_n. By performing a second knot surgery in one of the above nuclei, say N_1, with a knot K', we obtain an irreducible 4-manifold $Z_{n,K'}$ that is homeomorphic to Z_n. By varying our choice of the second knot K', we can realize infinitely many pairwise nondiffeomorphic 4-manifolds, either symplectic or nonsymplectic. $\square$

Now we can prove Theorem 1.2 in the introduction. For each integer $k \geq 1$, let $M_k = Z_{8k-4}$. Then M_k is a simply connected spin symplectic 4-manifold having the ∞^2-property such that

$$\chi_h(M_k) = \frac{64}{3}(2k-1)^3(4096k^4 - 8192k^3 + 6656k^2 - 2632k + 429),$$
$$c_1^2(M_k) = 512(2k-1)^3(1536k^4 - 3072k^3 + 2464k^2 - 952k + 149).$$

We note that $c_1^2(M_k)/\chi_h(M_k)$ is an increasing function of k when $k \geq 1$. The first two values are

$$\frac{c_1^2(M_1)}{\chi_h(M_1)} = \frac{1000}{119} \approx 8.403, \quad \frac{c_1^2(M_2)}{\chi_h(M_2)} = \frac{64808}{7263} \approx 8.923,$$

and $c_1^2(M_k)/\chi_h(M_k) \to 9$ as $k \to \infty$.

4. Geography of spin symplectic 4-manifolds

We start this section by stating the following theorem which was proved by the first two authors.

Theorem 4.1. (cf. [1]) *Let Z be a closed spin symplectic 4-manifold that contains a symplectic torus T of self-intersection 0. Let $\nu(T)$ be a tubular neighborhood of T and $\partial\nu(T)$ its boundary. Suppose that the homomorphism $\pi_1(\partial\nu(T)) \to \pi_1(Z \setminus \nu(T))$ induced by the inclusion is trivial. Then for any pair of integers (χ, c) satisfying*

$$\chi \geq 1, \quad 0 \leq c \leq 8\chi \quad \textit{and} \quad c - 8\chi \equiv 0 \pmod{16}, \tag{8}$$

there exists a closed spin symplectic 4-manifold Y with $\pi_1(Y) = \pi_1(Z)$,

$$\chi_h(Y) = \chi_h(Z) + \chi \quad \textit{and} \quad c_1^2(Y) = c_1^2(Z) + c.$$

Remark 4.2. Note that the conclusion of Theorem 4.1 holds trivially when $\chi = c = 0$ by setting $Y = Z$. Thus we can drop $\chi \geq 1$ condition in (8) since $0 \leq c \leq 8\chi$ already implies that $\chi \geq 0$ with $\chi = 0$ only when $c = 0$.

The main result of this section is the following technical theorem, which gives new upper bounds on Λ_p (cf. Definition 1.3) that are fairly easy to compute.

Theorem 4.3. *Let $p \geq 0$ be an even integer. If k is any positive integer satisfying*

$$p \leq \frac{128}{3}(2k-1)^3(256k^4 - 512k^3 + 368k^2 - 112k + 9), \tag{9}$$

then we have

$$\Lambda_p \leq -10p - 1 + 128(2k-1)^3(1536k^4 - 3072k^3 + 2464k^2 - 952k + 149). \tag{10}$$

Proof. We apply Theorem 4.1 to the pair $(Z, T) = (M_k, T_2)$, where T_2 is a torus fiber of Gompf nucleus N_2 in $M_k = Z_{8k-4}$ as in the proof of Theorem 3.2. For each (χ, c) satisfying (8), we obtain a simply connected spin symplectic 4-manifold Y with

$$e(Y) = e(M_k) + 12\chi - c \quad \text{and} \quad \sigma(Y) = \sigma(M_k) + c - 8\chi,$$

where

$$e(M_k) = 256(2k-1)^3(1024k^4 - 2048k^3 + 1728k^2 - 728k + 131),$$

$$\sigma(M_k) = \frac{1024}{3}(2k-1)^3(256k^4 - 512k^3 + 368k^2 - 112k + 9).$$

The intersection form of Y is given by $pE_8 \oplus qH$, where

$$p = \frac{1}{8}\sigma(Y) = \frac{1}{8}(\sigma(M_k) + c - 8\chi) \equiv 0 \pmod{2},$$

$$q = b_2^-(Y) = \frac{1}{2}(e(M_k) - \sigma(M_k)) + 10\chi - c - 1 \equiv 1 \pmod{2}.$$

Since $c - 8\chi \le 0$ in (8), we must have $p \le \frac{1}{8}\sigma(M_k)$, which is exactly (9). Solving for (χ, c) in terms of p, q and k, we get

$$\chi = \frac{1}{2}(8p + q + 1) - \chi_h(M_k),$$
$$c = 40p + 4q + 4 - c_1^2(M_k).$$

Since $c \ge 0$ in (8), we must have $q \ge -10p - 1 + \frac{1}{4}c_1^2(M_k)$.

Now recall from [1] that Y is the symplectic normal sum of (M_k, T_2) and a suitable symplectic pair. Since we can perform a knot surgery in the Gompf nucleus

$$N_1 \subset [E(2n^3)_K \setminus (\nu(\Sigma_K) \cup N_2)] \subset [M_k \setminus \nu(T_2)] \subset Y,$$

Y has the ∞^2-property just like M_k. Therefore, for every even integer p satisfying $0 \le p \le \frac{1}{8}\sigma(M_k)$, we have $\Lambda_p \le -10p - 1 + \frac{1}{4}c_1^2(M_k)$, which is exactly (10). □

For example, (9) is satisfied when $p = 0$ and $k = 1$. Thus Theorem 4.3 implies that $\Lambda_0 \le 15999$. By Freedman's classification theorem in [5], any closed simply connected 4-manifold with intersection form qH (see (4)) must be homeomorphic to $q(S^2 \times S^2)$, the connected sum of q copies of $S^2 \times S^2$. Hence we have proved the following.

Corollary 4.4. *$q(S^2 \times S^2)$ has ∞^2-property for every odd integer $q \ge 15999$.*

It has been proved in [1] that $q(S^2 \times S^2)$ has ∞^2-property for every odd integer $q \ge 275$, so Corollary 4.4 is not new. Indeed, the upper bound for Λ_p in [1] is stronger than (10) for very small values of p. However, (10) is strictly stronger than the upper bound in [1] for large values of p, as was observed in Corollary 1.5 in the introduction.

Remark 4.5. We note that the simply connected spin symplectic Y's constructed in the proof of Theorem 4.3 populate an infinite number of wedge-like regions above the line $c_1^2 = 8.76\chi_h$ that were not covered in [21].

5. Arbitrary fundamental group

Throughout, let G be a finitely presented group. The following theorem was proved by Gompf in [8].

Theorem 5.1. (Gompf) *Let G be a finitely presented group. Then there exists a spin symplectic 4-manifold S^G with $\pi_1(S^G) = G$, $c_1^2(S^G) = 0$, and $\chi_h(S^G) > 0$. Moreover, S^G contains a symplectic torus T of self-intersection 0 such that the inclusion induced homomorphism $\pi_1(T) \to \pi_1(S^G)$ is trivial.* □

Remark 5.2. As in the proof of Theorem 4.1 in [8], we let S^G be the symplectic normal sum of $\Sigma_g \times \mathbb{T}^2$ and ℓ copies of the K3 surface $E(2)$ along suitably chosen self-intersection 0 tori. Here, Σ_g is a closed genus g Riemann surface. The positive integers g and ℓ cannot be expressed in simple formulas, but they can be algorithmically determined from the presentation of G. We note that

$$e(S^G) = 24\ell, \quad \sigma(S^G) = -16\ell, \quad \chi_h(S^G) = 2\ell.$$

If $\pi_1(S^G) = G$ is residually finite, then S^G is irreducible as well (cf. [12], [10]).

By combining Theorems 1.2 and 5.1, we can now prove the following.

Corollary 5.3. *Let G be any finitely presented group. Then there exists an infinite family of closed spin symplectic 4-manifolds $\{M_k^G \mid k \in \mathbb{N}\}$ such that $\pi_1(M_k^G) = G$ and $0 < c_1^2(M_k^G)/\chi_h(M_k^G) < 9$ for every $k \in \mathbb{N}$, and*

$$\lim_{k\to\infty} \frac{c_1^2(M_k^G)}{\chi_h(M_k^G)} = 9.$$

If G is residually finite, then M_k^G is irreducible for every $k \in \mathbb{N}$. Let

$$\tau_k = 128(2k-1)^3 - 1 \quad \textit{and} \quad J = \{1, 2, 3, \dots, \tau_k\}.$$

Then, for every $k \in \mathbb{N}$, M_k^G contains τ_k disjoint symplectic torus submanifolds T_j $(j \in J)$ of self-intersection 0 that represent τ_k linearly independent homology classes. Moreover, the inclusion induced homomorphism

$$\pi_1(\partial\nu(T_j)) \to \pi_1(M_k^G \setminus \nu(T_j)) \tag{11}$$

is trivial for every $j \in J$, and

$$\pi_1(M_k^G \setminus \nu(\textstyle\bigcup_{j\in J} T_j)) = G. \tag{12}$$

Proof. Recall from Theorem 3.2 that $M_k = Z_{8k-4}$ contains $2(8k-4)^3 = \tau_k + 1$ disjoint symplectic torus submanifolds T_j $(j = 1, \dots, \tau_k + 1)$ of self-intersection 0 that represent $\tau_k + 1$ linearly independent homology classes and satisfy

$$\pi_1(M_k \setminus \nu(\textstyle\bigcup_{j=1}^{\tau_k+1} T_j)) = 1. \tag{13}$$

Let M_k^G be the symplectic normal sum of the pairs (M_k, T_{τ_k+1}) and (S^G, T):

$$M_k^G = M_k \#_{T_{\tau_k+1}=T} S^G = \big[M_k \setminus \nu(T_{\tau_k+1})\big] \cup \big[S^G \setminus \nu(T)\big]. \tag{14}$$

Since M_k and S^G are both spin, M_k^G can be given a spin structure.

The unused tori T_j $(j = 1, \dots, \tau_k)$ in $[M_k \setminus \nu(T_{\tau_k+1})]$ descend to disjoint symplectic torus submanifolds of self-intersection 0 in M_k^G that still represent τ_k linearly independent homology classes. For every $j = 1, \dots, \tau_k$, (11) can be factored as the composition of two inclusion induced homomorphisms

$$\pi_1(\partial\nu(T_j)) \to \pi_1(M_k \setminus \nu(\textstyle\bigcup_{j=1}^{\tau_k+1} T_j)) \to \pi_1(M_k^G \setminus \nu(T_j)).$$

This composition is trivial since the middle group is trivial by (13).

It follows from (13) that $\pi_1(M_k \setminus \nu(T_{\tau_k+1})) = 1$. From Theorem 5.1, we deduce that $\pi_1(S^G)/\langle \pi_1(T)\rangle = \pi_1(S^G)$. Applying Seifert-Van Kampen theorem to (14), we conclude that

$$\pi_1(M_k^G) = \frac{\pi_1(S^G \setminus \nu(T))}{\langle \pi_1(\partial\nu(T))\rangle} = \frac{\pi_1(S^G)}{\langle \pi_1(T)\rangle} = \pi_1(S^G) = G.$$

To prove (12), we observe that

$$M_k^G \setminus \nu(\textstyle\bigcup_{j\in J} T_j) = \big[M_k \setminus \nu(\bigcup_{j=1}^{\tau_k+1} T_j)\big] \cup \big[S^G \setminus \nu(T)\big].$$

Seifert-Van Kampen theorem and (13) imply that

$$\pi_1(M_k^G \setminus \nu(\bigcup_{j\in J} T_j)) = \frac{\pi_1(S^G \setminus \nu(T))}{\langle \pi_1(\partial\nu(T))\rangle} = G.$$

If $\pi_1(M_k^G) = G$ is residually finite, then the minimality of M_k^G implies irreducibility (cf. [12], [10]). We compute that

$$\chi_h(M_k^G) = \chi_h(M_k) + \chi_h(S^G) = \chi_h(M_k) + 2\ell,$$
$$c_1^2(M_k^G) = c_1^2(M_k) + c_1^2(S^G) = c_1^2(M_k).$$

Since $\chi_h(S^G) = 2\ell > 0$, $\chi_h(M_k) > 0$ and $c_1^2(M_k) > 0$, we have

$$0 < \frac{c_1^2(M_k^G)}{\chi_h(M_k^G)} < \frac{c_1^2(M_k)}{\chi_h(M_k)} < 9$$

for every $k \in \mathbb{N}$. Since $\chi_h(S^G) = 2\ell$ does not depend on k, we have

$$\lim_{k\to\infty} \frac{c_1^2(M_k^G)}{\chi_h(M_k^G)} = \lim_{k\to\infty} \frac{c_1^2(M_k)}{\chi_h(M_k) + 2\ell} = \lim_{k\to\infty} \frac{c_1^2(M_k)}{\chi_h(M_k)} = 9. \qquad \square$$

Finally, we note that we can apply Theorem 4.1 to the symplectic pair (M_k^G, T_2) and obtain closed spin symplectic 4-manifolds with fundamental group G that fill in all (χ, c) points satisfying $c - 8\chi \equiv 0 \pmod{16}$ and

$$c_1^2(M_k^G) \le c \le c_1^2(M_k^G) - 8\chi_h(M_k^G) + 8\chi$$

for some $k \in \mathbb{N}$. If G is residually finite, then all these manifolds will be irreducible.

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