

Locality of continuous Hamiltonian flows and Lagrangian intersections with the conormal of open subsets

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ABSTRACT. In this paper, we prove that if a continuous Hamiltonian flow fixes the points in an open subset U of a symplectic manifold (M, ω) , then its associated Hamiltonian is constant at each moment on U . As a corollary, we prove that the Hamiltonian of compactly supported continuous Hamiltonian flows is unique both on a compact M with contact-type boundary ∂M and on a non-compact manifold bounded at infinity. An essential tool for the proof of the locality is the Lagrangian intersection theorem for the conormals of open subsets proven by Kasturirangan and the author [KO], combined with Viterbo's scheme that he introduced in the proof of uniqueness of the Hamiltonian on a closed manifold [V2]. We also prove the converse of the theorem which localizes a previously known global result in symplectic topology.

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1. Introduction

Let ϕ be a Hamiltonian diffeomorphism on a symplectic manifold (M, ω) . Hofer's L^∞ -norm of Hamiltonian diffeomorphisms is defined by

$$\|\phi\| = \inf_{H \mapsto \phi} \|H\|_\infty$$

where $H \mapsto \phi$ means that $\phi = \phi_H^1$ is the time-one map of Hamilton's equation

$$\dot{x} = X_H(t, x)$$

and the norm $\|H\|$ is defined by

$$\|H\|_\infty = \max_{t \in [0, 1]} \text{osc } H_t \quad (1)$$

where

$$\text{osc } H_t = \max_x H_t - \min_x H_t.$$

We equip the group $\text{Homeo}(X)$ of homeomorphisms with the metric defined as

$$\bar{d}(h, k) = \max_{x \in X} (d(h(x), k(x)) + d(h^{-1}(x), k^{-1}(x)))$$

where d is the distance induced by any given Riemannian metric. With this metric, $\text{Homeo}(X)$ becomes a topological group which is a complete metric space. For two given continuous paths $\lambda, \mu : [a, b] \rightarrow \text{Homeo}(M)$, we define their distance by

$$\bar{d}(\lambda, \mu) = \max_{t \in [a, b]} \bar{d}(\lambda(t), \mu(t)). \quad (2)$$

Definition 1.1 (Continuous Hamiltonian flow). A continuous map $\lambda : \mathbb{R} \rightarrow \text{Homeo}(M)$ is called a *continuous Hamiltonian flow* if there exists a sequence of smooth Hamiltonians $H_i : \mathbb{R} \times M \rightarrow \mathbb{R}$ satisfying the following :

- (1) $\phi_{H_i} \rightarrow \lambda$ locally uniformly on $\mathbb{R} \times M$.
- (2) the sequence H_i is Cauchy in the C^0 -topology and so has a limit H lying in C^0 .

We call a continuous path $\lambda : [a, b] \rightarrow \text{Homeo}(M)$ a *continuous Hamiltonian path* if it satisfies the same conditions with \mathbb{R} replaced by $[a, b]$, and the limit C^0 -function H a *continuous Hamiltonian*. In any of these cases, we say that the pair (λ, H) is the *C^0 -hamiltonian limit* of (ϕ_{H_i}, H_i) , and write

$$\text{hlim}_{i \rightarrow \infty} (\phi_{H_i}, H_i) \rightarrow (\lambda, H)$$

or sometimes even $\text{hlim}_{i \rightarrow \infty} (\phi_{H_i}, H_i) = \lambda$.

We denote by $\mathcal{P}_{\infty; [a, b]}^{\text{ham}}(\text{Sympeo}(M, \omega), id)$ the set of continuous Hamiltonian paths defined on $[a, b]$. When $[a, b] = [0, 1]$ or when we do not specify the domain of λ , we often just write $\mathcal{P}_{\infty}^{\text{ham}}(\text{Sympeo}(M, \omega), id)$ for the corresponding set of continuous Hamiltonian paths.

For the rest of the paper until the last section, we will assume that M is closed.

A proof of the following theorem was provided by Viterbo in [V2] for the C^0 -hamiltonian limits. An immediate corollary of the theorem is that each continuous Hamiltonian path carries a unique Hamiltonian. This uniqueness question was asked by the author in [OM].

Theorem 1.2 (Viterbo, [V2]). *Let M be a closed symplectic manifold. Suppose $H_i : [0, 1] \times M \rightarrow \mathbb{R}$ is a sequence of smooth normalized Hamiltonian functions converging to a continuous function $H : Y \rightarrow \mathbb{R}$ in C^0 -topology. Then if $\phi_{H_i} \rightarrow id$ uniformly on $[0, 1] \times M$, we have $H \equiv 0$.*

The main purpose of the present paper is to prove the following local version of Theorem 1.2 and its converse.

Theorem 1.3 (C^0 -Locality). *Let $U \subset M$ be an open subset. Suppose $H_i : [0, 1] \times M \rightarrow \mathbb{R}$ is a sequence of smooth normalized Hamiltonian functions such that both ϕ_{H_i} and H_i converge in C^0 -topology. Denote their limits by λ and H respectively. Then the followings are equivalent*

- (1) $\lambda_t(x) = x$ for all $(t, x) \in [0, 1] \times U$,
- (2) $H(t, x) \equiv c(t)$ on U for a continuous function $c : [0, 1] \rightarrow \mathbb{R}$ depending only on t .

This locality of Hamiltonians is an easy and well-known fact for the case of smooth (or more generally C^1) Hamiltonian paths. The case $U = M$ is precisely the uniqueness question asked in [OM] and answered by Viterbo. Our proof uses the crucial idea from [V2] which reduces the uniqueness proof to a certain Lagrangian intersection theorem.

One rather immediate consequence of the locality theorem above is the following uniqueness theorem for *compactly supported* continuous Hamiltonian flows. We refer to [OM] for the precise definition of compactly supported topological Hamiltonian flows (i.e., $L^{(1, \infty)}$ -case) which can be adapted to the current continuous Hamiltonian flows (i.e., C^0 -case).

Theorem 1.4 (C^0 -Uniqueness on open manifolds). *Let M be a compact symplectic manifold with contact-type boundary ∂M or a noncompact manifold bounded at infinity. Suppose $H_i : [0, 1] \times M \rightarrow \mathbb{R}$ is a sequence of smooth Hamiltonian functions such that*

- (1) *there exists a compact set $K \subset \text{Int } M$ such that*

$$\text{supp } H_i \subset K$$

- (2) *H_i uniformly converges to a continuous function $H : M \rightarrow \mathbb{R}$.*

Then if $\phi_{H_i} \rightarrow id$ uniformly on $[0, 1] \times M$, we have $H \equiv 0$.

The special case of two-dimensional compact surface with smooth boundary is an immediate consequence of the uniqueness result on closed surfaces by the (space)-doubling argument. But this doubling argument cannot be applied to the high dimensional cases. The presence of the geometric conditions ‘with contact-type boundary’ or ‘bounded an infinity’ in the hypothesis is an artifact of our proof that relies on the Floer homology theory of conormals from [KO]. It is not obvious to the author whether the same uniqueness holds on arbitrary open symplectic manifolds which are not necessarily bounded at infinity.

The main idea of our proof of the locality theorem is again to reduce its proof to a version of Lagrangian intersection theorem. Viterbo used the global version of Lagrangian intersection theorem on the cotangent bundle from [H1], [LS], [Gr]. For the current locality proof, the Lagrangian intersection theorem that we use is the *relative* version of Lagrangian intersection theorem that Kasturirangan and the author proved in [KO] using the study of Floer homology of conormals of open subsets :

Theorem 1.5 (Kasturirangan-Oh, [KO]). *Let $U \subset N$ be any open subset with smooth boundary ∂U , and $H : T^*N \times [0, 1] \rightarrow \mathbb{R}$ be any smooth Hamiltonian. Then*

$$\phi_H^1(o_N) \cap \nu^*\bar{U} \neq \emptyset$$

where $\nu^*\bar{U}$ is the conormal of \bar{U} .

We refer readers to section 2.1 for the definition of the conormal $\nu^*\bar{U}$. Once we have this local Lagrangian intersection theorem, the proof will follow a localized version of Viterbo's argument [V2] after proving a simple lemma in Morse theory. Here we use the constructions of the looping process of Hamiltonian paths and of Lagrangian suspension together with the arguments in geometric measure theory in a more systematic way than Viterbo did in [V2].

Specialized to the case $U = M$ and stripping the usage of conormals of general open subsets but restricting to the zero section of the whole manifold in the Lagrangianization, our proof of the uniqueness theorem, when restricted to the closed case, is essentially the same as the one given by Viterbo in [V2] except that we put his scheme in general perspective of C^0 -symplectic topology and C^0 -Hamiltonian dynamics, and added some more details in the proof so that we can safely apply them to the current local setting. The more general case of $L^{(1,\infty)}$ -Hamiltonian flows is a subject of future study.

We will assume that M is closed throughout the paper until section 9 where we explain how to generalize the main theorem to (M, ω) that is bounded at infinity in the sense of Gromov [Gr].

I like to thank S. Müller and C. Viterbo for useful communications regarding the proof of Lemma 6.3 in Appendix.

Notations

- (1) $S^1 = \mathbb{R}/\mathbb{Z}$, $S^1(2) = \mathbb{R}/2\mathbb{Z}$
- (2) $\pi : T^*N \rightarrow N$, the canonical projection of cotangent bundle.
- (3) $\tilde{\pi}_1 : T^*(S^1(2) \times Y) \rightarrow T^*S^1(2)$, $\tilde{\pi}_2 : T^*(S^1(2) \times Y) \rightarrow T^*Y$ the obvious projections.
- (4) $\pi_1 : S^1(2) \times Y \rightarrow S^1(2)$, $\pi_2 : S^1(2) \times Y \rightarrow Y$, the obvious projections.
- (5) $\iota_Y : [0, 1] \times Y \rightarrow T^*Y$, the map defined by $\iota_Y(t, y) = o_y$ where o_y the zero at $y \in T_y^*Y$. Or when $Y \subset (X, \omega)$ is a Lagrangian submanifold, ι_Y also denotes the corresponding map $[0, 1] \times Y \rightarrow X$ induced by the inclusion.
- (6) $i_Y : Y \rightarrow T^*Y$, X the obvious inclusion maps of Y .

2. Preliminaries

2.1. Conormal of open subsets

In this section, we recall the definition of the *conormal* of an open subset and state some results from a Morse theory, which will be used in the proof of the locality theorem later.

Definition 2.1. Let $U \subset M$ be an open subset with smooth boundary \bar{U} . We define the *conormal* of ∂U by

$$o_U \coprod \nu_-^*(\partial U) := \nu^*\bar{U}$$

Here we define $\nu_-^*(\partial U)$ to be the “negative” part of the conormal bundle of ∂U , i.e.,

$$\nu_-^*(\partial U) = \{\alpha \in \nu^*(\partial U) \mid \alpha(\bar{n}) \leq 0, \bar{n} \text{ outward normal to } \partial U\}.$$

In the case $U = (-1, 1) \subset \mathbb{R}$, the conormal to $\bar{U} = [-1, 1]$ is the one pictured below

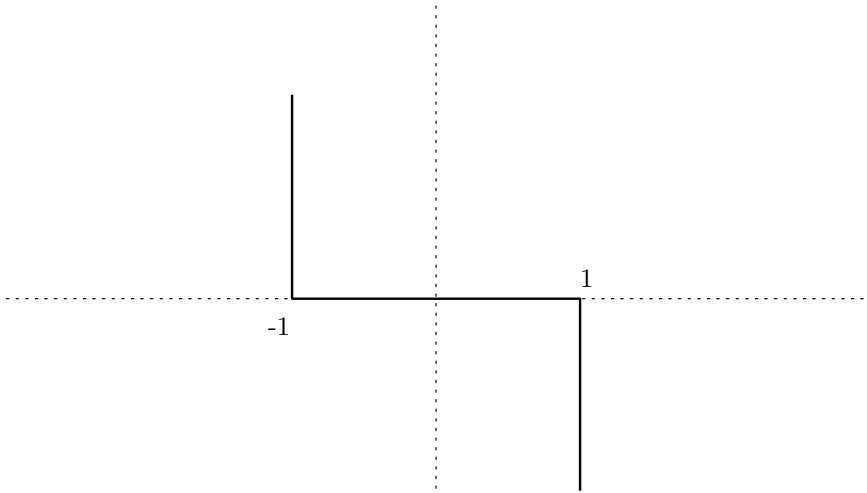


FIGURE 1. “Conormal” to $[-1, 1]$ in \mathbb{R}

The following lemma will be used in the proof of the main theorem. For the simplicity but without loss of any generality, we assume that \bar{U} is compact.

Lemma 2.2. *Let U be an open subset of Y with smooth boundary and let V be an open subset of Y such that*

$$V \subset \bar{V} \subset U.$$

Then there exists a smooth Morse function $k : \bar{U} \rightarrow \mathbb{R}$ such that

(1) we have

$$-dk(\vec{n})(y) > 0 \quad \text{on } \partial U \quad \text{or equivalently} \quad \text{Graph}(-dk) \cap \nu_-^*(\partial U) = \emptyset$$

(2) the function $k : U \rightarrow \mathbb{R}$ has no critical points on $\bar{U} \setminus \bar{V}$.

Proof. We choose any function k constant on ∂U such that (1) holds on ∂U . Then perturb the function to a Morse function away from a neighborhood of ∂U . Finally we compose the function k with a diffeomorphism of U that is supported away from ∂U and moves all the critical points into V . This finishes the proof. \square

2.2. Continuous Hamiltonian flows and their Hamiltonians

Following [OM], we denote by

$$\mathcal{P}^{ham}(Symplectic(M, \omega), id)$$

the set of smooth Hamiltonian paths $\lambda : [0, 1] \rightarrow Symplectic(M, \omega)$ with $\lambda(0) = id$, and equip it with the *Hamiltonian topology*. We refer to [OM], [Oh2] for an extensive study of this topology in the $L^{(1, \infty)}$ -context and [Mu] in the L^∞ or C^0 -context. In this paper, we will exclusively use the C^0 -Hamiltonian topology.

It is equivalent to the metric topology induced by the metric

$$d_{ham_\infty}(\lambda, \mu) := \bar{d}(\lambda, \mu) + \text{length}_\infty(\lambda^{-1}\mu)$$

where length_∞ is the L^∞ -Hofer length given by

$$\text{length}_\infty(\lambda) = \|H\|_\infty, \quad \lambda = \phi_H$$

(Proposition 3.10 [OM]), and \bar{d} is the C^0 metric on $\mathcal{P}(Homeo(M), id)$. We consider the *developing map*

$$\text{Dev} : \mathcal{P}^{ham}(Symplectic(M, \omega), id) \rightarrow C_m^\infty([0, 1] \times M, \mathbb{R}) :$$

This is defined by the assignment of the normalized generating Hamiltonian H of λ , when $\lambda = \phi_H : t \mapsto \phi_H^t$. We also consider the inclusion map

$$\iota_{ham} : \mathcal{P}^{ham}(Symplectic(M, \omega), id) \rightarrow \mathcal{P}(Symplectic(M, \omega), id) \rightarrow \mathcal{P}(Homeo(M, \omega), id).$$

Following [OM], we call the product map $(\iota_{ham}, \text{Dev})$ the *unfolding map* and denote the image thereof by

$$\mathcal{Q} := \text{Image}(\iota_{ham}, \text{Dev}) \subset \mathcal{P}(Homeo(M), id) \times C_m^0([0, 1] \times M, \mathbb{R}). \quad (3)$$

Then both maps Dev and ι_{ham} are Lipschitz with respect to the metric d_{ham} by definition and so the unfolding map canonically extends to the closure $\bar{\mathcal{Q}}$ in that we have the following continuous projections

$$\bar{\iota}_{ham} : \bar{\mathcal{Q}} \rightarrow \mathcal{P}(Homeo(M), id) \quad (4)$$

$$\bar{\text{Dev}} : \bar{\mathcal{Q}} \rightarrow C_m^0([0, 1] \times M, \mathbb{R}). \quad (5)$$

We would like to note that by definition we also have the extension of the evaluation map $ev_1 : \mathcal{P}^{ham}(Symp(M, \omega), id) \rightarrow Symp(M, \omega) \rightarrow Homeo(M)$ to

$$\bar{ev}_1 : \text{Image}(\bar{t}_{ham}) \rightarrow Homeo(M). \quad (6)$$

In this context, Theorem 1.2 is equivalent to saying that the map \bar{t}_{ham} is one-one.

Now we recall the notion of *continuous Hamiltonian paths* studied in [OM], [V2], [Mu].

Definition 2.3 (Definition 3.18 [OM], [Mu]). We define the set $\mathcal{P}_\infty^{ham}(Sympeo(M), id)$ by

$$\mathcal{P}_\infty^{ham}(Sympeo(M, \omega), id) := \text{Image}(\bar{t}_{ham}) \subset \mathcal{P}(Homeo(M), id)$$

and call any element thereof a *continuous Hamiltonian path*. We also define the group of C^0 -Hamiltonian homeomorphisms, denoted by $Hameo_\infty(M, \omega)$, to be

$$Hameo_\infty(M, \omega) := \text{Image}(\bar{ev}_1).$$

Similarly to the case of the interval $[0, 1]$, we can define a continuous Hamiltonian path on $[a, b]$ with $b > a$

$$\lambda : [a, b] \rightarrow Hameo_\infty(M, \omega)$$

to be a path such that

$$\lambda \circ (\lambda(a))^{-1} \in \mathcal{P}_{\infty;[a,b]}^{ham}(Sympeo(M, \omega), id) \quad (7)$$

where $\mathcal{P}_{\infty;[a,b]}^{ham}(Sympeo(M, \omega), id)$ is defined in the same way as $\mathcal{P}_\infty^{ham}(Sympeo(M, \omega), id)$ with $[0, 1]$ replaced by $[a, b]$.

We consider smooth functions $\zeta : [0, 1] \rightarrow [0, 1]$ satisfying

$$\zeta(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \varepsilon_0 \\ 1 & \text{for } 1 - \varepsilon_0 \leq t \leq 1 \end{cases} \quad (8)$$

and

$$\zeta'(t) \geq 0 \quad \text{for all } t \in [0, 1],$$

for some small $\varepsilon_0 > 0$, and denote the reparameterized Hamiltonian by H^ζ defined by

$$H^\zeta(t, x) = \zeta'(t)H(\zeta(t), x)$$

which generates the Hamiltonian isotopy $t \mapsto \phi_H^{\zeta(t)}$ in general. Using this reparametrization, we can always make the Hamiltonian path *boundary flat* and its Hamiltonian vanishes near $t = 0, 1$.

We will always assume this boundary-flatness until section 9, where we will explain how to treat the general case.

Now we introduce the notion of *continuous Hamiltonians*.

Definition 2.4. We denote

$$\mathcal{H}_m^0([0, 1] \times M) := \text{Im}(\overline{\text{Dev}})$$

and call an element in $\mathcal{H}_m^0([0, 1] \times M)$ a normalized *continuous Hamiltonian*.

By definition, if $H \in \mathcal{H}_m^0([0, 1] \times M)$, we have a sequence of smooth Hamiltonian functions H_i such that

- (1) $\phi_{H_i}^t$ converges uniformly over $[0, 1] \times M$.
- (2) H_i converges in $C_m^0([0, 1] \times M, \mathbb{R})$ to H .

3. The first half of Theorem 1.3

In this section, we prove a half of Theorem 1.3. This is the localized version of the corresponding theorem from [V1], [HZ], [OM] where $U = M$ is considered.

Theorem 3.1. *Let $U \subset M$ be an open subset. Suppose $H_i : [0, 1] \times M \rightarrow \mathbb{R}$ is a sequence of smooth normalized Hamiltonian functions such that both ϕ_{H_i} and H_i converge in C^0 -topology. Denote by λ and H the corresponding C^0 -limits respectively. Then if $H(t, x) \equiv c(t)$ on U for a continuous function $c : [0, 1] \rightarrow \mathbb{R}$ depending only on t , then $\lambda \equiv id$ on $[0, 1] \times U$.*

Proof. We first note that $H(t, x) \equiv c(t)$ on U is equivalent to $\|H\|_{\infty, U} = 0$.

It is enough to prove that for any given $\delta > 0$, we have

$$\lambda(t, x) = x \quad \text{for all } x \in U_\delta \tag{9}$$

where U_δ is the subset

$$U_\delta = \{x \in U \mid d(x, \partial U) > \delta\}.$$

Suppose (9) does not hold and so there exists $t_0 \in [0, 1]$ and $x_0 \in U_\delta$ such that $\lambda(t_0, x_0) \neq x_0$. Then there exists $\varepsilon_0 > 0$ such that the closed ball $B_{\varepsilon_0}(x_0)$ satisfies

$$\lambda_{t_0}(B_{\varepsilon_0}(x_0)) \cap B_{\varepsilon_0}(x_0) = \emptyset.$$

Furthermore, since λ is continuous and $\lambda_0 = id$ there exists $t_1 > 0$ such that

$$t_1 := \max\{t \in [0, 1] \mid \lambda_s(B_{\varepsilon_0}(x_0)) \subset \bar{U} \text{ for all } 0 \leq s \leq t\}.$$

Then we can choose another smaller ball $B_{\varepsilon_1}(x_1) \subset B_{\varepsilon_0}(x_0)$ such that

$$\lambda_{t_1}(B_{\varepsilon_1}(x_1)) \cap B_{\varepsilon_1}(x_1) = \emptyset \tag{10}$$

and

$$\lambda_t(B_{\varepsilon_1}(x_1)) \subset U \tag{11}$$

for all $0 \leq t \leq t_1$ in addition.

By the uniform convergence of ϕ_{H_i} , we still have

$$\phi_{H_i}^t(B_{\varepsilon_1}(x_1)) \cap B_{\varepsilon_1}(x_1) = \emptyset$$

for all $0 \leq t \leq t_1$ if we choose a sufficiently large i . Choose an open subset $V \subset U$ such that $\bar{V} \subset U$ and

$$V \supset \bigcup_{0 \leq t \leq t_1} \phi_{H_i}^t(B_{\varepsilon_1}(x_1)).$$

We multiply to H_i a cut-off function $\chi : Y \rightarrow \mathbb{R}$ satisfying

$$\chi(x) = \begin{cases} 1 & x \in \bigcup_{0 \leq t \leq t_1} \phi_{H_i}^t(B_{\varepsilon_1}(x_1)) \\ 0 & x \in Y \setminus V \end{cases}$$

and define a new Hamiltonian $F_i = \chi H_i$. Then we still have

$$\phi_{F_i}^t(B_{\varepsilon_1}(x_1)) \cap B_{\varepsilon_1}(x_1) = \emptyset$$

for all $0 \leq t \leq t_1$ because the flow of F_i and that of H_i coincide on $\bigcup_{0 \leq t \leq t_1} \phi_{H_i}^t(B_{\varepsilon_1}(x_1))$ since $\chi \equiv 1$ thereon. Therefore we obtain

$$\|F_i^{t_1}\|_{\infty} \geq e(B_{\varepsilon_1}(x_1)) > 0 \quad (12)$$

where $F_i^{t_1}$ is the reparameterized F_i given by

$$F_i^{t_1}(t, x) = t_1 F_i(t_1 t, x)$$

and $e(B_{\varepsilon_1}(x_1))$ is the displacement energy of $B_{\varepsilon_1}(x_1)$. (See [H2], [LM] for its definition.) The positivity of $e(B_{\varepsilon_1}(x_1))$ follows from the energy-capacity inequality from [LM]. On the other hand, obviously we have

$$F_i^{t_1} = \chi H_i^{t_1} \rightarrow \chi H^{t_1}$$

in C^0 -topology and hence (12) implies

$$\|\chi H^{t_1}\|_{\infty} > \frac{1}{2} e(B_{\varepsilon_1}(x_1)) > 0 \quad (13)$$

for all sufficiently large i 's. However we have

$$\begin{aligned} \|\chi H^{t_1}\|_{\infty} &= \max_{t \in [0, 1]} \left(\max_{x \in M} \text{osc}(\chi H(t_1 t, x)) \right) \\ &= \max_{t \in [0, t_1]} \left(\max_{x \in M} \text{osc}(\chi H(t, x)) \right) \\ &\leq \max_{t \in [0, 1]} \left(\max_{x \in U} \text{osc} H^{t_1}(t, x) \right) = \|H\|_{\infty, U} = 0 \end{aligned} \quad (14)$$

where the inequality holds since $\text{supp } \chi \subset U$ for all $t \in [0, 1]$ for a sufficiently i and $0 \leq \chi(t) \leq 1$.

Obviously (13) and (14) contradict to each other. This finishes the proof. \square

For the rest of the paper, we will provide the other half of the proof of Theorem 1.3.

4. Continuous exact Lagrangian isotopy

In this section, we denote a general symplectic manifold by (X, ω) , instead of using the letter M . We also denote a closed manifold by Y . When we are given a Lagrangian embedding of Y into (X, ω) , we often do not distinguish Y from the image in X of the given embedding.

Let $\psi : [0, 1] \times Y \rightarrow X$ be an isotopy of embeddings $\psi_t : Y \rightarrow X$. The pull-back $\psi^*\omega$ is decomposed to

$$\psi^*\omega = dt \wedge \alpha + \beta \quad (15)$$

where α and β are forms on Y satisfying

$$\frac{\partial}{\partial t} \lrcorner \alpha = 0 = \frac{\partial}{\partial t} \lrcorner \beta.$$

The following definition is given in [Gr].

Definition 4.1. An isotopy $\psi : [0, 1] \times Y \rightarrow X$ is called a Lagrangian isotopy (respectively, an *exact Lagrangian isotopy*) if in the above decomposition $\beta = 0$ and $i_t^*\alpha$ is closed (respectively, exact) for all $t \in [0, 1]$.

We remark that in this definition, the condition $\beta = 0$ automatically implies closedness of $i_t^*\alpha$. Therefore the latter condition is redundant for the definition of Lagrangian isotopy. We however put it there separately to make comparison with the case of exact Lagrangian isotopy.

The following lemma is well-known and is an easy consequence of the definition whose proof we omit.

Lemma 4.2. *If ψ is an exact Lagrangian isotopy, then there exists a function $h : [0, 1] \times Y \rightarrow \mathbb{R}$ such that $\psi^*\omega = dt \wedge dh$ i.e.,*

$$\psi^*\omega + dt \wedge (-dh) = 0.$$

Furthermore if h' is another such function, we have $h' - h$ is a function depending only on t , but not on Y .

We call any such h a *hamiltonian* of ψ . Here we like to alert readers that we use the lower case ‘h’ to call the function h the ‘hamiltonian’ associated to ψ to distinguish it from the Hamiltonian functions defined on the ambient symplectic manifold X .

According to this lemma, for a given exact Lagrangian isotopy ψ its associated hamiltonian is not unique. As we will see later, the suspension of an exact Lagrangian isotopy depends not just on the isotopy ψ but also on its hamiltonian. Because of this, we will regard the pair

$$(h, \psi) : [0, 1] \times Y \rightarrow \mathbb{R} \times X$$

as an exact Lagrangian isotopy. With a suitable normalization on the hamiltonians associated to ψ , the choice of h can be made unique in the smooth category.

The following extension lemma is also well-known whose proof is omitted.

Lemma 4.3. *For any exact Lagrangian isotopy $\psi : [0, 1] \times Y \rightarrow X$ of Lagrangian embeddings, there exists a Hamiltonian $H : [0, 1] \times X \rightarrow \mathbb{R}$ such that*

$$\psi(t, x) = \phi_H^t(\psi(0, x)) = \phi_H^t(\psi_0(x)). \quad (16)$$

When the exact Lagrangian isotopy (h, ψ) is given, we can always adjust H away from the support of the isotopy so that H satisfies the normalization condition

$$\int_X H_t \Omega_\omega = 0$$

where Ω_ω is the Liouville volume form.

And if the isotopy is boundary flat, then h and H can be made boundary flat. Namely, we may assume that there exists $\varepsilon_0 > 0$ such that

$$H \equiv 0 \quad \text{for } t \in [0, \varepsilon_0] \cup [1 - \varepsilon_0, 1]. \quad (17)$$

Definition 4.4. Let $Y \subset X$ be a Lagrangian submanifold and (h, ψ) an exact Lagrangian isotopy. The *time-reversal* of (h, ψ) is the pair $(\tilde{h}, \tilde{\psi})$ defined by

$$\tilde{\psi}(t, x) = \psi(1 - t, x), \quad \tilde{h}(t, x) = -h(1 - t, x).$$

Obviously for a boundary flat (h, ψ) , the concatenated isotopy

$$(h, \psi) * (\tilde{h}, \tilde{\psi})$$

defines a *smooth* closed embedding $S^1(2) \times Y \rightarrow \mathbb{R} \times S^1(2) \times X$, where we represent S^1 by $\mathbb{R}/2\mathbb{Z}$. We call this concatenated isotopy an *odd double* of the isotopy (h, ψ) and denote it by (h^{od}, ψ^{od}) .

Now we are ready to give the definition of the *odd-double suspension* of an exact Lagrangian isotopy (h, ψ) .

Definition 4.5. The odd-double suspension of an exact Lagrangian isotopy $(h, \psi) : [0, 1] \times Y \rightarrow \mathbb{R} \times X$ is defined by the embedding

$$\begin{aligned} \iota_{(h, \psi)} : S^1(2) \times Y &\rightarrow T^*S^1(2) \times X \\ \iota_{(h, \psi)}(\theta, x) &= \begin{cases} (\theta, -h(\theta, x), \psi(\theta, x)) & \text{for } \theta \in [0, 1] \\ (\theta, h(2 - \theta, x), \psi(2 - \theta, x)) & \text{for } \theta \in [1, 2]. \end{cases} \end{aligned} \quad (18)$$

We also denote the obvious double $h * \tilde{h}$ of the hamiltonian h by h^{od} .

Now we state a proposition which is an essential ingredient in the proof of Theorem 1.3. We provide its proof in the appendix. (See also Theorem 6.1.B [P].)

Proposition 4.6. *Let (h, ψ) be an exact Lagrangian isotopy. Suppose that the image of ψ is contained in $U \subset X$. Equip $T^*S^1(2) \times X$ with the symplectic form*

$$d\theta \wedge db + \omega_X$$

*where (θ, b) is the canonical coordinates of $T^*S^1(2)$. Then the Lagrangian embedding $\iota_{(h, \psi)} : S^1(2) \times Y \rightarrow T^*S^1(2) \times X$ is isotopic to the embedding $o_{S^1(2)} \times i_Y : S^1(2) \times Y \subset T^*S^1(2) \times X$ by a Hamiltonian flow supported in $T^*S^1(2) \times U$.*

We assume that for any $s \in [0, 1]$, $h(s, \cdot)$ assumes the value zero so that

$$\min_x h_s \leq 0 \leq \max_x h_s. \quad (19)$$

This can be done, for example, by normalizing h_s so that

$$\frac{\max_x h_s + \min_x h_s}{2} = 0. \quad (20)$$

In particular, this normalization makes

$$\min_x h_s < 0 < \max_x h_s \quad (21)$$

unless $h_s \equiv 0$. We call any such hamiltonian h satisfying (20) *average normalized*.

We denote by $L_s = \phi_s(Y)$ the family of Lagrangian submanifolds associated to the isotopy. Without loss of generality, we will assume that ψ is boundary flat. We introduce the following definition

Definition 4.7. A pair (h, ψ) such that $\psi : [0, 1] \times Y \rightarrow X$ is a continuous map and $h : [0, 1] \times Y \rightarrow \mathbb{R}$ an C^0 -function is called a *continuous exact Lagrangian isotopy* if there exists a sequence of smooth exact Lagrangian isotopies (h_i, ψ_i) such that $\psi_i \rightarrow \psi$ in the C^0 -topology and $h_i \rightarrow h$ in the C^0 -topology. We call any such sequence (h_i, ψ_i) a *smoothing sequence* of (h, ψ) .

Similarly as the definition of $\mathcal{H}_m^0([0, 1] \times M)$, we give the following Lagrangian counterpart thereof.

Definition 4.8. We define the subset

$$h_{av}^0([0, 1] \times Y) \subset C^0([0, 1] \times L)$$

to be the set of the above C^0 -limits of normalized sequences of smooth exact Lagrangian isotopies (h_i, ψ_i) . We call any element $h \in h_{av}^0([0, 1] \times Y)$ a (time-dependent) C^0 -hamiltonian function on Y .

We define the norms

$$\|h\|_\infty = \max_{t \in [0, 1]} \left(\max_{x \in Y} h_t - \min_{x \in Y} h_t \right)$$

and for a given open subset U , we define

$$\|h\|_{\infty, U} = \max_{t \in [0, 1]} \left(\max_{x \in U} h_t - \min_{x \in U} h_t \right)$$

for an average-normalized hamiltonian h .

The following theorem is the Lagrangian version of Theorem 1.3, which will be proved in the rest of the paper.

Theorem 4.9. *Let (X, ω) be a symplectic manifold and $Y \subset X$ be a closed Lagrangian submanifold. We denote by*

$$\iota_Y : [0, 1] \times Y \rightarrow [0, 1] \times X$$

the obvious inclusion map. Let $(h_i, \psi_i) : [0, 1] \times Y \rightarrow \mathbb{R} \times X$ be a sequence of exact Lagrangian isotopies that are boundary flat. Let $U \subset Y$ be an open subset with smooth boundary ∂U .

Suppose that (h_i, ψ_i) converges to (h, ψ) in C^0 -topology on Y . Then if $\psi \equiv \iota_Y$ on $[0, 1] \times U$, we have $h(t, y) \equiv c(t)$ for $y \in U$ where $c : [0, 1] \rightarrow \mathbb{R}$ is a continuous function depending only on t .

5. Lagrangian intersections and analysis of Calabi currents

We assume that Y is oriented throughout this section. The proof for the unoriented case will be similar if one uses *odd currents* [dR] whose details we omit because it will not be used in the proofs of the main theorems in this paper.

We first state the following lemma whose proof will be given in section 7 in the more general $L^{(1, \infty)}$ -context.

Lemma 5.1. *Let $\alpha = h^{od}d\theta$ be the Calabi current associated to the continuous exact Lagrangian isotopy (h, ψ) as defined above. Suppose $\|h\|_{\infty, U} \neq 0$. Then the current α on $S^1(2) \times Y$ is not closed on U .*

The following definition will be useful for our purpose.

Definition 5.2. A diffeomorphism $\phi : S^1(2) \times Y \rightarrow S^1(2) \times Y$ is called *fiberwise-like* with respect to the projection $\pi_2 : S^1(2) \times Y \rightarrow Y$, if it has the form

$$\phi(\theta, y) = (\theta, Y_\theta(y))$$

for a 2-periodic family of diffeomorphisms $Y_\theta : Y \rightarrow Y$ with $Y_0 = id_Y$. We define its support by

$$\text{supp } \phi = \bigcup_{\theta \in [0, 2]} \text{supp } Y_\theta.$$

Obviously the space $C^0(S^1(2) \times Y)$ is invariant under a full diffeomorphism group of $S^1(2) \times Y$.

For each fiberwise-like diffeomorphism $\phi : S^1(2) \times Y \rightarrow S^1(2) \times Y$, we consider the push-forward $\phi_*\alpha$ as a current and decompose it into

$$\phi_*\alpha = \beta_\phi + (\phi_*\alpha - \beta_\phi) \tag{22}$$

where β_ϕ satisfies $\frac{\partial}{\partial \theta} \lrcorner \beta_\phi = 0$. Note that $\phi_*\alpha - \beta_\phi$ can be written as

$$\phi_*\alpha - \beta_\phi = \left(\frac{\partial}{\partial \theta} \lrcorner \phi_*\alpha \right) \wedge d\theta.$$

The following can be proved in a straightforward calculation and so its proof is omitted. We however point out that $\phi_*\alpha$ defines a well-defined n -current on $S^1(2) \times Y$.

Lemma 5.3. *Let ϕ be a fiberwise-like diffeomorphism. Under the decomposition (22), we have*

$$\phi_*\alpha - \beta_\phi = h_\phi^{od} d\theta$$

where h_ϕ^{od} is an C^0 -function which is given by the formula

$$h_\phi^{od}(\theta, y) := h^{od}(\phi^{-1}(\theta, y)) = h^{od}(\theta, Y_\theta^{-1}(y)). \quad (23)$$

Using this preparation, we state the following proposition whose proof will be given in section 8 in the more general $L^{(1,\infty)}$ -context. We also refer the readers to Lemma 2.5 [V2] for the corresponding global statement.

Proposition 5.4. *Let $U \subset Y$ be an open subset. Let $\alpha = h^{od} d\theta$ be the above mentioned current on $S^1(2) \times Y$. Suppose that α is not closed on U . Then there exists a fiberwise-like diffeomorphism $\phi : S^1(2) \times Y \rightarrow S^1(2) \times Y$ supported on $S^1(2) \times U$ such that we have some $y \in U$ such that*

$$\int_{\pi_2^{-1}(y)} h_\phi^{od} d\theta \neq 0.$$

Furthermore ϕ can be chosen arbitrarily C^∞ -close to the identity map and $\text{supp } \phi \subset U$.

With Lemma 5.1 and Proposition 5.4 assumed, we now start the proof of Theorem 4.9.

For this purpose, we closely analyze the function $P_{h_\phi^{od}} : Y \rightarrow \mathbb{R}$ on U which is defined by

$$P_{h_\phi^{od}}(y) = \int_{\pi_2^{-1}(y)} h_\phi^{od} d\theta \quad (24)$$

for $y \in Y$. This function is obviously continuous on Y for a C^0 -function h and for a diffeomorphism ϕ . Here we like to note that due to the doubling process, the function P given by

$$P_{h^{od}}(y) = \int_{\pi_2^{-1}(y)} h^{od} d\theta$$

itself identically vanish, which prompts one to deform it by a diffeomorphism or to deform the fibrations as put in [V2].

The following will be used in our proof in an essential way.

Lemma 5.5. *Let $\alpha = h_\phi^{od} d\theta$ with $h \in C^0$ and $\|h_\phi^{od}\|_{\infty, U} \neq 0$. Then $P_{h_\phi^{od}} : Y \rightarrow \mathbb{R}$ is a non-zero continuous function such that*

(1) *the set*

$$U_\phi := \{y \in \bar{U} \mid P_{h_\phi^{od}}(y) \neq 0\}$$

is a non-empty open subset of \bar{U} .

(2) *$P_{h_\phi^{od}} \equiv 0$ on $\bar{U} \setminus \text{supp } \phi \supset \partial U$ where $\phi(\theta, y) = (\theta, Y_\theta(y))$.*

Proof. We start with the formula

$$P_{h_\phi^{od}}(y) = \int_{S^1(2)} h^{od}(\phi^{-1}(\theta, y)) d\theta = \int_{S^1(2)} h^{od}(\theta, Y_\theta^{-1}(y)) d\theta$$

which follows from (23). Obviously $P_{h_\phi^{od}}$ is a continuous function on Y which satisfies $P_{h_\phi^{od}} \not\equiv 0$ by Proposition 5.4.

Next, we recall that

$$\int_0^2 h^{od}(\theta, y) d\theta = \int_0^1 h(\theta, y) d\theta + \int_1^2 (-h(2 - \theta, y)) d\theta \equiv 0$$

for all $y \in Y$. Therefore we derive

$$P_{h_\phi^{od}}(y) = \int_0^2 h^{od}(\theta, Y_\theta^{-1}(y)) d\theta = 0$$

whenever $y \in U \setminus \text{supp } \phi$, i.e., $Y_\theta^{-1}(y) = y$ for all $\theta \in [0, 2]$.

This finishes the proof. \square

The following variant of Hörmander's construction (see (2.5.4) [Hö]) will be useful for our discussion coming henceforth.

Definition 5.6. Consider the cotangent bundle T^*N of a smooth manifold. Let $\psi_i : N_i \rightarrow T^*N$ be two Lagrangian embeddings of manifolds N_i , $i = 1, 2$. We introduce the *difference set* denoted by $\psi_1 \ominus \psi_2$. The set $\psi_1 \ominus \psi_2$ is defined by

$$\psi_1 \ominus \psi_2 = \{(q, p) \in T^*N \mid (q, p) = \psi_1(n_1) - \psi_2(n_2), q \in N, \psi_i(n_i) \in N_i \cap T_q^*N\} \quad (25)$$

Now consider the symplectic diffeomorphism

$$T^*\phi^{-1} : T^*(S^1(2) \times Y) \rightarrow T^*(S^1(2) \times Y)$$

given by

$$T^*\phi^{-1}(\alpha_{(\theta, y)}) = (T_{\phi(\theta, y)}\phi^{-1})^*(\alpha_{(\theta, y)}) \in T_{\phi(\theta, y)}^*(S^1(2) \times Y).$$

The composition

$$\iota_{(h_i, \psi_i)}^\phi := T^*\phi^{-1} \circ \iota_{(h_i, \psi_i)} \circ \phi^{-1} : S^1(2) \times Y \rightarrow T^*(S^1(2) \times Y)$$

defines a Lagrangian embedding. Using the splitting $T^*(S^1(2) \times Y) = T^*S^1(2) \times T^*Y$ we write

$$\iota_{(h_i, \psi_i)}^\phi(\theta, y) = (\alpha_i^\phi(\theta, y), \beta_i^\phi(\theta, y)) \quad (26)$$

as an element in $T_{(\theta', y')}^*(S^1(2) \times Y)$ where

$$(\theta', y') = \pi(\iota_{(h_i, \psi_i)}^\phi(\theta, y)).$$

Here we note that $\theta' = \theta$ as ϕ has the form $\phi(\theta, y) = (\theta, Y_\theta(y))$. We denote by $\tilde{\pi}_i$, $i = 1, 2$ the projections from $T^*(S^1(2) \times Y)$ to $T^*S^1(2)$ and T^*Y respectively.

We note the identity

$$\pi_2 \circ \pi \circ \iota_{(h_i, \psi_i)}^\phi = \pi \circ \psi_i.$$

since ψ_i converges to the zero section map $\iota_Y : S^1(2) \times Y \rightarrow T^*Y$ in C^0 topology, the map

$$\pi \circ \iota_{(h_i, \psi_i)}^\phi : S^1(2) \times Y \rightarrow S^1(2) \times Y$$

can be made arbitrarily close to the identity uniformly over i .

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Similarly we define a continuous embedding $\iota_{(h, \iota_Y)}^\phi : S^1(2) \times Y \rightarrow T^*(S^1(2) \times Y)$ by

$$\iota_{(h, \iota_Y)}^\phi(\theta, y) = (\theta, h \circ \phi^{-1}(\theta, y), o_y). \quad (27)$$

The following proposition shows that $\iota_{(h, \iota_Y)}^\phi$ is again a continuous *Lagrangian* embedding.

Proposition 5.7. *The smooth Lagrangian embedding $\iota_{(h_i, \psi_i)}^\phi$ converges to the continuous embedding $\iota_{(h, \iota_Y)}^\phi$ in C^0 -topology.*

Proof. For this, we compute the values of $\alpha_i^\phi(\theta, y)$ as an element in $T_{(\theta', y')}^*(S^1(2) \times Y)$ explicitly.

Lemma 5.8. *We have*

$$\alpha_i^\phi(\theta, y) = h_i^{od}(\phi^{-1}(\theta, y)) d\theta + \tilde{\pi}_1 \circ T^* \phi^{-1} \circ \psi_i^{od}(\phi^{-1}(\theta, y)). \quad (28)$$

Proof. We compute

$$\begin{aligned} \iota_{(h_i, \psi_i)}^\phi(\theta, y) \left(\frac{\partial}{\partial \theta} \right) &= T^* \phi^{-1} \circ \iota_{(h_i, \psi_i)}(\phi^{-1}(\theta, y)) \left(\frac{\partial}{\partial \theta} \right) \\ &= \iota_{(h_i, \psi_i)}(\phi^{-1}(\theta, y)) \left(T \phi^{-1} \left(\frac{\partial}{\partial \theta} \right) \right) \\ &= h_i^{od}(\phi^{-1}(\theta, y)) d\theta \left(T \phi^{-1} \left(\frac{\partial}{\partial \theta} \right) \right) + \psi_i^{od}(\phi^{-1}(\theta, y)) \left(T \phi^{-1} \left(\frac{\partial}{\partial \theta} \right) \right) \end{aligned}$$

where we used $\alpha_i = h_i^{od} d\theta$.

Using the identity $\phi^{-1}(\theta, y) = (\theta, Y_\theta^{-1}(\theta, y))$, the first term becomes

$$h_i^{od}(\phi^{-1}(\theta, y)) d\theta \left(\frac{\partial}{\partial \theta} \right) = h_i^{od}(\phi^{-1}(\theta, y))$$

which gives rise to the first term in (28).

The second term here gives rise to the second one in (28). Hence the proof. \square

Obviously the first term of (28) converges to $h^{od} \circ \phi^{-1} d\theta$. We now estimate the second term.

Lemma 5.9. *Denote by $\iota_Y : S^1(2) \times Y \rightarrow X$ the obvious inclusion-induced map given by $(\theta, y) \mapsto y$. Let $(h_i, \psi_i) \rightarrow (h, \iota_Y)$ in the hamiltonian topology and denote*

$$m_i(\phi) := \max_{(\theta, y)} \text{dist} \left(\tilde{\pi}_1 \circ T^* \phi^{-1}(\psi_i(\phi^{-1}(\theta, y))), o_{S^1} \right).$$

Then there exists a constant $C > 0$ independent of ϕ and i such that

$$m_i(\phi) \leq C d_{C^0}(\psi_i, \iota_Y) |\phi^{-1}|_{C^1} \quad (29)$$

for all i . In particular, $m_i(\phi) \rightarrow 0$ as $i \rightarrow \infty$.

Proof. We have

$$\begin{aligned}
 m_i(\phi) &\leq \max_{(\theta, y)} \left| \tilde{\pi}_1 \circ T^* \phi^{-1} \circ \psi_i(\phi^{-1}(\theta, y)) \left(\frac{\partial}{\partial \theta} \right) \right| \\
 &= \max_{(\theta, y)} \left| \psi_i(\phi^{-1}(\theta, y)) \left(T \phi^{-1} \left(\frac{\partial}{\partial \theta} \right) \right) \right| \\
 &\leq C \max_{(\theta, y)} |\psi_i(\theta, y)| |\phi^{-1}|_{C^1}
 \end{aligned}$$

for a universal constant $C > 0$ where $|\psi_i(\phi^{-1}(\theta, y))|$ is the norm taken as an element of $T_{y'}^*(Y)$ with

$$y' = \pi_2 \phi(\pi(\psi_i(\phi^{-1}(\theta, y)))) = Y_\theta(\pi(\psi_i(\theta, Y_\theta^{-1}(y)))).$$

Here we recall that we identify the exact Lagrangian isotopies $\psi_i : S^1(2) \times Y \rightarrow X$ with the corresponding isotopies in the Darboux chart T^*Y of $Y \subset X$. Under this identification, we can identify

$$\max_{(\theta, y)} |\psi_i(\theta, y)|$$

with $d_{C^0}(\psi_i, \iota_Y)$. This finishes the proof of the first statement. The last statement follows from (29) by the C^0 -convergence $\psi_i \rightarrow \iota_Y$ that we assumed.

This finishes the proof of the lemma. \square

Combining Lemma 5.8 and Lemma 5.9, we have finished the proof of Proposition 5.7. \square

6. Proof of Theorem 4.9

In this section, we will wrap-up the proof of Theorem 4.9 by producing a contradiction to Theorem 1.5, *if we assume* $\|h\|_{\infty, U} \neq 0$.

For the simplicity of notation, we denote

$$P_\phi = P_{h_\phi^{od}}.$$

The following is the local version of the key ingredient used in Viterbo's scheme [V2].

Proposition 6.1. *Suppose $\|h\|_{\infty, U} \neq 0$. Let ϕ be a fiberwise-like diffeomorphism as in Proposition 5.5. Then there exists a smooth function $f : S^1(2) \times Y \rightarrow \mathbb{R}$ such that*

$$\iota_{(h, \iota_Y)}^\phi \ominus df$$

does not intersect $\nu^(S^1(2) \times \bar{U})$.*

Once we have proved this proposition, an immediate corollary of Proposition 5.7 will be the following intersection result.

Corollary 6.2. *Suppose $\|h\|_{\infty, U} \neq 0$ and let f be the smooth function as in Proposition 6.1. Then there exists an $N \in \mathbb{N}$ for which the smooth Lagrangian embedding $\iota_{(h_N, \psi_N)}^\phi \ominus df$ cannot intersect $\nu^*(S^1(2) \times \bar{U})$.*

Proof. Since taking the difference $\ominus df$ is a continuous operation in C^0 -topology, Proposition 5.7 implies that $\iota_{(h_i, \psi_i)}^\phi \ominus df$ converges to $\iota_{(h, \iota_Y)}^\phi \ominus df$ in C^0 -topology. Therefore Proposition 6.1 and compactness of $S^1(2) \times Y$ imply that $\iota_{(h_N, \psi_N)}^\phi \ominus df$ do not intersect $\nu^*(S^1(2) \times \bar{U})$ if we choose a sufficiently large N . \square

With this corollary, we can wrap-up the proof of Theorem 4.9.

Proof of Theorem 4.9. If we assume $\|h\|_{\infty, U} \neq 0$, Proposition 6.1 holds and so Corollary 6.2 would give rise to a contradiction to Theorem 1.5 because $\iota_{(h_N, \psi_N)}^\phi \ominus df$ is Hamiltonian isotopic to the zero section : the latter follows since ϕ is isotopic to the identity on $S^1(2) \times Y$ and $\iota_{(h_N, \psi_N)}$ is Hamiltonian isotopic to the zero section. This will then finish the proof of Theorem 4.9. \square

The rest of the section will be occupied with the proof of Proposition 6.1.

Proof of Proposition 6.1. Motivated by Viterbo's scheme used in the proof of Proposition 2.1 [V2], we first solve the equation

$$\begin{cases} h_\phi^{\circ d}(\theta, y) - \frac{\partial \tilde{f}}{\partial \theta}(\theta, y) = P_\phi(y) \\ \tilde{f}(0, y) = 0 \end{cases} \quad (30)$$

for some \tilde{f} . This is uniquely solvable for given $y \in Y$ for a smooth function $\tilde{f}(\cdot, y)$ because we have

$$\int_0^1 (h_\phi^{\circ d}(\theta, y) - P_\phi(y)) d\theta = 0 \quad (31)$$

by the definition of P_ϕ . We denote by $\tilde{f} = \tilde{f}(\theta, y)$ such a solution. However \tilde{f} is a priori only continuous and hence we will suitably perturb P_ϕ so that the resulting solution becomes smooth. In this regard, the following lemma will be important. This is what Viterbo briefly mentioned in the proof of Lemma 2.4 [V2]. Because the lemma is not totally obvious, we will give its proof in the appendix for completeness' sake. To make comparison of our proof with those in [V2] easier, we will use the same letter ε for the perturbed function as in [V2].

Lemma 6.3. *There exists a continuous function $\varepsilon : S^1(2) \times Y \rightarrow \mathbb{R}$ such that*

- (1) *For all $y \in Y$, $\int_{S^1(2)} (h_\phi^{\circ d}(t, y) - \varepsilon(t, y)) dt = 0$,*
- (2) *$h_\phi^{\circ d} - \varepsilon$ is smooth.*
- (3) *We have the inequality*

$$\|P_\phi - \varepsilon\|_{C^0} < \frac{1}{30} \|P_\phi\|_{C^0}. \quad (32)$$

Now we consider the equation

$$\begin{cases} h_\phi^{\circ d}(\theta, y) - \frac{\partial g}{\partial \theta}(\theta, y) = \varepsilon(\theta, y) \\ g(0, y) = 0 \end{cases} \quad (33)$$

instead. Then the solution

$$g(\theta, y) = \int_0^\theta (h_\phi^{od}(t, y) - \varepsilon(t, y)) dt$$

becomes 2-periodic and so well-defined on $S^1(2) \times Y$ and becomes smooth by the choice of ε in Lemma 6.3.

We now choose δ so small independently of i , (e.g., the choice

$$\delta = \frac{1}{10} \|P_\phi\|_{C^0}$$

will do the purpose) that for the set

$$Z_{P_\phi}^\delta = \{(\theta, y) \mid |P_\phi(y)| \leq \delta\}$$

$Y \setminus \pi(Z_{P_\phi}^\delta)$ is still a non-empty open subset of Y contained in U : Recall that $P_\phi \equiv 0$ on $Y \setminus \text{supp } \phi$ and $\text{supp } \phi \subset U$.

Then we choose the perturbation ε of P as given Lemma 6.3 so that the inclusion

$$Z_\varepsilon := \{(\theta, y) \mid \varepsilon(\theta, y) = 0\} \subset \{(\theta, y) \mid |P_\phi(y)| < \delta\} = Z_{P_\phi}^\delta$$

holds. The function $h_\phi^{od}(\theta, y) - \frac{\partial \tilde{g}}{\partial t}(\theta, y) (= \varepsilon(\theta, y))$ is non-vanishing outside Z_ε , in particular outside $Z_{P_\phi}^\delta$.

Now we closely describe the difference set

$$\iota_{(h, \iota_Y)}^\phi \ominus dg.$$

We would like to note that we have the base point formula

$$\pi_2 \circ \pi(\iota_{(h, \iota_Y)}^\phi(\theta, y)) = y$$

and

$$\pi_1 \circ \pi(\iota_{(h, \iota_Y)}^\phi(\theta, y)) = \theta$$

since ϕ is fiberwise-like, i.e., $\phi(\theta, y) = (\theta, Y_\theta(y))$.

Therefore the projection in the direction T^*S^1 of $\iota_{(h, \iota_Y)}^\phi$ consists of the elements of the form

$$\alpha_g^\phi(\theta, y) = \tilde{\pi}_1 \circ (\iota_{(h, \iota_Y)}^\phi)(\theta, y) - \left(\frac{\partial g}{\partial \theta}(\theta, y) \right) d\theta \quad (34)$$

$$\begin{aligned} &= \alpha^\phi(\theta, y) - \left(\frac{\partial g}{\partial \theta}(\theta, y) \right) d\theta \\ &= \left(h_\phi^{od}(\theta, y) - \frac{\partial g}{\partial \theta}(\theta, y) \right) d\theta \\ &= \varepsilon(\theta, y) d\theta \end{aligned} \quad (35)$$

Here we use Lemma 5.8 for the second equality, and the identity

$$\varepsilon(\theta, y) = h_\phi^{od}(\theta, y) - \frac{\partial g}{\partial \theta}(\theta, y).$$

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On the other hand, $\bar{U} \setminus \pi(Z_{P_\phi}^\delta)$, an open subset of \bar{U} is disjoint from ∂U by Lemma 5.5. Therefore Lemma 2.2 enables us to find a smooth function k on \bar{U} that has no critical point on $\pi(Z_{P_\phi}^\delta) \supset \partial \bar{U}$ and

$$-dk(\bar{n}) > 0 \quad \text{on } \partial U. \quad (36)$$

By multiplying a large *positive* constant, if necessary, we may assume dk to be arbitrarily large on $\pi(Z_{P_\phi}^\delta)$ so that

$$|dk(y)| > |\tilde{\pi}_2 \circ (\iota_{(h, \iota_Y)}^\phi)(\theta, y) - d_y g(\theta, y)| \quad (37)$$

for $(\theta, y) \in Z_{P_\phi}^\delta \cap (S^1(2) \times \bar{U})$, and

$$\begin{aligned} \beta_k^\phi(\bar{n}) &= \tilde{\pi}_2((\iota_{(h, \iota_Y)}^\phi)(\theta, y))(\bar{n}) - (dk(y)(\bar{n}) + d_y g(\theta, y)(\bar{n})) \\ &= -dk(y)(\bar{n}) + \left(\tilde{\pi}_2 \circ (\iota_{(h, \iota_Y)}^\phi)(\theta, y)(\bar{n}) - d_y g(\theta, y)(\bar{n}) \right) > 0 \end{aligned} \quad (38)$$

on $\partial(S^1(2) \times \bar{U}) = S^1(2) \times \partial \bar{U}$.

In particular (37) implies

$$\beta_k^\phi(\theta, y) := \tilde{\pi}_2 \circ (\iota_{(h, \iota_Y)}^\phi)(\theta, y) - (dk(y) + d_y g(\theta, y)) \neq 0 \quad (39)$$

for $(\theta, y) \in Z_{P_\phi}^\delta$. And (38) implies that the projection of the image of $\iota_{(h, \iota_Y)}^\phi \ominus d(g+k)$ to T^*Y does not intersect $\nu^*(\partial U)$.

On the other hand, on $\bar{U} \setminus Z_{P_\phi}^\delta$, $\alpha_\varepsilon^\phi = \tilde{\pi}_1 \circ \iota_{(h, \iota_Y)}^\phi(\theta, y)$ does not vanish by definition of $Z_{P_\phi}^\delta$. Combining this with (38) and (39), we conclude that the difference set $\iota_{(h, \iota_Y)}^\phi \ominus d(g+k)$ does not intersect

$$\nu^*(S^1(2) \times \bar{U})$$

on $\bar{U} \cap Z_{P_\phi}^\delta$. Therefore we have proved that

$$\iota_{(h, \iota_Y)}^\phi \ominus df$$

does not intersect $\nu^*(S^1(2) \times \bar{U})$, if we choose $f = g + k$.

Finally to extend the function $f : \bar{U} \rightarrow \mathbb{R}$ to Y , we first extend the function f to a slightly bigger open subset $S^1(2) \times U' \supset S^1(2) \times \bar{U}$ and then apply the partitions of unity subordinate to the covering $\{U', Y \setminus \bar{U}\}$. This finishes the proof. \square

Wrap-up of the proof of Theorem 1.3. We set

$$\begin{aligned} (X, \omega) &= (M \times M, (-\omega_M) \oplus \omega_M), \quad Y = M \\ h_i(t, x) &= H_i(t, \phi_{H_i}^t(x)), \quad \psi_i(t, x) = (t, x, \phi_{H_i}^t(x)). \end{aligned}$$

Then the hypothesis (1) of Theorem 1.3 implies ψ_i uniformly converges to the embedding ι_Y on U . On the other hand, the hypotheses (1) and (2) therein, combined with the hypothesis that H_i is Cauchy in C^0 -topology, imply

$$\|\text{Tan}(H_i) - H_i\|_U \rightarrow 0; \quad \text{Tan}(H)(t, x) := H(t, \phi_H^t(x)). \quad (40)$$

as $i \rightarrow \infty$, and so $h_i = \text{Tan}(H_i)$ is Cauchy on $[0, 1] \times U$ as H_i is assumed so. Therefore the map $(h_i, \psi_i) : [0, 1] \times M \rightarrow [0, 1] \times M \times M$ defines an exact Lagrangian isotopy that satisfies all the hypotheses in Theorem 4.9, except the normalization condition (20). However this can be easily adjusted by adding a function c_i on $[0, 1]$ to h_i and considering $H_i - c_i$ instead. This does not affect ψ_i at all and so $(h_i - c_i, \psi_i)$ is still an exact Lagrangian isotopy. From this, we conclude $h - c_\infty \equiv 0$ and so $h \equiv c_\infty$. Hence $h(t, x, x) \equiv c_\infty(t)$ for all $x \in U$. This in turn proves that we have

$$\| \text{Tan}(H_i) - c_\infty \|_U \rightarrow 0. \quad (41)$$

Combining (40), (41), we have proved $H \equiv c_\infty$ on $[0, 1] \times U$, and finished the proof of Theorem 1.3. \square

7. Proof of Lemma 5.1

In this section, we prove Lemma 5.1. Because the proofs are not very different for the C^0 and $L^{(1, \infty)}$ cases, we prove this closedness for more general class of $L^{(1, \infty)}$ -functions h for the future study of $L^{(1, \infty)}$ -Hamiltonian flows.

Since $h \in L^{(1, \infty)} \subset L^1$, it follows that $\alpha = h^{od} d\theta$ defines a well-defined n -current on $S^1(2) \times Y$ (see [Fe]). For the simplicity, we assume that Y is orientable and fix a volume form Ω on Y . For the non-orientable case, exactly the same proof works by replacing the real-valued current by one with coefficients in a flat line bundle (or in the orientation sheaf) [dR].

It will be enough to find a smooth $(n-1)$ -form η on $S^1(2) \times Y$ such that we have

$$\int_{S^1(2) \times Y} \alpha \wedge d\eta \neq 0 \quad (42)$$

with $\text{supp } \eta \subset U$. For the $(n-1)$ form η , we can write

$$d\eta = d\theta \wedge \frac{\partial \eta}{\partial \theta} + d_Y \eta$$

where d_Y is the exterior derivative in the direction of Y and $\frac{\partial \eta}{\partial \theta} := \frac{\partial}{\partial \theta} \lrcorner d\eta$. We can write

$$d_Y \eta = f(\theta, y) \Omega \quad (43)$$

for some smooth function f on $S^1(2) \times U$ which must satisfy

$$\int_U f(\theta, y) \Omega = 0 \quad (44)$$

for all $\theta \in S^1$. By applying a simple argument from the Hodge theory, we can choose a family of $(n-1)$ -forms η_θ on U that satisfy (43) for each $\theta \in S^1$ and depend smoothly on θ .

We next note that since α has the form $\alpha = h^{od} d\theta$ we have

$$\alpha \wedge d\eta = \alpha \wedge d_Y \eta$$

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and so derive

$$\int_{S^1(2) \times U} \alpha \wedge d\eta = \int_{S^1(2) \times U} \alpha \wedge (f\Omega) = \int_{S^1(2) \times U} fh^{od} d\theta \wedge \Omega.$$

Therefore it is enough to choose η so that this integral does not vanish. We will use η of the form given by

$$\eta = \pi_2^*(\eta_\theta)$$

where η_θ are those chosen above. Now it remains to choose f above so that

$$\int_{S^1(2) \times U} fh^{od} d\theta \wedge \Omega \neq 0. \quad (45)$$

Since we have

$$0 < \|h\| = \int_0^1 \text{osc}(h_t) dt$$

by the hypothesis, there exists a measurable subset $A \subset [0, 1] \subset S^1$ with $0 < m(A)$ such that $\text{osc}(h_t)$ is defined on $A \times U$ and $\text{osc}(h_t) > 0$ for $t \in A$. Furthermore it is not difficult to check (see Theorem 2.6 in the original version of [Oh2] for its proof) that h_t is continuous on U at such $t \in A$. We consider the sets

$$\begin{aligned} U_{A,h}^+ &:= \{(t, y) \in A \times U \mid h(t, y) > 0\} \\ U_{A,h}^- &:= \{(t, y) \in A \times U \mid h(t, y) < 0\}. \end{aligned}$$

Note that because of the normalized condition (20) and $\text{osc}(h_t) > 0$ for $t \in A$, the sections

$$U_{A,h}^\pm(t) := U_{A,h}^\pm \cap (\{t\} \times U)$$

are non-empty open subsets of U and so have positive measure. Then it follows from the Fubini theorem that both $U_{A,h}^\pm$ have positive measure, say $m_\pm > 0$ respectively. By definition, $U_{A,h}^+ \cap U_{A,h}^- = \emptyset$. Furthermore, we also have

$$e_{A,h}^+ := \int_{U_{A,h}^+} h d\theta \wedge \Omega > 0 \quad (46)$$

$$e_{A,h}^- := \int_{U_{A,h}^-} (-h) d\theta \wedge \Omega > 0 \quad (47)$$

and

$$e_{A,h}^+(t) := \int_{U_{A,h}^+(t)} h_t \Omega > 0 \quad (48)$$

$$e_{A,h}^-(t) := \int_{U_{A,h}^-(t)} (-h_t) \Omega > 0 \quad (49)$$

for $t \in A$.

By the standard outer-measure property of the measure induced by the volume form $d\theta \wedge \Omega$, for any given $\varepsilon > 0$, we can find open subsets $V_\varepsilon^\pm \subset [0, 1] \times U$ such that

$$V_\varepsilon^\pm \supset U_{A,h}^\pm$$

and

$$m(V_\varepsilon^\pm) \leq m(U_{A,h}^\pm) + \varepsilon, \quad m(V_\varepsilon^+ \cap V_\varepsilon^-) \leq \varepsilon.$$

In particular we have

$$\begin{aligned} m(V_\varepsilon^+ \setminus \overline{V_\varepsilon^-}) &\geq m(U_{A,h}^+) - 2\varepsilon = m^+ - 2\varepsilon \\ m(V_\varepsilon^- \setminus \overline{V_\varepsilon^+}) &\geq m(U_{A,h}^-) - 2\varepsilon = m^- - 2\varepsilon \end{aligned} \tag{50}$$

By choosing ε so that $\varepsilon = \frac{1}{4} \min\{m^+, m^-\}$, it follows that both $V_\varepsilon^+ \setminus \overline{V_\varepsilon^-}$ and $V_\varepsilon^- \setminus \overline{V_\varepsilon^+}$ are two disjoint open subsets of positive measure.

Now we choose a partitions of unity $\{\chi_\varepsilon^+, \chi_\varepsilon^-, \chi_\varepsilon^0\}$ subordinate to the open covering

$$\{V_\varepsilon^+ \setminus \overline{V_\varepsilon^-}, V_\varepsilon^- \setminus \overline{V_\varepsilon^+}, M \setminus \overline{V_\varepsilon^+ \cup V_\varepsilon^-}\}.$$

We note that as $\varepsilon \rightarrow 0$, we have

$$\chi_\varepsilon^\pm \rightarrow \chi_{U_{A,h}^\pm} \quad \text{in } L^1$$

respectively, where χ_B denotes the characteristic function of the set B in general.

We consider the function f_ε of the form

$$f_\varepsilon(\theta, y) = a_\varepsilon^+(\theta, y)\chi_\varepsilon^+(\theta, y) - a_\varepsilon^-(\theta, y)\chi_\varepsilon^-(\theta, y)$$

for a suitable choice of positive functions $a_\varepsilon^\pm > 0$ so that

$$\int_{\{\theta\} \times U} f_\varepsilon(\theta, \cdot) \Omega = 0 \tag{51}$$

for all $\theta \in S^1$. We can choose

$$0 < a_\varepsilon^\pm(\theta, y) \leq C \tag{52}$$

for some $C > 0$ independent of ε : We have only to define

$$\begin{aligned} a_\varepsilon^+(\theta) &= \min \left\{ 1, \frac{\int_U \chi_\varepsilon^-(\theta, y) \Omega}{\int_U \chi_\varepsilon^+(\theta, y) \Omega} \right\} \\ a_\varepsilon^-(\theta) &= a_\varepsilon^+(\theta) \left(\frac{\int_U \chi_\varepsilon^+(\theta, y) \Omega}{\int_U \chi_\varepsilon^-(\theta, y) \Omega} \right). \end{aligned} \tag{53}$$

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Then we note that as $\varepsilon \rightarrow 0$, the quotient $\frac{\int_U \chi_\varepsilon^-(\theta, y) \Omega}{\int_U \chi_\varepsilon^+(\theta, y) \Omega}$ converges to $\frac{m(U_{A,h}^-) \setminus m(U_{A,h}^+)}{m(U_{A,h}^+) \setminus m(U_{A,h}^-)}$ and hence

$$0 < a_\varepsilon^+(\theta) \leq \frac{m(U_{A,h}^-) \setminus m(U_{A,h}^+)}{m(U_{A,h}^+) \setminus m(U_{A,h}^-)} + 1 \quad (54)$$

$$0 < a_\varepsilon^-(\theta) \leq \frac{m(U_{A,h}^-) \setminus m(U_{A,h}^+)}{m(U_{A,h}^+) \setminus m(U_{A,h}^-)} + 1. \quad (55)$$

We just choose C to be

$$C = \max \left\{ \frac{m(U_{A,h}^-) \setminus m(U_{A,h}^+)}{m(U_{A,h}^+) \setminus m(U_{A,h}^-)} + 1, \frac{m(U_{A,h}^-) \setminus m(U_{A,h}^+)}{m(U_{A,h}^+) \setminus m(U_{A,h}^-)} + 1 \right\}.$$

This in turn implies

$$\begin{aligned} \int_{S^1 \times U} \chi_A(\theta) f_\varepsilon h^{od} d\theta \wedge \Omega &= \int_{[0,1] \times U} \chi_A(\theta) f_\varepsilon h d\theta \wedge \Omega \\ &\rightarrow \int_{[0,1] \times U} h(a_{(+,A)} \chi_{U_{A,h}^+} - a_{(-,A)} \cdot \chi_{U_{A,h}^-}) d\theta \wedge \Omega \end{aligned}$$

as $\varepsilon \rightarrow 0$, where we denote by $a_{(\pm, A)}$ the L^1 -limits of a_ε^\pm . The last integral is strictly positive because both $a_{(\pm, A)}$ cannot vanish simultaneously by the choice of a_ε^\pm made in (53).

Therefore if $\varepsilon > 0$ is sufficiently small, (42) holds for η given by $\eta = \pi_Y^*(\eta_\theta)$ where η_θ is the form satisfying (43) for $\varphi_\varepsilon(\theta) f_\varepsilon$ with $\varphi_\varepsilon = \varphi_\varepsilon(\theta)$ being a suitable smooth L^1 -approximation of the characteristic function χ_A on S^1 . This proves that the current α is not closed and so finishes the proof of Lemma 5.1.

8. Proof of Lemma 5.4 : Problem of mass transport

Let $\alpha = h d\theta$ as in section 5. Again we will prove this for h lying in $L^{(1, \infty)}$. (In this section, for the simplicity of notations, we will just denote h for h^{od} .)

We re-state here in its contrapositive form, which is more close to the statement in Lemma 2.5 [V2].

Lemma 8.1. *Let α be as above. Suppose*

$$\int_{\pi_Y^{-1}(y)} \phi_* \alpha = 0$$

for all $y \in U$ and for any fiberwise-like diffeomorphism $\phi : S^1(2) \times Y \rightarrow S^1(2) \times Y$ with $\text{supp } \phi \subset U$. Then α is closed as an n -current on $S^1(2) \times U$.

We recall that for any fiberwise-like diffeomorphism the map $Y_\theta : Y \rightarrow Y$ is a diffeomorphism for each $\theta \in S^1$, where $Y_\theta(y) = Y(\theta, y)$ in the representation of ϕ ,

$$\phi(\theta, y) = (\theta, Y(\theta, y)).$$

The rest of the section will be occupied with the proof of this lemma.

Let η' be an $(n-1)$ form on U and consider the pull-back form $\pi^*\eta'$ on $S^1(2) \times U$. Then the Fubini theorem implies

$$\int_{S^1(2) \times U} \phi_* \alpha \wedge \pi^* d\eta' = \int_U \left(\int_{S^1} \phi_* \alpha \right) \wedge d_Y \eta' = 0.$$

Therefore we have

$$\begin{aligned} 0 &= \int_{S^1(2) \times U} \alpha \wedge \phi^* \pi^* (d\eta') = \int_{S^1(2) \times U} \alpha \wedge (\pi \circ \phi)^* d\eta' \\ &= \int_{S^1(2) \times U} \alpha \wedge Y_\theta^* (d\eta') = \int_{S^1(2) \times U} \alpha \wedge d(Y_\theta^* \eta') \end{aligned}$$

Now the proof of Lemma 8.1 will be finished by the following proposition

Proposition 8.2. *Let $\alpha = h d\theta$ be the above section with coefficients in $L^{(1,\infty)}$ considered as an n -current considered above on $S^1(2) \times U$. Suppose that*

$$\int_{S^1(2) \times U} \alpha \wedge d(Y_\theta^* \eta') = 0 \quad (56)$$

for all smooth $(n-1)$ forms η' on U and for all fiberwise-like diffeomorphism $\phi : S^1(2) \times Y \rightarrow S^1(2) \times Y$ with $\phi(\theta, y) = (\theta, Y(\theta, y))$, where $Y_\theta : Y \rightarrow Y$ is the diffeomorphism $y \mapsto Y(\theta, y)$ such that $\text{supp } Y_\theta \subset U$ for all $\theta \in S^1(2)$. Then α is closed on $S^1(2) \times U$.

Proof. We have to show

$$\int_{S^1(2) \times Y} \alpha \wedge d\eta = 0 \quad (57)$$

for all smooth $(n-1)$ -form η supported in $S^1(2) \times U$.

Here we note that closedness (57) of a current is a local property and so it is enough to check the closedness in a coordinate neighborhood $(a, b) \times B \subset S^1(2) \times U$ of a given point $(e, y) \in S^1(2) \times U$. By restricting to an even smaller neighborhood $(c, d) \times C \subset [c, d] \times C \subset (a, b) \times B$ of point (e, y) , we may assume that $d\eta$ is supported on $(c, d) \times C \cup (Y \setminus \bar{B})$. We will assume this for the rest of the proof.

We decompose

$$\eta = \eta_Y + d\theta \wedge \zeta_Y$$

where η_Y and ζ_Y are forms satisfying

$$\frac{\partial}{\partial \theta} \lrcorner \eta_Y = \frac{\partial}{\partial \theta} \lrcorner \zeta_Y = 0.$$

Since $\alpha = h d\theta$, it follows

$$\alpha \wedge d\eta = \alpha \wedge d_Y(\eta_Y).$$

Now we would like to compare $d_Y(Y_\theta^*(\eta'))$ and $d_Y(\eta_Y)$. Both are n -forms supported in $S^1(2) \times U$ with coefficients depending on (θ, y) . More precisely, we have

$$d_Y(\eta_Y) = f(\theta, y)\Omega, \quad \int_Y f(\theta, y)\Omega = 0$$

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and

$$d_Y(Y_\theta^*(\eta')) = Y_\theta^*(d_Y(\eta')) = Y_\theta^*(g\Omega), \quad \int_Y g(y)\Omega = 0.$$

For given f as above, we would like to represent $f(\theta, y)\Omega$ as $Y_\theta^*(g\Omega)$, i.e., we would like to solve the equation

$$f(\theta, y)\Omega = Y_\theta^*(g\Omega) \quad \left(= g(Y(\theta, y))|\det(dY_\theta)|\Omega \right) \quad (58)$$

for a suitable choice of ϕ (and so Y_θ) and $g : Y \rightarrow \mathbb{R}$.

We will apply Moser's trick to solve this equation. Since we can make the coefficients at a fixed point θ the same on both sides, it is enough to consider its derivative. We write (58) as

$$(Y_\theta)_*(f(\theta, \cdot)\Omega) = g(y)\Omega.$$

Differentiating this with respect to θ , we get

$$(Y_\theta)_* \left(\frac{\partial f}{\partial \theta} \Omega - d_Y(X_\theta]f\Omega) \right) = 0$$

where X_θ is the vector field generating the family $Y_\theta : Y \rightarrow Y$ of diffeomorphisms. Since Y_θ is a diffeomorphism supported in U , we obtain

$$\frac{\partial f}{\partial \theta} \Omega - d_Y(X_\theta]f\Omega) = 0 \quad (59)$$

with $\text{supp } X_\theta \subset U$. Let β_θ be an $(n-1)$ form on Y such that

$$\frac{\partial f}{\partial \theta} \Omega = d_Y \beta_\theta \quad (60)$$

which exists because

$$\int_Y \frac{\partial f}{\partial \theta} \Omega = \frac{\partial}{\partial \theta} \int_Y f \Omega = 0.$$

We may assume that β_θ depends smoothly on θ . Substituting (60) into (59), the problem is reduced to solving the algebraic equation

$$\beta_\theta = fX_\theta]\Omega \quad (61)$$

in terms of X_θ . This equation can be solved pointwise whenever f does not vanish.

By adjusting $f\Omega$ outside $[a, b] \times B$ so that $\int_Y f\Omega = 0$ is satisfied, we may safely assume that

$$f \neq 0 \quad \text{on } (c, d) \times C. \quad (62)$$

Then we can solve (61) for X_θ on $(c, d) \times B$ which will have support on

$$(c, d) \times C \cup (U \setminus \overline{B}).$$

Now by making the interval (c, d) smaller if necessary, we can integrate

$$\dot{y} = X_\theta(y), \quad y(e) = y$$

to obtain a flow Y_θ that solves (58) on $(c, d) \times C$, if we choose the function g above so that its values are given by

$$g(y) = f(e, y) |\det(dY_e(y))|.$$

Therefore the hypothesis (56) implies that α is closed on $(c, d) \times C$. Since this holds at any given point $(e, y) \in S^1(2) \times U$, α is a closed current on $S^1(2) \times U$. This finishes the proof of Lemma 8.1. Now the statement in Lemma 5.4 concerning the C^0 -closeness to the identity is immediate from the proof, because ϕ can be obtained by integrating a vector field X_θ on a short ‘time’ interval starting from $\theta = e$ in this proof.

The above also proves that the diffeomorphism ϕ constructed above has the form $\phi(\theta, y) = (\theta, Y_\theta(y))$. Hence the proof of Lemma 5.4. \square

9. The case of general boundary and etc.

In this section, we explain how we derive the same result for a general open subset whose boundary is not necessarily smooth and how to treat general C^0 -Hamiltonian path which is not necessarily boundary-flat. We then explain how we can generalize to open subsets $U \subset M$ where M is not necessarily closed. Finally we explain the proof of the uniqueness of compactly supported Hamiltonians, Theorem 1.2, either on M with contact-type boundary or on M which is bounded at infinity.

For the first point, we just choose an increasing sequence $U_l \subset U$ of open subsets such that

$$U = \bigcup_l U_l$$

and each U_l has smooth boundary ∂U_l . Obviously if $\phi_H = \lambda \equiv id$ on U , so is on U_i and so we obtain

$$H(t, x) \equiv c_l(t)$$

on U_l for each l . Since $U_l \subset U_{l'}$ for $l' \geq l$, we must have $c_l \equiv c_{l'}$ for all l, l' . Defining c to be the common function, we have proved $H(t, x) \equiv c(t)$ on U and so on \bar{U} by continuity of H .

For the second point, we imitate the idea of Viterbo from [Erratum, [V2]]. Let H_i be a sequence of Hamiltonians, which are not necessarily boundary-flat, such that ϕ_{H_i} and H_i converges in C^0 -topology.

Then for any reparameterization H_i^ζ defined by $H^\zeta(t, x) = \zeta'(t)H(\zeta(t), x)$ with each fixed function $\zeta : [0, 1] \rightarrow [0, 1]$ satisfying $\zeta' = 0$ near $t = 0, 1$, both $\phi_{H_i^\zeta}$ and H_i^ζ converge in C^0 -topology. Furthermore $\phi_{H_i^\zeta}$ converge to id since $\phi_{H_i^\zeta}$ is nothing but the time reparameterization $\phi_{H_i}^{\zeta(t)}$ and ζ fixed. Therefore we have

$$H^\zeta \equiv c_\zeta \quad \text{on } U.$$

We note that for any given $t_0 \in [0, 1]$ we can choose some ζ_{t_0} so that $\zeta_{t_0}(t_0) = t_0$ and $\zeta'_{t_0}(t_0) = 1$. By considering such ζ , we prove that

$$H(t_0, \cdot) = c_{\zeta_{t_0}}(t_0)$$

OH

and hence $H(t, x) \equiv c_{\zeta_t}(t)$. We just define $c : [0, 1] \rightarrow [0, 1]$ to be

$$c(t) := c_{\zeta_t}(t).$$

Since H is continuous, this c is a continuous function on $[0, 1]$. This takes care of the case of arbitrary C^0 -Hamiltonian paths and finishes the proof of Theorem 1.3.

For the point of generalizing the locality to an arbitrary complete symplectic manifold *bounded at infinity* in the sense of Gromov [Gr] but not necessarily closed, we have only to point out that the proof of the relative version in [KO] of Lagrangian intersection result between the Hamiltonian deformation of the zero section $o_{S^1(2) \times M}$ with the conormal

$$\nu^*(S^1(2) \times \bar{U})$$

goes through verbatim, once we have an a-priori C^0 -estimate for the relevant Floer trajectory moduli space, which makes the Floer homology between $\nu^*(S^1(2) \times \bar{U})$ and the zero section $o_{S^1(2) \times Y}$ well-defined and invariant under the compactly supported Hamiltonian isotopy of $o_{S^1(2) \times Y}$ with $Y = M$ by the standard continuation argument. Here we also use the fact that the intersection

$$\nu^*(S^1(2) \times \bar{U}) \cap o_{S^1(2) \times Y}$$

is compact. We leave the details to interested readers.

An immediate consequence of this locality is the following uniqueness theorem of compactly supported topological Hamiltonians introduced in [OM].

Theorem 9.1. *Let M be a symplectic manifold which is bounded at infinity. Suppose $H_i : [0, 1] \times M \rightarrow \mathbb{R}$ is a sequence of smooth Hamiltonian functions such that*

- (1) *there exists a compact set $K \subset \text{Int } M$ such that*

$$\text{supp } H_i \subset K$$

- (2) *H_i converges to $H : Y \rightarrow \mathbb{R}$ in C^0 -topology.*

Then if $\phi_{H_i} \rightarrow \text{id}$ uniformly on $[0, 1] \times M$, we have $H \equiv 0$.

Proof. We choose U so that ∂U is smooth, \bar{U} compact and $K \subset U$. Then we apply the locality theorem. \square

Finally the uniqueness result for compact (M, ω) with contact-type boundary ∂M is reduced to the above case of complete manifold bounded at infinity by attaching the cylinder $\mathbb{R}_+ \partial M$ to its boundary. This is standard and so left to readers for the details. It is not clear to the author whether or not the uniqueness still holds for general (M, ω) with smooth boundary, not necessarily of contact-type, for which there does not seem to exist an obvious way of symplectically embedding (M, ω) into a symplectic manifold bounded at infinity.

10. Appendix

10.1. Exact Lagrangian property of the isotopy (64)

In this appendix, we prove the exactness of the Lagrangian embedding $\iota_{(h,\psi)}$ given in (18) and prove Proposition 4.6. This is a consequence of Theorem 6.1.B [P] whose proof is left as an exercise in [P] with the indication that the exactness of $i_s^*\alpha$ will follow from some homological argument. Here we give a computational proof somewhat different from the proof suggested in [P] and also explicitly write down the anti-derivative of $i_s^*\alpha$. The outcome is quite interesting and suggestive, and gives rise to a slight generalization of Theorem 6.1.B [P] in that it applies to any Hamiltonian paths, not just to Hamiltonian loops, to which homological argument may not apply.

Consider an isotopy of Hamiltonian paths i.e., a two parameter family of Hamiltonian diffeomorphisms

$$(s, \theta) \in [0, 1] \times [a, b] \mapsto \psi_s^\theta \in \text{Ham}(X, \omega).$$

We denote by $H = H(s, \theta, x)$ the *normalized* Hamiltonian generating the θ -isotopy $\theta \mapsto \psi_s^\theta$ and $K = K(s, \theta, x)$ the one generating the s -isotopy $s \mapsto \psi_s^\theta$. In other words,

$$\frac{\partial \psi_s^\theta}{\partial \theta} \circ (\psi_s^\theta)^{-1} = X_H, \quad \frac{\partial \psi_s^\theta}{\partial s} \circ (\psi_s^\theta)^{-1} = X_K.$$

Then the following identity has been proved in [B], [P], [Oh1]

$$\frac{\partial K}{\partial \theta} = \frac{\partial H}{\partial s} - \{K, H\} \tag{63}$$

for the normalized Hamiltonian H .

With this preparation, we prove the following theorem.

Theorem 10.1 (Compare with Theorem 6.1.B [P]). *Let $Y \subset X$ be a Lagrangian embedding and consider a two parameter family ψ_s^θ of Hamiltonian diffeomorphisms as above. Then the isotopy $f : [0, 1] \times [a, b] \times Y \rightarrow T^*[a, b] \times X =: M$ defined by*

$$f(s, \theta, x) = (\theta, -H(s, \theta, \psi_s^\theta(x)), \psi_s^\theta(x)) \tag{64}$$

is an exact Lagrangian isotopy in the sense of Definition 4.1. Furthermore, we have

$$i_s^*\alpha = d(-K \circ f)_s. \tag{65}$$

We like to mention that here s replaces the role of t and $[a, b] \times Y$ that of Y in Definition 4.1.

Proof. First we compute

$$f^*(d\theta \wedge db + \omega_X).$$

Since $\sigma_{[a,b]} \times i_Y \subset T^*[a, b] \times X$ is a Lagrangian embedding and ψ_s^θ is symplectic, it is easy to check that

$$f^*(d\theta \wedge db + \omega_X) = \alpha \wedge ds$$

OH

for some one-form α on $[0, 1] \times ([a, b] \times Y)$. We have

$$\alpha = \frac{\partial}{\partial s} \lrcorner f^*(d\theta \wedge db + \omega_X)$$

which we now evaluate. First we have

$$\alpha(\xi) = (d\theta \wedge db + \omega_X) \left(\frac{\partial f}{\partial s}, Tf(\xi) \right)$$

for general tangent vector $\xi \in T([a, b] \times Y)$. Straight computations give rise to the following formulae :

$$\begin{aligned} Tf \left(\frac{\partial}{\partial s} \right) &= \left(\frac{\partial H}{\partial s} - \{K, H\} \right) \frac{\partial}{\partial b} \oplus X_K \circ \psi_s^\theta \\ &= \frac{\partial K}{\partial \theta} \frac{\partial}{\partial b} \oplus X_K \circ \psi_s^\theta \\ Tf \left(\frac{\partial}{\partial \theta} \right) &= \frac{\partial}{\partial \theta} \oplus -\frac{\partial H}{\partial \theta} \frac{\partial}{\partial b} \oplus X_H \circ \psi_s^\theta \\ Tf(\eta) &= -d_X H(T\psi_s^\theta(\eta)) \frac{\partial}{\partial b} \oplus T\psi_s^\theta(\eta) \end{aligned}$$

where η is a vector field whose projection to $[0, 1] \times [a, b]$ is zero. For the first equality, we used the identity $dH(X_K) = \{H, K\}$ and for the second equality, we have used (63). From these formulae, we derive

$$\begin{aligned} i_s^* \alpha &= i_s^* \left(\frac{\partial}{\partial s} \lrcorner (d\theta \wedge db + \omega_X) \right) = - \left(\left(\frac{\partial K}{\partial \theta} \circ f_s \right) d\theta \circ Tf_s + d_X K \circ Tf_s \right) \\ &= -f_s^* \left(\frac{\partial K}{\partial \theta} d\theta + d_X K \right) = -f_s^* dK = d(-K \circ f)_s \end{aligned}$$

Therefore the form $i_s^* \alpha$ is exact and hence f defines a Lagrangian isotopy in the sense of Definition 4.1. Furthermore we have also proved (65). This finishes the proof. \square

Now we are ready to give the proof of Proposition 4.6 by applying this theorem to to the embedding $\iota_{(h, \psi)}$.

Proof of Proposition 4.6. To apply this theorem to the embedding $\iota_{(h, \psi)}$, we replace $[a, b]$ by $[0, 2]$ replace $[a, b]$ by $[0, 2]$ and apply the theorem to $\psi_s^\theta = \phi_H^{s\chi(\theta)}$ where $\chi : [0, 2] \rightarrow [0, 1]$ is a function defined by

$$\chi(s) = \begin{cases} \zeta(s) & \text{for } s \in [0, 1] \\ \zeta(2-s) & \text{for } s \in [1, 2]. \end{cases} \quad (66)$$

Here $\zeta : [0, 1] \rightarrow [0, 1]$ is a reparameterization function as defined in (8).

Recall the Hamiltonian generating in θ -direction, which we still denote by H with some abuse of notations, is given by

$$H(s, \theta, x) = s\chi'(\theta)H(s\chi(\theta), x).$$

Now we have only to check that the Hamiltonian $K : [0, 1] \times [0, 2] \times Y \rightarrow \mathbb{R}$ generating in s -direction above is 2-periodic, i.e., satisfies

$$K(s, 0, x) = K(s, 2, x) \quad (67)$$

for the present case. But a straightforward calculation differentiating ψ_s^θ in s gives rise to

$$K(s, \theta, x) = \chi(\theta)H(s\chi(\theta), x)$$

which obviously satisfies (67) by the definition (66) of χ . The condition on the support stated in Proposition 4.6 is apparent from the definition of f . This finishes the proof. \square

10.2. Proof of Lemma 6.3

In this appendix, we prove Lemma 6.3. We restate the lemma here.

Lemma 10.2. *There exists a continuous function $\varepsilon : S^1(2) \times Y \rightarrow \mathbb{R}$ such that*

- (1) *For all $y \in Y$, $\int_{S^1(2)} (h_\phi^{od}(t, y) - \varepsilon(t, y)) dt = 0$,*
- (2) *$h_\phi^{od} - \varepsilon$ is smooth.*
- (3) *We have the inequality*

$$\|P_\phi - \varepsilon\|_{C^0} < \frac{1}{30} \|P_\phi\|_{C^0}.$$

Proof. Consider the solution \tilde{f} for the equation (33), which can be written as

$$\tilde{f}(\theta, y) = \int_0^\theta (h_\phi^{od}(t, y) - P_\phi(y)) dt.$$

From this expression, it follows that \tilde{f} is C^1 in θ and C^0 in y . Therefore we can choose a smooth approximation $f_{sm} : S^1(2) \times Y \rightarrow \mathbb{R}$ so that both norms

$$\left| \frac{\partial \tilde{f}}{\partial \theta} - \frac{\partial f_{sm}}{\partial \theta} \right|_{C^0}, \quad |\tilde{f} - f_{sm}|_{C^0} \quad (68)$$

as small as we want. Denote $\xi = \tilde{f} - f_{sm}$, and define

$$\varepsilon(\theta, y) := P_\phi(y) + \frac{\partial \xi}{\partial \theta}. \quad (69)$$

We now show that ε will do our purpose. First (1) follows since we have

$$h_\phi^{od} - \varepsilon = (h_\phi^{od} - P_\phi) - \frac{\partial \xi}{\partial \theta}$$

and $h - \varepsilon$ satisfies the vanishing of the corresponding integral.

For (2), we recall

$$h_\phi^{od} - P - \frac{\partial \xi}{\partial \theta} = \frac{\partial \tilde{f}}{\partial \theta} - \frac{\partial \xi}{\partial \theta} = \frac{\partial}{\partial \theta} (\tilde{f} - \xi) = \frac{\partial f_{sm}}{\partial \theta}$$

which is smooth.

For (3), we note

$$\varepsilon - P_\phi = \frac{\partial \xi}{\partial \theta} = \frac{\partial}{\partial \theta}(\tilde{f} - f_{sm})$$

which can be made as small as we want by choosing f_{sm} so that the norms in (68) become small. This finishes the proof. \square

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