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New symplectic 4-manifolds with nonnegative signature

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Abstract. We construct new families of symplectic 4-manifolds with nonnegative signature that are interesting with respect to the geography problem. In particular, we construct an irreducible symplectic 4-manifold that is homeomorphic to $m\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$ for each odd integer m satisfying $m \geq 49$.

1. Introduction

The geography problem for simply connected symplectic 4-manifolds with negative signature is fairly well understood. We refer the reader to the recent works [1, 3, 13] for the current status. In stark contrast, the geography problem for simply connected symplectic 4-manifolds with nonnegative signature is poorly understood. The existing literature [12, 14, 15, 17, 18] are far from capturing all possible (χ_h, c_1^2) coordinates, even if we allow nontrivial fundamental groups.

In this paper, we construct several new families of symplectic 4-manifolds with positive signature that can be used as building blocks for constructing simply connected symplectic 4-manifolds with nonnegative signature. As one such application, we construct simply connected nonspin irreducible symplectic 4-manifolds with signature σ within the range $0 \leq \sigma \leq 4$. These have the smallest Euler characteristics amongst all known simply connected 4-manifolds with nonnegative signature which are currently known to possess more than one smooth structure. The remaining cases corresponding to signature $\sigma \geq 5$ are dealt with in the sequel [2].

Our paper is organized as follows. In Sections 2 and 3, we construct families of symplectic 4-manifolds similar to $H(n)$ in [4, 17] which lie slightly below the Bogomolov-Miyaoka-Yau line, $c_1^2 = 9\chi_h$, using branched covering techniques. In Section 4, we construct two simply connected irreducible symplectic 4-manifolds with positive signature which will serve as useful building blocks. In Section 5, we construct families of simply connected nonspin irreducible symplectic 4-manifolds with signature equal to 0, 1, 2, 3 or 4 having relatively small Euler characteristics.

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2. First family with positive signature

Let g be a positive integer. Think of a closed genus g Riemann surface Σ_g as two concentric spheres with $g+1$ tubes connecting them. Consider an orientation-preserving self-diffeomorphism $\gamma : \Sigma_g \to \Sigma_g$ which is the rotation of this surface by angle $\frac{2\pi}{g+1}$ which has 4 fixed points (the axis of rotation goes through two points on each sphere) and has order $g+1$. Given a positive integer i satisfying $1 \le i \le g+1$, let

$$\Gamma_i = \mathrm{graph}(\gamma^i) = \{(x, \gamma^i(x)) \mid x \in \Sigma_g\} \subset \Sigma_g \times \Sigma_g.$$

Since $\gamma^{g+1} = \mathrm{id}$, the graph Γ_{g+1} is the diagonal of $\Sigma_g \times \Sigma_g$.

Lemma 2.1. *Let ω be a symplectic form on Σ_g, and let $f : \Sigma_g \to \Sigma_g$ be an orientation-preserving self-diffeomorphism. Then the graph of f, $\{(x, f(x)) \mid x \in \Sigma_g\} \subset \Sigma_g \times \Sigma_g$, is a symplectic submanifold with respect to a product symplectic form $\tilde{\omega} = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega$ on $\Sigma_g \times \Sigma_g$, where $\mathrm{pr}_j : \Sigma_g \times \Sigma_g \to \Sigma_g$ is the projection map onto the j-th factor.*

Proof. Consider the embedding $h : \Sigma_g \to \Sigma_g \times \Sigma_g$ given by $h(x) = (x, f(x))$. It is enough to show that $h^*\tilde{\omega}$ is a positive multiple of ω. Given a point $p \in \Sigma_g$, choose local coordinates (x_1, x_2) in a neighborhood U of p and (y_1, y_2) in a neighborhood V of $f(p)$ such that

$$\begin{aligned} \omega|_U &= \xi(x_1, x_2)\, dx_1 \wedge dx_2, \\ \omega|_V &= \eta(y_1, y_2)\, dy_1 \wedge dy_2, \end{aligned}$$

where ξ and η are strictly positive functions defined on U and V, respectively. On the graph of f, we have

$$(y_1, y_2) = f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2)).$$

It follows that

$$\begin{aligned} (h^*\tilde{\omega})|_U &= \xi\, dx_1 \wedge dx_2 \,+\, (\eta \circ f)\, df_1 \wedge df_2 \\ &= \xi\, dx_1 \wedge dx_2 \,+\, (\eta \circ f) \left(\frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 \right) \wedge \left(\frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 \right) \\ &= (\xi + (\eta \circ f) \det(Df))\, dx_1 \wedge dx_2 \\ &= \left(1 + \frac{(\eta \circ f) \det(Df)}{\xi} \right) \omega|_U, \end{aligned}$$

where $Df = (\partial f_i / \partial x_j)$ denotes the 2×2 matrix of partial derivatives of f with respect to the above coordinates. Since f is orientation-preserving, we always have $\det(Df) > 0$. Hence we have shown that $h^*\tilde{\omega}$ is equal to some positive function times ω. □

Thus Γ_i is a symplectic submanifold of $\Sigma_g \times \Sigma_g$ with respect to $\tilde{\omega}$ for each $1 \le i \le g+1$. We have $[\Gamma_i]^2 = L(\gamma^i, \gamma^i) = \deg(\gamma^i) e(\Sigma_g) = 2 - 2g$ for each $1 \le i \le g+1$. (Here, L denotes the Lefschetz coincidence number and e denotes the Euler characteristic.) Also note that the graphs $\Gamma_1, \ldots, \Gamma_{g+1}$ intersect at 4 points. If we blow up at these 4 intersection points, the proper transform B of the union $\Gamma_1 \cup \cdots \cup \Gamma_{g+1}$ consists of $g+1$ disjoint surfaces and its homology class is divisible by $g+1$. If PD denotes the Poincaré duality isomorphism, then

$$PD(c_1((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2)) = -2(g-1)\big([\Sigma_g \times \{\mathrm{pt}\}] + [\{\mathrm{pt}'\} \times \Sigma_g]\big) - \sum_{j=1}^{4}[E_j],$$

where E_j is the exceptional sphere of the j-th blow-up for $j = 1, \ldots, 4$. The homology class of B in $H_2((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2; \mathbb{Z})$ is given by

$$[B] = (g+1)\Big([\Sigma_g \times \{\mathrm{pt}\}] + [\{\mathrm{pt}'\} \times \Sigma_g] - \sum_{j=1}^{4}[E_j]\Big).$$

Let X_g be the $(g+1)$-fold branched cover of $(\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2$ branched along this proper transform B. Using formulas in Section 7.1 of [8], we compute that

$$\begin{aligned}
e(X_g) &= (g+1)e((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2) - g(g+1)e(\Sigma_g) \\
&= 2(g+1)(3g^2 - 5g + 4), \\
c_1^2(X_g) &= (g+1)\Big(c_1((\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2) - \frac{g}{g+1}[B]\Big)^2 \\
&= 2(g+1)(7g^2 - 8g + 2), \\
\sigma(X_g) &= \frac{c_1^2(X_g) - 2e(X_g)}{3} = \frac{2}{3}(g+1)(g^2 + 2g - 6), \\
\chi_h(X_g) &= \frac{e(X_g) + \sigma(X_g)}{4} = \frac{1}{6}(g+1)(10g^2 - 13g + 6).
\end{aligned}$$

Consequently, we have

$$\lim_{g \to \infty} \frac{c_1^2(X_g)}{\chi_h(X_g)} = \frac{84}{10}.$$

Table 1 lists the characteristic numbers of X_g when $1 \leq g \leq 10$.

Table 1

g	1	2	3	4	5	6	7	8	9	10
$e(X_g)$	8	36	128	320	648	1148	1856	2808	4040	5588
$\sigma(X_g)$	-4	4	24	60	116	196	304	444	620	836
$\chi_h(X_g)$	1	10	38	95	191	336	540	813	1165	1606
$c_1^2(X_g)$	4	84	328	820	1644	2884	4624	6948	9940	13684

As explained in Section 2 of [17], the composition of maps

$$X_g \longrightarrow (\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2 \longrightarrow \Sigma_g \times \Sigma_g \xrightarrow{\mathrm{pr}_1} \Sigma_g \tag{1}$$

gives a Lefschetz fibration of X_g over Σ_g. Here, pr_1 denotes the projection onto the first factor. A regular fiber of this fibration is a cyclic $(g+1)$-fold cover of Σ_g branched over $g+1$ points. Thus a regular fiber is a surface of genus $\frac{1}{2}g(3g+1)$. The proper transform of each graph Γ_i gives rise to a section of (1) whose image is a genus g surface of self-intersection -2.

For each positive integer $n \geq 2$, let $\varphi_n : \Sigma_k \to \Sigma_g$ be an n-fold unbranched covering of Σ_g, where $k = n(g-1)+1$. Let $X_g(n)$ denote the total space of the pull back of fibration (1) via φ_n. A regular fiber of $X_g(n) \to \Sigma_k$ is again a surface of genus $\frac{1}{2}g(3g+1)$. The new fibration has a section whose image is a genus k surface of self-intersection $-2n$. Since $X_g(n)$ can be viewed as an n-fold unbranched cover of X_g, we have $e(X_g(n)) = n \cdot e(X_g)$, $\sigma(X_g(n)) = n \cdot \sigma(X_g)$, $\chi_h(X_g(n)) = n \cdot \chi_h(X_g)$, and $c_1^2(X_g(n)) = n \cdot c_1^2(X_g)$.

As in Section 2 of [18], we can also pull back the branched covering

$$X_g \longrightarrow (\Sigma_g \times \Sigma_g)\#4\overline{\mathbb{CP}}^2 \longrightarrow \Sigma_g \times \Sigma_g$$

via the map $\varphi_n \times \varphi_n : \Sigma_k \times \Sigma_k \to \Sigma_g \times \Sigma_g$, and obtain a new symplectic 4-manifold $\tilde{X}_g(n^2)$, which is a $(g+1)$-fold branched cover of $\Sigma_k \times \Sigma_k$ and an n^2-fold unbranched cover of X_g. The composition

$$\tilde{X}_g(n^2) \longrightarrow \Sigma_k \times \Sigma_k \xrightarrow{\mathrm{pr}_1} \Sigma_k$$

gives a new Lefschetz fibration whose regular fiber has genus $1 + \frac{1}{2}n(g+1)(3g-2)$. This fibration has a section whose image is a genus k surface of self-intersection $-2n$. We have $e(\tilde{X}_g(n^2)) = n^2 \cdot e(X_g)$, $\sigma(\tilde{X}_g(n^2)) = n^2 \cdot \sigma(X_g)$, $\chi_h(\tilde{X}_g(n^2)) = n^2 \cdot \chi_h(X_g)$, and $c_1^2(\tilde{X}_g(n^2)) = n^2 \cdot c_1^2(X_g)$.

For the applications in Sections 4 and 5, we will only need the symplectic 4-manifold X_2. The other symplectic 4-manifolds that were constructed in this section are used in the sequel [2] to construct simply connected symplectic 4-manifolds with signature $\sigma \geq 5$.

3. Second family with positive signature

Let g be a positive integer. As in Exercise IV.5.6 of [11], we can think of the genus g surface Σ_g as a $4g$-gon with diametrically opposite edges identified so that the word given by the boundary of the $4g$-gon is

$$a_1 a_2 \cdots a_{2g} a_1^{-1} a_2^{-1} \cdots a_{2g}^{-1}.$$

Divide this $4g$-gon into two $(2g+1)$-gons by cutting along a diagonal d such that the boundaries of the resulting two $(2g+1)$-gons give the words

$$a_1 a_2 \cdots a_{2g} d \quad \text{and} \quad a_1^{-1} a_2^{-1} \cdots a_{2g}^{-1} d^{-1}.$$

Viewing each $(2g+1)$-gon as a regular polygon, we can rotate each $(2g+1)$-gon by angle $\frac{2\pi}{2g+1}$, and then reglue them to obtain an orientation-preserving self-diffeomorphism

$\delta : \Sigma_g \to \Sigma_g$ of order $2g+1$ with 3 fixed points. For $1 \le i \le 2g+1$, let Δ_i be graph$(\delta^i) \subset \Sigma_g \times \Sigma_g$. As in Section 2, $[\Delta_i]^2 = 2-2g$ for $1 \le i \le 2g+1$, and each Δ_i is a symplectic submanifold of $\Sigma_g \times \Sigma_g$, which is equipped with a product symplectic form $\mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega$. Note that $\Delta_1, \dots, \Delta_{2g+1}$ intersect in 3 points. If we blow up at these 3 points, the homology class of the proper transform D of the union $\Delta_1 \cup \cdots \cup \Delta_{2g+1}$ will be divisible by $2g+1$. Let Z_g be the $(2g+1)$-fold branched cover of $(\Sigma_g \times \Sigma_g)\#3\overline{\mathbb{CP}}^2$ branched along the proper transform D. We compute that

$$\begin{aligned}
e(Z_g) &= (2g+1)e((\Sigma_g \times \Sigma_g)\#3\overline{\mathbb{CP}}^2) - 2g(2g+1)e(\Sigma_g) \\
&= (2g+1)(8g^2-12g+7), \\
c_1^2(Z_g) &= (2g+1)\Big(c_1((\Sigma_g \times \Sigma_g)\#3\overline{\mathbb{CP}}^2) - \frac{2g}{2g+1}[D]\Big)^2 \\
&= 5(2g+1)(4g^2-4g+1), \\
\sigma(Z_g) &= \frac{c_1^2(Z_g) - 2e(Z_g)}{3} = \frac{1}{3}(2g+1)(4g^2+4g-9), \\
\chi_h(Z_g) &= \frac{e(Z_g)+\sigma(Z_g)}{4} = \frac{1}{3}(2g+1)(7g^2-8g+3).
\end{aligned}$$

We conclude that

$$\lim_{g\to\infty} \frac{c_1^2(Z_g)}{\chi_h(Z_g)} = \frac{60}{7} \approx 8.571428571.$$

Table 2 lists the characteristic numbers of Z_g when $1 \le g \le 10$.

Table 2

g	1	2	3	4	5	6	7	8	9	10
$e(Z_g)$	9	75	301	783	1617	2899	4725	7191	10393	14427
$\sigma(Z_g)$	-1	25	91	213	407	689	1075	1581	2223	3017
$\chi_h(Z_g)$	2	25	98	249	506	897	1450	2193	3154	4361
$c_1^2(Z_g)$	15	225	875	2205	4455	7865	12675	19125	27455	37905

When $g=2$, we obtain the 4-manifold $H(1)$ on the BMY line in [17]. The $g=1$ case is also interesting in terms of the geography of small 4-manifolds. It is obtained from $(T^2 \times T^2)\#3\overline{\mathbb{CP}}^2$ by taking a 3-fold branched cover.

As in Section 2, there is a Lefschetz fibration

$$Z_g \longrightarrow (\Sigma_g \times \Sigma_g)\#3\overline{\mathbb{CP}}^2 \longrightarrow \Sigma_g \times \Sigma_g \xrightarrow{\mathrm{pr}_1} \Sigma_g. \tag{2}$$

A regular fiber of this fibration is a $(2g+1)$-fold cover of Σ_g branched over $2g+1$ points. Hence a regular fiber is a surface of genus $4g^2$. The proper transform of each graph Δ_i gives rise to a section of (2) whose image is a genus g surface of self-intersection -1.

By pulling back (2) via the unbranched covering $\varphi_n : \Sigma_k \to \Sigma_g$ as in Section 2, we obtain a new Lefschetz fibration $Z_g(n) \to \Sigma_k$, where $k = n(g-1)+1$. A regular fiber is a genus $4g^2$ surface and there is a genus k section of self-intersection $-n$. We have $e(Z_g(n)) = n \cdot e(Z_g)$, $\sigma(Z_g(n)) = n \cdot \sigma(Z_g)$, $\chi_h(Z_g(n)) = n \cdot \chi_h(Z_g)$, and $c_1^2(Z_g(n)) = n \cdot c_1^2(Z_g)$.

As in Section 2, we can also construct a family of symplectic 4-manifolds $\tilde{Z}_g(n^2)$ satisfying $e(\tilde{Z}_g(n^2)) = n^2 \cdot e(Z_g)$, $\sigma(\tilde{Z}_g(n^2)) = n^2 \cdot \sigma(Z_g)$, $\chi_h(\tilde{Z}_g(n^2)) = n^2 \cdot \chi_h(Z_g)$, and $c_1^2(\tilde{Z}_g(n^2)) = n^2 \cdot c_1^2(Z_g)$. There exists a Lefschetz fibration $\tilde{Z}_g(n^2) \to \Sigma_k$ whose regular fiber has genus $1+n(4g^2-1)$. This fibration has a section whose image is a genus k surface of self-intersection $-n$.

The symplectic 4-manifolds $Z_g(n)$ and $\tilde{Z}_g(n^2)$ will be useful in the sequel [2] when we construct simply connected symplectic 4-manifolds with signature $\sigma \geq 5$.

4. Simply connected building blocks

The goal of this section is to prove that there exist exotic irreducible smooth structures on $47\mathbb{CP}^2\#45\overline{\mathbb{CP}}^2$ and $51\mathbb{CP}^2\#47\overline{\mathbb{CP}}^2$.

Theorem 4.1. *There exists a closed simply connected minimal symplectic* 4*-manifold* M *such that* $e(M) = 94$ *and* $\sigma(M) = 2$*. Moreover,* M *contains a symplectic torus* T *with self-intersection* 0 *satisfying* $\pi_1(M \setminus T) = 1$.

Proof. Our 4-manifold M will be a symplectic sum (cf. [7]) of two symplectic 4-manifolds along genus 9 surfaces of self-intersection 0. Let X_2 be the symplectic 4-manifold that we constructed in Section 2. Recall that X_2 is the total space of a genus 7 Lefschetz fibration over a genus 2 surface. Also recall that this Lefschetz fibration has a section whose image has self-intersection -2 in X_2. Thus we may take a regular fiber and a section and then symplectically resolve their intersection to obtain a genus 9 symplectic surface $\Sigma_9' \subset X_2$ with self-intersection 0.

Next let $Y_7(1)$ be the minimal symplectic 4-manifold constructed in Section 2 of [3]. Recall from [3] that $Y_7(1)$ has the same cohomology ring as the connected sum $11(S^2 \times S^2)$ and that $Y_7(1)$ is obtained from $\Sigma_2 \times \Sigma_7$ by performing 18 Luttinger surgeries. As observed in [3], the geometrically dual symplectic surfaces $\Sigma_2 \times \{\mathrm{pt}\}$ and $\{\mathrm{pt}'\} \times \Sigma_7$ in $\Sigma_2 \times \Sigma_7$ descend to geometrically dual symplectic surfaces in $Y_7(1)$. If we symplectically resolve the intersection between these two surfaces, then we obtain a genus 9 symplectic surface of self-intersection 2 in $Y_7(1)$. Symplectically blowing up twice, we obtain a genus 9 symplectic surface Σ_9'' of self-intersection 0 in $Y_7(1)\#2\overline{\mathbb{CP}}^2$.

Now let M be the symplectic sum $X_2\#_{\Sigma_9'=\Sigma_9''}(Y_7(1)\#2\overline{\mathbb{CP}}^2)$. We compute that

$$\begin{aligned} e(M) &= e(X_2) + e(Y_7(1)\#2\overline{\mathbb{CP}}^2) - 2e(\Sigma_9) = 36 + 26 - 2(-16) = 94, \\ \sigma(M) &= \sigma(X_2) + \sigma(Y_7(1)\#2\overline{\mathbb{CP}}^2) = 4 - 2 = 2. \end{aligned}$$

Since X_2 is a relatively minimal Lefschetz fibration over a surface of positive genus, X_2 is minimal by Theorem 1.4 in [19]. Also note that the pair $(Y_7(1)\#2\overline{\mathbb{CP}}^2, \Sigma_9'')$ is a relatively minimal pair by Corollary 3 in [10]. It now follows from Usher's theorem in [20] that the symplectic sum M is minimal.

To prove $\pi_1(M)=1$, we proceed as follows. First the long exact homotopy sequence of a fibration (cf. Proposition 8.1.9 in [8]) implies that the inclusion induced homomorphism $\pi_1(\Sigma_9') \to \pi_1(X_2)$ is surjective. Since the exceptional sphere of a blow-up intersects Σ_9'' once transversally, any meridian of Σ_9'' is nullhomotopic in the complement of a tubular neighborhood $\nu\Sigma_9''$. Hence we conclude that

$$\pi_1\big((Y_7(1)\#2\overline{\mathbb{CP}}^2)\setminus \nu\Sigma_9''\big) = \pi_1(Y_7(1)\#2\overline{\mathbb{CP}}^2) = \pi_1(Y_7(1)).$$

Next we choose standard presentations

$$\pi_1(\Sigma_2\times\{\mathrm{pt}\}) \,=\, \langle a_1,b_1,a_2,b_2 \mid [a_1,b_1][a_2,b_2]=1\rangle,$$

$$\pi_1(\{\mathrm{pt}'\}\times\Sigma_7) \,=\, \langle c_1,d_1,\dots,c_7,d_7 \mid \prod_{j=1}^{7}[c_j,d_j]=1\rangle.$$

From [3], we know that the inclusion induced homomorphism

$$\pi_1\big((\Sigma_2\times\{\mathrm{pt}\})\cup(\{\mathrm{pt}'\}\times\Sigma_7)\big) \longrightarrow \pi_1(Y_7(1)) \tag{3}$$

is also surjective. Moreover, we also know that $\pi_1(Y_7(1))/\langle\alpha\rangle = 1$, where α is the image of any one of the generators c_1, d_1, c_2, d_2 of $\pi_1(\{\mathrm{pt}'\}\times\Sigma_7)$ under homomorphism (3).

Let $(\Sigma_9')^{\|}$ and $(\Sigma_9'')^{\|}$ denote parallel copies of Σ_9' and Σ_9'' in the boundaries $\partial(\nu\Sigma_9')$ and $\partial(\nu\Sigma_9'')$, respectively. When forming the symplectic sum M, we choose the gluing diffeomorphism such that α, viewed as an element of $\pi_1((\Sigma_9'')^{\|})$, is mapped to an element of $\pi_1((\Sigma_9')^{\|})$ that is represented by a non-separating vanishing cycle in the fiber of the Lefschetz fibration $X_2\to\Sigma_2$. Thus $\alpha=1$ in $\pi_1(M)$, which then implies that the inclusion induced homomorphism

$$\pi_1\big((Y_7(1)\#2\overline{\mathbb{CP}}^2)\setminus\nu\Sigma_9''\big) \longrightarrow \pi_1(M) \tag{4}$$

is trivial. Note that the inclusion induced homomorphism $\pi_1((\Sigma_9')^{\|})\to\pi_1(M)$ is also trivial since it can be factored through homomorphism (4) after $(\Sigma_9')^{\|}$ is identified with $(\Sigma_9'')^{\|}$. The meridians of Σ_9' are also trivial in $\pi_1(M)$ because they are identified with the meridians of Σ_9'', which are trivial. It follows that the inclusion induced homomorphism $\pi_1(X_2\setminus\nu\Sigma_9')\to\pi_1(M)$ is trivial as well, and by Seifert-Van Kampen theorem, we conclude that $\pi_1(M)=1$.

Finally, note that $Y_7(1)$ contains 10 pairs of geometrically dual Lagrangian tori. The images of these 20 tori in the blow-up $Y_7(1)\#2\overline{\mathbb{CP}}^2$ are disjoint from Σ_9'', and thus they lie in M. Let T denote one of these 20 Lagrangian tori. By perturbing the symplectic form on M, we can turn T into a symplectic submanifold of M.

To show $\pi_1(M\setminus T)=1$, it will be convenient to fix T, say $T=a_1'\times c_3''$. Here, a_1' and c_3'' are parallel copies of a_1 and c_3 as defined in [5]. Then $\pi_1(M\setminus T)$ is normally generated by meridians of T, which are all conjugate to the commutator $[b_1^{-1},d_3]$. Note that the generators b_1 and d_3 are still trivial in $\pi_1(M \setminus T)$ since the Luttinger surgery relations in Section 2 of [3] still hold true in $\pi_1(M \setminus T)$. It follows that meridians of T are trivial and hence $\pi_1(M \setminus T) = 1$. $\square$

Theorem 4.2. *There exists a closed simply connected minimal symplectic* 4*-manifold* N *such that* $e(N) = 100$ *and* $\sigma(N) = 4$*. Moreover,* N *contains a symplectic torus* $\tilde{T}$ *with self-intersection* 0 *satisfying* $\pi_1(N \setminus \tilde{T}) = 1$*.*

Proof. Let $Y_9(1)$ be the minimal symplectic 4-manifold constructed in Section 2 of [3]. $Y_9(1)$ has the same cohomology ring as the connected sum $15(S^2 \times S^2)$, and $Y_9(1)$ is obtained from $\Sigma_2 \times \Sigma_9$ by performing 22 Luttinger surgeries. The symplectic submanifold $\{\mathrm{pt}'\} \times \Sigma_9 \subset \Sigma_2 \times \Sigma_9$ descends to a symplectic submanifold of self-intersection 0 in $Y_9(1)$, which we will denote by Σ_9.

Let $\Sigma_9' \subset X_2$ be the genus 9 symplectic submanifold of self-intersection 0 in the proof of Theorem 4.1. Let N be the symplectic sum $X_2 \#_{\Sigma_9' = \Sigma_9} Y_9(1)$. Usher's theorem (cf. [20]) again implies that N is minimal. We compute that

$$\begin{aligned} e(N) &= e(X_2) + e(Y_9(1)) - 2e(\Sigma_9) = 36 + 32 - 2(-16) = 100, \\ \sigma(N) &= \sigma(X_2) + \sigma(Y_9(1)) = 4 + 0 = 4. \end{aligned}$$

To prove $\pi_1(N) = 1$, first choose a standard presentation

$$\pi_1(\Sigma_9) = \langle c_1, d_1, \ldots, c_9, d_9 \mid \prod_{j=1}^{9} [c_j, d_j] = 1 \rangle.$$

From [3], we know that $\pi_1(Y_9(1))/\langle \alpha \rangle = 1$, where α is the image of any one of the four generators c_1, d_1, c_2, d_2 of $\pi_1(\Sigma_9)$ under the homomorphism $\pi_1(\Sigma_9) \to \pi_1(Y_9(1))$ induced by inclusion. We also know that a meridian of Σ_9 is conjugate to the image of $[a_1, b_1][a_2, b_2]$ in $\pi_1(Y_9(1) \setminus \nu\Sigma_9)$, where a_i, b_i $(i = 1, 2)$ are the standard generators of $\pi_1(\Sigma_2 \times \{\mathrm{pt}\})$. Since $\alpha = 1$ implies $a_i = b_i = 1$ $(i = 1, 2)$, we deduce that $\pi_1(Y_9(1) \setminus \nu\Sigma_9)/\langle \alpha \rangle = 1$.

When forming the symplectic sum N, we choose the gluing diffeomorphism such that α, viewed as an element of $\pi_1(\Sigma_9^{\|})$, is mapped to an element of $\pi_1((\Sigma_9')^{\|})$ that is represented by a non-separating vanishing cycle in the fiber of the Lefschetz fibration $X_2 \to \Sigma_2$. Thus $\alpha = 1$ in $\pi_1(N)$, which then implies that the inclusion induced homomorphism

$$\pi_1(Y_9(1) \setminus \nu\Sigma_9) \longrightarrow \pi_1(N) \tag{5}$$

is trivial. Note that the inclusion induced homomorphism $\pi_1((\Sigma_9')^{\|}) \to \pi_1(N)$ is also trivial since it can be factored through homomorphism (5) after $(\Sigma_9')^{\|}$ is identified with $\Sigma_9^{\|}$. The meridians of Σ_9' are also trivial in $\pi_1(N)$ because they are identified with the meridians of Σ_9, which are trivial. It follows that the inclusion induced homomorphism $\pi_1(X_2 \setminus \nu\Sigma_9') \to \pi_1(N)$ is trivial as well, and by Seifert-Van Kampen theorem, we conclude that $\pi_1(N) = 1$.

Finally, note that $Y_9(1)$ contains 14 pairs of geometrically dual Lagrangian tori that are all disjoint from Σ_9. Let $\tilde{T}$ denote one of these 28 Lagrangian tori, say $a_1' \times c_3''$.

By perturbing the symplectic form on N, we can turn $\tilde{T}$ into a symplectic submanifold of N. We can deduce that $\pi_1(N \setminus \tilde{T}) = 1$ in exactly the same way as in the proof of Theorem 4.1. □

By Rohlin's theorem (cf. [16]), M and N must have odd intersection form since their signatures are not divisible by 16. Hence by Freedman's classification theorem for simply connected topological 4-manifolds (cf. [6]), M and N must be homeomorphic to $47\mathbb{CP}^2\#45\overline{\mathbb{CP}}^2$ and $51\mathbb{CP}^2\#47\overline{\mathbb{CP}}^2$, respectively.

Remark 4.3. Note that a simply connected minimal symplectic 4-manifold is always irreducible (cf. [9]). By replacing $Y_7(1)$ in the construction of M with $Y_7(m)$ for positive integers $m \geq 2$ (cf. [3]), we obtain an infinite family of pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds that are homeomorphic to $47\mathbb{CP}^2\#45\overline{\mathbb{CP}}^2$. Similarly, replacing $Y_9(1)$ in the construction of N with $Y_9(m)$ gives an infinite family of pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds homeomorphic to $51\mathbb{CP}^2\#47\overline{\mathbb{CP}}^2$.

5. Nonspin 4-manifolds with nonnegative signature

In this section, we use the two simply connected symplectic 4-manifolds constructed in Section 4 to build infinite families of other examples. For convenience we introduce the following definition.

Definition 5.1. Let X be a smooth 4-manifold. We say that X has ∞-*property* if there exist an irreducible symplectic 4-manifold and infinitely many pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds, all of which are homeomorphic to X.

Remark 4.3 says that $47\mathbb{CP}^2\#45\overline{\mathbb{CP}}^2$ and $51\mathbb{CP}^2\#47\overline{\mathbb{CP}}^2$ both have ∞-property. Recall from [1] that $m\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$ has ∞-property for m odd and $m \geq 91$. We also recall Theorem 2 in [3] which is restated below.

Theorem 5.2 (cf. [3])**.** *Let X be a closed symplectic 4-manifold and suppose that X contains a symplectic torus T of self-intersection 0 such that the homomorphism $\pi_1(T) \to \pi_1(X)$ induced by the inclusion is trivial. Then for any pair (χ, c) of nonnegative integers satisfying*

$$0 \leq c \leq 8\chi - 1, \tag{6}$$

there exists a symplectic 4-manifold Y with $\pi_1(Y) = \pi_1(X)$,

$$\chi_h(Y) = \chi_h(X) + \chi \quad \textit{and} \quad c_1^2(Y) = c_1^2(X) + c.$$

Moreover, Y has an odd indefinite intersection form, and if X is minimal then Y is minimal as well. □

A brief synopsis of the proof of Theorem 5.2 is as follows. For each pair of nonnegative integers (χ, c) satisfying (6), [1] and [3] explicitly construct an odd minimal symplectic 4-manifold W with $\chi_h(W) = \chi$, $c_1^2(W) = c$, and a symplectic torus $T' \subset W$ of self-intersection 0. Our Y is then the symplectic sum $X\#_{T=T'}W$.

If we strengthen the hypothesis of Theorem 5.2 a bit, then we can add the line $c = 8\chi$ to wedge-like region (6).

Theorem 5.3. *Let X be a closed symplectic 4-manifold that contains a symplectic torus T of self-intersection 0. Let νT be a tubular neighborhood of T and $\partial(\nu T)$ its boundary. Suppose that the homomorphism $\pi_1(\partial(\nu T)) \rightarrow \pi_1(X \setminus \nu T)$ induced by the inclusion is trivial. Then for any pair of integers (χ, c) satisfying*

$$\chi \geq 1 \quad \textit{and} \quad 0 \leq c \leq 8\chi, \tag{7}$$

there exists a symplectic 4-manifold Y with $\pi_1(Y) = \pi_1(X)$,

$$\chi_h(Y) = \chi_h(X) + \chi \quad \textit{and} \quad c_1^2(Y) = c_1^2(X) + c.$$

Moreover, if X is minimal then Y is minimal as well. If $c < 8\chi$, or if $c = 8\chi$ and X has an odd intersection form, then the corresponding Y has an odd indefinite intersection form.

Proof. In light of Theorem 5.2, it only remains to check the case when $c = 8\chi$ and $\chi \geq 1$. For each integer $\chi \geq 1$, it will be enough to exhibit a closed symplectic 4-manifold W having the following properties.

(i) $\chi_h(W) = \chi$ and $c_1^2(W) = 8\chi$.
(ii) W contains a symplectic torus T' of self-intersection 0 such that the complement $W \setminus T'$ does not contain any symplectic sphere of self-intersection -1.
(iii) $\pi_1(W \setminus \nu T')/K = 1$, where K is the normal subgroup of $\pi_1(W \setminus \nu T')$ that is generated by the image of the homomorphism $\pi_1(\partial(\nu T')) \rightarrow \pi_1(W \setminus \nu T')$ induced by the inclusion.

The desired 4-manifold Y is then the symplectic sum $X \#_{T=T'} W$. An easy application of Seifert-Van Kampen theorem gives us $\pi_1(Y) = \pi_1(X \setminus \nu T) = \pi_1(X)$. The minimality of Y follows from the minimality of X by Usher's theorem in [20]. Other properties of Y can also be immediately verified.

Now we proceed to the construction of W. Given a positive integer χ, consider the closed minimal symplectic 4-manifold $Y_{\chi+1}(1)$ that was constructed in Section 2 of [3]. Recall that $Y_{\chi+1}(1)$ is obtained from the cartesian product $\Sigma_2 \times \Sigma_{\chi+1}$ by performing $2\chi + 6$ Luttinger surgeries. If we choose not to perform one of these $2\chi + 6$ Luttinger surgeries, i.e. we only perform $2\chi + 5$ Luttinger surgeries on $\Sigma_2 \times \Sigma_{\chi+1}$, then the resulting symplectic 4-manifold W is still minimal and satisfies

$$\chi_h(W) = \chi_h(Y_{\chi+1}(1)) = \chi \quad \text{and} \quad c_1^2(W) = c_1^2(Y_{\chi+1}(1)) = 8\chi.$$

For concreteness, suppose that we do not perform $(a_1' \times c_1', a_1', -1)$ Luttinger surgery in [3]. This means that the Luttinger surgery relation $a_1 = [b_1^{-1}, d_1^{-1}]$ no longer holds in $\pi_1(W)$. We let $T' = a_1' \times c_1'$. By perturbing the symplectic form on W, we can turn the Lagrangian torus $a_1' \times c_1'$ into a symplectic submanifold. Inside the quotient group $\pi_1(W \setminus \nu T')/K$, the images of the generators a_1 and c_1 are trivial. Combined with the remaining Luttinger relations from [3], which continue to hold in $\pi_1(W \setminus \nu T')/K$, this implies that the images of all other generators of $\pi_1(W \setminus \nu T')$ are trivial in the quotient group. $\square$

The following corollary gives infinitely many irreducible smooth structures on a large class of simply connected nonspin 4-manifolds with signature $\sigma = 0, 1, 2$.

Corollary 5.4. *Let m be an odd positive integer. If $m \geq 49$, then $m\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$ and $m\mathbb{CP}^2\#(m-1)\overline{\mathbb{CP}}^2$ have ∞-property. If $m \geq 47$, then $m\mathbb{CP}^2\#(m-2)\overline{\mathbb{CP}}^2$ has ∞-property.*

Proof. From Remark 4.3, we already know that $47\mathbb{CP}^2\#45\overline{\mathbb{CP}}^2$ has ∞-property. We apply Theorem 5.3 to the symplectic 4-manifold M in Theorem 4.1. Since $\chi_h(M) = 24$ and $c_1^2(M) = 194$, we obtain a simply connected minimal symplectic 4-manifold Y satisfying $\chi_h(Y) = \chi + 24$ and $c_1^2(Y) = c + 194$. By Freedman's theorem (cf. [6]), such Y is homeomorphic to $(2\chi + 47)\mathbb{CP}^2\#(10\chi - c + 45)\overline{\mathbb{CP}}^2$. By setting

$$c = 8\chi - s, \quad \text{where} \quad s \in \{0, 1, 2\},$$

in (7), we obtain a simply connected minimal symplectic 4-manifold Y that is homeomorphic to

$$(2\chi + 47)\mathbb{CP}^2\#(2\chi + 45 + s)\overline{\mathbb{CP}}^2$$

for each integer $\chi \geq 1$. Simply connected minimal symplectic 4-manifolds are also irreducible ([9]). By performing surgeries along a nullhomologous torus in Y as in [5] (cf. Remark 4.3), we obtain infinite families of pairwise non-diffeomorphic irreducible non-symplectic 4-manifolds homeomorphic to $(2\chi + 47)\mathbb{CP}^2\#(2\chi + 45 + s)\overline{\mathbb{CP}}^2$. $\square$

The next corollary gives infinitely many irreducible smooth structures on a large class of simply connected nonspin 4-manifolds with signature $\sigma = 3, 4$.

Corollary 5.5. *Let m be an odd positive integer. If $m \geq 53$, then $m\mathbb{CP}^2\#(m-3)\overline{\mathbb{CP}}^2$ has ∞-property. If $m \geq 51$, then $m\mathbb{CP}^2\#(m-4)\overline{\mathbb{CP}}^2$ has ∞-property.*

Proof. From Remark 4.3, we already know that $51\mathbb{CP}^2\#47\overline{\mathbb{CP}}^2$ has ∞-property. We now apply Theorem 5.3 to the symplectic 4-manifold N in Theorem 4.2. Since $\chi_h(N) = 26$ and $c_1^2(N) = 212$, we obtain a simply connected minimal symplectic 4-manifold Y satisfying $\chi_h(Y) = \chi + 26$ and $c_1^2(Y) = c + 212$. By Freedman's theorem, such Y is homeomorphic to $(2\chi + 51)\mathbb{CP}^2\#(10\chi - c + 47)\overline{\mathbb{CP}}^2$. By setting

$$c = 8\chi - s, \quad \text{where} \quad s \in \{0, 1\}, \tag{8}$$

in (7), we obtain a simply connected minimal symplectic 4-manifold Y that is homeomorphic to

$$(2\chi + 51)\mathbb{CP}^2\#(2\chi + 47 + s)\overline{\mathbb{CP}}^2$$

for each integer $\chi \geq 1$. The rest of the proof goes exactly the same way as the proof of Corollary 5.4. $\square$

Remark 5.6. In the proof of Corollary 5.5, if we take $s \in \{2,3,4\}$ in (8) instead, we obtain infinitely many irreducible smooth structures on

$$m\mathbb{CP}^2\#(m-2)\overline{\mathbb{CP}}^2, \quad m\mathbb{CP}^2\#(m-1)\overline{\mathbb{CP}}^2 \quad \text{or} \quad m\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2,$$

respectively, for each odd integer $m \geq 53$. At this moment, the authors do not know whether these exotic 4-manifolds are diffeomorphic to the corresponding 4-manifolds that were constructed in the proof of Corollary 5.4.

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