

## Differential characters as stacks and prequantization

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ABSTRACT. We generalize geometric prequantization of symplectic manifolds to differentiable stacks. Our approach is atlas-independent and provides a bijection between isomorphism classes of principal  $S^1$ -bundles (with or without connections) and second cohomology groups of certain chain complexes.

### 1. Introduction.

#### 1.1. Quantization and reduction.

Our original motivation for this project has to do with the “quantization commutes with reduction” principle in symplectic geometry. Let us explain the issues involved on a simple example.

Consider the complex irreducible representations of  $SU(2)$ . There is one in every dimension:  $V_0, V_1, \dots$ ;  $\dim_{\mathbb{C}} V_n = n$ . The multiplicity of the zero weight of a maximal torus  $T \simeq S^1$  of  $SU(2)$  in  $V_n$  is

$$\dim(V_n)^T = \begin{cases} 0 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

The representation  $V_n$  can be constructed by quantizing the complex projective plane  $\mathbb{C}P^1$ : Start with the pair  $(\mathbb{C}P^1, n\omega)$ , where  $\omega$  is the  $SU(2)$  invariant area form on  $\mathbb{C}P^1$  normalized so that  $\int_{\mathbb{C}P^1} \omega = 1$ . The action of  $SU(2)$  on the Kähler manifold  $(\mathbb{C}P^1, n\omega)$  is Hamiltonian with an equivariant moment map  $\mu_n : \mathbb{C}P^1 \rightarrow \mathfrak{su}(2)^*$ . The Kähler form  $n\omega$  determines a Hermitian holomorphic line bundle  $L_n \rightarrow \mathbb{C}P^1$  with a Hermitian connection. The connection and the moment map allow us to lift the action of  $\mathfrak{su}(2)$  on  $\mathbb{C}P^1$  to an action on  $L_n$ , which integrates to an action of  $SU(2)$ . The space of holomorphic sections  $H^0(\mathbb{C}P^1, L_n)$  is then a representation of  $SU(2)$  which happens to be  $V_n$ , and we may think of the Hilbert space  $H^0(\mathbb{C}P^1, L_n)$  as a quantization of  $(\mathbb{C}P^1, n\omega, \mu_n)$ . By the principle that “quantization commutes with reduction”

$$(V_n)^T = \text{Quant}(\mathbb{C}P^1 // T, (n\omega)_0),$$

where  $\mathbb{C}P^1 // T$  denotes the symplectic quotient  $\mu_n^{-1}(0)/T$  and  $(n\omega)_0$  the induced symplectic form on the quotient. It’s not hard to see that  $\mu_n^{-1}(0)$  is a single  $T$ -orbit, so the

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symplectic quotient  $\mathbb{C}P^1//T$  is a single point. The quantization of a point is a complex line bundle over the point, i.e.,  $\mathbb{C} \rightarrow \text{pt}$ . Clearly  $H^0(\text{pt}, \mathbb{C}) \simeq \mathbb{C}$ . This contradicts that  $(V_n)^T = 0$  for  $n$  even. What did we do wrong?

Several things. First of all the symplectic quotient  $\mathbb{C}P^1//T$  is not a point. Rather, it's a point with a trivial action of  $\mathbb{Z}/2$ . We think of it as a groupoid  $\mathbb{Z}/2 \rightrightarrows \text{pt}$ . A prequantization of this groupoid is a line bundle over  $\text{pt}$  with an action of  $\mathbb{Z}/2$  and a quantization is the space of  $\mathbb{Z}/2$ -invariant sections of this bundle. For  $n$  even, the restriction  $L_n|_{\mu_n^{-1}(0)}$  descends to a bundle over  $\text{pt}$  where  $\mathbb{Z}/2$  acts non-trivially on the fiber and so the space of invariant sections is zero. For  $n$  odd, the restriction  $L_n|_{\mu_n^{-1}(0)}$  descends to a bundle over  $\text{pt}$  with a trivial  $\mathbb{Z}/2$  action and so the space of invariant sections is  $\mathbb{C}$ . More abstractly, one can think of a point with  $\mathbb{Z}/2$ -action as (an atlas of) a stack  $[\text{pt}/(\mathbb{Z}/2)]$ . From this point of view the reduction procedure is just the restriction of  $\omega$  to  $\mu_n^{-1}(0)$ . Nothing happens on the quotient stage since  $[\mu_n^{-1}(0)/T] = [\text{pt}/(\mathbb{Z}/2)]$  as stacks.

Why are there two different ways of (pre)quantizing a point with a  $\mathbb{Z}/2$ -action? After all, there is only one symplectic form on the groupoid  $\mathbb{Z}/2 \rightrightarrows \text{pt}$  (or on  $T \rightrightarrows \mu_n^{-1}(0)$ ). This only looks puzzling if we think of prequantization as a construction

$$\text{integral symplectic form} \mapsto \text{Hermitian line bundle with connection.}$$

Such a point of view is rather misleading: the connection is not uniquely determined by its curvature. For example flat line bundles are classified by 1-dim representations of the fundamental group. Nor does the de Rham cohomology class of the symplectic form see all of the Chern class of the pre-quantum line bundle. The right input for the quantization procedure is the group  $DC_2^2$  of differential characters introduced by Cheeger-Simons [5] (the indices of the notation  $DC_2^2$  will be explained later). There are several definitions of differential characters. The simplest one says that a differential character of degree 2 (1 in Cheeger-Simons grading) is a pair  $(\omega, \chi)$ , where  $\omega$  is a differential 2-form and  $\chi : Z_1 \rightarrow \mathbb{R}/\mathbb{Z}$  is a character of the group of smooth singular 1-cycles. This pair should satisfy the following condition:

$$\chi(\partial S) \equiv \int_S \omega \pmod{\mathbb{Z}},$$

for any smooth singular 2-chain  $S$ . One should think of  $\omega$  as a symplectic form (provided it is non-degenerate). We use another definition of  $DC_2^2$  which identifies it with the second cohomology group of a certain complex  $DC_2^\bullet$  involving both differential forms and singular cochains (cf. 3.6). In any case the crucial fact about differential characters is that they classify isomorphism classes of Hermitian line bundles with connections on a manifold  $M$ . Namely given such a bundle we can put  $\omega$  to be the curvature of the connection and  $\chi$  to be its holonomy. This map is a bijection and we call the inverse map

$$DC_2^2(M) \rightarrow \{\text{isomorphism classes of Hermitian line bundles with connections on } M\}$$

the prequantization. The actual quantization involves a choice of polarization (for example a complex structure). Our goal is to understand in which sense the prequantization commutes with reduction. The present paper provides the first step in this direction

by explaining prequantization in the equivariant setting (i.e. on stacks). The reduction procedure is addressed in [12].

Before proceeding with details of our construction of equivariant prequantization, we would like (for technical reasons) to switch from the category of Hermitian line bundles to the equivalent category of principal  $S^1$ -bundles.

### 1.2. Prequantization as a functor.

Let  $\Gamma_0 \rightrightarrows \Gamma_1$  be a Lie groupoid (one can think, for example, of an action groupoid  $M \rightrightarrows G \times M$ ). An (equivariant) bundle on such a groupoid is a bundle on  $\Gamma_0$  together with an isomorphism between the two pull-backs of the bundle to  $\Gamma_1$ . This definition leads us to an observation that in order to understand equivariant prequantization we have to include morphisms (in particular isomorphisms) between bundles into our construction. So we promote the prequantization from a bijection

$$DC_2^2(M) \rightarrow \{\text{iso classes of Hermitian line bundles with connections on } M\}$$

to a functor (equivalence of categories)

$$\text{Preq}_M : \mathcal{DC}_2^2(M) \rightarrow \mathcal{DBS}^1(M) ,$$

where  $\mathcal{DBS}^1(M)$  is the category of principal  $S^1$  bundles with connections over a manifold  $M$  and  $\mathcal{DC}_2^2(M)$  is a category which has  $DC_2^2(M)$  (i.e. second cohomology group of the complex  $DC_2^\bullet(M)$ ) as the set of isomorphism classes of objects. It is easy to see what the category  $\mathcal{DC}_2^2(M)$  should be: its objects are closed 2-cochains in  $DC_2^\bullet(M)$  and the morphisms are 1-chains (see Section 3 for a precise definition). This is an example of a chain category, i.e. a category built from a complex of abelian groups.

Observe that both categories  $\mathcal{DC}_2^2(M)$  and  $\mathcal{DBS}^1(M)$  can be pulled back under smooth maps of manifolds (and in fact,  $\text{Preq}$  will intertwine the pull-back functors). Moreover (and this is crucial for our argument) either of these categories can be glued from its restrictions to open sets in a covering of  $M$ . In other words there exists a stack  $\mathcal{DC}_2^2$  over the category of manifolds with fiber over  $M$  being  $\mathcal{DC}_2^2(M)$ , and similar statement holds for  $\mathcal{DBS}^1(M)$ . Once we know that  $\mathcal{DBS}^1$  and  $\mathcal{DC}_2^2$  are stacks, the general abstract nonsense allows us to upgrade the functor  $\text{Preq}$  from manifolds to orbifolds, Lie groupoids, and in fact arbitrary stacks (see Prequantization Theorem). This solves the equivariant prequantization problem.

We should confess that we lifted the idea of applying chain categories to classification problems from a paper by Hopkins-Singer ([9]). However they don't describe descent properties of the categories involved and hence we could not quite fill-in the details of their proofs. One can consider the present paper as a set of exercises on some ideas of [9].

### 1.3. Chain stacks.

The stack  $\mathcal{DC}_2^2$  introduced above is an example of an Eilenberg-MacLane stack (cf. [18]), which means that isomorphism classes of objects of its fiber  $\mathcal{DC}_2^2(M)$  over a manifold  $M$  are in bijection with some kind of cohomology (in this case differential characters). More

familiar examples would be the stack  $\mathcal{H}^1(C^\bullet)$ , which computes the first cohomology of the singular cochain complex  $C^\bullet$ , or  $\mathcal{H}^1(\Omega^\bullet)$  computing the first de Rham cohomology. The importance of Eilenberg-MacLane stacks is that they allow one to define equivariant cohomology, or more generally, stack cohomology, in an intrinsic fashion (without reference to an atlas). For example, the first singular cohomology of an arbitrary stack  $\mathcal{W}$  is defined to be the group of isomorphism classes of objects of the functor category  $\text{Hom}(\mathcal{W}, \mathcal{H}^1(C^\bullet))$ . This definition produces correct answers for cohomology of manifolds and equivariant cohomology of Lie groupoids.

We would like to stress an important difference between what we call chain stacks and the general Eilenberg-MacLane stacks defined in abstract algebraic topology. A chain stack is a presheaf of categories explicitly constructed from a complex of presheaves. Being a stack is a condition on this presheaf of categories (and so on the original complex of presheaves). An abstract definition of an Eilenberg-MacLane stack involves stackification of a given presheaf of categories. Roughly, working with chain stacks is similar to working only with complexes of acyclic sheaves in sheaf cohomology.

Chain stacks provide a natural setup for proving classification theorems. By a classification theorem we mean a one-to-one correspondence between isomorphism classes of some kind of geometric objects on a manifold (for example, bundles with or without connections, gerbes, etc.) and some cohomology group of the manifold. Usually such theorems are proved using Čech-type argument and proofs become increasingly messy once one wants to deal with multiple covers or a group action. A better approach is to show that the stack of objects we want to classify is equivalent to a chain stack. Since stacks are local, one just has to consider the case of the manifold being a Euclidean space. Of course, the hard part is to invent the cohomology theory represented by a chain stack (see remarks after Theorem 4.2).

An observant reader would notice that cohomology groups in the above examples are all of low degrees. In fact higher cohomology is represented by higher stacks (cf. [15, 18]). A higher stack is a presheaf of higher categories, or more precisely, of simplicial sets, satisfying some descent condition. The results of the present paper generalize to higher gerbs and higher degree differential characters (cf. [13]). In fact the proofs become much cleaner when done in the language of simplicial sheaves. However we want the present paper to be accessible to wider audience and so we keep the exposition as elementary as possible (at the expense of reproving many things which are well-known and/or obvious in abstract algebraic topology/geometry)

#### 1.4. Past classification results.

Classification of principal  $S^1$ -bundles with or without connection has a long history going back to work of Weil [20], who showed that the first Chern class determines a line bundle up to an isomorphism. Kostant [10] described the prequantization map from differential 2-forms with integer periods to isomorphism classes of line bundles with Hermitian connections. He also quantified the failure of the prequantization map to be a bijection. There are two closely related modifications of prequantization which make it into

a bijection: Deligne cohomology (unpublished, see [3]) and Cheeger-Simons differential characters [5]. We use the latter in the form modified by Hopkins-Singer [9].

Recently there has been an effort to generalize classification theorems to equivariant setting. In the case of a group action Brylinski [4] proved a Weil-type theorem using good covers of the action groupoid, and then this approach was extended to Deligne cohomology by Gomi [7] (see also Lupercio-Urbe paper [14] for the case of a finite group actions). Prequantization in the sense of Kostant was generalized to arbitrary Lie groupoids by Behrend and Xu [2]. Note that in the groupoid case the prequantization map fails to be a bijection in a more complicated way than in the case of manifolds.

The main results of the present paper are Weil and prequantization theorems (4.1 and Prequantization Theorem, resp.) for general stacks (over  $\mathcal{M}an$ ). Our approach provides a new point of view on classification theorems, which is useful for two reasons: (1) the theory works for arbitrary stacks and is atlas-independent; (2) it provides a bijection between isomorphism classes of bundles (with or without connections) and cohomology groups of certain chain complexes (as opposed to sheaf cohomology groups).

### 1.5. Warning: connections on equivariant bundles.

One should be careful comparing various versions of equivariant classification theorems. For example, in the case of a Lie group  $G$  acting on a manifold  $M$  one can consider equivariant bundles with arbitrary,  $G$ -invariant, or  $G$ -basic connections. Recall that a connection is  $G$ -basic iff it is  $G$ -invariant and vanishes on vector fields generating the action of the Lie algebra of  $G$  on the total space of the bundle. In the present paper we only consider basic connections on our bundles. These are the connections which descend onto the quotient stack  $[M/G]$ , make “stacky” sense, are atlas independent, etc.. The cases of  $G$ -invariant and arbitrary connections on equivariant principal  $S^1$ -bundles are treated in [12].

### 1.6. Structure of the paper.

As mentioned above we wanted to keep the paper as elementary as possible. In particular, we reproduce many standard proofs from singular homology theory and theory of stacks. Our goal is to define cohomology stacks and to prove classification theorems for principal  $S^1$ -bundles with or without connections.

Section 2 contains a review of stacks with bundles being the main example.

Section 3 defines cohomology stacks for singular and de Rham cohomology, and differential characters.

Section 4 contains classification theorems for principal  $S^1$ -bundles.

## 2. Stacks of bundles.

This section contains a brief review of the basic theory of (differentiable) stacks. As examples we consider the stack of  $S^1$ -bundles and the stack of  $S^1$ -bundles with connections. We refer the reader to [2, 16, 17] for a detailed exposition. Laumon and Moret-Bailly’s

book [11] develops the theory of algebraic stacks translatable into the language of differential geometry.

### 2.1. Presheaves of groupoids.

We denote by  $\mathcal{M}an$  the category of differentiable manifolds (with smooth maps).

Recall that a groupoid is a category such that any morphism is invertible. A common notation for a groupoid is  $\Gamma_0 \rightleftarrows \Gamma_1$ , where  $\Gamma_0$  is the set of objects,  $\Gamma_1$  is the set of morphisms, and the two arrows represent source and target maps. A groupoid is differentiable (or a Lie groupoid) if  $\Gamma_0$  and  $\Gamma_1$  are manifolds and all structure maps (source, target, composition, inverse, identity) are smooth.

A (lax) presheaf of groupoids  $\mathcal{X}$  over  $\mathcal{M}an$  is a lax contravariant 2-functor from  $\mathcal{M}an$  to the 2-category of groupoids. This means we have a groupoid  $\mathcal{X}(M)$  for each manifold  $M$ , a functor  $f^* : \mathcal{X}(N) \rightarrow \mathcal{X}(M)$  for each smooth map  $f : M \rightarrow N$ , and a natural transformation  $f^* \circ g^* \cong (g \circ f)^*$  for each pair of composable smooth maps  $f$  and  $g$ .

For example, given a manifold  $M$ , one can define a presheaf of groupoids  $[M]$  as follows. The objects of  $[M]$  over a manifold  $N$  are smooth maps  $N \rightarrow M$  and the only morphisms are the identity ones. The pull-back functors are the usual pull-backs of smooth maps.

To avoid (lax) 2-functors one can, instead of presheaves of groupoids, consider an equivalent notion of categories fibered in groupoids over  $\mathcal{M}an$ . Given such a category, its fibers form a presheaf of groupoids, and conversely, one can build the total category from its fibers. See [16] for details.

Consider two presheaves  $\mathcal{X}$  and  $\mathcal{Y}$  of groupoids over  $\mathcal{M}an$ . The category  $\mathcal{H}om(\mathcal{X}, \mathcal{Y})$  of morphisms is defined as follows: the objects are maps  $\mathcal{X} \rightarrow \mathcal{Y}$  of presheaves and the morphisms are natural transformations. For example, 2-Yoneda Lemma says that, given a manifold  $M$  and an arbitrary presheaf of groupoids  $\mathcal{X}$ , the category  $\mathcal{H}om([M], \mathcal{X})$  is equivalent to  $\mathcal{X}(M)$ . The equivalence functor (on objects) is given by:

$$\text{Ob}(\mathcal{H}om([M], \mathcal{X})) \ni f \mapsto f(M \xrightarrow{\text{id}} M) \in \text{Ob}(\mathcal{X}(M)) .$$

We will see later generalizations of this equivalence with  $M$  replaced by a Lie groupoid or a stack.

In the present paper (presheaves of) groupoids play two roles. On one hand, geometric objects we are interested in, such as base and the total space of a bundle, covering of a manifold, etc., will be thought of as (Lie) groupoids, or (via 2-Yoneda Lemma) as presheaves of groupoids over  $\mathcal{M}an$ . On the other hand, categories of bundles, chains of various complexes, etc., naturally form presheaves of groupoids over  $\mathcal{M}an$ .

Some presheaves of groupoids are local (satisfy descent property) with respect to a topology on  $\mathcal{M}an$  – such presheaves are called stacks (see below for the precise definition). We use two pretopologies (in Grothendieck’s sense). One is given by open embeddings, the other by submersions. Since a submersion has local sections, these pretopologies define the same topology.

## 2.2. Example: $\mathcal{BS}^1$ .

A typical example of a presheaf of groupoids is provided by principal  $S^1$ -bundles. Namely, one defines a presheaf  $\mathcal{BS}^1$  of groupoids as follows:

- Given a manifold  $M$ , objects of  $\mathcal{BS}^1(M)$  are principal  $S^1$ -bundles  $P \rightarrow M$ . Abusing notation we often specify only the total space  $P$ .
- A morphism in  $\mathcal{BS}^1(M)$  from  $P$  to  $P'$  is a smooth map  $\phi : P \rightarrow P'$  such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\text{id}} & M \end{array}$$

We call such a  $\phi$  a bundle map.

- Given a smooth map  $f : M \rightarrow N$  the functor  $f^* : \mathcal{BS}^1(N) \rightarrow \mathcal{BS}^1(M)$  is the pull-back of  $S^1$ -bundles (note that pull-backs are only defined up to isomorphism).

Note that 2-Yoneda Lemma implies that the category  $\mathcal{BS}^1(M)$  of principal  $S^1$ -bundles on a manifold  $M$  is equivalent to the morphisms category  $\mathcal{H}om([M], \mathcal{BS}^1)$ . This is to be compared with the definition of the classifying space  $BS^1$  in topology:  $BS^1$  is a topological space such that the homotopy classes of maps  $M \rightarrow BS^1$  are in bijection with equivalence classes of  $S^1$ -bundles on  $M$ .

## 2.3. Equivariant objects of a presheaf.

Consider a Lie groupoid  $\Gamma_\bullet = \{\Gamma_0 \rightrightarrows \Gamma_1\}$ . The nerve of  $\Gamma_\bullet$  is the simplicial manifold

$$\Gamma_0 \begin{array}{c} \xleftarrow{\partial_1} \\ \xleftarrow{\partial_0} \end{array} \Gamma_1 \begin{array}{c} \xleftarrow{\partial_2} \\ \xleftarrow{\partial_0} \end{array} \Gamma_2 \begin{array}{c} \xleftarrow{\partial_3} \\ \xleftarrow{\partial_0} \end{array} \cdots$$

where

$$\Gamma_n = \underbrace{\Gamma_1 \times_{\Gamma_0} \Gamma_1 \times_{\Gamma_0} \cdots \times_{\Gamma_0} \Gamma_1}_n$$

and arrows are the canonical projections. We omit face maps (in particular, the identity map for the groupoid) in the above diagram. We also don't distinguish between a groupoid and the associated simplicial manifold and denote both by  $\Gamma_\bullet$ .

Now let  $\mathcal{X}$  be a presheaf of groupoids over  $\mathcal{M}an$ . We define a category  $\mathcal{X}(\Gamma_\bullet)$  of objects of  $\mathcal{X}$  on  $\Gamma_\bullet$  (or  $\Gamma_\bullet$ -equivariant objects of  $\mathcal{X}$ ) as follows:

- An object of  $\mathcal{X}(\Gamma_\bullet)$  is a pair  $(x, \phi)$ , where  $x$  is an object of  $\mathcal{X}(\Gamma_0)$  and  $\phi : \partial_0^* x \rightarrow \partial_1^* x$  is an isomorphism in  $\mathcal{X}(\Gamma_1)$  satisfying a cocycle condition on  $\mathcal{X}(\Gamma_2)$ .
- A morphism from  $(x, \phi)$  to  $(x', \phi')$  is a morphism  $\xi : x \rightarrow x'$  in  $\mathcal{X}(\Gamma_0)$  such that  $\phi' \circ \partial_0^* \xi = \partial_1^* \xi \circ \phi$  in  $\mathcal{X}(\Gamma_1)$ .

If the base groupoid is just a manifold  $M_\bullet = M \begin{smallmatrix} \xleftarrow{\text{id}} \\ \xrightarrow{\text{id}} \end{smallmatrix} M$  then the category  $\mathcal{X}(M_\bullet)$  is equivalent to the category  $\mathcal{X}(M)$ . Another important example is provided by an action of a Lie group  $G$  on a manifold  $M$ . The associated (action) groupoid is  $M \begin{smallmatrix} \xleftarrow{a} \\ \xrightarrow{p} \end{smallmatrix} G \times M$ , where the arrows  $a$  and  $p$  represent the action and the canonical projection respectively. The category  $\mathcal{X}(M \rightrightarrows G \times M)$  is the category of  $G$ -equivariant objects of  $\mathcal{X}(M)$ . For example  $\mathcal{BS}^1(M \rightrightarrows G \times M)$  is the category of  $G$ -equivariant  $S^1$ -bundles on  $M$ .

## 2.4. Descent.

We turn our attention to the topological structure of the site  $\mathcal{Man}$ . A presheaf of groupoids is a stack if it is local; in other words, if global objects are glued from local ones. To make it precise one uses equivariant objects for a covering groupoid.

Given a covering  $U \rightarrow M$  of a manifold  $M$ , we denote by  $U_\bullet$  the Lie groupoid  $U \times_M U \rightrightarrows U$ . The groupoid  $U_\bullet$  is called the covering groupoid (corresponding to the covering  $U \rightarrow M$ ). In the open subsets topology we have  $U = \bigsqcup_\alpha U_\alpha$ , where  $\{U_\alpha\}_\alpha$  is a covering of  $M$  by open subsets, and then  $U \times_M U = \bigsqcup_{\alpha, \beta} U_\alpha \cap U_\beta$ .

If  $\mathcal{X}$  is a presheaf of groupoids and  $U_\bullet$  is a covering groupoid, then the groupoid  $\mathcal{X}(U_\bullet)$  is called the descent data (of  $\mathcal{X}$  with respect to the covering  $U \rightarrow M$ ). The descent data is effective if it is equivalent to the groupoid  $\mathcal{X}(M)$  (more precisely, if the natural restriction functor  $\mathcal{X}(M) \rightarrow \mathcal{X}(U_\bullet)$  is an equivalence of categories). A presheaf of groupoids is a stack if its descent data is effective on any manifold with respect to any covering (it is enough to consider coverings by open subsets).

**Proposition 2.1.** *The presheaf  $\mathcal{BS}^1$  is a stack.*

*Proof.* Given a covering  $M = \bigcup U_\alpha$  of  $M$  by open subsets, an object of the descent data is a collection of bundles on the subsets  $U_\alpha$  together with bundle isomorphisms on the intersections  $U_\alpha \cap U_\beta$ . These isomorphisms satisfy cocycle conditions on triple intersections and therefore define an equivalence relation on the disjoint union of the total spaces. The equivalence classes of this relation form the total space of an  $S^1$ -bundle on  $M$ . It is easy to see that the gluing procedure is a functor quasi-inverse (inverse up to natural transformations) to the restriction functor.  $\square$

## 2.5. Differentiable stacks.

In the last several subsections we considered objects of a stack on manifolds and, more generally, on Lie groupoids. Now we want to explain what an object of a stack (say, an  $S^1$ -bundle) on another stack is. We start by recalling in what sense stacks are generalizations of Lie groupoids.

Given a Lie groupoid  $\Gamma_0 \rightrightarrows \Gamma_1$  (for example associated to an action  $M \rightrightarrows G \times M$ ), one can define the quotient stack  $[\Gamma_0/\Gamma_1]$  as the classifying stack of principal  $\Gamma_\bullet$ -bundles. Recall that a principal  $\Gamma_\bullet$ -bundle on a manifold  $M$  is a surjective submersion  $\pi : P \rightarrow M$  together with a right action of  $\Gamma_\bullet$  on  $P$  which commutes with the projection  $\pi$  and is free

and transitive on fibers. A right action of  $\Gamma_\bullet$  on  $P$  is given by an anchor map  $P \rightarrow \Gamma_0$  and an action map  $P \times_{\Gamma_0} \Gamma_1 \rightarrow P$  satisfying a set of axioms. We write the action map as  $(p, g) \mapsto pg$ . Since the action commutes with  $\pi$  (i.e.  $\pi(pg) = \pi(p)$ ), the assignment  $(p, g) \mapsto (p, pg)$  defines a map  $P \times_{\Gamma_0} \Gamma_1 \rightarrow P \times_M P$ . The action is free and transitive on fibers of  $\pi$  iff this map is a diffeomorphism.

The above discussion can be summarized by saying that a principal  $\Gamma_\bullet$ -bundle over  $M$  is a commutative diagram:

$$\begin{array}{ccccccc}
 M & \longleftarrow & P & \xleftarrow{\quad} & P \times_M P & \xleftarrow{\quad} & P \times_M P \times_M P \xleftarrow{\quad} \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \Gamma_0 & \xleftarrow{\quad} & \Gamma_1 & \xleftarrow{\quad} & \Gamma_2 \xleftarrow{\quad} \cdots
 \end{array} \tag{2.5.1}$$

Note that the top row is a covering groupoid for  $M$  (in the submersion topology). Also, locally on  $M$  there exists a section  $M \rightarrow \Gamma_0$  trivializing the bundle. One easily defines morphisms of  $\Gamma_\bullet$ -bundles and the pull-back functor with respect to smooth maps. Hence  $\Gamma_\bullet$ -bundles form a presheaf  $[\Gamma_0/\Gamma_1]$  of groupoids, and it is easy to check that  $[\Gamma_0/\Gamma_1]$  is a stack. For example,  $\mathcal{BS}^1 = [\text{pt}/S^1]$ .

A stack  $\mathcal{W}$  is differentiable if it is equivalent to a stack  $[\Gamma_0/\Gamma_1]$  for some groupoid  $\Gamma_\bullet$ . Equivalently,  $\mathcal{W}$  is differentiable if it has a smooth atlas (surjective representable morphism)  $[\Gamma_0] \rightarrow \mathcal{W}$ . Recall (cf. [8, 16]) that a morphism  $[X] \rightarrow \mathcal{V}$  from a manifold to a stack is surjective representable if for any morphism  $[Y] \rightarrow \mathcal{V}$  the stack  $[Y] \times_{\mathcal{V}} [X]$  is representable by a non-empty manifold. Given an atlas  $[\Gamma_0] \rightarrow \mathcal{W}$ , one defines  $\Gamma_1$  to be a representative of  $[\Gamma_0] \times_{\mathcal{W}} [\Gamma_0]$  (i.e.  $[\Gamma_1] \cong [\Gamma_0] \times_{\mathcal{W}} [\Gamma_0]$ ). Conversely, the trivial  $\Gamma_\bullet$ -bundle over  $\Gamma_0$  defines an atlas  $[\Gamma_0] \rightarrow [\Gamma_0/\Gamma_1]$ . Abusing terminology we will call the whole simplicial manifold  $\Gamma_\bullet$  the atlas.

The following standard proposition shows that morphisms of stacks are local (i.e. can be described using atlases).

**Proposition 2.2.** *If  $\mathcal{X}$  is an arbitrary stack and  $\mathcal{W}$  is a differentiable stack with an atlas  $\Gamma_\bullet$ , then the following three categories are equivalent to each other:  $\mathcal{H}om(\mathcal{W}, \mathcal{X})$ ,  $\mathcal{H}om([\Gamma_0/\Gamma_1], \mathcal{X})$ , and  $\mathcal{X}(\Gamma_\bullet)$ .*

*Proof.* The first two categories are obviously equivalent (since  $\mathcal{W}$  is equivalent to  $[\Gamma_0/\Gamma_1]$ ). It remains to be shown that they are equivalent to  $\mathcal{X}(\Gamma_\bullet)$ . We provide functors in both directions. First, precomposition with the atlas map  $[\Gamma_0] \rightarrow \mathcal{W}$  defines a functor from  $\mathcal{H}om(\mathcal{W}, \mathcal{X})$  to  $\mathcal{H}om([\Gamma_0], \mathcal{X}) \xrightarrow{2\text{-Yoneda}} \mathcal{X}(\Gamma_0)$  together with a natural transformation (satisfying a natural cocycle condition) between the two pull-backs of this functor to  $\mathcal{H}om([\Gamma_0] \times_{\mathcal{W}} [\Gamma_0], \mathcal{X}) \cong \mathcal{X}(\Gamma_1)$ . This data is the same as a functor from  $\mathcal{H}om(\mathcal{W}, \mathcal{X})$  to  $\mathcal{X}(\Gamma_\bullet)$ . Up to this point the argument works for any presheaf of categories  $\mathcal{X}$ ; however existence of a quasi-inverse functor  $\mathcal{X}(\Gamma_\bullet) \rightarrow \mathcal{H}om([\Gamma_0/\Gamma_1], \mathcal{X})$  requires the assumption that  $\mathcal{X}$  is a stack. Let us describe this functor on objects (extension to morphisms is clear). Given an object of  $\mathcal{X}(\Gamma_\bullet)$  and a  $\Gamma_\bullet$ -bundle over a manifold  $M$ , we want to produce an object of  $\mathcal{X}(M)$ . So we pull back  $\mathcal{X}(\Gamma_\bullet)$  to  $\mathcal{X}(P_\bullet)$  along the vertical projection

in (2.5.1). The result is a descent data object of  $\mathcal{X}$  for the covering groupoid  $P_\bullet$  of  $M$ , which determines an object of  $\mathcal{X}(M)$  since  $\mathcal{X}$  is a stack.  $\square$

The above proposition motivates the following definition, which extends the notion of an object of a stack on a manifold (or on a groupoid). Given two stacks  $\mathcal{X}$  and  $\mathcal{W}$  we define the category  $\mathcal{X}(\mathcal{W})$  of objects of  $\mathcal{X}$  on  $\mathcal{W}$  to be the category  $\mathcal{H}om(\mathcal{W}, \mathcal{X})$ . Note that up to an equivalence this category depends only on the equivalence classes of  $\mathcal{W}$  and  $\mathcal{X}$ . If  $\mathcal{W}$  is differentiable with an atlas  $\Gamma_\bullet$ , then the category  $\mathcal{X}(\mathcal{W})$  is equivalent to the category of  $\Gamma_\bullet$ -equivariant objects of  $\mathcal{X}$ . The new definition however is independent of the atlas (up to an equivalence).

## 2.6. Stackification and $\mathcal{BS}_{\text{triv}}^1$ .

Given a presheaf of groupoids  $\mathcal{X}_0$  over  $Man$ , there exists a unique (up to equivalence) stack  $\mathcal{X}$  together with a functor  $s : \mathcal{X}_0 \rightarrow \mathcal{X}$  satisfying the following universal property: precomposition with  $s$  is an equivalence of categories  $\mathcal{H}om(\mathcal{X}, \mathcal{Y}) \cong \mathcal{H}om(\mathcal{X}_0, \mathcal{Y})$  for any stack  $\mathcal{Y}$ . One can construct a stackification as the union of the descent data of  $\mathcal{X}_0$  with respect to all coverings up to equivalence corresponding to refinements of coverings.

As an example consider a full sub-presheaf  $\mathcal{BS}_{\text{triv}}^1$  of  $\mathcal{BS}^1$  consisting of trivial bundles, i.e. for a manifold  $M$  the only object of  $\mathcal{BS}_{\text{triv}}^1(M)$  is the trivial bundle  $M \times S^1$ . Recall that a sub-presheaf of groupoids is called full if the set of morphisms between any two objects is the same as in the original presheaf. In particular, the morphisms in  $\mathcal{BS}_{\text{triv}}^1(M)$  are functions  $M \rightarrow S^1$ . The presheaf  $\mathcal{BS}_{\text{triv}}^1$  is not a stack but we have the following

**Proposition 2.3.** *The stackification of  $\mathcal{BS}_{\text{triv}}^1$  is  $\mathcal{BS}^1$ .*

*Proof.* Any  $S^1$ -bundle is locally trivial.  $\square$

## 2.7. The classifying stack $\mathcal{DBS}^1$ .

Now we want to equip our bundles with connections. So let  $\mathcal{DBS}^1$  be the following presheaf of groupoids over  $Man$ :

- Given a manifold  $M$ , objects of  $\mathcal{DBS}^1(M)$  are principal  $S^1$ -bundles with connections over  $M$ , i.e. pairs  $(P, A)$ , where  $P \rightarrow M$  is a bundle and  $A \in \Omega^1(P)$  is a connection.
- Given two objects  $(P, A)$  and  $(P', A')$  of  $\mathcal{DBS}^1(M)$ , a morphism from  $(P, A)$  to  $(P', A')$  is a bundle map  $\phi : P \rightarrow P'$  such that  $\phi^* A' = A$ .
- Given a smooth map  $f : M \rightarrow N$  the functor  $f^* : \mathcal{DBS}^1(N) \rightarrow \mathcal{DBS}^1(M)$  is the pull-back of bundles with connections.

As with any presheaf of groupoids we can consider categories of equivariant objects of  $\mathcal{DBS}^1$ . For example, in the case of an action groupoid, the category  $\mathcal{DBS}^1(M \ltimes G \times M)$  is the category of  $G$ -equivariant principal  $S^1$ -bundles on  $M$  with basic connections (cf. subsection 1.5).

**Proposition 2.4.** *The presheaf  $\mathcal{DBS}^1$  is a stack.*

*Proof.* Similar to the case of  $\mathcal{BS}^1$ . Note that gluing isomorphisms preserve connections. Hence connections on local bundles determine a connection on the global bundle.  $\square$

### 2.8. The presheaf $\mathcal{DBS}_{\text{triv}}^1$ .

Similar to  $\mathcal{BS}_{\text{triv}}^1$ , we introduce a full presheaf  $\mathcal{DBS}_{\text{triv}}^1$  of  $\mathcal{DBS}^1$  consisting of trivial bundles with connections, that is, for a manifold  $M$ , objects of  $\mathcal{DBS}_{\text{triv}}^1(M)$  are pairs  $(M \times S^1, \alpha + d\theta)$ , where  $\alpha \in \Omega^1(M)$ . The morphisms in  $\mathcal{DBS}_{\text{triv}}^1$  are locally constant functions  $M \rightarrow S^1$ .

**Proposition 2.5.** *The stackification of  $\mathcal{DBS}_{\text{triv}}^1$  is  $\mathcal{DBS}^1$ .*

*Proof.* Any  $S^1$ -bundle is locally trivial and any connection on a trivial bundle is of the form  $\alpha + d\theta$ .  $\square$

## 3. Chain stacks.

In this section we introduce chain stacks associated to complexes of presheaves. In particular we define differential characters.

### 3.1. Categories built from complexes.

Let  $A^\bullet = \{A^\bullet \xrightarrow{d} A^{\bullet+1}\}$  be a complex of abelian groups. All complexes appearing in this paper are assumed to have  $A^n = 0$  for  $n < 0$ . We fix an integer  $n \geq 0$  and define a category  $\mathcal{H}^n(A^\bullet)$  as follows:

- Objects are  $n$ -cocycles:  $z \in A^n, dz = 0$ .
- Morphisms are  $(n-1)$ -cochains up to  $(n-2)$ -cochains. The set of morphisms from  $z$  to  $z'$  is

$$\{b \in A^{n-1} \mid db = z' - z\} / (b \sim b + dc, c \in A^{n-2}).$$

Composition of morphisms is the addition in  $A^{n-1}$ .

The map  $\mathcal{H}^n$  can be extended to a functor from the category of complexes (and chain maps) to the 2-category of categories: given a map  $f : A^\bullet \rightarrow B^\bullet$  we have a functor  $\mathcal{H}^n(f) : \mathcal{H}^n(A^\bullet) \rightarrow \mathcal{H}^n(B^\bullet)$  defined as follows:

- On objects:  $\mathcal{H}^n(f)(z) = f(z)$  for  $z \in A^n, dz = 0$ .
- On morphisms:  $\mathcal{H}^n(f)([b]) = [f(b)]$  for  $[b] \in A^{n-1}/dA^{n-2}$ .

Moreover, let  $k : A^\bullet \rightarrow B^{\bullet-1}$  be a chain homotopy between two chain maps  $f, g : A^\bullet \rightarrow B^\bullet$  (i.e.  $g - f = dk + kd$ ). Then we have a natural transformation  $\mathcal{H}^n(k) : \mathcal{H}^n(f) \rightarrow \mathcal{H}^n(g)$  given by  $(\mathcal{H}^n(k))(z) = [k(z)]$  for  $z \in A^n, dz = 0$ .

Let us make a few remarks about this construction:

- The category  $\mathcal{H}^n(A^\bullet)$  is a groupoid.
- The set of objects of  $\mathcal{H}^n(A^\bullet)$  is an abelian group.
- The set of isomorphism classes of objects of  $\mathcal{H}^n(A^\bullet)$  is the  $n^{\text{th}}$  cohomology group  $H^n(A^\bullet)$  of the complex  $A^\bullet$ .
- The automorphism group of any object of  $\mathcal{H}^n(A^\bullet)$  is  $H^{n-1}(A^\bullet)$ .

- The category  $\mathcal{H}^0(A^\bullet)$  is discrete (the only morphisms are identity ones): we assume our complexes to be trivial in negative degrees.

It follows that, if  $f : A^\bullet \rightarrow B^\bullet$  is a quasi-isomorphism (i.e. induces isomorphism in cohomology), then  $\mathcal{H}^n(f)$  is an equivalence of categories. The reason is that a functor from a groupoid to a groupoid is an equivalence if it preserves isomorphism classes of objects and their automorphism groups.

Now let  $F^\bullet$  be a complex of presheaves of abelian groups over  $\mathcal{Man}$ . This means that we have a complex  $F^\bullet(M)$  of abelian groups for each manifold  $M$  and a pull-back map  $F^\bullet(N) \rightarrow F^\bullet(M)$  for each smooth map  $M \rightarrow N$ . We can apply the functor  $\mathcal{H}$  to  $F^\bullet$  and obtain a presheaf  $\mathcal{H}(F^\bullet)$  of groupoids.

### 3.2. Equivariant cochains.

Given a Lie groupoid  $\Gamma_\bullet$  and a complex  $F^\bullet$  of presheaves of abelian groups we consider the double complex  $F^\bullet(\Gamma_\bullet)$  with the differentials  $d : F^\bullet(\Gamma_\bullet) \rightarrow F^{\bullet+1}(\Gamma_\bullet)$  and  $\delta : F^\bullet(\Gamma_\bullet) \rightarrow F^\bullet(\Gamma_{\bullet+1})$ :

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots \\
 d \uparrow & & d \uparrow & & -d \uparrow & & d \uparrow \\
 F^2 & \xrightarrow{\delta} & F^2(\Gamma_0) & \xrightarrow{\delta} & F^2(\Gamma_1) & \xrightarrow{\delta} & F^2(\Gamma_2) \xrightarrow{\delta} \dots \\
 d \uparrow & & d \uparrow & & -d \uparrow & & d \uparrow \\
 F^1 & \xrightarrow{\delta} & F^1(\Gamma_0) & \xrightarrow{\delta} & F^1(\Gamma_1) & \xrightarrow{\delta} & F^1(\Gamma_2) \xrightarrow{\delta} \dots \\
 d \uparrow & & d \uparrow & & -d \uparrow & & d \uparrow \\
 F^0 & \xrightarrow{\delta} & F^0(\Gamma_0) & \xrightarrow{\delta} & F^0(\Gamma_1) & \xrightarrow{\delta} & F^0(\Gamma_2) \xrightarrow{\delta} \dots
 \end{array}$$

$$\Gamma_0 \begin{array}{c} \xleftarrow{\partial_1} \\ \xrightarrow{\partial_0} \end{array} \Gamma_1 \begin{array}{c} \xleftarrow{\partial_2} \\ \xrightarrow{\partial_0} \end{array} \Gamma_2 \begin{array}{c} \xleftarrow{\partial_3} \\ \xrightarrow{\partial_0} \end{array} \dots$$

The differential  $d$  is given by the original differential in the complex of presheaves  $F^\bullet$  and the differential  $\delta$  is constructed using the structure of a simplicial manifold on the nerve of  $\Gamma_\bullet$ :

$$\delta = \sum_i (-1)^i \partial_i^*$$

The cohomology of the total complex  $(F(\Gamma)_{\text{tot}}^\bullet, d_{\text{tot}})$  associated to the above double complex is called the  $\Gamma_\bullet$ -equivariant cohomology of  $F^\bullet$  (cf. [1]).

Presently we have two categories associated to a complex of presheaves  $F^\bullet$ , a Lie groupoid  $\Gamma_\bullet$ , and an integer  $n \geq 0$ : the category  $(\mathcal{H}^n(F^\bullet))(\Gamma_\bullet)$  of  $\Gamma_\bullet$ -equivariant objects of the presheaf  $\mathcal{H}^n(F^\bullet)$  (cf. 2.3) and the chain category of the total complex  $\mathcal{H}^n(F(\Gamma)_{\text{tot}}^\bullet)$ . These categories are not equivalent for a general  $n$ , but  $n = 0$  and  $n = 1$  are exceptional cases.

**Proposition 3.1.** *The following categories are isomorphic:*

$$\begin{aligned} (\mathcal{H}^0(F^\bullet))(\Gamma_\bullet) &= \mathcal{H}^0(F(\Gamma)_{\text{tot}}^\bullet) \\ (\mathcal{H}^1(F^\bullet))(\Gamma_\bullet) &= \mathcal{H}^1(F(\Gamma)_{\text{tot}}^\bullet) \end{aligned} \tag{3.2.1}$$

*Proof.* The isomorphisms follow directly from definitions. Here are the details.

Degree 0 case. Both  $(\mathcal{H}^0(F^\bullet))(\Gamma_\bullet)$  and  $\mathcal{H}^0(F(\Gamma)_{\text{tot}}^\bullet)$  are discrete categories with the set of objects  $\{z \in F^0(\Gamma_0) \mid dz = 0, \partial_0^* z = \partial_1^* z\}$  and no morphisms except identity ones.

Degree 1 case. Objects of the category  $\mathcal{H}^1(F(\Gamma)_{\text{tot}}^\bullet)$  are pairs  $(z, t) \in F^1(\Gamma_0) \oplus F^0(\Gamma_1)$  such that

$$dz = 0 \tag{3.2.2}$$

$$\partial_0^* z - \partial_1^* z = dt \tag{3.2.3}$$

$$\partial_0^* t - \partial_1^* t + \partial_2^* t = 0 \tag{3.2.4}$$

Now observe that (3.2.2) means  $z$  is an object of  $(\mathcal{H}^1(F^\bullet))(\Gamma_0)$ , (3.2.3) means  $t$  is a morphism from  $\partial_0^* z$  to  $\partial_1^* z$  in  $(\mathcal{H}^1(F^\bullet))(\Gamma_1)$ , and (3.2.4) is the cocycle condition for this morphism. In other words, the pair  $(z, t)$  is an object of  $(\mathcal{H}^1(F^\bullet))(\Gamma_\bullet)$ . A morphism from  $(z, t)$  to  $(z', t')$  in either  $\mathcal{H}^1(F(\Gamma)_{\text{tot}}^\bullet)$  or  $(\mathcal{H}^1(F^\bullet))(\Gamma_\bullet)$  is an element  $b \in F^0(\Gamma_0)$  such that  $db = z - z'$ ,  $\partial_0^* b - \partial_1^* b = t - t'$ .  $\square$

### 3.3. Example: singular cochains.

Given a manifold  $M$ , we consider the complex  $C_\bullet(M)$  of smooth singular chains: the group  $C_n(M)$  is a free abelian group generated by all smooth maps from the  $n$ -simplex to  $M$ , and the differential is the boundary operator. The dual complex is the complex of smooth singular cochains  $C^\bullet(M)$ . Cochains can be naturally pulled back with respect to smooth maps, and we obtain a complex  $C^\bullet$  of presheaves of abelian groups.

It is easy to see that  $C^0$  and  $H^0(C^\bullet)$  are sheaves. On the other hand,  $H^1(C^\bullet)$  is not a sheaf. In fact not even  $C^1$  is a sheaf (this is a well-known technical issue in singular (co)homology theory). However we have the following

**Proposition 3.2** (Mayer-Vietoris). *The presheaf of groupoids  $\mathcal{H}^1(C^\bullet)$  is a stack.*

The presheaf of isomorphism classes of objects of  $\mathcal{H}^1(C^\bullet)$  is the first singular cohomology presheaf. One can phrase the above proposition as “first cohomology is local if one thinks of it as a stack”.

*Proof.* We have to show that given an open covering  $U = \bigsqcup_\alpha U_\alpha \xrightarrow{v} M$  of a manifold  $M$  with the associated (Čech) groupoid  $U_\bullet$ , the restriction map is an equivalence between groupoids  $(\mathcal{H}^1(C^\bullet))(M) = \mathcal{H}^1(C^\bullet(M))$  and  $(\mathcal{H}^1(C^\bullet))(U_\bullet)$ . By Proposition 3.1 we

can replace the latter with  $\mathcal{H}^1(C(U)_{\text{tot}}^\bullet)$ . It remains to show that the restriction map  $v^* : C^\bullet(M) \rightarrow C^\bullet(U)$  followed by the inclusion  $\epsilon$  of the complex  $C^\bullet(U)$  into the double complex  $C(U)_{\text{tot}}^\bullet$  induces an equivalence of categories  $\mathcal{H}^1(C^\bullet(M)) \rightarrow \mathcal{H}^1(C(U)_{\text{tot}}^\bullet)$ . This follows from a version of the Mayer-Vietoris theorem saying that  $\epsilon \circ v^*$  is a quasi-isomorphism.

Let us explain the details of the proof showing that the restriction functor  $\mathcal{H}^1(\epsilon \circ v^*) : \mathcal{H}^1(C^\bullet(M)) \rightarrow \mathcal{H}^1(C(U)_{\text{tot}}^\bullet)$  is an equivalence. The argument is standard in singular homology theory but we want to rephrase it in terms of chain categories and functors.

*Step 1.* We consider a complex  $C^\bullet(M, U)$  of “small” cochains, i.e. cochains that take values only on those simplices  $\Delta \rightarrow M$  that can be lifted to  $\Delta \rightarrow U$  (in other words such that the image of  $\Delta$  is contained in a single open set  $U_\alpha$ ). The standard subdivision argument (see [19]) shows that the restriction map  $C^\bullet(M) \rightarrow C^\bullet(M, U)$  has an inverse up to homotopy, which according to Subsection 3.1 corresponds to the restriction functor  $\mathcal{H}^1(C^\bullet(M)) \rightarrow \mathcal{H}^1(C^\bullet(M, U))$  being equivalence of categories. More precisely, in [19] a homotopy between small and arbitrary chains is given by subdivision. We use the dual homotopy between cochains. In any case it remains to be shown that the restriction  $\mathcal{H}^1(C^\bullet(M, U)) \rightarrow \mathcal{H}^1(C(U)_{\text{tot}}^\bullet)$  is an equivalence.

*Step 2.* Observe that  $\mathcal{H}^1(C(U)_{\text{tot}}^\bullet)$  is the chain category of the total complex of the following double complex

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \uparrow d & & \uparrow -d & & \uparrow d & \\
 C^2(U) & \xrightarrow{\delta} & C^2(U \times_M U) & \xrightarrow{\delta} & C^2(U \times_M U \times_M U) & \xrightarrow{\delta} & \dots \\
 \uparrow d & & \uparrow -d & & \uparrow d & & \\
 C^1(U) & \xrightarrow{\delta} & C^1(U \times_M U) & \xrightarrow{\delta} & C^1(U \times_M U \times_M U) & \xrightarrow{\delta} & \dots \\
 \uparrow d & & \uparrow -d & & \uparrow d & & \\
 C^0(U) & \xrightarrow{\delta} & C^0(U \times_M U) & \xrightarrow{\delta} & C^0(U \times_M U \times_M U) & \xrightarrow{\delta} & \dots
 \end{array} \tag{3.3.1}$$

The crucial fact about singular cochains is that the rows of the double complex (3.3.1) are acyclic except at the first column, where the horizontal cohomology is equal to  $C^\bullet(M, U)$ . More precisely, consider the extended  $p^{\text{th}}$  row:

$$0 \rightarrow C^p(M, U) \xrightarrow{v^*} C^p(U) \xrightarrow{\delta} C^p(U \times_M U) \xrightarrow{\delta} C^p(U \times_M U \times_M U) \xrightarrow{\delta} \dots \tag{3.3.2}$$

Here the second arrow is the pull-back under the covering map  $U \xrightarrow{v} M$ . It follows from the definition of  $U \times_M U$  that

$$\delta \circ v^* = 0 \tag{3.3.3}$$

and hence (3.3.2) is a complex. We claim that this complex is acyclic, in fact homotopic to 0-complex. It is easier to describe the dual homotopy for the dual complex of small

singular chains

$$0 \leftarrow C_p(M, U) \xleftarrow{v_*} C_p(U) \xleftarrow{\delta^*} C_p(U \times_M U) \xleftarrow{\delta^*} C_p(U \times_M U \times_M U) \xleftarrow{\delta^*} \dots \quad (3.3.4)$$

The following construction was explained to us by Matthew Ando. Choose a section  $\tau$  of the map  $v_*$  ( $\tau$  exists since we consider  $U$ -small simplices) and consider a map from the total space of the complex (3.3.4) to itself given by

$$\begin{aligned} \rho(\sigma) &= \tau(\sigma) \in C_p(U) && \text{if } \sigma \in C_p(M, U) \\ \rho(\sigma) &= (\tau \circ v_* \circ \text{pr}_*(\sigma)) \times \sigma \in C_p(\underbrace{U \times_M \dots \times_M U}_{q+1}) && \text{if } \sigma \in C_p(\underbrace{U \times_M \dots \times_M U}_q) \end{aligned}$$

where  $\text{pr} : U \times_M \dots \times_M U \rightarrow U$  is the projection map (onto any of the  $U$ -factors). One easily checks that

$$\begin{aligned} v_* \circ \rho &= \text{id} && \text{on } C_p(M, U) \\ \rho \circ v^* + \delta^* \circ \rho &= \text{id} && \text{on } C_p(U) \\ \rho \circ \delta^* + \delta^* \circ \rho &= \text{id} && \text{on } C_p(U \times_M \dots \times_M U) \end{aligned}$$

which means that the complex (3.3.4) (and thus the dual complex (3.3.2)) is homotopic to 0.

*Step 3.* Recall that  $\epsilon$  is the inclusion of the first column in the double complex (3.3.1). Because of (3.3.3) the map  $\epsilon \circ v^* : C^\bullet(M, U) \rightarrow C(U)_{\text{tot}}^\bullet$  is a morphism of complexes. Step 1 reduces the Theorem to the statement that the corresponding functor

$$\mathcal{H}^1(\epsilon \circ v^*) : \mathcal{H}^1(C^\bullet(M, U)) \rightarrow \mathcal{H}^1(C(U)_{\text{tot}}^\bullet)$$

is an equivalence of categories. Explicitly, we have to show (a) the functor is full and faithful and (b) any object in the range is isomorphic to an object in the image. Both statements follow from the acyclicity of the complex (3.3.2):

(a) Let  $x_1, x_2$  be objects in the image of  $\mathcal{H}^1(\epsilon \circ v^*)$ . This means  $x_i = v^*(y_i) \in C^1(U)$ , where  $y_i \in C^1(M, U)$ ,  $dy_i = 0$ . Now we have

$$\begin{aligned} \text{Hom}_{\mathcal{H}^1(C(U)_{\text{tot}}^\bullet)}(x_1, x_2) &= \{b \in C^0(U) \mid db = x_2 - x_1, \delta b = 0\} = \\ &= \{b = v^*(a) \mid a \in C^0(M, U), da = y_2 - y_1\} = \text{Hom}_{\mathcal{H}^1(C^\bullet(M, U))}(y_1, y_2) \end{aligned}$$

where the second equality is due to acyclicity of the complex (3.3.2) with  $p = 0, 1$ .

(b) Consider an object of  $\mathcal{H}^1(C(U)_{\text{tot}}^\bullet)$ , i.e. an element  $(z, t) \in C^1(U) \oplus C^0(U \times_M U)$ , such that  $\delta t = 0$ ,  $dz = 0$ ,  $dt = \delta z$ . Since the complex (3.3.2) with  $p = 0$  is acyclic, there is an element  $c \in C^0(U)$  such that  $\delta c = t$ . Now we think of  $-c$  as an isomorphism in  $\mathcal{H}^1(C(U)_{\text{tot}}^\bullet)$  from  $(z, t)$  to  $(z - dc, 0)$ . Moreover,  $\delta(z - dc) = 0$  and acyclicity of the complex (3.3.2) with  $p = 1$  implies  $(z - dc) = \epsilon \circ v^*(y)$  for some  $y \in C^1(M, U)$ ,  $dy = 0$ . In other words,  $(z - dc, 0)$  is in the image of the functor  $\mathcal{H}^1(\epsilon \circ v^*)$ .

This completes the proof of the Proposition.  $\square$

### 3.4. De Rham stack.

Similarly to singular cochains, one can consider the presheaf  $\Omega^\bullet$  of differential forms with smooth coefficients.

**Proposition 3.3.** *One has:*

- (1)  $\Omega^p$  is a sheaf for any  $p$ .
- (2)  $\mathcal{H}^0(\Omega^\bullet)$  is a sheaf (discrete stack).
- (3)  $\mathcal{H}^1(\Omega^\bullet)$  is a stack.

*Proof.* The proof is similar to the proof of Proposition 3.3. An important difference is that every differential form can be pulled back along a covering map  $U = \bigsqcup_\alpha U_\alpha \xrightarrow{v} M$ . Hence we don't need the first step of that proof in the present situation and, moreover, we get the additional result (1). The rest of the proof is the same once one shows that Mayer-Vietoris principle holds for  $\Omega^p$ , i.e. that the complex

$$0 \rightarrow \Omega^p(M) \xrightarrow{v^*} \Omega^p(U) \xrightarrow{\delta} \Omega^p(U \times_M U) \xrightarrow{\delta} \Omega^p(U \times_M U \times_M U) \xrightarrow{\delta} \dots \quad (3.4.1)$$

is acyclic. The proof of Mayer-Vietoris principle for singular chains relied on the existence of a section of the push-forward map  $v_*$ . In the context of differential forms the role of such a section is played by a partition of unity. We recall the standard argument below.

First note that  $\underbrace{U \times_M \dots \times_M U}_q = \bigsqcup_{\alpha_1 \dots \alpha_q} U_{\alpha_1} \cap \dots \cap U_{\alpha_q}$ . Given a differential form  $\omega \in \Omega^p(\underbrace{U \times_M \dots \times_M U}_q)$  we denote by  $\omega_{\alpha_1 \dots \alpha_q}$  its restriction to  $U_{\alpha_1} \cap \dots \cap U_{\alpha_q}$ . Now

let  $1 = \sum_\alpha \eta_\alpha$  be a partition of unity subordinate to the covering  $U = \bigsqcup_\alpha U_\alpha \xrightarrow{v} M$  and consider a map from the total space of the complex (3.4.1) to itself given by

$$(\rho(\omega))_{\alpha_1 \dots \alpha_q} = \sum_{\alpha_0} \eta_{\alpha_0} \omega_{\alpha_0 \alpha_1 \dots \alpha_q}$$

Then

$$\begin{aligned} \rho \circ v^* &= \text{id} && \text{on } \Omega^p(M) \\ v^* \circ \rho + \rho \circ \delta &= \text{id} && \text{on } \Omega^p(U) \\ \delta \circ \rho + \rho \circ \delta &= \text{id} && \text{on } \Omega^p(U \times_M \dots \times_M U) \end{aligned}$$

which means that the complex (3.4.1) is homotopic to 0.  $\square$

### 3.5. (Pre)sheaf (hyper)cohomology on stacks.

Let  $F^\bullet$  be a complex of presheaves and  $\mathcal{W}$  be a stack. There are several ways to define cohomology of  $F^\bullet$  on  $\mathcal{W}$ . The most abstract definition uses injective resolution, which is not very useful for actual calculations. A more explicit construction (generalizing Čech cohomology) exists for differentiable stacks. One starts with an atlas  $\Gamma_\bullet \rightarrow \mathcal{W}$  and defines  $H^n(\mathcal{W}, F^\bullet)$  as the total cohomology of the double complex  $F^\bullet(\Gamma_\bullet)$ . It can be shown (see e.g. [1]) that if  $F^\bullet$  is Čech-acyclic for covers of manifolds, then  $H^n(\mathcal{W}, F^\bullet)$  does not

depend on the choice of the atlas (for a general  $F^\bullet$ , the double complex will be the first term of a spectral sequence converging to  $H^n(\mathcal{W}, F^\bullet)$ ).

An intrinsic definition of hyper-cohomology uses the notion of an Eilenberg-MacLane stack  $\mathcal{K}(F^\bullet, n)$ , which is characterized by its universal property (cf. [18]): for any manifold  $M$  the isomorphism classes of objects of  $(\mathcal{K}(F^\bullet, n))(M)$  are in (functorial in  $M$ ) bijection with  $H^n(F^\bullet(M))$ . Theorems 3.3 and 3.4 say that, if  $F^\bullet = C^\bullet$  or  $\Omega^\bullet$ , and  $n = 1$  (or 0), then  $\mathcal{K}(F^\bullet, n)$  is equivalent to  $\mathcal{H}^n(F^\bullet)$  (or, even better,  $\mathcal{H}^n(F^\bullet)$  provides an explicit construction of  $\mathcal{K}(F^\bullet, n)$ ). Given an Eilenberg-MacLane stack  $\mathcal{K}(F^\bullet, n)$  the category  $\mathcal{H}^n(\mathcal{W}, F^\bullet) := \mathcal{H}om(\mathcal{W}, \mathcal{K}(F^\bullet, n))$  is called the cohomology category of the complex  $F^\bullet$  on the stack  $\mathcal{W}$ , and the set of isomorphism classes of objects of this category the cohomology  $H^n(\mathcal{W}, F^\bullet)$  of  $F^\bullet$  on  $\mathcal{W}$ . This definition assumes nothing about the stack  $\mathcal{W}$ .

If  $n > 1$ , then the presheaves of groupoids  $\mathcal{H}^n(C^\bullet)$  and  $\mathcal{H}^n(\Omega^\bullet)$  are not stacks (and neither  $\mathcal{K}(C^\bullet, n)$  nor  $\mathcal{K}(\Omega^\bullet, n)$  exists as a stack). For example, the automorphism group of the 0-object of  $\mathcal{H}^2(C^\bullet)$  is the first singular cohomology group and so is not local. The reason is clear from the above proofs of locality: to make double complexes work we need to consider cochains of all degrees lower than  $n$  even if we are only interested in  $n^{\text{th}}$  cohomology. This leads one to think of, say,  $H^n(C^\bullet(M))$  as the set of isomorphism classes of objects of an  $n$ -category having cochains of degree  $k \leq n$  as  $(n - k)$ -morphisms. So for each manifold we have a  $n$ -category  $\mathcal{H}^n(C^\bullet(M))$ , albeit a very simple one – with invertible morphisms and strict associativity in all degrees, and this sheaf of  $n$ -categories satisfies descent condition. We refer the reader to [15, 18] for the theory of higher stacks. In particular, Theorems 3.3 and 3.4 generalize to higher degrees:  $\mathcal{K}(C^\bullet, n)$  is equivalent to  $\mathcal{H}^n(C^\bullet)$  and  $\mathcal{K}(\Omega^\bullet, n)$  is equivalent to  $\mathcal{H}^n(\Omega^\bullet)$  for any  $n$ .

In the present paper we avoid higher stacks and hence higher cohomology. There is a situation however in which one can describe the second cohomology group of a complex of presheaves in terms of the usual (1-)stacks. This happens if  $H^0(M)$  is trivial for any  $M$ . In that case the 2-categories involved are 2-discreet and thus equivalent to 1-categories. An example of such situation is provided by differential characters.

### 3.6. Differential characters.

Recall that  $C^\bullet$  is the smooth singular cochain complex. Let  $C_{\mathbb{R}}^\bullet = C^\bullet \otimes_{\mathbb{Z}} \mathbb{R}$  and, for consistency, denote  $C^\bullet$  by  $C_{\mathbb{Z}}^\bullet$ . The integration provides an inclusion of the de Rham complex  $\Omega^\bullet$  into  $C_{\mathbb{R}}^\bullet$ .

Given an integer  $s \geq 0$  Hopkins-Singer [9] define the following complex of presheaves of abelian groups over  $\mathcal{Man}$ :

$$DC_s^n = \{(c, h, \omega) \mid \omega = 0 \text{ if } n < s\} \subset C_{\mathbb{Z}}^n \times C_{\mathbb{R}}^{n-1} \times \Omega^n$$

with the differential

$$d(c, h, \omega) = (dc, \omega - c - dh, d\omega)$$

The importance of this complex is that the cohomology group  $H^k(DC_k^\bullet(M))$  is isomorphic to the group of Cheeger-Simons differential characters [5] of degree  $k$  on a manifold  $M$ .

Recall that such a differential character is a pair  $(\omega, \chi)$ , where  $\omega \in \Omega^k(M)$  is a differential  $k$ -form and  $\chi : Z_{k-1}(M) \rightarrow \mathbb{R}/\mathbb{Z}$  is a character of the group of smooth singular  $(k-1)$ -cycles. This pair should satisfy the following condition:

$$\chi(\partial S) \equiv \int_S \omega \pmod{\mathbb{Z}} \quad (3.6.1)$$

for any smooth singular  $k$ -chain  $S$ . The isomorphism between  $H^k(DC_k^\bullet(M))$  and the group of differential characters is given by the map  $[(c, h, \omega)] \mapsto (\omega, \chi)$ , where  $\chi$  is the restriction of  $h \in C_{\mathbb{R}}^{k-1}(M)$  to  $Z_{k-1}(M)$  modulo  $\mathbb{Z}$ . The condition (3.6.1) ensures that  $c = \omega - dh$  is an integral cochain.

We are going to show that  $H^2(DC_1^\bullet)$  and  $H^2(DC_2^\bullet)$  classify  $S^1$ -bundles and  $S^1$ -bundles with connections respectively. So in the cases we are interested in we always have  $s > 0$ .

Following our usual procedure we replace the second cohomology with the presheaf of groupoids  $\mathcal{DC}_s^2 := \mathcal{H}^2(DC_s^\bullet)$ . At first sight  $\mathcal{DC}_s^2$  has no chance of being a stack since we discarded 0-cochains. Fortunately, if  $s > 0$ , then  $DC_s^\bullet$  has trivial 0<sup>th</sup> cohomology because

$$d(c, 0, 0) = (dc, -c, 0) \quad \text{for } (c, 0, 0) \in DC_s^0(M), \quad s > 0$$

This vanishing allows us to prove the following crucial result.

**Proposition 3.4.** *If  $s > 0$  then the presheaf of groupoids  $\mathcal{DC}_s^2$  is a stack.*

We call  $\mathcal{DC}_s^2$  the stack of differential characters (of degree 2).

*Proof.* Let us consider two double complexes associated to the complex  $DC_s^\bullet$  and a covering  $v : U \rightarrow M$ . The first is the usual one computing the descent data

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots & & (3.6.2) \\ & & \uparrow d & & \uparrow d & & \uparrow -d & & \\ 0 & \longrightarrow & DC_s^2(M) & \xrightarrow{v^*} & DC_s^2(U) & \xrightarrow{\delta} & DC_s^2(U \times_M U) & \xrightarrow{\delta} & \dots \\ & & \uparrow d & & \uparrow d & & \uparrow -d & & \\ 0 & \longrightarrow & DC_s^1(M) & \xrightarrow{v^*} & DC_s^1(U) & \xrightarrow{\delta} & DC_s^1(U \times_M U) & \xrightarrow{\delta} & \dots \\ & & \uparrow d & & \uparrow d & & \uparrow -d & & \\ 0 & \longrightarrow & DC_s^0(M) & \xrightarrow{v^*} & DC_s^0(U) & \xrightarrow{\delta} & DC_s^0(U \times_M U) & \xrightarrow{\delta} & \dots \end{array}$$

and the second is

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow d & & \uparrow d & & \uparrow -d \\
 0 & \longrightarrow & DC_s^2(M) & \xrightarrow{v^*} & DC_s^2(U) & \xrightarrow{\delta} & DC_s^2(U \times_M U) \xrightarrow{\delta} \cdots \\
 & & \uparrow d & & \uparrow d & & \uparrow -d \\
 0 & \longrightarrow & \widetilde{DC}_s^1(M) & \xrightarrow{v^*} & \widetilde{DC}_s^1(U) & \xrightarrow{\delta} & \widetilde{DC}_s^1(U \times_M U) \xrightarrow{\delta} \cdots
 \end{array} \tag{3.6.3}$$

Here  $\widetilde{DC}_s^1(\cdot) = DC_s^1(\cdot)/d DC_s^0(\cdot)$ . We denote  $[x] \in \widetilde{DC}_s^1(\cdot)$  the class in of  $x \in DC_s^1(\cdot)$ .

Rows of (3.6.2) are acyclic since they are direct sums of rows of the double complexes associated to  $C_{\mathbb{Z}}^\bullet$ ,  $C_{\mathbb{R}}^\bullet$ , and  $\Omega^\bullet$ . However in order to prove the Proposition we need the acyclicity of the rows of (3.6.3). Only the last row is new. So let  $[x] \in \widetilde{DC}_s^1(\cdot)$ ,  $\delta[x] = [0]$ . This means that  $\delta x = dt$  for some  $t$ . By anticommutativity of the double complex we have  $d\delta t = -\delta^2 x = 0$ . But columns of (3.6.2) have trivial 0<sup>th</sup> cohomology. Hence  $\delta t = 0$  and so, 0<sup>th</sup> row of (3.6.2) being acyclic,  $t = \delta s$  for some  $s$ . It follows that  $\delta(x + ds) = 0$  and the acyclicity of the 1<sup>st</sup> row of (3.6.2) implies  $x + ds = \delta y$  for some  $y$ . Then  $[x] = \delta[y]$ , which shows that the last row of (3.6.3) is acyclic. Of course one can replace this diagram chasing with the long exact sequence in cohomology associated to the short exact sequence  $0 \rightarrow DC_s^0(\bullet) \xrightarrow{d} DC_s^1(\bullet) \rightarrow \widetilde{DC}_s^1(\bullet) \rightarrow 0$  of rows (i.e.  $\delta$ -complexes). However we prefer explicit calculations with chains because they are easily translated into the language of chain categories.

The rest of the proof is the same as for the singular and the de Rham complexes.  $\square$

### 3.7. Equivariant differential characters.

Let  $\Gamma_\bullet$  be a Lie groupoid. By definition (or rather, by an argument similar to the proof of Proposition 3.1), the category  $(\mathcal{H}^2(DC_s^\bullet))(\Gamma_\bullet)$  is isomorphic to the second cohomology category of the total complex of (3.6.3). However, for computational as well as aesthetic purposes, we would like to use the standard double complex  $DC_s^\bullet(\Gamma_\bullet)$ . Fortunately one has the following result.

**Proposition 3.5.** *If  $s > 0$  then the following categories are equivalent*

$$(\mathcal{H}^2(DC_s^\bullet))(\Gamma_\bullet) = \mathcal{H}^2(DC_s(\Gamma)_{\text{tot}}^\bullet) \tag{3.7.1}$$

*Proof.* The proof is a combination of the proofs of Propositions 3.1 and 3.6 and relies on the fact that  $H^0(DC_s^\bullet(M)) = 0$  for any manifold  $M$ .  $\square$

## 4. Chern functor and prequantization.

In this section we prove classification theorems for principal  $S^1$ -bundles with or without connections.

#### 4.1. Chern class.

There is a natural functor  $\text{Ch}_{\text{triv}} : \mathcal{BS}_{\text{triv}}^1 \rightarrow \mathcal{DC}_1^2$  defined as follows.

- The category  $\mathcal{BS}_{\text{triv}}^1(M)$  has unique object  $M \times S^1$ . We put

$$\text{Ch}_{\text{triv}}(M \times S^1) = \underline{0} = (0, 0, 0) \in \mathcal{DC}_1^2(M) .$$

- A morphism in  $\mathcal{BS}_{\text{triv}}^1(M)$  is a smooth function  $f : M \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$ . We pick a lift  $\tilde{f} : M \rightarrow \mathbb{R}$  ( $\tilde{f}$  is not required to be smooth) and put

$$\text{Ch}_{\text{triv}}(f) = [(d(\tilde{f} - f), -\tilde{f}, -df)] \in \text{Hom}_{\mathcal{DC}_1^2(M)}(\underline{0}, \underline{0}) = H^1(\mathcal{DC}_1^\bullet(M)) .$$

The value  $\text{Ch}_{\text{triv}}(f)$  does not depend on the choice of  $\tilde{f}$ .

If  $M$  is contractible (for example  $M = \mathbb{R}^n$ ) then we have

$$H^2(\mathcal{DC}_1^\bullet(M)) = 0 ,$$

$$H^1(\mathcal{DC}_1^\bullet(M)) = \Omega^0(M)/\{\text{const functions } M \rightarrow \mathbb{Z}\} = \{C^\infty \text{ functions } M \rightarrow S^1\} ,$$

which means that  $\text{Ch}_{\text{triv}}$  is an equivalence of categories from  $\mathcal{BS}_{\text{triv}}^1(\mathbb{R}^n)$  to  $\mathcal{DC}_1^2(\mathbb{R}^n)$ . Therefore (by the universal property of stackification)  $\text{Ch}_{\text{triv}}$  induces an equivalence  $\text{Ch}$  from the stackification of  $\mathcal{BS}_{\text{triv}}^1$  (i.e.  $\mathcal{BS}^1$ ) to the stack  $\mathcal{DC}_1^2$ . Note that the functor  $\text{Ch}$  is defined only up to a natural transformation.

To summarize we have the following theorem.

**Theorem 4.1.** *There is an equivalence of stacks  $\text{Ch} : \mathcal{BS}^1 \rightarrow \mathcal{DC}_1^2$ . In particular:*

- For any stack  $\mathcal{W}$  the functor  $\text{Ch}$  induces an equivalence of categories from  $\mathcal{BS}^1(\mathcal{W})$  to  $\mathcal{DC}_1^2(\mathcal{W}) = (\mathcal{H}^2(\mathcal{DC}_1^\bullet))(\mathcal{W})$ ;
- For any manifold  $M$  the functor  $\text{Ch}$  induces a bijection from the set of isomorphism classes of principal  $S^1$ -bundles on  $M$  to  $H^2(\mathcal{DC}_1^\bullet(M))$ ;
- For any groupoid  $\Gamma_\bullet$  the functor  $\text{Ch}$  induces a bijection from the set of isomorphism classes of  $\Gamma_\bullet$ -equivariant principal  $S^1$ -bundles to the second total cohomology group of the double complex  $\mathcal{DC}_1^\bullet(\Gamma_\bullet)$ ;
- For a Lie group  $G$  acting on a manifold  $M$  the functor  $\text{Ch}$  induces a bijection from the set of isomorphism classes of  $G$ -equivariant principal  $S^1$ -bundles on  $M$  to the equivariant cohomology group  $H_G^2(\mathcal{DC}_1^\bullet(M))$ .

Here by equivariant cohomology we mean the simplicial model of equivariant cohomology, that is  $H_G^2(\mathcal{DC}_1^\bullet(M)) := H^2(\mathcal{DC}_1^\bullet(M \rtimes G \times M))$ . The reason for separating this case is that there are other models (especially, for a compact group action), but of course before using them one has to prove they give the same answer.

#### 4.2. Weil Theorem.

Let us explain how the above result implies Weil's classification theorem for principal  $S^1$ -bundles on a manifold  $M$ . Recall that

$$\mathcal{DC}_1^n(M) = \{(c, h, \omega) \mid \omega = 0 \text{ if } n = 0\} \subset C_{\mathbb{Z}}^n(M) \times C_{\mathbb{R}}^{n-1}(M) \times \Omega^n(M) .$$

We claim that the projection  $p : DC_1^2(M) \rightarrow C_{\mathbb{Z}}^2(M)$  induces an isomorphism of the second cohomology groups. Both surjectivity and injectivity follow from the de Rham Theorem which says that the inclusion (given by integration) of the complex of differential forms into the complex of singular cochains with real coefficients induces isomorphism in cohomology. In plain words it means that, given  $a \in C_{\mathbb{R}}^n(M)$  such that  $da \in \Omega^{n+1}(M)$ , there exist  $\alpha \in \Omega^n(M)$  and  $b \in C_{\mathbb{R}}^{n-1}(M)$  such that  $a = \alpha + db$ . We call  $\alpha$  a de Rham representative of (the class) of  $a$ .

Now let  $\omega \in \Omega^2(M)$  be a de Rham representative of a cocycle  $c \in C_{\mathbb{Z}}^2(M) \subset C_{\mathbb{R}}^2(M)$ , i.e.  $c = \omega - dh$ ,  $h \in C_{\mathbb{R}}^1$ . Then the cochain  $(c, h, w)$  is closed in  $DC_1^2(M)$  and provides an extension of  $c$ . This proves surjectivity of  $p$ . To prove injectivity consider  $x \in DC_1^2(M)$  such that  $dx = 0$ ,  $p(x) = db$ , or  $x = (db, h, d(b+h))$ . Let  $\alpha \in \Omega^1(M)$  be a de Rham representative of  $b+h$ , i.e.  $b+h = \alpha - df$ ,  $f \in C_{\mathbb{R}}^0$ . Then  $x = d(b, f, \alpha)$ , which implies that  $p$  is injective on cohomology.

Composing the projection  $p$  with the bijection from the manifold case of Theorem 4.1 we obtain Weil's theorem.

**Theorem 4.2.** *There is a bijection (the first Chern class) from the set of isomorphism classes of principal  $S^1$ -bundles on a manifold  $M$  to  $H^2(M, \mathbb{Z})$ .*

A few remarks are in order.

First, there are natural group structures on the sets  $H^2(M, \mathbb{Z})$ ,  $H^2(DC_1^\bullet(M))$ , and on the sets of objects of the categories  $\mathcal{BS}^1(M)$  and  $\mathcal{DC}_1^2(M)$ . A straightforward refinement of the above argument (considering sheaves of groupoids with additive structure on objects, their descent and stackifications, etc.) shows that the first Chern class is a group homomorphism.

Second, the reader could wonder why do we deal with the complicated complex  $DC_1^\bullet$  when the final answer involves only singular cochains. The reason is that we would like to get a *local* proof of Weil's theorem. In other words we consider the equivalence of stacks  $\mathcal{Ch}$  to be more fundamental than the global bijection between isomorphism classes of objects. This local equivalence makes no sense for the complex  $C_{\mathbb{Z}}^\bullet$  of singular cochains since  $\mathcal{H}^2(C_{\mathbb{Z}}^\bullet)$  is not a stack (it is a 2-stack).

Finally, there is a sheaf-theoretic proof of Weil's theorem. One identifies  $S^1$ -bundles with the first Čech cohomology of the sheaf  $\underline{S}^1$  of smooth  $S^1$ -valued functions and then uses the short exact sequence of sheaves  $0 \rightarrow \underline{\mathbb{Z}} \rightarrow \underline{\mathbb{R}} \rightarrow \underline{S}^1 \rightarrow 0$  to show that  $\check{H}^1(M, \underline{S}^1) = \check{H}^2(M, \underline{\mathbb{Z}}) = H^2(M, \mathbb{Z})$ . Let us explain the relation of this argument with our approach through cochain stacks. Note that even though  $\underline{S}^1$  is a sheaf (hence a stack), the way it is used is not local - one has to consider a covering of the manifold to evaluate the first cohomology of  $\underline{S}^1$ . In order to replace the first cohomology group by the group of global sections one has to shift degree by one. So we think of  $\underline{S}^1$  as morphisms in a category rather than objects. The resultant category is  $\mathcal{BS}_{\text{triv}}^1$ . Now we have to stackify it. The answer is  $\mathcal{DC}_1^2 \cong \mathcal{BS}^1$ . The exact sequence  $0 \rightarrow \underline{\mathbb{Z}} \rightarrow \underline{\mathbb{R}} \rightarrow \underline{S}^1 \rightarrow 0$  is hidden in the proof of the fact that  $\mathcal{DC}_1^2$  is a stack.



Hence

$$d(c_2 + \delta h_1) = \delta \omega_1 \in \Omega^2(\Gamma_1)$$

and, by the de Rham Theorem, we can find  $h'_2 \in C_{\mathbb{R}}^0(\Gamma_1)$  and  $\omega'_2 \in \Omega^1(\Gamma_1)$  such that

$$c_2 + \delta h_1 = \omega'_2 - dh'_2 \quad (4.3.2)$$

i.e.  $d(c_2, h'_2, \omega'_2) = \delta(c_1, h_1, \omega_1)$ . A similar argument shows that  $c_3 - \delta h'_2 \in \Omega^0(\Gamma_2)$ . But then, by the Theorem assumption,  $c_3 - \delta h'_2 = \delta f$  for some  $f \in \Omega^0(\Gamma_1)$ . Finally we put  $h_2 = h'_2 + f$ ,  $\omega_2 = \omega'_2 + df$  to get a  $d_{\text{tot}}$ -closed cochain  $((c_1, h_1, \omega_1), (c_2, h_2, \omega_2), (c_3, 0, 0))$  in  $(DC_1(\Gamma))_{\text{tot}}^2$ . This proves surjectivity of  $p$ .

To prove injectivity of  $p$  (on cohomology) we have to show (cf. 4.2) that any cocycle  $x$  in  $(DC_1(\Gamma))_{\text{tot}}^2$  of the form  $x = ((db_1, h_1, \omega_1), (\delta b_1 - db_2, h_2, \omega_2), (\delta b_2, 0, 0))$  is in the image of  $d_{\text{tot}}$ . Repeating the argument in subsection 4.2 we can assume that  $b_1 = h_1 = \omega_1 = 0$ . Then  $d_{\text{tot}}x = 0$  implies  $d(h_2 - b_2) = \omega_2$  and  $\delta(h_2 - b_2) = 0$ . Hence  $(h_2 - b_2) \in \Omega^0(\Gamma_1)$  and  $\delta(h_2 - b_2) = 0$ . Therefore, by the Theorem assumption,  $h_2 - b_2 = \delta f$  for some  $f \in \Omega^0(\Gamma_0)$ . This means  $((0, 0, 0), (-db_2, h_2, \omega_2), (\delta b_2, 0, 0)) = d_{\text{tot}}((0, f, df), (b_2, 0, 0))$ , which proves the injectivity of  $p$  on cohomology and hence the Theorem.  $\square$

#### 4.4. Prequantization.

We turn to bundles with connections and consider a functor  $\text{DCh}_{\text{triv}} : \mathcal{DBS}_{\text{triv}}^1 \rightarrow \mathcal{DC}_2^2$  given by (cf. 4.1):

- $\text{DCh}_{\text{triv}}((M \times S^1, a + d\theta)) = (0, a, da)$  on objects;
- $\text{DCh}_{\text{triv}}(f) = [(d(\tilde{f} - f), -\tilde{f}, 0)] \in DC_2^1/d(DC_2^0)$  on morphisms. Here we think of a morphism in  $\mathcal{DBS}_{\text{triv}}^1$  from  $(M \times S^1, a + d\theta)$  to  $(M \times S^1, a' + d\theta)$  as a smooth function  $f : M \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$  such that  $df = a' - a$  and let  $\tilde{f} \in C_{\mathbb{R}}^0(M)$  be a lift of  $f$ .

As in 4.1, a simple cohomology calculation in  $DC_2^\bullet(\mathbb{R}^n)$  shows that  $\text{DCh}_{\text{triv}}$  restricts to an equivalence of categories between  $\mathcal{DBS}_{\text{triv}}^1(\mathbb{R}^n)$  and  $\mathcal{DC}_2^2(\mathbb{R}^n)$ , and hence induces an equivalence of stacks  $\text{DCh} : \mathcal{DBS}^1 \rightarrow \mathcal{DC}_2^2$ . The quasi-inverse functor  $\text{Preq}$  (defined up to a natural transformation) is called the prequantization.

To summarize we have the following theorem.

**Theorem 4.4** (Prequantization Theorem). *There is an equivalence of stacks given by  $\text{Preq} : \mathcal{DC}_2^2 \rightarrow \mathcal{DBS}^1$  from the stack of differential characters to the stack of principal  $S^1$ -bundles with connections. In particular, isomorphism classes of principal  $S^1$ -bundles with connections are classified by differential characters. More precisely:*

- For any stack  $\mathcal{W}$  the functor  $\text{Preq}$  induces an equivalence of categories from  $\mathcal{DC}_2^2(\mathcal{W}) = (\mathcal{H}^2(DC_2^\bullet))(\mathcal{W})$  to  $\mathcal{DBS}^1(\mathcal{W})$ ;
- For any manifold  $M$  the functor  $\text{Preq}$  induces a bijection from  $H^2(DC_2^\bullet(M))$  to the set of isomorphism classes of principal  $S^1$ -bundles with connections on  $M$ ;
- For any groupoid  $\Gamma_\bullet$  the functor  $\text{Preq}$  induces a bijection from the second total cohomology group of the double complex  $DC_2^\bullet(\Gamma_\bullet)$  to the set of isomorphism classes of  $\Gamma_\bullet$ -equivariant principal  $S^1$ -bundles with basic connections;

- For a Lie group  $G$  acting on a manifold  $M$  the functor  $\text{Preq}$  induces a bijection from the equivariant cohomology group  $H_G^2(DC_2^\bullet(M))$  to the set of isomorphism classes of  $G$ -equivariant principal  $S^1$ -bundles with  $G$ -basic connections on  $M$ .

To make things a bit more explicit let us write down cochains of  $DC_2^\bullet(\Gamma_\bullet)$  of total degree  $n \leq 2$  on a groupoid  $\Gamma_\bullet$  (cf. (4.3.1) for cochains of  $DC_1^\bullet(\Gamma_\bullet)$ ):

$$\begin{array}{ccccc}
 & \uparrow d & & & \\
 & \{(c_1, h_1, \omega_1)\} & \xrightarrow{\delta} & & \\
 & \uparrow d & & \uparrow -d & \\
 & \{(b_1, g_1, 0)\} & \xrightarrow{\delta} & \{(c_2, h_2, 0)\} & \xrightarrow{\delta} & \\
 & \uparrow d & & \uparrow -d & & \uparrow d & \\
 \{(a_1, 0, 0)\} & \xrightarrow{\delta} & \{(b_2, 0, 0)\} & \xrightarrow{\delta} & \{(c_3, 0, 0)\} & \xrightarrow{\delta} & 
 \end{array}$$

Here  $a_i, b_i, c_i$ , are singular cochains with integer coefficients,  $g_i, h_i$  are singular cochains with real coefficients, and  $\omega_1$  is a differential form. As in the manifold case (cf. subsection 3.6 and the Introduction), the map

$$[((c_1, h_1, \omega_1), (c_2, h_2, 0), (c_3, 0, 0))] \mapsto (\omega_1, (h_1 \bmod \mathbb{Z}, h_2 \bmod \mathbb{Z}))$$

provides an isomorphism between the second cohomology group  $H_{\text{tot}}^2(DC_2^\bullet(\Gamma_\bullet))$  and the group of differential characters, i.e., pairs  $(\omega, \chi)$ , where  $\omega \in \Omega^2(\Gamma_0) \subset \Omega^2(\Gamma_\bullet)$  and  $\chi : Z_1(\Gamma_\bullet) \rightarrow \mathbb{R}/\mathbb{Z}$  is a character of the group of smooth singular 1-cycles on the groupoid  $\Gamma_\bullet$ , satisfying a condition similar (and generalizing) (3.6.1). We refer the reader to [12] (where a more general case is considered) for details, pictures, and examples.

#### 4.5. Equivariant Kostant theorem.

Let  $\Gamma_\bullet$  be a Lie groupoid. We denote by  $\Omega_{\mathbb{Z}, \text{cl}, \text{bas}}^2(\Gamma_\bullet)$  the group of closed 2-forms on  $\Gamma_0$  which are integral (i.e. have integral periods, or equivalently, represent the image of an integral cohomology class in de Rham cohomology), and basic (i.e. are in the kernel of  $\partial : \Omega^2(\Gamma_0) \rightarrow \Omega^2(\Gamma_1)$ ). Suppose  $\omega_1 \in \Omega_{\mathbb{Z}, \text{cl}, \text{bas}}^2(\Gamma_\bullet)$ . Then  $(\omega_1, 0, 0) \in \Omega^2(\Gamma_\bullet) = \Omega^2(\Gamma_0) \times \Omega^1(\Gamma_1) \times \Omega^0(\Gamma_2)$  represents the image of an integral cohomology class on  $\Gamma_\bullet$  in de Rham cohomology (here we used de Rham theorem on  $\Gamma_\bullet$ , cf. [1], to identify de Rham and real-valued singular cohomology). In other words, there exists an integral cochain  $(c_1, c_2, c_3) \in C_{\mathbb{Z}}^2(\Gamma_0) \times C_{\mathbb{Z}}^1(\Gamma_1) \times C_{\mathbb{Z}}^0(\Gamma_2)$  and a real cochain  $(h_1, h_2) \in C_{\mathbb{R}}^1(\Gamma_0) \times C_{\mathbb{R}}^0(\Gamma_1)$  such that  $d_{\text{tot}}(h_1, h_2) = (\omega_1, 0, 0) - (c_1, c_2, c_3)$ , where  $d_{\text{tot}}$  is the total differential in the double complex of smooth singular chains on  $\Gamma_\bullet$ . This means that the projection map

$$\begin{aligned}
 \eta : H_{\text{tot}}^2(DC_2^\bullet(\Gamma_\bullet)) &\rightarrow \Omega_{\mathbb{Z}, \text{cl}, \text{bas}}^2(\Gamma_\bullet) \\
 \eta([((c_1, h_1, \omega_1), (c_2, h_2, 0), (c_3, 0, 0))]) &= \omega_1
 \end{aligned}$$

is surjective, and it is easy to see that the kernel of  $\eta$  is isomorphic to the cohomology group  $H_{\text{tot}}^1(C_{\mathbb{R}/\mathbb{Z}}^\bullet(\Gamma_\bullet))$  of smooth singular 1-cochains with values in  $\mathbb{R}/\mathbb{Z}$  by the map

$$[(c_1, h_1, 0), (c_2, h_2, 0), (c_3, 0, 0)] \mapsto [(h_1 \bmod \mathbb{Z}, h_2 \bmod \mathbb{Z})] .$$

Hence we obtain a short exact sequence

$$0 \rightarrow H_{\text{tot}}^1(C_{\mathbb{R}/\mathbb{Z}}^\bullet(\Gamma_\bullet)) \rightarrow H_{\text{tot}}^2(DC_2^\bullet(\Gamma_\bullet)) \rightarrow \Omega_{\mathbb{Z}, \text{cl}, \text{bas}}^2(\Gamma_\bullet) \rightarrow 0 . \quad (4.5.1)$$

In the manifold case this sequence appeared in the original Cheeger and Simons paper [5]. Since  $H_{\text{tot}}^2(DC_2^\bullet(\Gamma_\bullet))$  classifies  $\Gamma_\bullet$ -equivariant principal  $S^1$ -bundles with basic connections (cf. Theorem 4.4), we can interpret (4.5.1) as follows:

**Theorem 4.5.** *Let  $\Gamma_\bullet$  be a Lie groupoid. Given an integral closed basic 2-form  $\omega$  on  $\Gamma_0$  there exists a  $\Gamma_\bullet$ -equivariant principal  $S^1$ -bundle with a basic connection whose curvature is  $\omega$ . Moreover, the set of isomorphism classes of such bundles with connections is in bijection with  $H_{\text{tot}}^1(C_{\mathbb{R}/\mathbb{Z}}^\bullet(\Gamma_\bullet))$ .*

This theorem is due to Kostant [10] in the manifold case. A more general version (about equivariant bundles with arbitrary connections) was proved in [2] for proper groupoids, and in [12] for arbitrary groupoids.

As explained in the Introduction and remarks after Theorem 4.2, we believe that the abstract version of Theorem 4.4 is more fundamental than Cheeger-Simons- and Kostant-type interpretations because (1) it describes the category of bundles with connections (rather than isomorphism classes of objects) and (2) it is a local statement. On the other hand, a concrete description in terms of cohomology groups or differential characters is obviously useful in calculations and/or explicit geometric interpretations.

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