

Virtual crossings, convolutions and a categorification of the $SO(2N)$ Kauffman polynomial

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ABSTRACT. We suggest a categorification procedure for the $SO(2N)$ one-variable specialization of the two-variable Kauffman polynomial. The construction has many similarities with the HOMFLY-PT categorification: a planar graph formula for the polynomial is converted into a complex of graded vector spaces, each of them being the homology of a \mathbb{Z}_2 -graded differential vector space associated to a graph and constructed using matrix factorizations. This time, however, the elementary matrix factorizations are not Koszul; instead, they are convolutions of chain complexes of Koszul matrix factorizations.

We prove that the homotopy class of the resulting complex associated to a diagram of a link is invariant under the first two Reidemeister moves and conjecture its invariance under the third move.

1. Introduction

1.1. $SO(2N + 2)$ Kauffman polynomial and graded complexes

The $SO(2N + 2)$ Kauffman polynomial [4]

$$P_L(q) \in \mathbb{Z}[q^{\pm 1}]$$

is an invariant of an unoriented framed link $L \in S^3$, which satisfies the skein relation

$$P \left[\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right] - P \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right] = (q - q^{-1}) \left(P \left[\begin{array}{c} | \\ | \end{array} \right] \left(\left[\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right] - P \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right] \right) \right), \quad (1.1)$$

the change of framing relation

$$P \left[\begin{array}{c} | \\ \bigcirc \\ | \end{array} \right] = q^{2N+1} P \left[\begin{array}{c} | \\ | \end{array} \right] \quad (1.2)$$

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(throughout the paper we assume that links are endowed with the blackboard framing), and the multiplicativity property

$$P_{L_1 \sqcup L_2}(q) = P_{L_1}(q)P_{L_2}(q), \tag{1.3}$$

where $L_1 \sqcup L_2$ is the disjoint union of the links L_1 and L_2 . The conditions (1.1)–(1.3) determine $P_L(q)$ uniquely [4], [6]. In particular, the Kauffman polynomial of the unknot is equal to

$$P_{\text{unkn}}(q) = \frac{q^{2N+1} - q^{-2N+1}}{q - q^{-1}} + 1. \tag{1.4}$$

We will describe a conjectural categorification of the SO(2N+2) Kauffman polynomial. Namely, to a link diagram L we associate a chain complex $C^\bullet(L)$ of $\mathbb{Z}_2 \times \mathbb{Z}$ -graded \mathbb{Q} -vector spaces

$$\cdots \rightarrow C^n(L) \rightarrow C^{n+1}(L) \rightarrow \cdots, \quad C^n(L) = \bigoplus_{\substack{i \in \mathbb{Z} \\ j \in \mathbb{Z}_2}} C_{\{i\}}^{n,j}(L). \tag{1.5}$$

We prove that, as an object in the homotopy category of complexes of $\mathbb{Z}_2 \times \mathbb{Z}$ -graded vector spaces, the complex $C^\bullet(L)$ behaves nicely under the first two Reidemeister moves:

Theorem 1.1. *The complex $C^\bullet(L)$ is homotopy invariant up to degree shifts under the first Reidemeister move:*

$$C^\bullet \left[\left[\text{Reidemeister 1 move} \right] \right] \simeq C^\bullet \left[\left[\text{straight line} \right] \right] \{-2N-1\} \langle 1 \rangle [-1], \tag{1.6}$$

Here $[\cdot]$, $\langle \cdot \rangle$, and $\{\cdot\}$ denote the shifts in the homological degree of the complex (1.5), in the \mathbb{Z}_2 -degree and in the \mathbb{Z} -degree respectively.

Theorem 1.2. *The complex $C^\bullet(L)$ is homotopy invariant under the second Reidemeister move:*

$$C^\bullet \left[\left[\text{Reidemeister 2 move} \right] \right] \simeq C^\bullet \left[\left[\text{two parallel lines} \right] \right]. \tag{1.7}$$

Further, we conjecture that

Conjecture 1.3. *The complex $C^\bullet(L)$ is homotopy invariant under the third Reidemeister move:*

$$C^\bullet \left[\begin{array}{c} \text{Diagram 1} \end{array} \right] \simeq C^\bullet \left[\begin{array}{c} \text{Diagram 2} \end{array} \right]. \quad (1.8)$$

Conjecture 1.4. *The whole complex $C^\bullet(L)$ has a homogeneous \mathbb{Z}_2 -degree*

$$\deg_{\mathbb{Z}_2} C^\bullet(L) = n_L \pmod{2}, \quad (1.9)$$

where n_L is the number of crossings in the diagram L .

We will show at the end of subsection 1.3 that Conjecture 1.3 implies that the graded Euler characteristic of the complex $C^\bullet(L)$

$$\chi_q(C^\bullet(L)) = \sum_{\substack{i, n \in \mathbb{Z} \\ j \in \mathbb{Z}_2}} (-1)^{j+n} q^i \dim C_{\{i\}}^{n,j}(L) \quad (1.10)$$

equals the Kauffman polynomial:

$$\chi_q(C^\bullet(L)) = P_L(q). \quad (1.11)$$

Our approach to the construction of the categorification complex $C^\bullet(L)$ is similar to that of [8]: it follows the alternating sum formula for the polynomial $P_L(q)$, which expresses it in terms of polynomial invariants of planar graphs. The categorification of elementary open graphs is provided by matrix factorizations, and this time the basic polynomial is $W(x, y) = xy^2 + x^{2N+1}$, as suggested by Gukov and Walcher [2].

1.2. Closed graphs and the alternating sum formula

Kauffman and Vogel [6] extended the Kauffman polynomial from links to knotted 4-valent graphs. They defined the invariant of graphs by presenting the 4-vertex as a linear combination of crossings and their resolutions:

$$\begin{aligned} P \left[\begin{array}{c} \text{4-vertex} \end{array} \right] &= -P \left[\begin{array}{c} \text{Crossing 1} \end{array} \right] + qP \left[\begin{array}{c} \text{Crossing 2} \end{array} \right] + q^{-1}P \left[\begin{array}{c} \text{Resolution 1} \end{array} \right] \\ &= -P \left[\begin{array}{c} \text{Crossing 1} \end{array} \right] + qP \left[\begin{array}{c} \text{Resolution 2} \end{array} \right] + q^{-1}P \left[\begin{array}{c} \text{Crossing 2} \end{array} \right] + q^{-1}P \left[\begin{array}{c} \text{Resolution 1} \end{array} \right]. \end{aligned} \quad (1.12)$$

The second equality follows from the skein relation (1.1) and expresses the 90° rotational symmetry of the 4-vertex.

The formula (1.12) presents the Kauffman polynomial of a knotted graph as a linear combination of Kauffman polynomials of links constructed by resolving each 4-vertex in three possible ways, thus reducing the computation of the Kauffman polynomial of a graph to that of links.

The relation (1.12) can also be played backwards, that is, a crossing can be presented as a linear combination

$$P \left[\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right] = qP \left[\begin{array}{c} \frown \\ \smile \end{array} \right] - P \left[\begin{array}{c} \diagup \diagdown \\ \bullet \\ \diagdown \diagup \end{array} \right] + q^{-1}P \left[\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right]. \quad (1.13)$$

In this form, it allows us to express the Kauffman polynomial of a link in terms of the polynomials of planar graph diagrams. Namely, let L be a link diagram with n_L enumerated crossings. To a multi-index $\mathbf{r} = (r_1, \dots, r_{n_L})$, $r_i \in \{-1, 0, 1\}$, we associate a planar graph diagram $\Gamma_{\mathbf{r}}$ constructed by resolving all crossings of L : the i -th crossing is resolved in r_i -th way. Then, according to eq.(1.13),

$$P_L(q) = (-1)^{n_L} \sum_{\mathbf{r}} (-1)^{|\mathbf{r}|} q^{-|\mathbf{r}|} P_{\Gamma_{\mathbf{r}}}(q), \quad (1.14)$$

where $|\mathbf{r}| = \sum_{i=1}^{n_L} r_i$. We call 4-valent graphs $\Gamma_{\mathbf{r}}$ *closed*.

1.3. The outline of the categorification

Similar to the HOMFLY-PT polynomial case, the categorification of the Kauffman polynomial is based on turning the alternating sign sum (1.14) into a complex of graded vector spaces. To each closed graph $\Gamma_{\mathbf{r}}$ we associate a $\mathbb{Z}_2 \times \mathbb{Z}$ -graded vector space $C(\Gamma_{\mathbf{r}})$ (in fact, we conjecture that $\deg_{\mathbb{Z}_2} C(\Gamma_{\mathbf{r}}) = 0$; the appearance of the \mathbb{Z}_2 degree n_L in the formula (1.9) is due to the \mathbb{Z}_2 degree shift in eq.(1.16)). We assemble all spaces $C(\Gamma_{\mathbf{r}})$ into an n_L -complex by placing them according to their multi-indices \mathbf{r} (interpreted as n_L -dimensional coordinates) at the nodes of a \mathbb{Z}^{n_L} -lattice and defining appropriate differential maps between adjacent spaces. Namely, if three graphs $\Gamma_{\smile}, \Gamma_{\bullet}, \Gamma_{\frown}$ result from the same resolution of all crossings of L , except for an i -th crossing $\diagup \diagdown$, where they differ according to their indices, then we construct the linear maps forming the chain complex

$$C(\Gamma_{\smile}) \{1\} \xrightarrow{\chi_{\text{in},i}} C(\Gamma_{\bullet}) \xrightarrow{\chi_{\text{out},i}} C(\Gamma_{\frown}) \{-1\}. \quad (1.15)$$

$$C^\bullet \left(\left(\text{Hopf link} \right) \right) = \left[\begin{array}{ccccc}
 C \left(\left(\text{Hopf link} \right) \right) \{2\} & \xrightarrow{\chi_{\text{in},u}} & C \left(\left(\text{Hopf link} \right) \right) \{1\} & \xrightarrow{\chi_{\text{out},u}} & C \left(\left(\text{Hopf link} \right) \right) \\
 \downarrow -\chi_{\text{in},d} & & \downarrow \chi_{\text{in},d} & & \downarrow -\chi_{\text{in},d} \\
 C \left(\left(\text{Hopf link} \right) \right) \{1\} & \xrightarrow{\chi_{\text{in},u}} & C \left(\left(\text{Hopf link} \right) \right) \{0\} & \xrightarrow{\chi_{\text{out},u}} & C \left(\left(\text{Hopf link} \right) \right) \{-1\} \\
 \downarrow -\chi_{\text{out},d} & & \downarrow \chi_{\text{out},d} & & \downarrow -\chi_{\text{out},d} \\
 C \left(\left(\text{Hopf link} \right) \right) & \xrightarrow{\chi_{\text{in},u}} & C \left(\left(\text{Hopf link} \right) \right) \{-1\} & \xrightarrow{\chi_{\text{out},u}} & C \left(\left(\text{Hopf link} \right) \right) \{-2\}
 \end{array} \right],$$

FIGURE 1. The categorification complex of the Hopf link

The complexes (1.15) represent the action of n_L mutually anti-commuting differentials, each differential being related to a resolution of a particular crossing of L . The categorification complex $C^\bullet(L)$ results from collapsing the n_L -complex into a single complex by taking the sum of individual differentials and shifting the \mathbb{Z}_2 -degree of the whole complex by n_L units. The homological degree of each space $C(\Gamma_{\mathbf{r}})$ is set to be equal to $|\mathbf{r}|$.

For example, the complex of the Hopf link is presented in Fig. 1. There the maps $\chi_{*,u}$ and $\chi_{*,d}$ are related to the upper and lower crossing of the Hopf link diagram.

In order to construct the spaces $C(\Gamma_{\mathbf{r}})$ and the morphisms between them, we break the link diagram L and the corresponding closed graphs $\Gamma_{\mathbf{r}}$ into simple pieces by cutting across each edge of the diagram L . The diagram L splits into elementary tangles $\begin{array}{l} \diagup \\ \diagdown \end{array}$, while the closed graphs $\Gamma_{\mathbf{r}}$ split into elementary *open graphs* \smile , \blacklozenge and \smile . To each elementary open graph γ we associate a matrix factorization $\hat{\gamma}$, and to an elementary

tangle $\widehat{\times}$ we associate a complex of matrix factorizations

$$\widehat{\times} = \left(\widehat{\smile} \{1\} \xrightarrow{\chi_{\text{in}}} \widehat{\times} \xrightarrow{\chi_{\text{out}}} \widehat{\smile} \{-1\} \right) \langle 1 \rangle \quad (1.16)$$

with a special choice of morphisms χ_{in} and χ_{out} , which are local versions of the morphisms (1.15). The homological degree of the middle matrix factorization in this complex is set to zero. If a closed graph $\Gamma_{\mathbf{r}}$ consists of n_L elementary open graphs γ_i , then we set

$$C(\Gamma_{\mathbf{r}}) = \text{H}_{\text{MF}}(\widehat{\Gamma}_{\mathbf{r}}), \quad \widehat{\Gamma}_{\mathbf{r}} = \bigotimes_{i=1}^{n_L} \widehat{\gamma}_i, \quad (1.17)$$

$$C^{\bullet}(L) = \text{H}_{\text{MF}}(\widehat{L}), \quad \widehat{L} = \bigotimes_{i=1}^{n_L} \widehat{\times}_i, \quad (1.18)$$

and H_{MF} is the matrix factorization homology (its definition is given in subsection 2.1; the precise definition of the tensor product appearing in the formulas for $\widehat{\Gamma}_{\mathbf{r}}$ and \widehat{L} will be given in subsection 2.4). As a tensor product of complexes (1.16), $C^{\bullet}(L)$ will have an expected structure of the total complex of the n_L -dimensional cube-like complex.

Now we can explain how Conjecture 1.3 implies the equality (1.11) between the graded Euler characteristic of the categorification complex and the Kauffman polynomial. Since the skein relation (1.1) and the framing relation (1.2) define a topological invariant of framed links uniquely, it is sufficient to show that the graded Euler characteristic $\chi_q(C^{\bullet}(L))$ satisfies the same relations. The framing relation follows directly from eq.(1.6). The skein relation follows from the relation (1.13), and $\chi_q(C^{\bullet}(L))$ satisfies the latter relation in view of the main categorification formula (1.16).

1.4. A virtual crossing and a convolution

The first challenge of the categorification of the Kauffman polynomial is to choose the right matrix factorization $\widehat{\times}$ and morphisms $\chi_{\text{in}}, \chi_{\text{out}}$. It turns out that in contrast to the $\text{SU}(N)$ and HOMFLY-PT cases we can not present $\widehat{\times}$ as a Koszul matrix factorization (its rank is not a power of 2). Instead, $\widehat{\times}$ is realized as a convolution of a complex of Koszul matrix factorizations.

First of all, we introduce a new type of 4-vertex for graphs: a *virtual crossing* \times . Its matrix factorization $\widehat{\times}$ is just a tensor product of two 1-arc matrix factorizations

(similarly to $\widehat{)}(\widehat{}$ and $\widehat{)}(\widehat{}$), as if the segments of $\widehat{\times}$ did not cross. Then we define two special morphisms

$$\widehat{)}(\widehat{ \xrightarrow{F} \widehat{\times} \xrightarrow{G} \widehat{)}(\widehat{ , \quad (1.19)}$$

which we call *saddle* morphisms, and we prove that

$$GF \simeq 0. \quad (1.20)$$

The category of matrix factorizations is triangulated, which means that to a morphism $A \xrightarrow{f} B$ between two matrix factorizations one can associate a matrix factorization $\text{Cone}_{\text{MF}}(f)$ called the cone of f . The cone comes with a pair of natural morphisms

$$B \rightarrow \text{Cone}_{\text{MF}}(f) \rightarrow A \langle 1 \rangle, \quad (1.21)$$

which form two sides of the exact triangle of f .

A similar construction can be applied to a chain of two morphisms $A \xrightarrow{f} B \xrightarrow{g} C$, such that $gf \simeq 0$. The resulting matrix factorization is called a *convolution* and we denote it by a frame box around the chain: $\boxed{A \xrightarrow{f} B \xrightarrow{g} C}$ (note that here we omitted the secondary homomorphism in order to simplify the diagrams; the role of secondary morphisms will be explained in Section 5). The module of the convolution is a sum of modules $A \oplus B \oplus C$. The convolution comes with two natural morphisms, which form a chain complex of matrix factorizations:

$$C \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \boxed{A \xrightarrow{f} B \xrightarrow{g} C} \xrightarrow{(1 \ 0 \ 0)} A. \quad (1.22)$$

This allows us to define $\widehat{\times}$ as a convolution of the chain (1.19)

$$\widehat{\times} = \boxed{\widehat{)}(\widehat{ \{-1\} \xrightarrow{F} \widehat{\times} \langle 1 \rangle \xrightarrow{G} \widehat{)}(\widehat{ \{1\}}} \quad (1.23)$$

and use the natural morphisms of (1.22) as χ_{in} and χ_{out} , so that the definition (1.16) of the categorification of a crossing becomes

$$\widehat{\times} = \left(\widehat{\smile} \{1\} \xrightarrow{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \left[\widehat{\smile} \{-1\} \xrightarrow{F} \widehat{\times} \langle 1 \rangle \xrightarrow{G} \widehat{\smile} \{1\} \right] \xrightarrow{(1 \ 0 \ 0)} \widehat{\smile} \{-1\} \right) \langle 1 \rangle. \quad (1.24)$$

In view of the explicit form (1.22) of the morphisms χ_{in} and χ_{out} , we may depict the diagram (1.24) informally as

$$\widehat{\times} = \left(\begin{array}{c} \widehat{\smile} \{-1\} \xrightarrow{1} \widehat{\smile} \{-1\} \\ \downarrow F \\ \widehat{\times} \langle 1 \rangle \\ \downarrow G \\ \widehat{\smile} \{1\} \end{array} \right) \langle 1 \rangle \quad (1.25)$$

If we substitute the definition (1.23) into the diagram of Fig. 1, then the complex for the Hopf link takes the form depicted in Fig. 2.

We omitted there degree shifts and secondary homomorphisms and used an abbreviated notation

$$\left(* \right) = H_{\text{MF}} \left(\boxed{*} \right), \quad (1.26)$$

where $*$ is a chain of matrix factorization morphisms.

1.5. A categorification complex of a virtual link

Since we have introduced the matrix factorization $\widehat{\times}$ for a virtual crossing, it is natural to extend the definition (1.18) of the categorification complex of a link to virtual links.

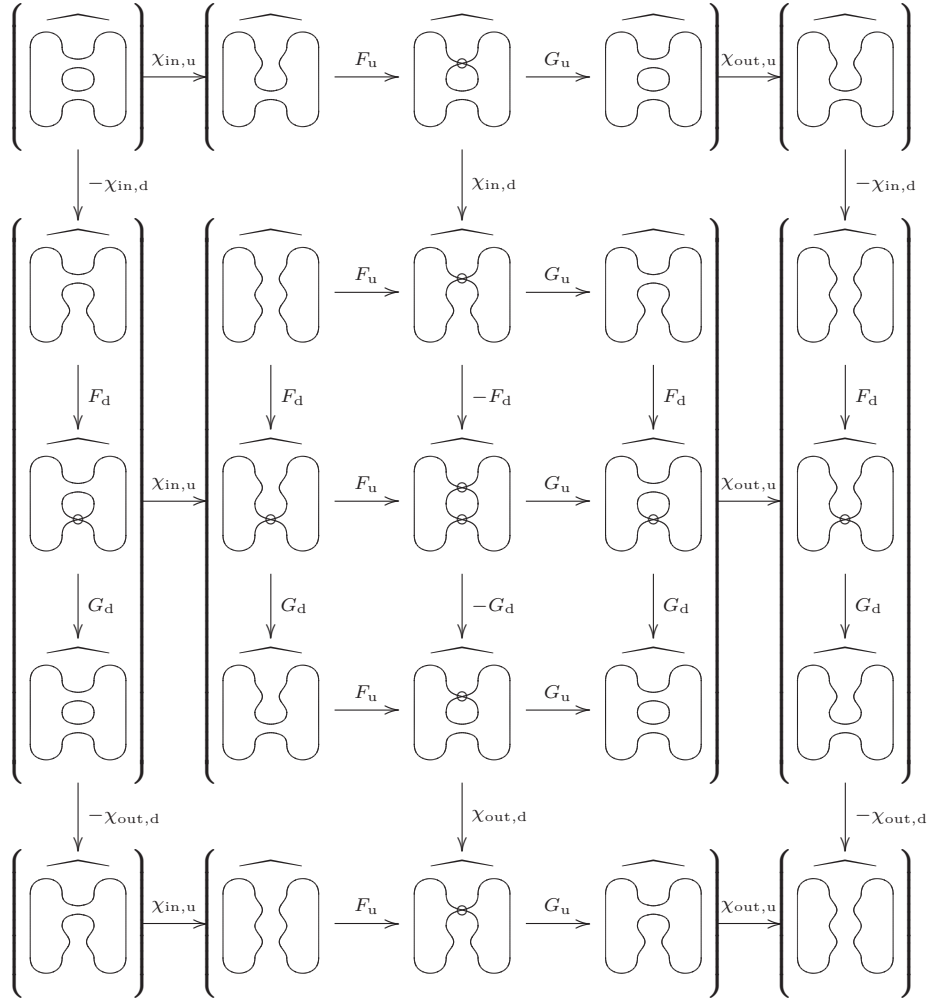


FIGURE 2. Detailed structure of the Hopf link categorification complex

Virtual links were invented by L. Kauffman [5] and their topological meaning was clarified by G. Kuperberg [10] (for the details see V. Manturov's book [11]). From the combinatorial point of view, they are equivalence classes of virtual link diagrams. A virtual link diagram is a 4-valent planar graph with two types of 4-valent vertices: an ordinary crossing \times and a virtual crossing \otimes . Two diagrams are equivalent (and thus

present the same virtual link), if one can be transformed into another by a sequence of special moves: ordinary Reidemeister moves, virtual Reidemeister moves and semi-virtual Reidemeister moves.

The virtual Reidemeister moves are the ordinary Reidemeister moves, in which all crossings are replaced by virtual crossings. A semi-virtual Reidemeister move is a third Reidemeister move, in which two crossings are replaced by virtual crossings:

(1.27)

Let L be a virtual link diagram with n_L 4-vertices. We break it into pieces γ_i ($1 \leq i \leq n_L$) by cutting across its edges. We call these pieces elementary tangles. The elementary tangles γ_i coming from the breakup of L are either crossings \times or virtual crossings \otimes . We define the categorification complex $C^\bullet(L)$ of a virtual link diagram L similar to eq.(1.18):

$$C^\bullet(L) = \mathbf{H}_{\text{MF}}(\hat{L}), \quad \hat{L} = \bigotimes_{i=1}^{n_L} \hat{\gamma}_i. \quad (1.28)$$

Theorem 1.5. *If two virtual link diagrams L and L' are related by a virtual or a semi-virtual Reidemeister move, then their complexes $C^\bullet(L)$ and $C^\bullet(L')$ are homotopy equivalent.*

We will prove this theorem in subsection 7.4.

2. Matrix factorizations and graphs

2.1. A category of matrix factorizations

Let us recall the basic definitions and notations related to matrix factorizations. For a polynomial W in a polynomial ring R , we define a (homotopy) category of matrix factorizations $\text{MF}_{R,W}$ (or simply MF_W). Its objects are \mathbb{Z}_2 -graded free R -modules $M = M_0 \oplus M_1$ equipped with a *twisted differential* $D \in \text{End}_R(M)$ such that

$$\deg_{\mathbb{Z}_2} D = 1, \quad D^2 = W 1. \quad (2.1)$$

Thus D splits into a sum of two homomorphisms $D = P + Q$

$$M_1 \xrightarrow{P} M_0 \xrightarrow{Q} M_1, \quad PQ = QP = W \mathbf{1}. \quad (2.2)$$

If $W \neq 0$, then the conditions (2.1) imply

$$\text{rank } M_0 = \text{rank } M_1. \quad (2.3)$$

If $W = 0$, however, then the relation (2.3) no longer holds.

Sometimes, when $\mathbf{R} = \mathbb{Q}[\mathbf{x}]$, $\mathbf{x} = x_1, \dots, x_n$, we may use a notation $M_{\mathbf{x}'}$ (for a subset $\{\mathbf{x}'\} \subset \{\mathbf{x}\}$) instead of simply M . The advantage is that we can then denote by $M_{\mathbf{y}'}$ the matrix factorization obtained from $M_{\mathbf{x}'} = M$ by renaming some of the variables \mathbf{x} , namely \mathbf{x}' , into \mathbf{y}' .

The twisted differential D generates a differential d ($d^2 = 0$) acting on \mathbf{R} -homomorphisms $F \in \text{Hom}_{\mathbf{R}}(M, M')$ between the underlying \mathbf{R} -modules M and M' by the formula

$$dF = [D_{M, M'}, F]_s, \quad (2.4)$$

where we use the notation

$$D_{M, M'} = D_M + D_{M'}, \quad (2.5)$$

while $[\cdot, \cdot]_s$ is the \mathbb{Z}_2 -graded commutator. In our conventions a composition of two homomorphisms is zero, unless the target of the first one matches the domain of the second one. Then we define the \mathbb{Z}_2 -graded extensions as the homology of d :

$$\text{Ext}^\bullet(M, M') = \ker d / \text{im } d, \quad \text{Ext}^\bullet(M, M') = \text{Ext}^0(M, M') \oplus \text{Ext}^1(M, M'), \quad (2.6)$$

and the morphisms between M and M' in the category $\text{MF}_{\mathbf{R}, W}$ are defined as

$$\text{Hom}_{\text{MF}}(M, M') = \text{Ext}^0(M, M'). \quad (2.7)$$

If $P = 1$, $Q = W \mathbf{1}$ or $P = W \mathbf{1}$, $Q = 1$, then the identity endomorphism in $\text{End}(M)$ becomes a trivial element in $\text{Ext}^0(M, M)$, so the corresponding matrix factorization is isomorphic to the zero object in MF_W . We call the latter matrix factorizations *contractible*.

If $W = 0$, then $D^2 = 0$ and we can define the homology of the matrix factorization M as

$$\mathbf{H}_{\text{MF}}^\bullet(M) = \ker D / \text{im } D. \quad (2.8)$$

If in addition to that $\mathbf{R} = \mathbb{Q}$, then

$$M \simeq \mathbf{H}_{\text{MF}}^\bullet(M) \quad (2.9)$$

as objects in the category $\text{MF}_{\mathbb{Q},0}$ of \mathbb{Z}_2 -graded differential vector spaces.

The translation functor $\langle 1 \rangle$ turns the matrix factorization (2.2) into

$$M \langle 1 \rangle = \left(M_1 \xrightarrow{-Q} M_0 \xrightarrow{-P} M_1 \right), \quad (2.10)$$

the double translation $\langle 2 \rangle$ acts as the identity and

$$\text{Hom}_{\text{MF}}(M, M' \langle 1 \rangle) = \text{Ext}^1(M, M'). \quad (2.11)$$

For $M \in \text{Ob}(\text{MF}_{\mathbb{R},W})$ and $M' \in \text{Ob}(\text{MF}_{\mathbb{R}',W'})$ we define a tensor product

$$M \otimes M' \in \text{Ob}(\text{MF}_{\mathbb{R} \otimes \mathbb{R}', W+W'}) \quad (2.12)$$

as a matrix factorization, whose module is the tensor product of modules $M \otimes M'$ and whose twisted differential is the graded sum of twisted differentials

$$D \otimes 1 + (-1)^{\text{deg}_{\mathbb{Z}_2}} 1 \otimes D'. \quad (2.13)$$

Similarly, for $M \in \text{Ob}(\text{MF}_{\mathbb{R},W})$, $M' \in \text{Ob}(\text{MF}_{\mathbb{R},W'})$ we define a tensor product

$$M \otimes_{\mathbb{R}} M' \in \text{Ob}(\text{MF}_{\mathbb{R},W+W'}) \quad (2.14)$$

as a matrix factorization, whose module is the tensor product of modules $M \otimes_{\mathbb{R}} M'$ and whose twisted differential is (2.13).

The conjugate of a matrix factorization (2.2) is the matrix factorization M^* of $-W$:

$$M_1^* \xrightarrow{Q^*} M_0^* \xrightarrow{-P^*} M_1^* \quad (2.15)$$

It satisfies the property that for any matrix factorization M' of W ,

$$\text{Ext}^\bullet(M, M') \cong \mathbf{H}_{\text{MF}}^\bullet(M' \otimes_{\mathbb{R}} M^*) \quad (2.16)$$

(note that according to (2.14) $M' \otimes_{\mathbb{R}} M^* \in \text{Ob}(\text{MF}_{\mathbb{R},0})$, so the *r.h.s.* is well-defined).

For a homomorphism $\mathbb{R} \xrightarrow{h} \mathbb{R}'$ between two polynomial algebras there is a functor

$$\text{MF}_{\mathbb{R},W} \xrightarrow{\hat{h}} \text{MF}_{\mathbb{R}',h(W)} \quad (2.17)$$

which takes a matrix factorization M to $M \otimes_{\mathbb{R}} \mathbb{R}'$.

There is a homomorphism

$$\mathbb{R} \xrightarrow{\hat{\quad}} \text{End}_{\text{MF}}(M), \quad (2.18)$$

which turns an element $r \in R$ into a multiplication by r :

$$\hat{r}(v) = rv, \quad v \in M. \quad (2.19)$$

If $R = \mathbb{Q}[\mathbf{x}]$, $\mathbf{x} = (x_1, \dots, x_m)$, then the homomorphism (2.18) factors through the Jacobi algebra

$$J_W = \mathbb{Q}[\mathbf{x}] / (\partial_{x_1} W, \dots, \partial_{x_m} W), \quad (2.20)$$

that is, the homomorphism (2.18) is a composition of homomorphisms

$$\begin{array}{ccc} & \widehat{} & \\ & \curvearrowright & \\ \mathbb{Q}[\mathbf{x}] & \longrightarrow & J_W \longrightarrow \text{End}_{\text{MF}}(M) . \end{array} \quad (2.21)$$

Finally, if R is a \mathbb{Z} -graded ring and W has an even homogeneous \mathbb{Z} -degree $\deg_{\mathbb{Z}} W = 2N$, then one can define a category of \mathbb{Z} -graded matrix factorizations of W . Its objects are matrix factorizations with $(\mathbb{Z}_2 \times \mathbb{Z})$ -graded modules M , whose twisted differentials have a homogeneous \mathbb{Z} -degree $\deg_{\mathbb{Z}} D = N$.

2.2. Koszul matrix factorizations

Many of our constructions are based upon Koszul matrix factorizations. For two polynomials $p, q \in R$ the *Koszul matrix factorization* $K(p; q)$ of $W = pq$ is a free \mathbb{Z}_2 -graded R -module $R_0 \oplus R_1$ of rank $(1, 1)$ with the twisted differential D , whose action is

$$R_1 \xrightarrow{p} R_0 \xrightarrow{q} R_1. \quad (2.22)$$

It is easy to verify that

$$\hat{p}, \hat{q} \simeq 0, \quad (2.23)$$

that is, both \hat{p} and \hat{q} are homotopic to 0.

If both polynomials p, q have homogeneous (although, maybe, unequal) \mathbb{Z} -degrees, then the Koszul matrix factorization (2.22) can be made \mathbb{Z} -graded by shifting the degree of R_1 by

$$k_{p,q} = \frac{1}{2} (\deg_{\mathbb{Z}} p - \deg_{\mathbb{Z}} q). \quad (2.24)$$

Hence when dealing with graded Koszul matrix factorizations we use the notation

$$K(p; q) = (R_1 \{k_{p,q}\} \xrightarrow{p} R_0 \xrightarrow{q} R_1 \{k_{p,q}\}), \quad (2.25)$$

where $\{*\}$ denotes the \mathbb{Z} -grading shift. $K(p; q) \{k\}$ denotes the Koszul matrix factorization in which the \mathbb{Z} -grading of both modules R_0, R_1 is shifted by extra k units relative to that of (2.25). Obviously,

$$K(p; q) \{k\} \langle 1 \rangle \cong K(-q; -p) \{k_{p,q} + k\}, \quad (K(p; q) \{k\})^* \cong K(q; -p) \{-k\}. \quad (2.26)$$

For two columns

$$\mathbf{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}, \quad p_i, q_i \in \mathbb{R}, \quad i = 1, \dots, n, \quad (2.27)$$

we define the Koszul matrix factorization $K(\mathbf{p}; \mathbf{q})$ of the polynomial $\sum_{i=1}^n p_i q_i$ as the tensor product of the elementary ones:

$$K(\mathbf{p}; \mathbf{q}) = \bigotimes_{i=1}^n K(p_i; q_i). \quad (2.28)$$

The case $n = 2$ plays an important role in our computations, so in order to set the basis notations, we present it explicitly. Thus consider a Koszul matrix factorization

$$K \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} = (R_1 \xrightarrow{p_1} R_0 \xrightarrow{q_1} R_1) \otimes (R'_1 \xrightarrow{p_2} R'_0 \xrightarrow{q_2} R'_1) \quad (2.29)$$

We choose the generators of rank 2 submodules of \mathbb{Z}_2 -degree 0 and 1 according to the splitting

$$R_0^2 = R_{00} \oplus R_{11}, \quad R_1^2 = R_{01} \oplus R_{10}, \quad \text{where } R_{ij} = R_i \otimes R'_j. \quad (2.30)$$

Then the Koszul matrix factorization (2.29) has the form

$$R_1^2 \xrightarrow{P} R_0^2 \xrightarrow{Q} R_1^2 \quad (2.31)$$

where

$$P = \begin{pmatrix} p_2 & p_1 \\ q_1 & -q_2 \end{pmatrix}, \quad Q = \begin{pmatrix} q_2 & p_1 \\ q_1 & -p_2 \end{pmatrix}. \quad (2.32)$$

In fact, the columns of ring elements (2.27) should be considered as elements of \mathbb{R}^n and $(\mathbb{R}^n)^*$:

$$\mathbf{p} \in \mathbb{R}^n, \quad \mathbf{q} \in (\mathbb{R}^n)^*, \quad W = \mathbf{q} \lrcorner \mathbf{p}. \quad (2.33)$$

The implication is that a particular choice of polynomials p_i and q_i representing \mathbf{p} and \mathbf{q} depends on the choice of free generators of \mathbb{R}^n . A change of generators modifies p_i 's

and q_i 's, but the Koszul matrix factorization $K(\mathbf{p}; \mathbf{q})$ remains the same. In particular, it is invariant under the following row transformations, acting on i -th and j -th rows of the columns (2.27):

$$\begin{pmatrix} p_i \\ p_j \end{pmatrix}, \begin{pmatrix} q_i \\ q_j \end{pmatrix} \xrightarrow{[i,j]\lambda} \begin{pmatrix} p_i \\ p_j + \lambda p_i \end{pmatrix}, \begin{pmatrix} q_i - \lambda q_j \\ q_j \end{pmatrix}, \quad \lambda \in \mathbb{R}. \quad (2.34)$$

A conjugation of this transformation by the translation functor $\langle 1 \rangle$ (which may act by switching p_j and q_j) yields another useful equivalence:

$$\begin{pmatrix} p_i \\ p_j \end{pmatrix}, \begin{pmatrix} q_i \\ q_j \end{pmatrix} \xrightarrow{[i,j]'\lambda} \begin{pmatrix} p_i \\ p_j \end{pmatrix}, \begin{pmatrix} q_i - \lambda p_j \\ q_j + \lambda p_i \end{pmatrix}, \quad \lambda \in \mathbb{R}. \quad (2.35)$$

For an invertible element $\lambda \in \mathbb{R}$ there is another equivalence transformation acting on the i -th row of \mathbf{p} and \mathbf{q} :

$$(p_i, q_i) \xrightarrow{[i]\lambda} (\lambda p_i, \lambda^{-1} q_i) \quad (2.36)$$

2.3. Two theorems

The following two theorems present us with convenient technical tools for establishing equivalences between Koszul matrix factorizations.

Theorem 2.1. *If $p_1, \dots, p_n \in \mathbb{R}$ form a regular sequence and*

$$\sum_{i=1}^n p_i q_i = \sum_{i=1}^n p_i q'_i, \quad (2.37)$$

then the following two Koszul matrix factorizations are isomorphic:

$$K(\mathbf{p}; \mathbf{q}) \cong K(\mathbf{p}; \mathbf{q}') \quad (2.38)$$

Proof. We prove the theorem by induction over n . If $n = 1$, then the claim is obvious. Suppose that the theorem holds for $n = k$ and consider the case $n = k + 1$. In view of the condition (2.37),

$$p_{k+1}(q'_{k+1} - q_{k+1}) = - \sum_{i=1}^k p_i(q'_i - q_i). \quad (2.39)$$

The definition of a regular sequence implies that p_{k+1} is not a zero divisor in the quotient $\mathbb{R}/(p_1, \dots, p_k)$, hence it follows from eq.(2.39) that

$$q'_{k+1} - q_{k+1} \in \mathbb{R}(p_1, \dots, p_k), \quad (2.40)$$

that is, there exist $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that

$$q'_{k+1} - q_{k+1} = \sum_{i=1}^k \lambda_i p_i. \quad (2.41)$$

Therefore, a composition of transformations $[i, k+1]_{\lambda_i}'$ for $1 \leq i \leq k$ turns $\mathbf{K}(\mathbf{p}; \mathbf{q})$ into a Koszul matrix factorization

$$\mathbf{K} \begin{pmatrix} p_1, & q_1 - \lambda_1 p_{k+1} \\ \vdots & \vdots \\ p_k, & q_k - \lambda_k p_{k+1} \\ p_{k+1}, & q'_{k+1} \end{pmatrix} \cong \mathbf{K} \begin{pmatrix} p_1, & q_1 - \lambda_1 p_{k+1} \\ \vdots & \vdots \\ p_k, & q_k - \lambda_k p_{k+1} \end{pmatrix} \otimes_{\mathbb{R}} \mathbf{K}(p_{k+1}; q'_{k+1}) \quad (2.42)$$

Since

$$\sum_{i=1}^k p_i (q_i - \lambda_i p_{k+1}) = \sum_{i=1}^k p_i q'_i, \quad (2.43)$$

then by the inductive assumption

$$\mathbf{K} \begin{pmatrix} p_1, & q_1 - \lambda_1 p_{k+1} \\ \vdots & \vdots \\ p_k, & q_k - \lambda_k p_{k+1} \end{pmatrix} \cong \mathbf{K} \begin{pmatrix} p_1, & q'_1 \\ \vdots & \vdots \\ p_k, & q'_k \end{pmatrix} \quad (2.44)$$

and the tensor product in the *r.h.s.* of eq.(2.42) is isomorphic to $\mathbf{K}(\mathbf{p}; \mathbf{q}')$, which proves the theorem. \square

Consider three polynomials: $\bar{W} \in \mathbb{Q}[\mathbf{x}]$ and $p, q \in \mathbb{Q}[\mathbf{x}, y]$. Let us fix a positive integer n such that

$$\deg_y p < n. \quad (2.45)$$

Let M be a matrix factorization of the polynomial

$$W = \bar{W} - (y^n - p) q. \quad (2.46)$$

In view of the condition (2.45), the quotient

$$M' = M / (y^n - p) M \quad (2.47)$$

is a free module over $\mathbb{Q}[\mathbf{x}]$. Hence it represents a matrix factorization of the polynomial \bar{W} over that algebra.

A tensor product of M with a Koszul matrix factorization of a special form

$$\bar{M} = \mathbf{K}(y^n - p; q) \otimes_{\mathbb{Q}[\mathbf{x}, y]} M \quad (2.48)$$

is also a matrix factorization of \bar{W} .

Theorem 2.2. M' and \bar{M} are homotopy equivalent as matrix factorizations over $\mathbb{Q}[\mathbf{x}]$ via a natural homotopy equivalence morphism

$$\bar{M} \xrightarrow{f_{\simeq}} M' . \quad (2.49)$$

Proof. The module of the tensor product $\bar{M} = \mathbf{K}(y^n - p; q) \otimes_{\mathbb{Q}[\mathbf{x}, y]} M$ is the sum of modules $M \oplus M \langle 1 \rangle$, and its twisted differential is the sum of twisted differentials of the modules and the homomorphisms in the diagram

$$-D \circlearrowleft M \langle 1 \rangle \xrightleftharpoons[q]{y^n - p} M \circlearrowright D , \quad (2.50)$$

where D is the twisted differential of M .

The matrix factorization \bar{M} contains a subfactorization

$$\bar{M}' = \mathbf{K}(1; (y^n - p) q) \otimes_{\mathbb{Q}[\mathbf{x}, y]} M, \quad (2.51)$$

as demonstrated by the following diagram

$$\begin{array}{ccc} \bar{M}' & & M \langle 1 \rangle \xrightleftharpoons[(y^n - p) q]{1} M \\ \downarrow & & \downarrow 1 \quad \downarrow y^n - p \\ \bar{M} & & M \langle 1 \rangle \xrightleftharpoons[q]{y^n - p} M \end{array} \quad (2.52)$$

Its top line represents \bar{M}' , its bottom line represents \bar{M} and the vertical homomorphisms represent the injection.

It is obvious from the diagram (2.52) that $\bar{M}/\bar{M}' = M/(y^n - p)M = M'$. The Koszul matrix factorization $\mathbf{K}(1; (y^n - p) q)$ is contractible, hence \bar{M}' is also contractible and $\bar{M} \cong \bar{M}/\bar{M}'$. Together with the previous isomorphism, this proves the homotopy equivalence $\bar{M} \cong M'$. The equivalence is established by the quotient map

$$\bar{M} \xrightarrow{f_{\simeq}} \bar{M}/\bar{M}' = M' , \quad (2.53)$$

which is natural in M . □

Different homotopy equivalence maps f_{\simeq} commute with each other. Namely, consider five polynomials: $\bar{W} \in \mathbb{Q}[\mathbf{x}]$, $p_1 \in \mathbb{Q}[\mathbf{x}, y_1]$ and $p_2, q_1, q_2 \in \mathbb{Q}[\mathbf{x}, y_1, y_2]$ with the conditions

$$\deg_{y_1} p_1 < n_1, \quad \deg_{y_2} p_2 < n_2. \quad (2.54)$$

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Let M be a matrix factorization of the polynomial

$$W = \bar{W} - (y_1^{n_1} - p_1) q_1 - (y_2^{n_2} - p_2) q_2. \quad (2.55)$$

Its tensor product with two Koszul matrix factorizations

$$\bar{M} = \mathbf{K}(y_1^{n_1} - p_1; q_1) \otimes_{\mathbb{Q}[\mathbf{x}, y_1, y_2]} \mathbf{K}(y_2^{n_2} - p_2; q_2) \otimes_{\mathbb{Q}[\mathbf{x}, y_1, y_2]} M \quad (2.56)$$

is a matrix factorization of \bar{W} and we consider it over the algebra $\mathbb{Q}[\mathbf{x}]$. A double application of Theorem 2.2 leads to the homotopy equivalence

$$\bar{M} \simeq M', \quad (2.57)$$

where

$$M' = M / ((y_1^{n_1} - p_1)M + (y_2^{n_2} - p_2)M) \quad (2.58)$$

is a matrix factorization of \bar{W} over $\mathbb{Q}[\mathbf{x}]$. However, one could establish the homotopy equivalence (2.57) by either applying Theorem 2.2 first to y_1, p_1, q_1 and then to y_2, p_2, q_2 or in the opposite order, thus obtaining two homotopy equivalence homomorphisms

$$\begin{array}{ccc} & \xrightarrow{f_{\simeq, 2} f_{\simeq, 1}} & \\ \bar{M} & \xrightarrow{\hspace{2cm}} & M' \\ & \xleftarrow{f_{\simeq, 1} f_{\simeq, 2}} & \end{array} \quad (2.59)$$

Theorem 2.3. *The homotopy equivalence homomorphisms (2.59) are equal:*

$$f_{\simeq, 2} f_{\simeq, 1} = f_{\simeq, 1} f_{\simeq, 2}. \quad (2.60)$$

Proof. The matrix factorization \bar{M} in (2.56) has the structure

$$\bar{M} = \left(\begin{array}{ccc} & \xrightarrow{y_1^{n_1} - p_1} & \\ & \xleftarrow{q_1} & M \langle 1 \rangle \\ & & \uparrow q_2 \\ -(y_2^{n_2} - p_2) & -q_2 & y_2^{n_2} - p_2 \\ & & \downarrow q_2 \\ & \xrightarrow{y_1^{n_1} - p_1} & \\ & \xleftarrow{q_1} & M \langle 1 \rangle \end{array} \right) \quad (2.61)$$

This matrix factorization has a subfactorization \bar{M}' :

$$\bar{M}' = \left(\begin{array}{ccc} M & \xrightleftharpoons[q_1]{y_1^{n_1} - p_1} & M \langle 1 \rangle \\ \uparrow -q_2 & & \uparrow q_2 \\ M \langle 1 \rangle & \xrightleftharpoons[q_1]{y_1^{n_1} - p_1} & (y_1^{n_1} - p_1)M + (y_2^{n_2} - p_2)M \end{array} \right) \quad (2.62)$$

A double application of the proof of Theorem 2.2 to \bar{M} indicates that M' is contractible and both compositions $f_{\simeq,2}f_{\simeq,1}$ and $f_{\simeq,1}f_{\simeq,2}$ are canonical homomorphisms between a module and its quotient:

$$\bar{M} \begin{array}{c} \xrightarrow{f_{\simeq,2}f_{\simeq,1}} \\ \xrightarrow{f_{\simeq,1}f_{\simeq,2}} \end{array} \bar{M}/\bar{M}' = M' . \quad (2.63)$$

□

2.4. Graphs, tangles and associated matrix factorizations

In our categorification constructions, an *open graph* is a planar graph diagram with special univalent vertices called *legs*. \frown , \blacktriangleright and \blacktriangleleft are examples of open graphs. The legs are enumerated and oriented as ‘in’ or ‘out’, but sometimes we drop these decorations on the pictures.

If an open graph has no legs, then it is called *closed*. The graphs $\Gamma_{\mathbf{r}}$ resulting from the resolution of the crossings of \mathcal{L} are examples of closed graphs.

More generally, a graph-tangle is a planar graph diagram with legs and also with special 4-valent vertices called *crossings*. These vertices are decorated with the under-over selection for the incident edges. A link diagram \mathcal{L} and an elementary tangle \times are examples of graph-tangles, the first one being, in fact, closed.

There are two operations on the sets of closed graphs (and similarly on graph-tangles). The first operation is a disjoint union

$$(\gamma_1, \gamma_2) \xrightarrow{\sqcup} \gamma_1 \sqcup \gamma_2 \quad (2.64)$$

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We call two legs of an open graph or a graph-tangle $\hat{\gamma}$ close if they can be connected by a line in the complement of $\hat{\gamma}$. The second operation joins two close legs i and j with opposite orientations:

$$\gamma \xrightarrow{\#_{i,j}} \#_{i,j}(\gamma) . \quad (2.65)$$

Our categorification is a map from open graphs to matrix factorizations and from graph-tangles to complexes of matrix factorizations

$$\gamma \mapsto \hat{\gamma}, \quad \tau \mapsto \hat{\tau} . \quad (2.66)$$

The categorification map should convert the operations (2.64) and (2.65) into appropriate algebraic counterparts.

First of all, we choose a basic set of variables $\mathbf{x} = (x_1, \dots, x_m)$ and a basic polynomial $W(\mathbf{x}) \in \mathbb{Q}[\mathbf{x}]$. To a set l of n enumerated oriented legs (of an open graph or a graph-tangle) we associate a polynomial of nm variables

$$\mathbf{x}_l = \mathbf{x}_1, \dots, \mathbf{x}_n, \quad \mathbf{x}_i = x_{1,i}, \dots, x_{m,i}, \quad (2.67)$$

according to the formula

$$W_l(\mathbf{x}_l) = \sum_{i=1}^n \mu_i W(\mathbf{x}_i), \quad (2.68)$$

where μ_i is the orientation of the i -th leg: $\mu_i = 1$ if it is oriented outwards, and $\mu_i = -1$ if it is oriented inwards.

To an open graph γ we associate a matrix factorization

$$\hat{\gamma} \in \text{Ob}(\text{MF}_{W_l(\gamma)}), \quad (2.69)$$

where $l(\gamma)$ is the set of oriented legs of γ . Similarly, to a graph-tangle τ we associate a complex of matrix factorizations up to homotopy:

$$\hat{\tau} \in \text{Ob}(\mathbf{K}(\text{MF}_{W_l(\tau)})). \quad (2.70)$$

If Γ is a closed graph then $l(\Gamma) = \emptyset$, so $W_l(\Gamma) = 0$, and as a ‘matrix factorization’, $\hat{\Gamma}$ is isomorphic to its own homology

$$\hat{\Gamma} \simeq C(\Gamma), \quad (2.71)$$

where by definition

$$C(\Gamma) = \text{H}_{\text{MF}}(\hat{\Gamma}) \quad (2.72)$$

(cf. eq.(1.17)).

Finally, we specify the functoriality rules for the maps (2.66). To a disjoint union of open graphs $\gamma_1 \sqcup \gamma_2$ we associate the tensor product of matrix factorizations

$$\widehat{\gamma_1 \sqcup \gamma_2} = \hat{\gamma}_1 \otimes \hat{\gamma}_2, \quad (2.73)$$

and to the joining of legs we associate the identification of the variables \mathbf{x}_i and \mathbf{x}_j :

$$\widehat{\#_{i,j}(\gamma)} = \widehat{h_{i,j}(\hat{\gamma})}, \quad (2.74)$$

where $\widehat{h_{i,j}}$ is the functor (2.17) corresponding to the algebra homomorphism

$$\mathbb{Q}[\mathbf{x}_{I(\gamma)}] \xrightarrow{h_{i,j}} \mathbb{Q}[\mathbf{x}_{I(\gamma)}]/(x_{1,i} - x_{1,j}, \dots, x_{m,i} - x_{m,j}). \quad (2.75)$$

It is important to note that the *r.h.s.* of eq.(2.74) should be considered as a matrix factorization over the ring $\mathbb{Q}[\mathbf{x}_{I(\#_{i,j}(\gamma))}]$, that is, the multiplication by \mathbf{x}_i and \mathbf{x}_j is not a part of the module structure, since these variables are no longer related to legs. Nevertheless, $\hat{\mathbf{x}}_i$ and $\hat{\mathbf{x}}_j$ remain collections of endomorphisms of $\widehat{\#_{i,j}(\gamma)}$, and

$$\hat{\mathbf{x}}_i = \hat{\mathbf{x}}_j \quad (2.76)$$

in view of the quotient (2.75).

The functorial properties of the categorification map allow us to define it first for the elementary open graphs and graph-tangles, which are just single vertices together with their legs, and for the 1-arc graph --- , and then derive the categorification of more complicated objects by cutting them into elementary pieces and assembling the corresponding matrix factorizations with the help of eqs.(2.73) and (2.75). The matrix factorization --- is determined by the principal property of --- : gluing it to a leg of another graph γ would produce the same graph. In other words, if γ has n legs enumerated by numbers $1 \dots n$, then

$$\#_{i,n+1} \left(\gamma \sqcup \begin{array}{c} n+1 \\ \text{---} \\ n+2 \end{array} \right) = \gamma', \quad 1 \leq i \leq n, \quad (2.77)$$

where γ' is the same graph as γ , except the i -th leg of γ is labeled as $(n+2)$ -th. Then the formulas (2.73) and (2.75) require that

$$\hat{\gamma}_{\mathbf{x}_i} \otimes_{\mathbb{Q}[\mathbf{x}_i]} \left(\widehat{\begin{array}{c} i \\ \text{---} \\ n+2 \end{array}} \right) \simeq \hat{\gamma}_{\mathbf{x}_{n+2}}. \quad (2.78)$$

Note that the property (2.78) of the 1-arc matrix factorization allows us to present the *r.h.s.* of eq.(2.74) as a tensor product with 1-arc matrix factorization, so that eq.(2.74) can be written as

$$\widehat{\#_{i,j}(\gamma)} \simeq \hat{\gamma} \otimes_{\mathbb{Q}[x_i, x_j]} \left(\widehat{i \text{ --- } j} \right). \quad (2.79)$$

This formula is similar to definition of the Hochschild homology of a bimodule, $\left(\widehat{i \text{ --- } j} \right)$ playing the role of the algebra resolution.

3. Arc matrix factorizations

3.1. Basic polynomial

We begin the categorification of the SO(2N+2) Kauffman polynomial by choosing the basic algebra $\mathbb{Q}[x, y]$ and the basic polynomial

$$W(x, y) = xy^2 + w(x), \quad w(x) = x^{2N+1}, \quad (3.1)$$

which was previously considered by Gukov and Walcher [2]. This polynomial has a homogeneous q -degree

$$\deg_q W(x, y) = 4N + 2, \quad (3.2)$$

if we set

$$\deg_q x = 2, \quad \deg_q y = 2N, \quad (3.3)$$

and this allows us to use graded matrix factorizations for categorification. In particular, the twisted differential should have a homogeneous q -degree

$$\deg_q D = 2N + 1. \quad (3.4)$$

The basic polynomial (3.1) is an odd function of its combined argument:

$$W(-x, -y) = -W(x, y). \quad (3.5)$$

This allows us to impose an additional constraint on the categorification map (2.66): if γ' is obtained from γ by reversing the orientation of the i -th leg, then $\hat{\gamma}'$ is obtained from $\hat{\gamma}$ by changing the signs of the variables x_i, y_i , that is,

$$\hat{\gamma}'_{x_i, y_i} = \hat{\gamma}_{-x_i, -y_i}. \quad (3.6)$$

Therefore, from now on we assume that all legs of graphs and tangles are oriented outwards, unless specified otherwise, and whenever we need to join two legs, we change the

orientation of one of them with the help of eq.(3.6). In other words, the homomorphism $h_{i,j}$ in the joining formula (2.74) should be modified from eq.(2.75) to

$$\mathbb{Q}[\mathbf{x}_{l(\gamma)}] \xrightarrow{h_{i,j}} \mathbb{Q}[\mathbf{x}_{l(\gamma)}]/(x_i + x_j, y_i + y_j), \quad (3.7)$$

because i -th and j -th legs are both oriented outwards.

We introduce a convenient notation for iterated differences of $w(x)$, which we define recursively as

$$w(x_1, \dots, x_{n-1}, x_n) = \frac{w(x_1, \dots, x_{n-2}, x_{n-1}) - w(x_1, \dots, x_{n-2}, x_n)}{x_{n-1} - x_n}. \quad (3.8)$$

For example,

$$w(x_1, x_2) = \frac{w(x_1) - w(x_2)}{x_1 - x_2}, \quad w(x_1, x_2, x_3) = \frac{w(x_1, x_2) - w(x_1, x_3)}{x_2 - x_3}. \quad (3.9)$$

Note that all these polynomials are symmetric functions of their arguments.

3.2. Jacobi algebra

The Jacobi algebra of W is defined as the quotient

$$J_W = \mathbb{Q}[x, y]/(\partial_x W, \partial_y W) = \mathbb{Q}[x, y]/(y^2 + (2N + 1)x^{2N}, xy). \quad (3.10)$$

It is graded with a q -graded basis

$$1, x \dots x^{2N}, y, \quad \deg_q x^i = 2i, \quad \deg_q y = 2N. \quad (3.11)$$

Hence the dimensions of q -graded subspaces of J_W are

$$\dim J_{W, \{2i\}} = \begin{cases} 1 & \text{if } 0 \leq i \leq 2N \text{ and } i \neq N, \\ 2 & \text{if } i = N, \end{cases} \quad (3.12)$$

and the graded dimension of J_W is equal to the Kauffman polynomial of the unknot (1.4) up to a degree shift:

$$\dim_q J_W = \frac{q^{4N+2} - 1}{q^2 - 1} + q^{2N} = q^{2N} P_{\text{unkn}}(q). \quad (3.13)$$

This agreement indicates that eq.(3.1) is a suitable choice for the basic polynomial.

Let

$$1^*, x^* \dots (x^{2N})^*, y^* \in J_W^* \quad (3.14)$$

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denote the dual basis of (3.11), and for $v \in J_W$ let $\hat{v} : J_W \xrightarrow{v} J_W$ denote the linear operator of multiplication by v . Then the algebra multiplication map

$$J_W \otimes J_W \xrightarrow{\mathfrak{m}} J_W \quad (3.15)$$

can be presented as

$$\mathfrak{m} = \sum_{i=0}^{2N} \hat{x}^i \otimes (x^i)^* + \hat{y} \otimes y^*. \quad (3.16)$$

The Jacobi algebra J_W acquires a Frobenius algebra structure, if we choose the Frobenius trace to be

$$\mathrm{Tr}_F = -\frac{1}{2(2N+1)} (x^{2N})^* \quad (3.17)$$

(the origin of the normalization factor will become clear later). Then the co-multiplication map

$$J_W \xrightarrow{\Delta} J_W \otimes J_W \quad (3.18)$$

can be presented as

$$\Delta = 2 \left(\hat{y} \otimes y - (2N+1) \sum_{i=0}^{2N} \hat{x}^i \otimes x^{2N-i} \right). \quad (3.19)$$

3.3. The 1-arc Koszul matrix factorization and the unknot space

To a 1-arc open graph we associate a Koszul matrix factorization

$$\widehat{1 \text{ --- } 2} = \mathbf{K} \begin{pmatrix} y_2 + y_1, & x_2 (y_2 - y_1) \\ x_2 + x_1, & y_1^2 + w(-x_1, x_2) \end{pmatrix}. \quad (3.20)$$

of the polynomial

$$W_2 = W(x_1, y_1) + W(x_2, y_2). \quad (3.21)$$

The left column of this Koszul matrix factorization contains sums rather than differences, because in our conventions both legs of the 1-arc graph $1 \text{ --- } 2$ are oriented outwards. Theorem 2.2 indicates that the matrix factorization (3.20) satisfies the property (2.78). Also note that in view of eq.(2.23),

$$\hat{x}_2 \simeq -\hat{x}_1, \quad \hat{y}_2 \simeq -\hat{y}_1. \quad (3.22)$$

According to eq.(2.26), the dual to the 1-arc matrix factorization is the matrix factorization of the 1-arc graph in which both legs are oriented inwards:

$$\left(\widehat{1 \text{ --- } 2} \right)^* \cong \widehat{1 \text{ } \curvearrowright \text{ } 2} \{2N\}. \quad (3.23)$$

The unknot diagram \bigcirc can be constructed by joining legs 1 and 2 of the arc $1 \text{ --- } 2$, hence, in accordance with eqs.(1.17) and (2.74), the corresponding space C_{unkn} is the homology of the differential vector space $\widehat{\bigcirc}$ constructed by setting

$$-x_2 = x_1 = x, \quad -y_2 = y_1 = y \quad (3.24)$$

in the Koszul matrix factorization (3.20):

$$\begin{array}{ccccc} R_{00} & \xrightarrow{y^2+w'(x)} & R_{01}\{1-2N\} & & R_{00} \\ \oplus & & \oplus & & \oplus \\ R_{11}\{-2N\} & \xrightarrow{2xy} & R_{10}\{-1\} & \xrightarrow{2xy} & R_{11}\{-2N\} \\ & & \xrightarrow{-(y^2+w'(x))} & & \end{array} \quad (3.25)$$

This is a Koszul complex $K \left(\begin{array}{c} 2xy \\ y^2 + w'(x) \end{array} \right)$. Since the polynomials $xy, y^2 + w'(x)$ form a regular sequence, the homology of the complex (3.25) appears only in the lower right corner and, in fact, as a q -graded vector space it is isomorphic to the Jacobi algebra (3.10) up to a q -degree shift:

$$H_{\text{MF}}^i \left(\widehat{\bigcirc} \right) \cong \begin{cases} J'_W, & \text{if } i = 0, \\ 0, & \text{if } i = 1, \end{cases} \quad \text{where } J'_W = J_W\{-2N\}. \quad (3.26)$$

Thus

$$C_{\text{unkn}} \cong J'_W, \quad \deg_{\mathbb{Z}_2} C_{\text{unkn}} = 0. \quad (3.27)$$

It follows from this equation and from eq.(3.13) that the relation (1.11) holds for the crossingless diagram of the unknot.

C_{unkn} is isomorphic to J'_W not only as a graded vector space but also as a module over J_W whose action on C_{unkn} is generated by

$$\hat{x} = \hat{x}_1 = -\hat{x}_2, \quad \hat{y} = \hat{y}_1 = -\hat{y}_2. \quad (3.28)$$

This module structure allows us to introduce a convenient basis of C_{unkn} . Let the first basis element e be a non-zero element of the 1-dimensional lowest q -degree subspace

$$C_{\text{unkn},\{-2N\}} \subset C_{\text{unkn}}, \quad (3.29)$$

then the rest of the basis is generated by the action of the elements of J_W on e :

$$e, \hat{x}(e), \hat{x}^2(e) \dots \hat{x}^{2N}(e), \hat{y}(e) \in C_{\text{unkn}}. \quad (3.30)$$

Later we will endow C_{unkn} with an algebra structure, and this will turn (3.27) into an algebra isomorphism. Then e will be chosen to be the unit of J_W .

Finally, we describe the endomorphism algebra of the 1-arc matrix factorization (3.20). The partial derivatives

$$\partial_x W = y^2 + (2N + 1)x^{2N}, \quad \partial_y W = 2xy \quad (3.31)$$

form a regular sequence, hence

$$\text{End}_{\text{MF}}(\widehat{1 \text{ --- } 2}) = \text{Ext}^0(\widehat{1 \text{ --- } 2}, \widehat{1 \text{ --- } 2}) \cong J_W, \quad \text{Ext}^1(\widehat{1 \text{ --- } 2}, \widehat{1 \text{ --- } 2}) = 0. \quad (3.32)$$

This isomorphism follows directly from (3.26). Indeed, according to the relations (2.16), (3.23) and to the joining legs formula (2.79),

$$\begin{aligned} \text{Ext}^i(\widehat{1 \text{ --- } 2}, \widehat{1 \text{ --- } 2}) &= \text{H}_{\text{MF}}^i(\widehat{1 \text{ --- } 2} \otimes_{\mathbb{R}} (\widehat{1 \text{ --- } 2})^*) = \text{H}_{\text{MF}}^i(\widehat{1 \text{ --- } 2} \otimes_{\mathbb{R}} \widehat{\text{---}}) \{2N\} \\ &= \text{H}_{\text{MF}}^i(\widehat{\text{---}}) \{2N\} = \begin{cases} J_W & \text{if } i = 0, \\ 0 & \text{if } i = 1, \end{cases} \end{aligned} \quad (3.33)$$

where $\mathbb{R} = \mathbb{Q}[x_1, y_1, x_2, y_2]$. The isomorphism $J_W \xrightarrow{\cong} \text{End}_{\text{MF}}(\widehat{1 \text{ --- } 2})$ can be generated by the homomorphism $\widehat{\text{---}}$ in two different ways:

$$(x, y) \mapsto (\hat{x}_1, \hat{y}_1) \quad \text{or} \quad (x, y) \mapsto (\hat{x}_2, \hat{y}_2). \quad (3.34)$$

Both versions differ by a sign in view of eq.(3.22).

The calculations (3.33) leading to eq.(3.32) can be generalized to find the morphisms between multi-arc graphs.

Theorem 3.1. *Consider a system of $2m$ distinct 1-vertices (legs). Let γ and γ' be two open graphs which are the unions of m 1-arc graphs connecting these $2m$ legs. Then*

$$\text{Ext}^0(\gamma, \gamma') = (J_W)^{\otimes m_0} \{2N(m - m_0)\}, \quad \text{Ext}^1(\gamma, \gamma') = 0, \quad (3.35)$$

where m_0 is the number of circles formed by joining γ and γ' together through their common legs.

Proof. Let γ_- be the graph γ , in which we reversed the orientations of all legs, so that they are now oriented inwards. Then

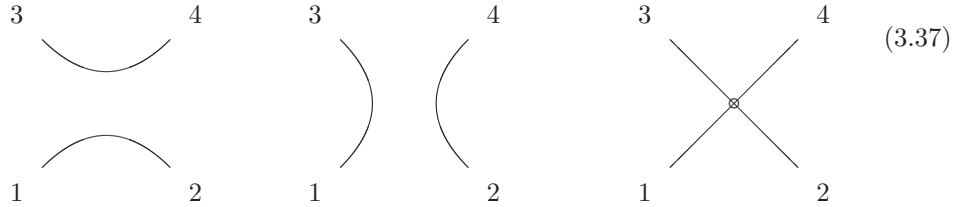
$$\text{Ext}^i(\hat{\gamma}, \hat{\gamma}') = \mathbf{H}_{\text{MF}}^i(\hat{\gamma}' \otimes_{\mathbb{R}} (\hat{\gamma})^*) = \mathbf{H}_{\text{MF}}^i(\hat{\gamma}' \otimes_{\mathbb{R}} \hat{\gamma}_-)\{2Nm\}, \quad (3.36)$$

and the formula (3.35) follows from the basic property of the 1-arc matrix factorization and from eq.(3.27). \square

3.4. 2-arc Koszul matrix factorizations

3.4.1. Category, endo-functors and saddle morphisms

Four legs 1, 2, 3, 4 can be connected by two arcs in three different ways:



The third connection requires the arcs to intersect in the plane of the diagram, however when we assign the matrix factorization to the open graph \times , we treat this intersection as non-existent (that is, virtual), while considering \times to be a disjoint union of two arcs similar to $) ($ and \frown .

The matrix factorizations $\widehat{) (}$, $\widehat{\times}$, $\widehat{\frown}$ belong to the same category of matrix factorizations of the polynomial

$$W_4(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^4 W(x_i, y_i) \quad (3.38)$$

over the algebra

$$\mathbb{R} = \mathbb{Q}[\mathbf{x}, \mathbf{y}], \quad \mathbf{x} = x_1, \dots, x_4, \quad \mathbf{y} = y_1, \dots, y_4. \quad (3.39)$$

The structure of 2-arc matrix factorizations is simple: in view of eq.(2.73) they are tensor products of matrix factorizations corresponding to constituent arcs. In particular, the virtual nature of the crossing \times implies that

$$\widehat{\times} = \widehat{1 - 4} \otimes \widehat{2 - 3}. \quad (3.40)$$

The symmetric group S_4 acts on the set of graphs (3.37) by permuting their legs, *e.g.*

$$\text{)} (\xrightarrow{\sigma_{12}} \text{)} . \quad (3.41)$$

The polynomial (3.38) is invariant under the simultaneous permutations of the variables \mathbf{x} and \mathbf{y} , hence the elements $\sigma \in S_4$ act as endo-functors on the category MF_{W_4} .

Theorem 3.1 describes the spaces $\text{Ext}^i(\hat{\gamma}, \hat{\gamma}')$ for 2-arc graphs γ and γ' :

$$\text{Ext}^0(\hat{\gamma}, \hat{\gamma}') = \begin{cases} J_W \otimes J_W & \text{if } \gamma = \gamma', \\ J_W\{2N\} & \text{if } \gamma \neq \gamma', \end{cases} \quad \text{Ext}^1(\hat{\gamma}, \hat{\gamma}') = 0. \quad (3.42)$$

Suppose that $\gamma \neq \gamma'$. According to eq.(3.42),

$$\dim \text{Ext}_{\{2N\}}^0(\hat{\gamma}, \hat{\gamma}') = 1. \quad (3.43)$$

Therefore, up to a constant factor, there exists a unique non-trivial morphism

$$\hat{\gamma} \xrightarrow{F} \hat{\gamma}', \quad \deg_q F = 2N. \quad (3.44)$$

We call it *the saddle morphism* and we denote it either as F , or G , or H depending on the choice of γ and γ' .

The uniqueness of the saddle morphism (up to a constant factor) for a given pair of graphs $\gamma \neq \gamma'$ implies that the elements of S_4 transform saddle morphisms into saddle morphisms (these functors are images of group elements, hence they are invertible and can not transform a saddle morphism into a trivial morphism).

3.4.2. Proper 2-arc Koszul matrix factorizations

We have to know an explicit presentation of a saddle morphism to perform computations with it. According to eqs.(3.20) and (3.40), the 2-arc matrix factorizations $\hat{\gamma}$ are presented as 4-row Koszul matrix factorizations, so their modules have rank (8,8), which makes explicit formulas for morphisms rather unwieldy. However, as we shall see, all 2-arc matrix factorizations can be presented as tensor products

$$\hat{\gamma} = K_{\text{cmn}} \otimes_{\mathbb{R}} \hat{\gamma}_{\overline{\mathbb{p}}}, \quad (3.45)$$

where K_{cmn} is a ‘common’ 2-row Koszul matrix factorization (the same for all $\hat{\gamma}$), while $\hat{\gamma}_{\overline{\mathbb{p}}}$ are ‘proper’ 2-row Koszul matrix factorizations. The saddle morphisms act as identity

on K_{cmn} , that is

$$\begin{array}{ccc}
 \hat{\gamma} & \xrightarrow{F} & \hat{\gamma}' \\
 \parallel & & \parallel \\
 K_{\text{cmn}} \otimes_{\mathbb{R}} \hat{\gamma}_{\mathbb{P}} & \xrightarrow{1 \otimes_{\mathbb{R}} F_{\mathbb{P}}} & K_{\text{cmn}} \otimes_{\mathbb{R}} \hat{\gamma}'_{\mathbb{P}}
 \end{array} \tag{3.46}$$

for a suitable map $\hat{\gamma}_{\mathbb{P}} \xrightarrow{F_{\mathbb{P}}} \hat{\gamma}'_{\mathbb{P}}$.

In order to describe this construction explicitly, we introduce a ‘proper’ algebra

$$\mathbb{R}_{\mathbf{p}} = \mathbb{Q}[\mathbf{p}, \mathbf{q}, \mathbf{r}, C], \quad \text{where } \mathbf{p} = (p_1, p_2), \quad \mathbf{q} = (q_1, q_2), \quad \mathbf{r} = (r_1, r_2), \tag{3.47}$$

and a ‘proper’ polynomial

$$W_{4, \mathbf{p}} = p_1 q_2 r_2 + q_1 p_2 r_2 + r_1 p_2 q_2 + p_1 q_1 r_1 C. \tag{3.48}$$

We also define a homomorphism

$$\mathbb{R}_{\mathbf{p}} \xrightarrow{h_{\mathbf{p}}} \mathbb{R} \tag{3.49}$$

by the formulas

$$\begin{array}{lll}
 h_{\mathbf{p}}(p_1) = x_2 + x_4, & h_{\mathbf{p}}(q_1) = x_3 + x_4, & h_{\mathbf{p}}(r_1) = x_1 + x_4, \\
 h_{\mathbf{p}}(p_2) = y_2 + y_4, & h_{\mathbf{p}}(q_2) = y_3 + y_4, & h_{\mathbf{p}}(r_2) = y_1 + y_4.
 \end{array} \tag{3.50}$$

and

$$h_{\mathbf{p}}(C) = \tilde{C}(\mathbf{x}), \tag{3.51}$$

where

$$\tilde{C}(\mathbf{x}) = w(x_1, x_2, -x_3, x_4) + w(x_1, -x_1, x_2, x_4) + w(x_1, -x_1, -x_2, x_4) \tag{3.52}$$

and eq.(3.8) defines the polynomial w .

Let $S_3 \subset S_4$ be the subgroup permuting the legs $(1, 2, 3)$. Consider its permutation action on the triplet of variables $(\mathbf{r}, \mathbf{p}, \mathbf{q})$. Note that the formulas (3.50) are invariant with respect to the simultaneous action of S_3 on (x_1, x_2, x_3) , (y_1, y_2, y_3) and $(\mathbf{r}, \mathbf{p}, \mathbf{q})$.

The polynomial $W_{4, \mathbf{p}}$ is invariant under the action of S_3 on $(\mathbf{r}, \mathbf{p}, \mathbf{q})$, hence the elements $\sigma \in S_3$ act as endo-functors $\hat{\sigma}$ on the category of matrix factorizations $\text{MF}_{\mathbb{R}_{\mathbf{p}}, W_{4, \mathbf{p}}}$. In

particular, they permute the matrix factorizations

$$\widehat{\left. \vphantom{\int} \right)_p} = \mathbf{K} \begin{pmatrix} p_1, & q_2 r_2 + q_1 r_1 C \\ p_2, & q_2 r_1 + q_1 r_2 \end{pmatrix}, \quad (3.53)$$

$$\widehat{\left. \vphantom{\int} \right)_p} = \mathbf{K} \begin{pmatrix} r_1, & p_2 q_2 + p_1 q_1 C \\ r_2, & p_1 q_2 + p_2 q_1 \end{pmatrix}, \quad (3.54)$$

$$\widehat{\left. \vphantom{\int} \right)_p} = \mathbf{K} \begin{pmatrix} q_1, & p_2 r_2 + p_1 r_1 C \\ q_2, & p_1 r_2 + p_2 r_1 \end{pmatrix}, \quad (3.55)$$

equivariantly with respect to their action on the 2-arc graphs:

$$\hat{\sigma}(\hat{\gamma}_p) = \widehat{(\sigma(\gamma))}_p. \quad (3.56)$$

Define

$$\hat{\gamma}_{\mathbb{P}} = \hat{h}_p(\hat{\gamma}_p), \quad (3.57)$$

where \hat{h}_p is the functor (2.17) induced by the homomorphism (3.49).

Proposition 3.2. *The three 2-arc matrix factorizations factor as*

$$\hat{\gamma} = K_{\text{cmn}} \otimes_{\mathbf{R}} \hat{\gamma}_{\mathbb{P}}, \quad (3.58)$$

where

$$K_{\text{cmn}} = \mathbf{K} \begin{pmatrix} x_1 + x_2 + x_3 + x_4, & A(\mathbf{x}, \mathbf{y}) \\ y_1 + y_2 + y_3 + y_4, & B(\mathbf{x}, \mathbf{y}) \end{pmatrix} \quad (3.59)$$

and

$$\begin{aligned} A(\mathbf{x}, \mathbf{y}) &= -(y_3 + y_4)(y_1 + y_2 + y_4) - y_1 y_2 \\ &\quad + w(-x_1, x_3) + (x_2 + x_4)w(x_1, x_2, -x_3) \end{aligned} \quad (3.60)$$

$$\begin{aligned} &\quad - (x_2 + x_4)(x_1 + x_4)(w(x_1, -x_1, x_2, x_4) + w(x_1, -x_1, -x_2, x_4)), \\ B(\mathbf{x}, \mathbf{y}) &= x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4. \end{aligned} \quad (3.61)$$

Proof. We will prove this proposition for $\gamma = \textcircled{1}$ (the proofs for other graphs are similar). According to our definitions,

$$K_{\text{cmn}} \otimes_{\mathbb{R}} \widehat{\textcircled{1}}_{\bar{\mathbb{P}}} = \text{K} \begin{pmatrix} x_1 + x_2 + x_3 + x_4, & A(\mathbf{x}, \mathbf{y}) \\ y_1 + y_2 + y_3 + y_4, & B(\mathbf{x}, \mathbf{y}) \\ x_2 + x_4, & (y_1 + y_4)(y_3 + y_4) + (x_1 + x_4)(x_3 + x_4) \tilde{C}(\mathbf{x}) \\ y_2 + y_4, & (x_1 + x_4)(y_3 + y_4) + (y_1 + y_4)(x_3 + x_4) \end{pmatrix} \quad (3.62)$$

A straightforward computation shows that this is a matrix factorization of the polynomial $W_4(\mathbf{x}, \mathbf{y})$. On the other hand, the matrix factorization $\widehat{\textcircled{1}}$ is a 4-row Koszul matrix factorization of the same polynomial, which is transformed by the combination $[3, 1]_1 \circ [4, 2]_1$ of the transformations (2.34) into the following form

$$\begin{aligned} \widehat{\textcircled{1}} &= \text{K} \begin{pmatrix} x_1 + x_3, & y_1^2 + w(-x_1, x_3) \\ y_1 + y_3, & x_3(y_3 - y_1) \\ x_2 + x_4, & y_2^2 + w(-x_2, x_4) \\ y_2 + y_4, & x_4(y_4 - y_2) \end{pmatrix} \\ &= \text{K} \begin{pmatrix} x_1 + x_2 + x_3 + x_4, & y_1^2 + w(-x_1, x_3) \\ y_1 + y_2 + y_3 + y_4, & x_3(y_3 - y_1) \\ x_2 + x_4, & y_2^2 - y_1^2 + w(-x_2, x_4) - w(-x_1, x_3) \\ y_2 + y_4, & x_4(y_4 - y_2) - x_3(y_3 - y_1) \end{pmatrix}. \end{aligned} \quad (3.63)$$

The latter Koszul matrix factorization shares the same first column with the matrix factorization (3.62). Since the polynomials in that column form a regular sequence, the claim of the proposition follows from Theorem 2.2. \square

4. Saddle morphisms

4.1. A proper saddle morphism

According to the formulas (2.31) and (2.32), the proper matrix factorization $\hat{\gamma}_{\mathbb{P}}$ has an explicit form

$$\mathbb{R}_{\mathbb{P},1}^2 \xrightarrow{P_i} \mathbb{R}_{\mathbb{P},0}^2 \xrightarrow{Q_i} \mathbb{R}_{\mathbb{P},1}^2, \quad (4.1)$$

where the homomorphisms P_i and Q_i are presented by matrices

$$P_{\langle} = \begin{pmatrix} p_2 & p_1 \\ q_2 r_2 + q_1 r_1 C & -(q_2 r_1 + q_1 r_2) \end{pmatrix}, \quad Q_{\langle} = \begin{pmatrix} q_1 r_2 + q_2 r_1 & p_1 \\ q_2 r_2 + q_1 r_1 C & -p_2 \end{pmatrix}, \quad (4.2)$$

$$P_{\otimes} = \begin{pmatrix} r_2 & r_1 \\ p_2 q_2 + p_1 q_1 C & -(p_1 q_2 + p_2 q_1) \end{pmatrix}, \quad Q_{\otimes} = \begin{pmatrix} p_1 q_2 + p_2 q_1 & r_1 \\ p_2 q_2 + p_1 q_1 C & -r_2 \end{pmatrix}, \quad (4.3)$$

$$P_{\widehat{}} = \begin{pmatrix} q_2 & q_1 \\ p_2 r_2 + p_1 r_1 C & -(p_1 r_2 + p_2 r_1) \end{pmatrix}, \quad Q_{\widehat{}} = \begin{pmatrix} p_1 r_2 + p_2 r_1 & q_1 \\ p_2 r_2 + p_1 r_1 C & -q_2 \end{pmatrix}. \quad (4.4)$$

Consider the following homomorphism between two matrix factorization modules

$$\begin{array}{ccccc} \widehat{\langle}_{\mathfrak{p}} & \mathbb{R}_{\mathfrak{p},1}^2 & \xrightarrow{P_{\langle}} & \mathbb{R}_{\mathfrak{p},0}^2 & \xrightarrow{Q_{\langle}} & \mathbb{R}_{\mathfrak{p},1}^2 \\ \downarrow F_{\mathfrak{p}} & \downarrow F_1 & & \downarrow F_0 & & \downarrow F_1 \\ \widehat{\otimes}_{\mathfrak{p}} & \mathbb{R}_{\mathfrak{p},1}^2 & \xrightarrow{P_{\otimes}} & \mathbb{R}_{\mathfrak{p},0}^2 & \xrightarrow{Q_{\otimes}} & \mathbb{R}_{\mathfrak{p},1}^2 \end{array}, \quad (4.5)$$

where

$$F_0 = \begin{pmatrix} 0 & 1 \\ q_2^2 - q_1^2 C & 0 \end{pmatrix}, \quad F_1 = \begin{pmatrix} q_2 & -q_1 \\ q_1 C & -q_2 \end{pmatrix}. \quad (4.6)$$

By using expressions (4.2) and (4.3), it is easy to verify that the diagram (4.5) is commutative, that is, $[D, F_{\mathfrak{p}}] = 0$. This means that $F_{\mathfrak{p}}$ defines a matrix factorization morphism. The isomorphisms (3.45) allow us to extend $F_{\mathfrak{p}}$ to a morphism between full 2-arc matrix factorizations:

$$\widehat{\langle} \xrightarrow{F} \widehat{\otimes}, \quad \text{where } F = 1 \otimes_{\mathbb{R}} \widehat{h}_{\mathfrak{p}}(F_{\mathfrak{p}}) \quad (4.7)$$

(cf. diagram (3.46)).

Proposition 4.1. *The morphism (4.7) is a saddle morphism.*

Proof. It is easy to verify that

$$\deg_{\mathbb{Z}_2} F = \deg_{\mathbb{Z}_2} F_{\mathfrak{p}} = 0, \quad \deg_q F = \deg_q F_{\mathfrak{p}} = 2N, \quad (4.8)$$

so it remains to verify that $F \not\approx 0$. Indeed, the matrix F_0 has a unit entry, so the matrix of F also has a unit entry. However, all entries of the twisted differential matrices (4.2)

and (4.3) belong to the ideal generated by the variables $x_1, \dots, x_4, y_1, \dots, y_4$, hence there does not exist a homomorphism $X \in \text{Hom}_{\mathbb{R}} \left(\widehat{\quad} \left(\quad, \widehat{\otimes} \right) \right)$, such that $F = [D, X]$. \square

The saddle morphisms for other pairs of 2-arc graphs $\gamma \neq \gamma'$ can be obtained from eqs.(4.6) and (4.7) by the simultaneous permutation action of S_3 on the legs of the graphs and on the variables $(\mathbf{r}, \mathbf{p}, \mathbf{q})$.

4.2. Compositions of saddle morphisms

Let us find a composition of two saddle morphisms

$$\hat{\gamma} \xrightarrow{F} \hat{\gamma}' \xrightarrow{G} \hat{\gamma}'', \quad \gamma' \neq \gamma, \gamma'' \tag{4.9}$$

We should consider two different cases: $\gamma = \gamma''$ and $\gamma \neq \gamma''$. In both cases the symmetric group S_4 acts transitively on such triplets of 2-arc graphs $(\gamma, \gamma', \gamma'')$, so it is sufficient to perform the calculation just for one triplet, and the answer for others could be deduced by permutation of legs and variables.

Proposition 4.2. *The composition of two saddle morphisms*

$$\widehat{\quad} \left(\quad \xrightarrow{F} \widehat{\otimes} \xrightarrow{H} \widehat{\quad} \right) \tag{4.10}$$

is equal to

$$HF \simeq 2\hat{y}_1\hat{y}_2 - 2(2N+1) \sum_{i=0}^{2N} \hat{x}_1^i \hat{x}_2^{2N-i} \tag{4.11}$$

Proof. Consider the composition of proper saddle morphisms

$$\begin{array}{ccccc} \widehat{\quad} \left(\quad \right)_{\mathbf{p}} & & \mathbb{R}_{\mathbf{p},1}^2 & \xrightarrow{P) \left(\quad \right)} & \mathbb{R}_{\mathbf{p},0}^2 & \xrightarrow{Q) \left(\quad \right)} & \mathbb{R}_{\mathbf{p},1}^2 & & \\ & & \downarrow F_1 & & \downarrow F_0 & & \downarrow F_1 & & \\ & & \widehat{\otimes} \left(\quad \right)_{\mathbf{p}} & & \widehat{\otimes} \left(\quad \right)_{\mathbf{p}} & & \widehat{\otimes} \left(\quad \right)_{\mathbf{p}} & & \\ & & \downarrow H_{\mathbf{p}} & & \downarrow H_0 & & \downarrow H_1 & & \\ & & \widehat{\quad} \left(\quad \right)_{\mathbf{p}} & & \mathbb{R}_{\mathbf{p},1}^2 & \xrightarrow{P) \left(\quad \right)} & \mathbb{R}_{\mathbf{p},0}^2 & \xrightarrow{Q) \left(\quad \right)} & \mathbb{R}_{\mathbf{p},1}^2 \end{array} \tag{4.12}$$

Since $H_p = \hat{\sigma}_{12}(F_p)$, and σ_{12} switches \mathbf{r} and \mathbf{p} while leaving \mathbf{q} and C intact, we conclude from eq.(4.6) that

$$H_0 = F_0, \quad H_1 = F_1, \quad (4.13)$$

and matrix multiplication shows that

$$H_p F_p = (q_2^2 - q_1^2 C) 1. \quad (4.14)$$

Hence

$$HF = (\hat{y}_3 + \hat{y}_4)^2 - (\hat{x}_3 + \hat{x}_4)^2 \tilde{C}(\hat{\mathbf{x}}), \quad (4.15)$$

where \tilde{C} is defined by eq.(3.52). Now it is an elementary algebra exercise to transform the *r.h.s.* of this formula into the *r.h.s.* of eq.(4.11) with the help of eqs.(3.52), (3.1) and the following relations between the endomorphisms of $\widehat{\quad}(\quad)$:

$$\hat{x}_3 \simeq -\hat{x}_1, \quad \hat{y}_3 \simeq -\hat{y}_1, \quad \hat{x}_4 \simeq -\hat{x}_2, \quad \hat{y}_4 \simeq -\hat{y}_2, \quad (4.16)$$

$$\hat{y}_1^2 \simeq -(2N+1)\hat{x}_1^{2N}, \quad \hat{y}_2^2 \simeq -(2N+1)\hat{x}_2^{2N}. \quad (4.17)$$

□

Proposition 4.3. *The composition of two saddle morphisms*

$$\widehat{\quad}(\quad) \xrightarrow{F} \widehat{\quad}(\quad) \xrightarrow{G} \widehat{\quad}(\quad) \quad (4.18)$$

is null-homotopic:

$$GF \simeq 0. \quad (4.19)$$

Proof. By definition, eq.(4.19) means that there exists an R-module homomorphism

$$\widehat{\quad}(\quad) \xrightarrow{X} \widehat{\quad}(\quad), \quad (4.20)$$

such that

$$\deg_{\mathbb{Z}_2} X = 1, \quad \deg_q X = 2N - 1 \quad (4.21)$$

and

$$GF = -\{D, X\} \quad (4.22)$$

(we put a minus sign in for future convenience).

Consider the proper saddle morphisms

$$\begin{array}{ccccc}
 \widehat{\left. \right\} \left(\right)_p & \mathbb{R}_{p,1}^2 & \xrightarrow{P_{\left(\right)_\left(\right)}} & \mathbb{R}_{p,0}^2 & \xrightarrow{Q_{\left(\right)_\left(\right)}} & \mathbb{R}_{p,1}^2 \\
 \downarrow F_p & \downarrow F_1 & & \downarrow F_0 & & \downarrow F_1 \\
 \widehat{\left. \right\} \left(\right)_p & \mathbb{R}_{p,1}^2 & \xrightarrow{P_{\left(\right)_\left(\right)}} & \mathbb{R}_{p,0}^2 & \xrightarrow{Q_{\left(\right)_\left(\right)}} & \mathbb{R}_{p,1}^2 \\
 \downarrow G_p & \downarrow G_1 & & \downarrow G_0 & & \downarrow G_1 \\
 \widehat{\left. \right\} \left(\right)_p & \mathbb{R}_{p,1}^2 & \xrightarrow{P_{\left(\right)_\left(\right)}} & \mathbb{R}_{p,0}^2 & \xrightarrow{Q_{\left(\right)_\left(\right)}} & \mathbb{R}_{p,1}^2
 \end{array} \tag{4.23}$$

where the second morphism is presented by the matrices

$$G_0 = \begin{pmatrix} 0 & 1 \\ p_2^2 - p_1^2 C & 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} p_2 & -p_1 \\ p_1 C & -p_2 \end{pmatrix}, \tag{4.24}$$

in accordance with the relation $G_p = \hat{\sigma}_{23} \hat{\sigma}_{12}(F_p)$. On the diagram

$$\begin{array}{ccccc}
 \mathbb{R}_{p,1}^2 & \xrightarrow{P_{\left(\right)_\left(\right)}} & \mathbb{R}_{p,0}^2 & \xrightarrow{Q_{\left(\right)_\left(\right)}} & \mathbb{R}_{p,1}^2 \\
 \downarrow G_1 F_1 & \swarrow X_0 & \downarrow G_0 F_0 & \swarrow X_1 & \downarrow G_1 F_1 \\
 \mathbb{R}_{p,1}^2 & \xrightarrow{P_{\left(\right)_\left(\right)}} & \mathbb{R}_{p,0}^2 & \xrightarrow{Q_{\left(\right)_\left(\right)}} & \mathbb{R}_{p,1}^2
 \end{array} \tag{4.25}$$

representing the composition of saddle morphisms we choose an \mathbb{R}_p -module homomorphism

$$X_p \in \text{Hom}_{\mathbb{R}_p} \left(\widehat{\left. \right\} \left(\right)_p, \widehat{\left. \right\} \left(\right)_p \right), \quad X_p = X_0 + X_1 \tag{4.26}$$

presented by the matrices

$$X_0 = \begin{pmatrix} -q_2 & 0 \\ q_1 C & 0 \end{pmatrix}, \quad X_1 = \begin{pmatrix} 0 & 0 \\ p_1 C & p_2 \end{pmatrix}. \tag{4.27}$$

A direct computation shows that these matrices satisfy the relations

$$G_0 F_0 = -(P_{\left(\right)_\left(\right)} X_0 + X_1 Q_{\left(\right)_\left(\right)}), \quad G_1 F_1 = -(Q_{\left(\right)_\left(\right)} X_1 + X_0 P_{\left(\right)_\left(\right)}), \tag{4.28}$$

which means that X_p satisfies the properties

$$\deg_{\mathbb{Z}_2} X_p = 1, \quad \deg_q X_p = 2N - 1, \quad G_p F_p = -\{D, X_p\}, \quad (4.29)$$

and if we choose

$$X = 1 \otimes_{\mathbb{R}} \widehat{h}_p(X_p), \quad (4.30)$$

then X would satisfy (4.21) and (4.22). \square

4.3. Semi-closed and closed saddle morphisms

In order to exhibit the structure of the categorification complex $C^\bullet(L)$ and to prove its Reidemeister move invariance we have to find what happens to the saddle morphism when we join together one or two pairs of legs of the 2-arc open graphs. As a result of this joining, a 2-arc graph becomes either an 1-arc graph, or a 1-arc graph with a disjoint circle, or a circle, or a pair of circles. Hence the initial 2-arc matrix factorizations can be simplified by homotopy equivalence transformations, and we want to know how this simplification affects the saddle morphisms.

4.3.1. Semi-closed saddle morphisms

Let us join one pair of legs in the saddle morphisms. The endomorphism symmetry S_4 allows us to consider only three cases, namely, the saddle morphisms

$$\widehat{)}(\begin{array}{c} \xrightarrow{F} \\ \xleftarrow{H} \end{array} \widehat{\times} \xrightarrow{G} \widehat{)} \quad (4.31)$$

in which we join together legs 2 and 4. After that the diagram becomes

$$\widehat{)}\circlearrowleft \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{H} \end{array} \widehat{\circlearrowleft} \xrightarrow{G} \widehat{)}\circlearrowleft \quad (4.32)$$

its homomorphisms resulting from taking quotients of the saddle homomorphisms (4.31) by the ideal $(x_2 + x_4, y_2 + y_4)$ according to eq.(3.7). The second and the third graphs in the diagram (4.32) are just 1-arcs, so we pass to homotopy equivalent matrix factorizations and present this diagram as

$$\widehat{)} \otimes C_{\text{unkn}} \begin{array}{c} \xrightarrow{F'} \\ \xleftarrow{H'} \end{array} \widehat{)} \xrightarrow{G'} \widehat{)} \quad (4.33)$$

The first matrix factorization in this diagram splits into a direct sum of 1-arc matrix factorizations

$$\widehat{\frown} \otimes C_{\text{unkn}} = \bigoplus_{i=0}^{2N} \left(\widehat{\frown} \otimes \hat{x}^i(e) \right) \oplus \left(\widehat{\frown} \otimes \hat{y}(e) \right), \quad (4.34)$$

where $\hat{x}^i(e)$ and $\hat{y}(e)$ form the basis (3.30) of the space C_{unkn} . Hence we will express the semi-closed saddle morphisms F' and G' in terms of the endomorphisms of the 1-arc matrix factorization $\widehat{\frown}_1^3$. According to eq.(3.33),

$$\text{End}_{\text{MF}} \left(\widehat{\frown}_1^3 \right) = J_W, \quad (4.35)$$

where J_W is generated by

$$\hat{x} = \hat{x}_1 = -\hat{x}_3, \quad \hat{y} = \hat{y}_1 = -\hat{y}_3. \quad (4.36)$$

Recall that the space C_{unkn} has a special basis (3.30). Let us introduce the dual basis for the dual space C_{unkn}^* :

$$e^*, (\hat{x}(e))^*, (\hat{x}^2(e))^* \dots (\hat{x}^{2N}(e))^*, (\hat{y}(e))^* \in C_{\text{unkn}}^*. \quad (4.37)$$

Proposition 4.4. *With the appropriate choice of the basis element e , the homomorphisms F' and G' of the diagram (4.33) are homotopic to the following homomorphisms*

$$F' \simeq F_m = \left(\sum_{i=0}^{2N} \hat{x}^i \otimes (\hat{x}^i(e))^* + \hat{y} \otimes (\hat{y}(e))^* \right), \quad (4.38)$$

$$H' \simeq H_\Delta = 2 \left(\hat{y} \otimes \hat{y}(e) - (2N+1) \sum_{i=0}^{2N} \hat{x}^i \otimes \hat{x}^{2N-i}(e) \right). \quad (4.39)$$

The homomorphism G' is null-homotopic:

$$G' \simeq 0. \quad (4.40)$$

The indices m and Δ indicate that the homomorphisms F_m and H_Δ are related to the multiplication (3.16) and comultiplication (3.19) in J_W , as we will see shortly.

Proof. We derive the expressions for the semi-closed saddle morphisms not by a direct computation, but rather from three properties of the saddle morphisms (4.31). First, a saddle morphism has homogeneous q -degree $2N$, so

$$\deg_q F' = \deg_q H' = 2N. \quad (4.41)$$

and these morphisms must have a form

$$F' \simeq \sum_{i=0}^{2N} a_i \hat{x}^i \otimes (\hat{x}^i(e))^* + \tilde{a} \hat{y} \otimes (\hat{y}(e))^* + a'_1 \hat{y} \otimes (\hat{x}^N(e))^* + a'_2 \hat{x}^N \otimes (\hat{y}(e))^*, \quad (4.42)$$

$$H' \simeq \sum_{i=0}^{2N} b_i \hat{x}^i \otimes \hat{x}^{2N-i}(e) + \tilde{b} \hat{y} \otimes \hat{y}(e) + b'_1 \hat{y} \otimes \hat{x}^N(e) + b'_2 \hat{x}^N \otimes \hat{y}(e), \quad (4.43)$$

where the coefficients a 's and b 's are rational numbers.

The second property is the composition formula (4.11), which we apply to the composition $H'F'$:

$$H'F' \simeq 2\hat{y}_1\hat{y}_2 - 2(2N+1) \frac{\hat{x}_1^{2N+1} - \hat{x}_2^{2N+1}}{\hat{x}_1 - \hat{x}_2}. \quad (4.44)$$

It imposes relations among the coefficients of expressions (4.42), which imply that

$$F' \simeq aF_m, \quad H' \simeq bH_\Delta, \quad a, b \in \mathbb{Q}, \quad ab = 1. \quad (4.45)$$

Indeed, if we apply eq.(4.44) to $\widehat{\bigcirc} \otimes e$, then we find that $H' \simeq bH_\Delta$ and $a_0b = 1$. The other coefficients of (4.42) are determined by applying the relation (4.44) to $\widehat{\bigcirc} \otimes \hat{x}^i(e)$ and to $\widehat{\bigcirc} \otimes \hat{y}(e)$.

Since $ab = 1$, then $a \neq 0$ and we can rescale the basis element e so that $a = b = 1$. Thus we obtain eqs.(4.38) and (4.39).

Finally, Proposition 4.3 says that

$$G'F' \simeq 0. \quad (4.46)$$

At the same time, according to eq.(4.38), the morphism F' acts on the submodule $\widehat{\bigcirc} \otimes e \subset \widehat{\bigcirc} \otimes C_{\text{unkn}}$ as 1, so G' must be null-homotopic. \square

4.3.2. Closed saddle morphisms

Let us consider the result of joining legs 2 and 4, as well as legs 1 and 3 in the graphs of the diagram (4.31):

$$\widehat{\bigcirc\bigcirc} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{H} \end{array} \widehat{\bigcirc\bigcirc} \xrightarrow{G} \widehat{\bigcirc\bigcirc}. \quad (4.47)$$

The first graph in this diagram is a pair of disjoint circles, while the second and the third graphs are circles. Therefore, we can use the matrix factorization homotopy equivalence in order to present the diagram (4.47) as

$$C_{\text{unkn}} \otimes C_{\text{unkn}} \begin{array}{c} \xrightarrow{F''} \\ \xleftarrow{H''} \end{array} C_{\text{unkn}} \xrightarrow{G''} C_{\text{unkn}} . \quad (4.48)$$

The morphisms on this diagram can be obtained from the morphisms of the diagram (4.33) by imposing the joining relation $\hat{x}_1 = -\hat{x}_3$ in the expressions (4.38), (4.39) and (4.40). In fact, since \hat{x}_3 never appears there, the expressions remain the same. First of all, we find that

$$G'' = 0. \quad (4.49)$$

Second, we observe that the operators \hat{x} and \hat{y} appearing in the formulas for F_m and H_Δ are the same as the operators \hat{x} and \hat{y} of eqs.(3.16) and (3.19). Thus, if we establish a canonical isomorphism

$$C_{\text{unkn}} \cong J'_W = J_W\{-2N\} \quad (4.50)$$

by the basis elements correspondence

$$\hat{x}^i(e) \leftrightarrow x^i, \quad \hat{y}(e) \leftrightarrow y, \quad (4.51)$$

then the diagram (4.48) becomes

$$J'_W \otimes J'_W \begin{array}{c} \xrightarrow{m} \\ \xleftarrow{\Delta} \end{array} J'_W \xrightarrow{0} J'_W , \quad (4.52)$$

while the homomorphisms (4.38) and (4.39) take the form

$$F' \simeq F_m = \sum_{i=0}^{2N} \hat{x}^i \otimes (x^i)^* + \hat{y} \otimes y^*, \quad (4.53)$$

$$H' \simeq H_\Delta = 2 \left(\hat{y} \otimes y - (2N+1) \sum_{i=0}^{2N} \hat{x}^i \otimes x^{2N-i} \right). \quad (4.54)$$

5. Multi-dimensional Postnikov systems and their convolutions

The category of matrix factorizations is triangulated. Relation (4.19) means that the diagram (4.18) is a chain complex of matrix factorization morphisms. This allows us to define the matrix factorization $\widehat{\times}$ as its convolution. But first let us review the basic facts about Postnikov systems and their convolutions. The general theory of Postnikov

systems within triangulated categories is described in the book [1] (Chapter IV, exercises). We adapt its approach to the specific case of matrix factorizations.

Some proofs in this section are omitted, because they are standard exercises in homological algebra.

5.1. A multi-dimensional Postnikov system

Let us introduce multi-index notations: for a positive integer n and $v_i \in \mathbb{Z}$ we denote $\mathbf{v} = (v_1, \dots, v_n)$ and $|\mathbf{v}| = \sum_{i=1}^n v_i$. Also $\mathbf{v} \geq \mathbf{w}$ means that $v_i \geq w_i$ for all $i = 1 \dots n$, and $\mathbf{v} > \mathbf{w}$ means that $\mathbf{v} \geq \mathbf{w}$ and $\mathbf{v} \neq \mathbf{w}$. The zero vector is denoted $\mathbf{0} = (0, \dots, 0)$, and the vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ form the standard basis in the lattice \mathbb{Z}^n .

We introduce a new \mathbb{Z}^n grading, which we call *length*, with multi-degree

$$\mathbf{deg}_{\text{ing}} = (\text{deg}_1, \dots, \text{deg}_n), \tag{5.1}$$

and assume that $\mathbf{deg}_{\text{ing}} x = \mathbf{0}$ for all $x \in R$. We also introduce the total length degree:

$$\text{deg}_{\text{ing}} = |\mathbf{deg}_{\text{ing}}|. \tag{5.2}$$

A length-graded R-module A is called *bounded* if its grading expansion

$$A = \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} A_{\mathbf{k}}, \quad \mathbf{deg}_{\text{ing}} A_{\mathbf{k}} = \mathbf{k} \tag{5.3}$$

has only finitely many non-trivial modules $A_{\mathbf{k}}$. A homomorphism $X \in \text{Hom}_R(A, B)$ between two length-graded R-modules is called non-negative if it is a sum of homomorphisms of non-negative length degrees:

$$X = \sum_{\mathbf{k} \geq \mathbf{0}} X_{\mathbf{k}}, \quad \mathbf{deg}_{\text{ing}} X_{\mathbf{k}} = \mathbf{k}. \tag{5.4}$$

Let $\text{Hom}_{R; \geq 0}(A, B)$ denote the space of all such homomorphisms.

For a fixed polynomial $W \in R$ we define the category PMF_W^n of n -dimensional Postnikov systems. Its objects are length-graded bounded matrix factorizations A of W with non-negative twisted differentials D_A :

$$A = \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} A_{\mathbf{k}}, \quad D_A : A_{\mathbf{k}} \rightarrow \bigoplus_{\mathbf{l} \geq \mathbf{k}} A_{\mathbf{l}}, \quad D_A^2 = W 1_A. \tag{5.5}$$

For two such matrix factorizations A and B , we define the differential d acting on the space $\text{Hom}_{R; \geq 0}(A, B)$ by the formula (2.4):

$$d = [D_{A,B}, \cdot]_s. \tag{5.6}$$

Then we define the \mathbb{Z}_2 -graded space

$$\mathrm{Ext}_{\mathbb{P}}^{\bullet}(A, B) = \ker d / \mathrm{im} d, \quad (5.7)$$

and the space of PMF_W^n -morphisms between A and B :

$$\mathrm{Hom}_{\mathrm{PMF}}(A, B) = \mathrm{Ext}_{\mathbb{P}}^0(A, B). \quad (5.8)$$

The category PMF_W^n is triangulated. For a morphism $A \langle 1 \rangle \xrightarrow{f} B$ we define the cone $\mathrm{Cone}_{\mathrm{PMF}}(f)$ as a Postnikov system whose module is the sum $A \oplus B$ and whose twisted differential D is the sum of individual twisted differentials and f : $D = D_A + D_B + f$.

The tensor product (2.12) creates a bifunctor

$$\mathrm{PMF}_{\mathbb{R}_1, W_1}^{n_1} \times \mathrm{PMF}_{\mathbb{R}_2, W_2}^{n_2} \xrightarrow{\otimes} \mathrm{PMF}_{\mathbb{R}_1 \otimes \mathbb{R}_2, W_1 + W_2}^{n_1 + n_2}, \quad (5.9)$$

Convolution is a functor

$$\mathrm{PMF}_W^n \xrightarrow{\mathrm{Conv}_{\mathrm{MF}}(\cdot)} \mathrm{MF}_W \quad (5.10)$$

which turns a Postnikov system A into an ordinary matrix factorization $\mathrm{Conv}_{\mathrm{MF}}(A)$ by ‘forgetting’ about its length grading. It extends to a functor between the corresponding homotopy categories of complexes by acting on individual chain modules:

$$\mathbf{K}(\mathrm{PMF}_W^n) \xrightarrow{\mathrm{Conv}_{\mathrm{MF}}(\cdot)} \mathbf{K}(\mathrm{MF}_W). \quad (5.11)$$

In order to reveal the inner structure of a Postnikov system and its relation to an n -complex of matrix factorizations, we split the module and the twisted differential of a Postnikov system A according to the length-grading:

$$A = \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} A_{\mathbf{k}}, \quad \mathrm{deg}_{\mathrm{lng}} A_{\mathbf{k}} = \mathbf{k}, \quad D_A = \sum_{k \geq 0} X_k, \quad \mathrm{deg}_{\mathrm{lng}} X_k = k. \quad (5.12)$$

We further split the twisted differential D_A according to the domain and range with respect to the splitting of A :

$$D_A = \sum_{\mathbf{k} < \mathbf{l}} X_{\mathbf{l}, \mathbf{k}}, \quad X_{\mathbf{l}, \mathbf{k}} \in \mathrm{Hom}_{\mathbb{R}}(A_{\mathbf{k}}, A_{\mathbf{l}}), \quad (5.13)$$

so that

$$X_k = \sum_{\substack{\mathbf{l}, \mathbf{l}' \in \mathbb{Z}^n \\ |\mathbf{l}' - \mathbf{l}| = k}} X_{\mathbf{l}', \mathbf{l}}. \quad (5.14)$$

The \mathbb{R} -module map $X_{\mathbf{l}, \mathbf{k}}$ has the internal \mathbb{Z}_2 -degree 1.

If we split both sides of the basic relation

$$D_A^2 = W 1_A \tag{5.15}$$

according to the total length, then the length-0 part says that $X_0^2 = W 1_A$, or equivalently

$$X_{\mathbf{k},\mathbf{k}}^2 = W 1_{A_{\mathbf{k}}}. \tag{5.16}$$

This means that the modules $A_{\mathbf{k}}$ together with the twisted differentials $D_{\mathbf{k}} = X_{\mathbf{k},\mathbf{k}}$ form matrix factorizations of W . We call them *constituent* matrix factorizations of the Postnikov system A .

Let us introduce a differential

$$d_0 = [X_0, \cdot]_s, \tag{5.17}$$

acting on $\text{End}_{\mathbb{R}; \geq 0}(A)$. Then the length- k ($k \geq 1$) part of eq.(5.15) can be put in a form

$$d_0 X_k = - \sum_{l=1}^{k-1} X_{k-l} X_l. \tag{5.18}$$

In particular, the length-1 part says that $d_0 X_1 = 0$, which means that the homomorphism $F_{\mathbf{k},i} = X_{\mathbf{k}+\mathbf{e}_i,\mathbf{k}}$ is a morphism between the adjacent matrix factorizations $A_{\mathbf{k}} \langle 1 \rangle$ and $A_{\mathbf{k}+\mathbf{e}_i}$. Moreover, the length-2 part of eqs.(5.18) implies that

$$\{F_i, F_j\} \simeq 0, \quad \text{where } F_i = \sum_{\mathbf{k}} F_{\mathbf{k},i}, \tag{5.19}$$

which means that the morphisms F_i can be interpreted as differentials of an n -complex over the homotopy category of matrix factorizations. This complex is formed by placing the matrix factorizations $A_{\mathbf{k}}$ at the nodes of an n -dimensional lattice \mathbb{Z}^n in accordance with their multi-indices \mathbf{k} , so that each pair of adjacent matrix factorizations $A_{\mathbf{k}}$ and $A_{\mathbf{k}+\mathbf{e}_i}$ is connected by a differential $F_{\mathbf{k},i}$ (see the diagram inside the dotted box of eq.(5.23)).

We refer to F_i as *primary* homomorphisms (or differentials), since they determine the n -complex of matrix factorizations, and we call X_k ($k \geq 2$) *secondary* homomorphisms, because their role is in part to ‘correct’ the consequences of the distinction between the module chain differentials for which the relation $\{F_i, F_j\} = 0$ holds, and the matrix factorization chain differentials satisfying eq.(5.19).

Proposition 5.1. *If two Postnikov systems A and A' are isomorphic in the category PMF_W^n , then there exist homotopy equivalences*

$$A_{\mathbf{k}} \simeq A'_{\mathbf{k}} \quad \text{in } \text{MF}_W \tag{5.20}$$

making the following diagrams commutative:

$$\begin{array}{ccc}
 A_{\mathbf{k}} & \xrightarrow{F_{\mathbf{k},i}} & A_{\mathbf{k}+e_i} \\
 \downarrow \simeq & & \downarrow \simeq \\
 A'_{\mathbf{k}} & \xrightarrow{F'_{\mathbf{k},i}} & A'_{\mathbf{k}+e_i}
 \end{array} \tag{5.21}$$

Proof. If a homotopy equivalence between A and A' within PMF_W^n is established by the homomorphisms

$$A \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{f'} \end{array} B, \tag{5.22}$$

then the homotopy equivalences (5.20) are established by their length-0 parts f_0 and f'_0 . \square

We denote a Postnikov system built upon an n -complex by encircling it with a dotted frame:

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & \\
 \cdots & \longrightarrow & A_{\mathbf{k}} & \xrightarrow{F_{\mathbf{k},i}} & A_{\mathbf{k}+e_i} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
 \end{array} \tag{5.23}$$

Since the Postnikov system might not be determined by this complex uniquely, we may also add the arrows carrying the secondary morphisms to the picture. The convolution of a Postnikov system is denoted by a solid frame:

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & \\
 \cdots & \longrightarrow & A_{\mathbf{k}} & \xrightarrow{F_{\mathbf{k},i}} & A_{\mathbf{k}+e_i} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
 \end{array} \tag{5.24}$$

5.2. Building a Postnikov system from the bottom up

A Postnikov system A can be constructed from the bottom up in two stages. At first we choose its constituent matrix factorizations $A_{\mathbf{k}}$. At the second stage we solve the equations (5.18) inductively on the value of k starting at $k = 1$. The inductive procedure is possible, since the *r.h.s.* of eq.(5.18) contain only the homomorphisms of length up to $k - 1$.

Let us find out how various choices involved in this build-up affect the emerging Postnikov system. At the first stage we choose $A_{\mathbf{k}}$, $\mathbf{k} \in \mathcal{S}$, where $\mathcal{S} \subset \mathbb{Z}^n$ is a finite subset.

Theorem 5.2. *Let A be a Postnikov system presented by matrix factorizations $A_{\mathbf{k}}$, $\mathbf{k} \in \mathcal{S}$ and homomorphisms X_m , $m \geq 1$. Given a family of homotopy equivalent matrix factorizations*

$$A'_{\mathbf{k}} \simeq A_{\mathbf{k}}, \quad \mathbf{k} \in \mathcal{S}, \quad (5.25)$$

there exists a system of homomorphisms X'_m , $m \geq 1$ such that the Postnikov system A' presented by $A'_{\mathbf{k}}$ and X'_m is homotopy equivalent to A .

In other words, the set of equivalence classes of all Postnikov systems based on matrix factorizations $A_{\mathbf{k}}$ is determined by homotopy equivalence classes of $A_{\mathbf{k}}$.

Proof. Let $|\mathcal{S}|$ denote the number of elements in \mathcal{S} . We will prove the theorem by induction over $|\mathcal{S}|$. If $|\mathcal{S}| = 1$, then the claim is obvious. Suppose that the claim is true for sets of k elements. Consider a Postnikov system A consisting of $k + 1$ constituent modules. Since the set \mathcal{S} is finite, it contains an element \mathbf{k} such that $\mathbf{l} \not\geq \mathbf{k}$ for all $\mathbf{l} \in \mathcal{S}_- = \mathcal{S} \setminus \{\mathbf{k}\}$. Non-negativity of the twisted differential D of A implies that

$$X_{\mathbf{l}, \mathbf{k}} = 0, \quad \mathbf{l} \in \mathcal{S}_-. \quad (5.26)$$

Let A_- be the Postnikov system presented by matrix factorizations $A_{\mathbf{l}}$, $\mathbf{l} \in \mathcal{S}_-$ and homomorphisms $X_{\mathbf{l}', \mathbf{l}}$, $\mathbf{l}, \mathbf{l}' \in \mathcal{S}_-$. The sum $f = \sum_{\mathbf{l} \in \mathcal{S}_-} X_{\mathbf{k}, \mathbf{l}}$ defines a morphism between A_- and $A_{\mathbf{k}}$. In view of eq.(5.26), the Postnikov system cone of f is isomorphic to A :

$$\text{Cone}_{\text{PMF}}(f) \cong A. \quad (5.27)$$

By the induction assumption there exists a Postnikov system $A'_- \simeq A_-$, which is based upon the matrix factorizations $A'_{\mathbf{l}}$, $\mathbf{l} \in \mathcal{S}_-$. Fix equivalence homomorphisms

$$A'_- \xrightarrow{g_-} A_- , \quad A_{\mathbf{k}} \xrightarrow{g} A'_{\mathbf{k}} \quad (5.28)$$

and set $f' = gfg_-$. The cone of f' is a Postnikov system based upon matrix factorizations A'_1 , $\mathbf{1} \in \mathcal{S}$, so the claim of the theorem follows from the homotopy equivalence of the cones $\text{Cone}_{\text{PMF}}(f) \simeq \text{Cone}_{\text{PMF}}(f')$ and from the isomorphism (5.27). \square

At the second stage we choose the solutions of eqs.(5.18) inductively over k starting with $k = 1$. Let us fix the matrix factorizations A_k and a sequence of homomorphisms X_1, \dots, X_{k-1} which solve the equations (5.18) up to the length $k-1$. Consider the equation (5.18) at the length k . Note that its *r.h.s.* is d_0 -closed, however it is not necessarily d_0 -exact, so not all choices of homomorphisms up to length $k-1$ may lead to a Postnikov system. For a solution X_k of the equation (5.18), let $\mathcal{A}(X_k)$ denote the set of homotopy equivalence classes of Postnikov system, which can be built upon the sequence of homomorphisms X_1, \dots, X_{k-1}, X_k .

Let X_k and X'_k be two solution of eq.(5.18). Since the *r.h.s.* of this equation is fixed, then $d_0 X_k = d_0 X'_k$, so the difference $\Delta X_k = X'_k - X_k$ is d_0 -closed:

$$d_0 \Delta X_k = 0. \quad (5.29)$$

Theorem 5.3. *If ΔX_k is d_0 -exact (that is, if there exists a homomorphism Y_k ($\deg_{\mathbb{Z}_2} Y_k = 0$, $\deg_{\text{ing}} Y_k = k$) such that $\Delta X_k = d_0 Y_k$), then $\mathcal{A}(X_k) = \mathcal{A}(X'_k)$.*

Proof. We will prove the inclusion $\mathcal{A}(X_k) \subset \mathcal{A}(X'_k)$, the other inclusion $\mathcal{A}(X'_k) \subset \mathcal{A}(X_k)$ is established similarly.

Suppose that a Postnikov system $A \in \mathcal{A}(X_k)$ is presented by a twisted differential $D_A = \sum_{l \geq 0} X_l$. A non-negative homomorphism $Y = 1 + Y_k$ is invertible:

$$Y^{-1} = 1 + \sum_{i=1}^{\infty} (-1)^i Y_k^i. \quad (5.30)$$

Hence it establishes an isomorphism between the Postnikov systems with the twisted differential D_A and with the conjugated twisted differential $D'_A = Y^{-1} D_A Y$. It is easy to see that since Y and Y^{-1} are equal to identity up to the terms of length k , then the low length components of $D'_A = \sum_{l \geq 0} X'_l$ are

$$X'_0 = X_0, \quad \dots, \quad X'_{k-1} = X_{k-1}, \quad X'_k = X_k + d_0 Y_k. \quad (5.31)$$

Hence the Postnikov system A is also an element of $\mathcal{A}(X'_k)$. \square

Let us choose a particular solution $X_{k,0}$ of eq.(5.18). Then, in view of eq.(5.29), for any solution X_k of that equation we can define a class

$$[X_k] = X_k - X_{k,0} \in \text{Ext}_k^1(\bar{A}, \bar{A}), \quad (5.32)$$

where Ext_k^1 is the total length k summand of Ext^1 and \bar{A} is a length-graded matrix factorization, which is a direct sum of matrix factorizations $A_{\mathbf{k}}$:

$$\bar{A} = \bigoplus_{\mathbf{k} \in \mathbb{Z}^n} A_{\mathbf{k}} \quad (5.33)$$

(in other words, its module is A and its twisted differential is just X_0 , which is the length-0 part of D). Theorem 5.3 says that the set $\mathcal{A}(X_k)$ depends only on the class $[X_k]$.

Let us split a solution X_k into components according to eq. (5.14). An individual component $X_{\mathbf{l}, \mathbf{l}'}$ satisfies the equation

$$d_0 X_{\mathbf{l}, \mathbf{l}'} = - \sum_{\mathbf{l} < \mathbf{m} < \mathbf{l}'} X_{\mathbf{l}, \mathbf{m}} X_{\mathbf{m}, \mathbf{l}'} \quad (5.34)$$

and defines a class

$$[X_{\mathbf{l}, \mathbf{l}'}] = X_{\mathbf{l}, \mathbf{l}'} - X_{\mathbf{l}, \mathbf{l}; 0} \in \text{Ext}^1(A_{\mathbf{l}}, A_{\mathbf{l}'}). \quad (5.35)$$

According to Theorem (5.3), the set $\mathcal{A}(X_k)$ is determined by all classes $[X_{\mathbf{l}, \mathbf{l}'}]$.

Corollary 5.4. *If for a pair of multi-indices $\mathbf{l}, \mathbf{l}' \in \mathbb{Z}^n$, $|\mathbf{l}' - \mathbf{l}| = k$ we have*

$$\text{Ext}^1(A_{\mathbf{l}}, A_{\mathbf{l}'}) = 0, \quad (5.36)$$

then the set $\mathcal{A}(X_k)$ does not depend on the choice of a solution to eq. (5.34).

5.3. Corner subsystems and factorsystems

Fix a subset $\mathcal{S} \in \mathbb{Z}^n$. For a length-graded module A , an \mathcal{S} -cut submodule is defined as

$$A \supset A_{\mathcal{S}} = \bigoplus_{\mathbf{i} \in \mathcal{S}} A_{\mathbf{i}}. \quad (5.37)$$

For a homomorphism $f \in \text{Hom}_{\mathbb{R}}(A, B)$ between two length-graded modules, an \mathcal{S} -cut homomorphism $f_{\mathcal{S}} \in \text{Hom}_{\mathbb{R}}(A_{\mathcal{S}}, B_{\mathcal{S}})$ is defined as

$$f_{\mathcal{S}} = \sum_{\mathbf{i}, \mathbf{j} \in \mathcal{S}} f_{\mathbf{i}, \mathbf{j}}. \quad (5.38)$$

A subset $\mathcal{S} \in \mathbb{Z}^n$ is called *length-convex* (or simply convex) if for any triplet of multi-indices $\mathbf{i}, \mathbf{j} \in \mathcal{S}$, $\mathbf{k} \in \mathbb{Z}^n$ such that $\mathbf{i} < \mathbf{k} < \mathbf{j}$, it turns out that $\mathbf{k} \in \mathcal{S}$. If A is a Postnikov system with a twisted differential D , then it is easy to verify that $D_{\mathcal{S}}^2 = W1_{\mathcal{S}}$, so that $A_{\mathcal{S}}$ is a Postnikov system with a twisted differential $D_{\mathcal{S}}$.

Theorem 5.5. *Fix a convex subset $\mathcal{S} \in \mathbb{Z}^n$. The map $\mathcal{F}_{\mathcal{S}}$, which associates $A_{\mathcal{S}}$ to a Postnikov system A and the morphism $f_{\mathcal{S}}$ to a Postnikov system morphism f , is an endo-functor*

$$\mathrm{PMF}_W^n \xrightarrow{\mathcal{F}_{\mathcal{S}}} \mathrm{PMF}_W^n . \quad (5.39)$$

We call $\mathcal{F}_{\mathcal{S}}$ an \mathcal{S} -cut functor. It behaves properly under the tensor product functor (5.9).

Theorem 5.6. *Fix two convex subsets $\mathcal{S} \in \mathbb{Z}^n$ and $\mathcal{S}' \in \mathbb{Z}^{n'}$. Let A and f be an object and a morphism of PMF_W^n , and let B and g be an object and a morphism of $\mathrm{PMF}_{W'}^{n'}$. Then*

$$A_{\mathcal{S}} \otimes B_{\mathcal{S}'} = (A \otimes B)_{\mathcal{S} \times \mathcal{S}'}, \quad f_{\mathcal{S}} \otimes g_{\mathcal{S}'} = (f \otimes g)_{\mathcal{S} \times \mathcal{S}'}. \quad (5.40)$$

For a subset $\mathcal{S} \subset \mathbb{Z}^n$ denote the complementary subset $\bar{\mathcal{S}} = \mathbb{Z}^n \setminus \mathcal{S}$. Suppose that a pair of complementary subsets $\mathcal{S}, \bar{\mathcal{S}}$ satisfies the property

$$\mathbf{i} \not\prec \mathbf{j} \quad \text{for all } \mathbf{i} \in \mathcal{S}, \mathbf{j} \in \bar{\mathcal{S}}. \quad (5.41)$$

Then both \mathcal{S} and $\bar{\mathcal{S}}$ are convex. Moreover, $A_{\mathcal{S}}$ is a subsystem of A , and $A_{\bar{\mathcal{S}}}$ is a factorsystem: $A_{\bar{\mathcal{S}}} \cong A/A_{\mathcal{S}}$. Denote by $\chi_{\mathcal{S},\mathrm{in}}$ and $\chi_{\bar{\mathcal{S}},\mathrm{out}}$ the injection and surjection homomorphisms

$$A_{\mathcal{S}} \xrightarrow{\chi_{\mathcal{S},\mathrm{in}}} A, \quad A \xrightarrow{\chi_{\bar{\mathcal{S}},\mathrm{out}}} A_{\bar{\mathcal{S}}} = A/A_{\mathcal{S}} . \quad (5.42)$$

Theorem 5.7.

- The homomorphisms $\chi_{\mathcal{S},\mathrm{in}}$ and $\chi_{\bar{\mathcal{S}},\mathrm{out}}$ are Postnikov system morphisms. If $A \simeq A'$ in PMF_W^n then $\chi_{\mathcal{S},\mathrm{in}} \simeq \chi'_{\mathcal{S},\mathrm{in}}$ and $\chi_{\bar{\mathcal{S}},\mathrm{out}} \simeq \chi'_{\bar{\mathcal{S}},\mathrm{out}}$.
- The morphisms χ_{in} and χ_{out} are natural, that is, for any Postnikov system morphism $A \xrightarrow{f} B$ the following diagrams are commutative:

$$\begin{array}{ccc} A_{\mathcal{S}} & \xrightarrow{\chi_{\mathcal{S},\mathrm{in}}} & A \\ f_{\mathcal{S}} \downarrow & & \downarrow f \\ B_{\mathcal{S}} & \xrightarrow{\chi_{\mathcal{S},\mathrm{in}}} & B \end{array} \quad \begin{array}{ccc} A & \xrightarrow{\chi_{\bar{\mathcal{S}},\mathrm{out}}} & A_{\bar{\mathcal{S}}} \\ f \downarrow & & \downarrow f_{\bar{\mathcal{S}}} \\ B & \xrightarrow{\chi_{\bar{\mathcal{S}},\mathrm{out}}} & B_{\bar{\mathcal{S}}} \end{array} \quad (5.43)$$

- If $\mathcal{S}', \bar{\mathcal{S}}' \subset \mathbb{Z}^n$ is another pair of complementary subsets satisfying the condition (5.41) and $\mathcal{S}' \subset \mathcal{S}$, then

$$\chi_{\bar{\mathcal{S}},\mathrm{out}} \chi_{\mathcal{S}',\mathrm{in}} = 0. \quad (5.44)$$

The natural morphisms (5.42) behave properly under the tensor product functor (5.9)

Theorem 5.8. *For two pairs of complementary subsets $\mathcal{S}, \bar{\mathcal{S}} \in \mathbb{Z}^n$, $\mathcal{S}', \bar{\mathcal{S}}' \in \mathbb{Z}^{n'}$*

$$\chi_{\mathcal{S}, \text{in}} \otimes \chi_{\mathcal{S}', \text{in}} = \chi_{\mathcal{S} \times \mathcal{S}', \text{in}}, \quad \chi_{\bar{\mathcal{S}}, \text{out}} \otimes \chi_{\bar{\mathcal{S}}', \text{out}} = \chi_{\bar{\mathcal{S}} \times \bar{\mathcal{S}}', \text{out}}. \quad (5.45)$$

Two types of pairs of complementary subsets $\mathcal{S}, \bar{\mathcal{S}} \subset \mathbb{Z}^n$ satisfying the condition (5.41) are particularly important. For $\mathbf{k} \in (\mathbb{Z}_\infty)^{\times n}$, where $\mathbb{Z}_\infty = \mathbb{Z} \cup \{-\infty, +\infty\}$ is an ordered set, define two subsets of \mathbb{Z}^n :

$$(\leq \mathbf{k}) = \{\mathbf{j} \in \mathbb{Z}^n \mid \mathbf{j} \leq \mathbf{k}\}, \quad (\geq \mathbf{k}) = \{\mathbf{i} \in \mathbb{Z}^n \mid \mathbf{i} \geq \mathbf{k}\}. \quad (5.46)$$

The condition (5.41) is satisfied by the pair $\mathcal{S} = (\geq \mathbf{k})$, $\bar{\mathcal{S}} = \mathbb{Z}^n \setminus (\geq \mathbf{k})$ and by the pair $\mathcal{S} = \mathbb{Z}^n \setminus (\leq \mathbf{k})$, $\bar{\mathcal{S}} = (\leq \mathbf{k})$. We call $A_{(\geq \mathbf{k})}$ a *corner subsystem* and we call $A_{(\leq \mathbf{k})}$ a *corner factorsystem*.

5.4. Graded Postnikov systems

A special feature of the matrix factorizations appearing in the categorification of the SO(2N + 2) Kauffman polynomial is their q -grading, for which

$$\deg_q W = 4N + 2, \quad \deg_q D = 2N + 1. \quad (5.47)$$

The primary homomorphisms appearing in convolutions producing the matrix factorizations $\hat{\gamma}$ and $\hat{\Gamma}$ (associated with an open graph γ and a closed graph Γ) will be saddle morphisms, so according to eq.(3.44),

$$\deg_q X_{\mathbf{e}_i} = 2N. \quad (5.48)$$

To guarantee that $\deg_q D_A = 2N + 1$, we introduce a (relative) q -degree shift for the constituent modules $A_{\mathbf{k}}\{|\mathbf{k}|\}$ and impose a condition on the q -degree of the secondary homomorphisms $X_{\mathbf{k}}$:

$$\deg_q X_{\mathbf{k}} = 2N - |\mathbf{k}| + 1. \quad (5.49)$$

As a result, the space that describes the dependence of the set of Postnikov systems on the choice of solutions $X_{\mathbf{l}, \mathbf{l}}$ to eq.(5.34) is the subspace of $\text{Ext}^1(A_{\mathbf{l}}, A_{\mathbf{l}'})$ of q -degree $2N - |\mathbf{l}' - \mathbf{l}| + 1$:

$$\text{Ext}_{\{2N - |\mathbf{l}' - \mathbf{l}| + 1\}}^1(A_{\mathbf{l}}, A_{\mathbf{l}'}). \quad (5.50)$$

5.5. Examples of Postnikov systems and convolutions

We will consider three simple examples of Postnikov systems of length 0, 1 and 2 with $n = 1$.

A Postnikov system A of length 0 is just an ordinary matrix factorization, and its convolution is equal to that matrix factorization:

$$\boxed{A} = A. \quad (5.51)$$

Consider a morphism

$$A \xrightarrow{F} B, \quad \deg_{\mathbb{Z}_2} F = 0. \quad (5.52)$$

In order to construct a Postnikov system out of it, we shift the \mathbb{Z}_2 -degree of A :

$$\boxed{A \langle 1 \rangle \xrightarrow{F} B}. \quad (5.53)$$

Its convolution is isomorphic to the cone of the morphism (5.52):

$$\boxed{A \langle 1 \rangle \xrightarrow{F} B} = \text{Cone}_{\text{MF}}(F), \quad (5.54)$$

and the convolutions of natural morphisms

$$B \xrightarrow{\chi_{\text{in}}} \boxed{A \langle 1 \rangle \xrightarrow{F} B} \xrightarrow{\chi_{\text{out}}} A \langle 1 \rangle, \quad \chi_{\text{in}} = \begin{pmatrix} 0 \\ 1_B \end{pmatrix}, \quad \chi_{\text{out}} = \begin{pmatrix} 1_{A \langle 1 \rangle} & 0 \end{pmatrix} \quad (5.55)$$

form two sides of the exact triangle related to (5.52).

Now consider a chain of two morphisms

$$A \xrightarrow{F} B \xrightarrow{G} C, \quad \deg_{\mathbb{Z}_2} F = \deg_{\mathbb{Z}_2} G = 0, \quad GF \simeq 0. \quad (5.56)$$

The latter condition implies that there exists a homomorphism $X \in \text{Hom}_{\mathbb{R}}(A, C)$ such that

$$\deg_{\mathbb{Z}_2} X = 1, \quad GF = -\{D_{A,C}, X\}, \quad (5.57)$$

where $D_{A,C} = D_A + D_C$. If we shift the \mathbb{Z}_2 -degree of B , then we can form a Postnikov system based on the chain (5.56)

$$\boxed{A \xrightarrow{F} B \langle 1 \rangle \xrightarrow{G} C} \quad (5.58)$$

Its module is a sum of modules

$$A \oplus B \langle 1 \rangle \oplus C \tag{5.59}$$

and its twisted differential is a sum of the matrix factorization twisted differentials and of the homomorphisms F , G and X :

$$D = D_A - D_B + D_C + F + G + X. \tag{5.60}$$

The effect of the choice of X satisfying eq.(5.57) on the class of this Postnikov system is (relatively) parametrized by the elements of $\text{Ext}^1(A, C)$, so if

$$\dim \text{Ext}^1(A, C) = 0, \tag{5.61}$$

then the Postnikov system (5.58) is determined by the complex (5.56) uniquely.

The convolutions of two natural morphisms

$$C \xrightarrow{\chi_{\text{in}}} \boxed{A \xrightarrow{F} B \langle 1 \rangle \xrightarrow{G} C} \xrightarrow{\chi_{\text{out}}} A, \quad \chi_{\text{in}} = \begin{pmatrix} 0 \\ 0 \\ 1_C \end{pmatrix}, \quad \chi_{\text{out}} = \begin{pmatrix} 1_A & 0 & 0 \end{pmatrix} \tag{5.62}$$

form a chain complex:

$$\chi_{\text{out}} \chi_{\text{in}} = 0. \tag{5.63}$$

6. Categorification of the 4-vertex, graphs and tangles

6.1. A convolution of the chain of saddle morphisms

We are going to build a Postnikov system upon the chain complex (4.18), following the steps outlined in subsection 5.5. According to eq.(3.42), the subspace $\text{Ext}^1(\widehat{\)}, \widehat{\)}$ is trivial, hence the Postnikov system is determined uniquely. The secondary homomorphism X of (5.57) is provided by eqs.(4.30), (4.27). We denote the resulting Postnikov system as

$$\widehat{\)} = \boxed{\widehat{\)} \langle -1 \rangle \xrightarrow{F} \widehat{\)} \langle 1 \rangle \xrightarrow{G} \widehat{\)} \langle 1 \rangle} \tag{6.1}$$

The q -grading shifts in the first and the third modules are required in order to ensure that the twisted differential

$$D_{\text{conv}} = D_{\widehat{\)} \langle -1 \rangle} - D_{\widehat{\)} \langle 1 \rangle} + D_{\widehat{\)} \langle 1 \rangle} + F + G + X \tag{6.2}$$

has a homogeneous degree in accordance with eq.(3.4). For the same reason the homomorphism X must have a degree prescribed by eq.(5.49), which is that of eq.(4.21). Two markers \cdot in the notation $\widetilde{\times}$ are needed, because this Postnikov system is not invariant under the 90° rotation.

The uniqueness of the saddle morphism up to a constant factor and the fact that saddle morphisms determine the Postnikov system (6.1) uniquely allows us to define similar Postnikov systems for any triplet of distinct 2-arc graphs:

$$\boxed{\begin{array}{ccc} \widehat{\gamma} \{-1\} & \xrightarrow{F} & \widehat{\gamma}' \langle 1 \rangle & \xrightarrow{G} & \widehat{\gamma}'' \{1\} \\ & \dashrightarrow & X & \dashrightarrow & \end{array}}, \quad \gamma \neq \gamma' \neq \gamma''. \quad (6.3)$$

In particular,

$$\widetilde{\times} = \boxed{\begin{array}{ccc} \widehat{\cdot} & \xrightarrow{F'} & \widehat{\times} & \xrightarrow{G'} & \widehat{\cdot} \\ \cdot & \dashrightarrow & X' & \dashrightarrow & \cdot \end{array}}, \quad (6.4)$$

where F' and G' are corresponding saddle morphisms.

Theorem 6.1. *The convolution of the Postnikov system (6.1) is invariant under the 90° rotation, that is*

$$\boxed{\begin{array}{ccc} \widehat{\cdot} & \xrightarrow{F} & \widehat{\times} & \xrightarrow{G} & \widehat{\cdot} \\ \cdot & \dashrightarrow & X & \dashrightarrow & \cdot \end{array}} \simeq \boxed{\begin{array}{ccc} \widehat{\cdot} & \xrightarrow{F'} & \widehat{\times} & \xrightarrow{G'} & \widehat{\cdot} \\ \cdot & \dashrightarrow & X' & \dashrightarrow & \cdot \end{array}}. \quad (6.5)$$

Proof. The Postnikov system (6.1) factors similarly to matrix factorizations (3.45):

$$\widetilde{\times} = K_{\text{cmn}} \otimes_{\mathbb{R}} \widetilde{\times}_{\mathbb{P}}, \quad \text{where} \quad \widetilde{\times}_{\mathbb{P}} = \boxed{\begin{array}{ccc} \widehat{\cdot}_{\mathbb{P}} & \xrightarrow{F_{\mathbb{P}}} & \widehat{\times}_{\mathbb{P}} & \xrightarrow{G_{\mathbb{P}}} & \widehat{\cdot}_{\mathbb{P}} \\ \cdot_{\mathbb{P}} & \dashrightarrow & X_{\mathbb{P}} & \dashrightarrow & \cdot_{\mathbb{P}} \end{array}}. \quad (6.6)$$

Therefore, it suffices to prove the rotation invariance for the proper convolutions

$$\text{Conv}_{\text{MF}} \left(\widetilde{\times}_{\mathbb{P}} \right) \simeq \text{Conv}_{\text{MF}} \left(\widetilde{\cdot}_{\mathbb{P}} \right). \quad (6.7)$$

The convolution

$$\boxed{\begin{array}{ccc} \widehat{\cdot}_{\mathbb{P}} & \xrightarrow{F_{\mathbb{P}}} & \widehat{\times}_{\mathbb{P}} & \xrightarrow{G_{\mathbb{P}}} & \widehat{\cdot}_{\mathbb{P}} \\ \cdot_{\mathbb{P}} & \dashrightarrow & X_{\mathbb{P}} & \dashrightarrow & \cdot_{\mathbb{P}} \end{array}} \quad (6.8)$$

SO(2N) categorification

in the *l.h.s.* of eq.(6.7) has an explicit presentation

$$\mathbb{R}_1^6 \xrightarrow{P_{\text{conv}}} \mathbb{R}_0^6 \xrightarrow{Q_{\text{conv}}} \mathbb{R}_1^6, \quad (6.9)$$

where the homomorphisms P_{conv} and Q_{conv} are presented by the matrices

$$P_{\text{conv}} = \begin{pmatrix} P_{\langle} & 0 & 0 \\ F_1 & -Q_{\times} & 0 \\ X_1 & G_0 & P_{\smile} \end{pmatrix}, \quad Q_{\text{conv}} = \begin{pmatrix} Q_{\langle} & 0 & 0 \\ F_0 & -P_{\times} & 0 \\ X_0 & G_1 & Q_{\smile} \end{pmatrix} \quad (6.10)$$

in the basis corresponding to the splitting

$$\mathbb{R}_0^6 = \mathbb{R}_{\langle,0}^2 \oplus \mathbb{R}_{\times,1}^2 \oplus \mathbb{R}_{\smile,0}^2, \quad \mathbb{R}_1^6 = \mathbb{R}_{\langle,1}^2 \oplus \mathbb{R}_{\times,0}^2 \oplus \mathbb{R}_{\smile,1}^2. \quad (6.11)$$

The diagonal entries of the matrices (6.10) are given by formulas (4.2)–(4.4). We simplify this presentation (6.10) by using the fact that the matrices F_0 and G_0 have unit entries. Consider the isomorphism of two matrix factorizations (the bottom one being (6.9)) established by the commutative diagram

$$\begin{array}{ccccc} \mathbb{R}_1 \oplus \mathbb{R}_1 \oplus \mathbb{R}_1^4 & \xrightarrow{W_{4,p} \oplus 1 \oplus P_{r,\text{conv}}} & \mathbb{R}_0 \oplus \mathbb{R}_0 \oplus \mathbb{R}_0^4 & \xrightarrow{1 \oplus W_{4,p} \oplus Q_{r,\text{conv}}} & \mathbb{R}_1 \oplus \mathbb{R}_1 \oplus \mathbb{R}_1^4 & (6.12) \\ \downarrow f_{\simeq,1} & & \downarrow f_{\simeq,0} & & \downarrow f_{\simeq,1} \\ \mathbb{R}_1^6 & \xrightarrow{P_{\text{conv}}} & \mathbb{R}_0^6 & \xrightarrow{Q_{\text{conv}}} & \mathbb{R}_1^6 \end{array}$$

where

$$P_{r,\text{conv}} = \begin{pmatrix} p_2 & p_1 & 0 & 0 \\ q_2 & -q_1 & q_2 r_1 & q_1 r_1 \\ q_1 C & -q_2 & -q_2 r_2 & -q_1 r_2 \\ p_1 C & p_2 & p_2 r_2 + p_1 r_1 C & -(p_1 r_2 + p_2 r_1) \end{pmatrix}, \quad (6.13)$$

$$Q_{r,\text{conv}} = \begin{pmatrix} q_1 r_2 + q_2 r_1 & p_1 r_2 & p_1 r_1 & 0 \\ q_2 r_2 + q_1 r_1 C & -p_2 r_2 & -p_2 r_1 & 0 \\ -q_2 & p_2 & -p_1 & q_1 \\ q_1 C & p_1 C & -p_2 & -q_2 \end{pmatrix} \quad (6.14)$$

and

$$f_{\simeq,0} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & r_2 & r_1 & 0 \\ 0 & -r_1 & 0 & 1 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad f_{\simeq,1} = \begin{pmatrix} p_1 & 0 & 1 & 0 & 0 & 0 \\ -p_2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -q_2 & -q_1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (6.15)$$

It is easy to verify that the latter matrices are invertible and hence define an isomorphism f_{\simeq} of two matrix factorizations. The top matrix factorization decomposes into a sum of three matrix factorizations, the first two being

$$\mathbf{R}_1 \xrightarrow{W_{4,p}} \mathbf{R}_0 \xrightarrow{1} \mathbf{R}_1, \quad \mathbf{R}_1 \xrightarrow{1} \mathbf{R}_0 \xrightarrow{W_{4,p}} \mathbf{R}_1 \quad (6.16)$$

and the third (reduced) one being a rank-(4,4) matrix factorization

$$\mathbf{R}_1^4 \xrightarrow{P_{r,\text{conv}}} \mathbf{R}_0^4 \xrightarrow{Q_{r,\text{conv}}} \mathbf{R}_1^4. \quad (6.17)$$

The matrix factorizations (6.16) are contractible, hence the *l.h.s.* of eq.(6.7) is homotopy equivalent to (6.17).

We construct the presentation of the *r.h.s.* convolution of eq.(6.7) by using the endofunctor action of the symmetric group S_3 . Namely, $\sigma_{23} \in S_3$ transforms the triplet of graphs

$$\sigma_{23} \left(\begin{array}{c} \text{) } \\ \text{ (, } \otimes \text{ , } \text{) } \end{array} \right) = \left(\begin{array}{c} \text{) } \\ \text{ (, } \otimes \text{ , } \text{) } \end{array} \right). \quad (6.18)$$

Therefore

$$\hat{\sigma}_{23} \left(\begin{array}{c} \widetilde{\text{)}} \\ \widetilde{\text{ (, } \otimes \text{ , } \text{)}} \\ \mathbf{p} \end{array} \right) = \begin{array}{c} \widetilde{\text{)}} \\ \widetilde{\text{ (, } \otimes \text{ , } \text{)}} \\ \mathbf{p} \end{array}. \quad (6.19)$$

The functor $\hat{\sigma}_{23}$ acts by switching the variables \mathbf{p} and \mathbf{q} , while leaving \mathbf{r} intact, so the *r.h.s.* of eq.(6.7) can be presented similarly to (6.17) as

$$\mathbf{R}_1^4 \xrightarrow{P'_{r,\text{conv}}} \mathbf{R}_0^4 \xrightarrow{Q'_{r,\text{conv}}} \mathbf{R}_1^4, \quad (6.20)$$

where

$$P'_{r,\text{conv}} = \begin{pmatrix} q_2 & q_1 & 0 & 0 \\ p_2 & -p_1 & p_2 r_1 & p_1 r_1 \\ p_1 C & -p_2 & -p_2 r_2 & -p_1 r_2 \\ q_1 C & q_2 & q_2 r_2 + q_1 r_1 C & -q_1 r_2 - q_2 r_1 \end{pmatrix}, \quad (6.21)$$

$$Q'_{r,\text{conv}} = \begin{pmatrix} p_1 r_2 + p_2 r_1 & q_1 r_2 & q_1 r_1 & 0 \\ p_2 r_2 + p_1 r_1 C & -q_2 r_2 & -q_2 r_1 & 0 \\ -p_2 & q_2 & -q_1 & p_1 \\ p_1 C & q_1 C & -q_2 & -p_2 \end{pmatrix}. \quad (6.22)$$

Now the isomorphism (6.7) is established by the commutative diagram

$$\begin{array}{ccccc} \mathbb{R}_1^4 & \xrightarrow{P_{r,\text{conv}}} & \mathbb{R}_0^4 & \xrightarrow{Q_{r,\text{conv}}} & \mathbb{R}_1^4 \\ \downarrow g_{\cong,1} & & \downarrow g_{\cong,0} & & \downarrow g_{\cong,1} \\ \mathbb{R}_1^4 & \xrightarrow{P'_{r,\text{conv}}} & \mathbb{R}_0^4 & \xrightarrow{Q'_{r,\text{conv}}} & \mathbb{R}_1^4 \end{array} \quad (6.23)$$

in which the isomorphism matrices $g_{\cong,0}$, $g_{\cong,1}$ are

$$g_{\cong,0} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad g_{\cong,1} = \begin{pmatrix} 1 & 0 & r_1 & 0 \\ 0 & -1 & 0 & r_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (6.24)$$

□

6.2. The 4-vertex matrix factorization and the crossing complex

Now that we have proved the isomorphism (6.5), we choose the matrix factorization associated with the 4-vertex graph to be the convolution (6.5):

$$\widehat{\text{X}} = \text{Conv}_{\text{MF}} \left(\widehat{\text{X}} \right) = \boxed{\widehat{\text{X}} \langle \{-1\} \xrightarrow{F} \widehat{\text{X}} \langle 1 \rangle \xrightarrow{G} \widehat{\text{X}} \langle 1 \rangle} \quad (6.25)$$

or, equivalently,

$$\widehat{\times} = \text{Conv}_{\text{MF}} \left(\widetilde{\times} \right) = \boxed{\begin{array}{c} \widehat{\smile} \{ -1 \} \xrightarrow{F'} \widehat{\times} \langle 1 \rangle \xrightarrow{G'} \widehat{\frown} \{ 1 \} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \bar{X}' \end{array}}. \quad (6.26)$$

This matrix factorization has a 90° -rotation symmetry as the picture of the graph \times suggests.

The Postnikov system (6.1) has matrix factorizations $\widehat{\smile} \{ -1 \}$ and $\widehat{\frown} \{ 1 \}$ as its left and right corner subsystems, hence they are connected to it by natural morphisms of Postnikov systems

$$\widehat{\frown} \{ 1 \} \xrightarrow{\chi_{\text{in}}} \widehat{\smile} \{ -1 \} \xrightarrow{F} \widehat{\times} \langle 1 \rangle \xrightarrow{G} \widehat{\smile} \{ 1 \} \xrightarrow{\chi_{\text{out}}} \widehat{\frown} \{ -1 \}, \quad (6.27)$$

$$\chi_{\text{in}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \chi_{\text{out}} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad (6.28)$$

$$\deg_{\mathbb{Z}_2} \chi_{\text{in}} = \deg_{\mathbb{Z}_2} \chi_{\text{out}} = 0, \quad \deg_q \chi_{\text{in}} = \deg_q \chi_{\text{out}} = 0, \quad \chi_{\text{out}} \chi_{\text{in}} = 0. \quad (6.29)$$

The latter relation allows us to interpret the diagram (6.27) as a chain complex in the homotopy category of Postnikov system complexes $\mathbf{K}(\text{PMF}_{W_4}^1)$ and define

$$\begin{aligned} \widetilde{\times} &= \left(\widehat{\smile} \{ 1 \} \xrightarrow{\chi_{\text{in}}} \widetilde{\times} \xrightarrow{\chi_{\text{out}}} \widehat{\frown} \{ -1 \} \right) \langle 1 \rangle \\ &= \left(\widehat{\smile} \{ 1 \} \xrightarrow{\chi_{\text{in}}} \widehat{\smile} \{ -1 \} \xrightarrow{F} \widehat{\times} \langle 1 \rangle \xrightarrow{G} \widehat{\smile} \{ 1 \} \xrightarrow{\chi_{\text{out}}} \widehat{\frown} \{ -1 \} \right) \langle 1 \rangle. \end{aligned} \quad (6.30)$$

Now we define the complex of matrix factorizations $\widehat{\times}$ for the elementary crossing as the convolution functor (5.11) applied to the *r.h.s.* of eq.(6.30):

$$\begin{aligned} \widehat{\times} &= \text{Conv}_{\text{MF}} \left(\widetilde{\times} \right) = \text{Conv}_{\text{MF}} \left(\widehat{\smile} \{1\} \xrightarrow{X_{\text{in}}} \widetilde{\times} \xrightarrow{X_{\text{out}}} \widehat{\frown} \{-1\} \right) \langle 1 \rangle \\ &= \left(\widehat{\smile} \{1\} \xrightarrow{X_{\text{in}}} \boxed{\widehat{\smile} \{-1\} \xrightarrow{F} \widehat{\times} \langle 1 \rangle \xrightarrow{G} \widehat{\frown} \{1\}} \xrightarrow{X_{\text{out}}} \widehat{\frown} \{-1\} \right) \langle 1 \rangle. \end{aligned} \quad (6.31)$$

This definition coincides with eq.(1.16) and the resulting complex should be considered as an object in the homotopy category of matrix factorization complexes $\mathbf{K}(\text{MF}_{W_4})$.

The rotational invariance of $\widetilde{\times}$ allows us to present a 90° rotation version of eq.(6.31) as

$$\begin{aligned} \widehat{\times} &= \text{Conv}_{\text{MF}} \left(\widetilde{\times} \right) = \text{Conv}_{\text{MF}} \left(\widehat{\smile} \{1\} \xrightarrow{X_{\text{in}}} \widetilde{\times} \xrightarrow{X_{\text{out}}} \widehat{\frown} \{-1\} \right) \\ &= \left(\widehat{\smile} \{1\} \xrightarrow{X_{\text{in}}} \boxed{\widehat{\smile} \{-1\} \xrightarrow{F'} \widehat{\times} \langle 1 \rangle \xrightarrow{G'} \widehat{\frown} \{1\}} \xrightarrow{X_{\text{out}}} \widehat{\frown} \{-1\} \right) \langle 1 \rangle. \end{aligned} \quad (6.32)$$

6.3. A categorification complex

6.3.1. A matrix factorization of an open graph as a convolution

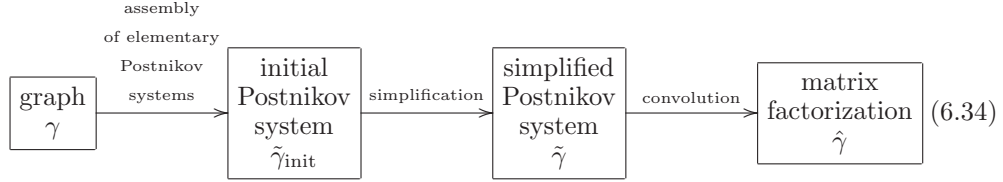
Let γ be an open graph with m 4-vertices. Its matrix factorization $\hat{\gamma}$ is constructed according to the standard procedure outlined in subsection 1.3: we cut all edges of γ connecting 4-vertices, so that γ splits into elementary pieces: circles, arcs and elementary 4-vertex graphs. To each piece we associate its matrix factorization according to eqs.(4.50), (3.20) and either eq.(6.25) or eq.(6.26). Then we join the pieces back together with the help of the operations (2.73) and (2.74).

The structure of $\hat{\gamma}$ can be described more precisely if this matrix factorization is constructed in a slightly different way. The convolution functor commutes with the gluing operations (2.73) and (2.74). Hence, instead of assembling the convolutions (6.25) and (6.26) we can assemble the original Postnikov systems (6.1) and (6.4). The assembly

of elementary pieces into the graph γ results in an m -dimensional Postnikov system $\tilde{\gamma}_{\text{init}}$, which will be simplified by homotopy equivalence transformations within the category of Postnikov systems, thus yielding $\tilde{\gamma}$. Finally we will apply the convolution functor:

$$\hat{\gamma} = \text{Conv}_{\text{MF}}(\tilde{\gamma}). \tag{6.33}$$

This procedure can be presented schematically by the diagram:



For $\mathbf{k} \in \mathbb{Z}^m$ ($-1 \leq k_i \leq 1$) let $\gamma_{\mathbf{k}}$ be an open graph constructed from γ by replacing each 4-vertex $\begin{matrix} \diagup & \diagdown \\ \diagdown & \diagup \end{matrix}$ ($1 \leq i \leq m$) with either $\begin{matrix}) & (\\ & \end{matrix}$, or $\begin{matrix} \diagdown & \diagup \\ \diagup & \diagdown \end{matrix}$, or $\begin{matrix} \diagup & \diagdown \\ \diagup & \diagdown \end{matrix}$ in accordance with the value of k_i (-1 , or 0 , or 1) and the choice (6.25) or (6.26) for the presentation of that 4-vertex. Initially, the constituent matrix factorizations of the m -dimensional Postnikov system $\tilde{\gamma}_{\text{init}}$ are matrix factorizations $\hat{\gamma}_{\text{init},\mathbf{k}}$, constructed by applying the assembly procedure to the 2-arc matrix factorizations $\begin{matrix}) & (\\ & \end{matrix}$, $\begin{matrix} \diagdown & \diagup \\ \diagup & \diagdown \end{matrix}$ and $\begin{matrix} \diagup & \diagdown \\ \diagup & \diagdown \end{matrix}$, which combines their graphs together into $\gamma_{\mathbf{k}}$. The homomorphisms between $\hat{\gamma}_{\text{init},\mathbf{k}}$ within $\tilde{\gamma}_{\text{init}}$ are exactly the homomorphisms of the Postnikov systems (6.1) and (6.3) (up to appropriate negative signs related to the \mathbb{Z}_2 -grading) applied to the corresponding factors of the tensor products which form $\hat{\gamma}_{\text{init},\mathbf{k}}$. In particular, the primary morphisms of $\tilde{\gamma}_{\text{init}}$ are saddle morphisms reconnecting the 2-arc subgraphs sitting inside the graphs $\gamma_{\mathbf{k}}$.

The graphs $\gamma_{\mathbf{k}}$ have a simple structure: they are disjoint unions of circles and arcs. Therefore each constituent matrix factorization $\hat{\gamma}_{\text{init},\mathbf{k}}$ is homotopy equivalent to the matrix factorization $\hat{\gamma}_{\mathbf{k}}$ which, according to our rules, is just a tensor product of circle spaces $C_{\text{unkn}} = J'_W$ and 1-arc matrix factorizations of the type (3.20). If we replace $\hat{\gamma}_{\text{init},\mathbf{k}}$ with $\hat{\gamma}_{\mathbf{k}}$, then the primary homomorphisms between $\hat{\gamma}_{\mathbf{k}}$ will be homotopy equivalent to saddle morphisms, semi-closed saddle morphisms (4.33) and closed saddle morphisms (4.52), applied to the matrix factorizations, corresponding to circle and 1-arc graphs, which are the connected components of the graphs $\gamma_{\mathbf{k}}$. Hence, according to theorems 5.2 and 5.3, the system $\tilde{\gamma}_{\text{init}}$ is homotopy equivalent in the category of Postnikov systems to a *simplified* system $\tilde{\gamma}$, whose constituent matrix factorizations are $\hat{\gamma}_{\mathbf{k}}$ and whose primary homomorphisms are saddle morphisms as well as their semi-closed and closed versions, acting on the matrix factorizations of connected components of the graphs $\gamma_{\mathbf{k}}$.

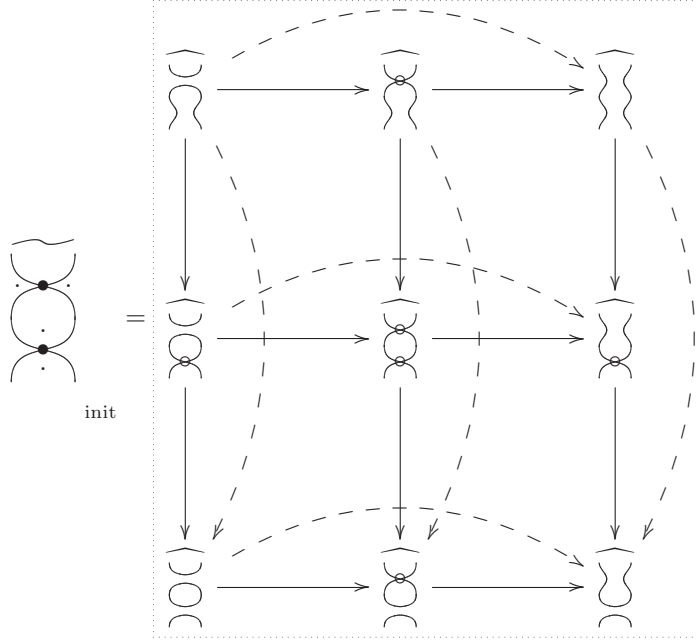



FIGURE 3. An example of an initial Postnikov system

The secondary homomorphisms of the Postnikov system $\tilde{\gamma}_{\text{init}}$ originate from the homomorphisms X of (6.1) and (6.3), hence they all have length 2. However, the homotopy equivalence transformation from $\tilde{\gamma}_{\text{init}}$ to $\tilde{\gamma}$ may generate secondary homomorphisms of higher length. The only restriction that we can impose on their structure so far follows from Theorem 3.1, according to which the spaces $\text{Ext}^1(A_1, A_{1'})$ are trivial for secondary homomorphisms of even length, so the homotopy equivalence class of $\tilde{\gamma}$ does not depend on the choice of a particular solution of eq.(5.34).

Thus we construct the graph matrix factorization $\hat{\gamma}$ by first reducing the Postnikov system $\tilde{\gamma}_{\text{init}}$ to $\tilde{\gamma}$ and then applying the convolution functor $\hat{\gamma} = \text{Conv}_{\text{MF}}(\tilde{\gamma})$, which just turns the dotted frame boxes into solid ones.

As an example of the simplification procedure, consider the initial Postnikov system for the graph  as depicted in Fig. 3 (this graph will appear in the proof of the second Reidemeister move invariance). There the solid arrows denote the saddle morphisms, while the dashed arrows denote the secondary morphisms X of (6.1). The horizontal arrow

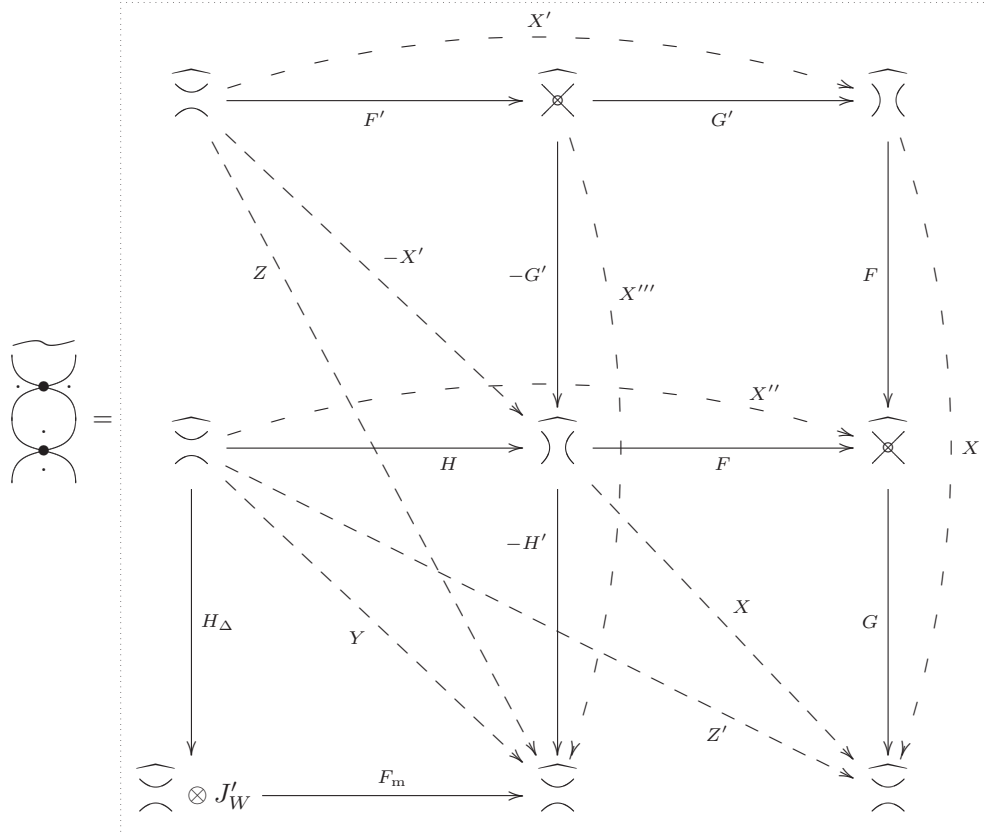


FIGURE 4. An example of a simplified Postnikov system

morphisms act on matrix factorizations coming from the top 4-vertex, while vertical ones act on the bottom matrix factorizations. Note that we omitted the degree shifts $\langle |\mathbf{k}| \rangle \{ |\mathbf{k}| \}$, which should accompany the constituent matrix factorizations $\hat{\gamma}_{\mathbf{k}}$ in the Postnikov system of Fig. 3.

The simplification of the constituent matrix factorizations and primary (that is, length 1) morphisms leads to the homotopy equivalent Postnikov system in Fig. 4. In that diagram we omitted two semi-closed saddle homomorphisms which are equal to zero (see eq. (4.40)). The homomorphisms F, F', G, G', H, H' are saddle morphisms related by the action of the symmetry group S_4 permuting the legs. X is the secondary homomorphism (4.27) and the homomorphisms X', X'' and X''' are related to it by the leg

permutation symmetry action. Y , Z and Z' are other secondary homomorphisms (we will not attempt to determine them).

The origin of the simplified constituent matrix factorizations and primary homomorphisms in the diagram of Fig. 4 is obvious. Let us explain the structure of the secondary homomorphisms. We follow the step-by-step procedure outlined after the Corollary 5.4. Namely, for any given $k \geq 2$ we first choose the homomorphisms of length k which satisfy the conditions (5.34) and then add to them the representatives of the spaces (5.50).

- ($k = 2$) We included all homomorphisms that might be needed to satisfy the conditions (5.34). The spaces $\text{Ext}^1(A_1, A_{1'})$ are trivial, so we do not have to add any extra homomorphisms. In particular, we set the homomorphism $\widehat{\text{X}} \longrightarrow \widehat{\text{X}}$ from $A_{0,-1}$ to $A_{1,0}$ equal to zero.
- ($k = 3$) We marked by Z and Z' the homomorphisms required to satisfy conditions (5.34). As for the spaces (5.50), it follows easily from eq.(3.35) that they are non-trivial only for two homomorphisms that have already been marked, hence we set homomorphisms $A_{-1,-1} \rightarrow A_{1,0}$ and $A_{0,-1} \rightarrow A_{1,1}$ equal to zero.
- ($k = 4$) We set the only homomorphism to zero. This choice satisfies eq.(5.34), and the corresponding space (5.50) is trivial

6.3.2. A complex of a tangle

Let τ be a tangle with n crossings. Its categorification complex $\hat{\tau}$ is a complex of matrix factorizations constructed by assembling the elementary crossing complexes (6.30). For $\mathbf{r} \in \mathbb{Z}^n$ ($-1 \leq r_i \leq 1$) let $\gamma_{\mathbf{r}}$ be an open graph constructed from τ by replacing each crossing X ($1 \leq i \leq n$) with either Y , or X , or Z (in accordance with the value of r_i (-1 , or 0 , or 1)). Then the complex $\hat{\tau}$ is formed by the matrix factorizations $\hat{\gamma}_{\mathbf{r}}$, each placed at the homological degree $|\mathbf{r}|$, the differential being the sum of natural morphisms χ_{in} and χ_{out} of (6.31).

We are going to construct a simplified homotopy equivalent version of the complex $\hat{\tau}$ by first constructing the complex of Postnikov systems, then simplifying them as in the previous subsection and finally applying the convolution functor to each Postnikov system forming the complex. Thus we start by applying the assembly process not to the matrix factorizations X but to their underlying Postnikov system complexes X of eq.(6.30). The result is a complex $\tilde{\tau}_{\text{init}}$ consisting of Postnikov systems $\tilde{\gamma}_{\text{init},\mathbf{r}}$, its differential being the sum of natural morphisms χ_{in} and χ_{out} .

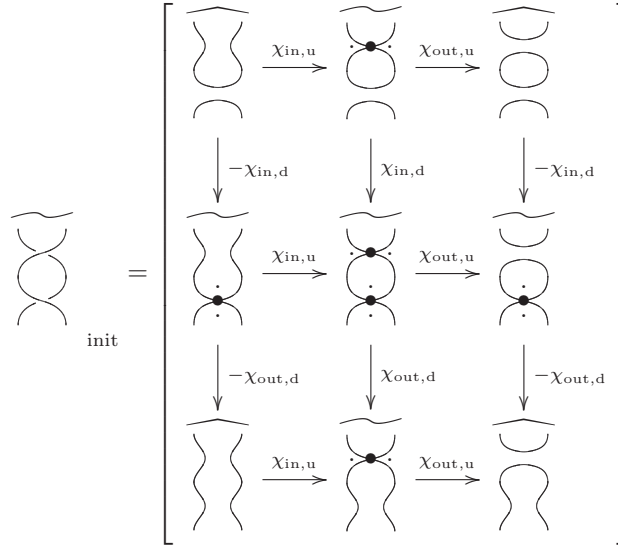


FIGURE 5. Initial complex for the tangle diagram in the second Reidemeister move

Since the left and the right Postnikov systems (consisting of a single constituent matrix factorization) in the complex (6.30) are convex cuts of the middle Postnikov system (the left one is a corner factorsystem and the right one is a corner subsystem), then all Postnikov systems $\tilde{\gamma}_{\text{init},\mathbf{r}}$ are convex cuts of the middle system $\tilde{\gamma}_{\text{init},\mathbf{0}}$. Let $\tilde{\gamma}_{\mathbf{0}}$ be a homotopy equivalent simplification of $\tilde{\gamma}_{\text{init},\mathbf{0}}$. Then its convex cuts $\tilde{\gamma}_{\mathbf{r}}$ are the simplification of $\tilde{\gamma}_{\text{init},\mathbf{r}}$, and all natural morphisms between them remain intact. Thus we obtain a complex of Postnikov systems $\tilde{\tau}$, which is a homotopy equivalent simplified version of $\tilde{\tau}_{\text{init}}$: its Postnikov systems are $\tilde{\gamma}_{\mathbf{r}}$ and its differential is still the sum of natural morphisms χ_{in} and χ_{out} . Finally, we apply the convolution functor: $\hat{\tau} = \text{Conv}_{\text{MF}}(\tilde{\tau})$.

As an example, let us consider the tangle $\begin{array}{c} \cup \\ \cap \end{array}$ appearing in the second Reidemeister move. The initial form of its Postnikov systems complex is depicted in Fig. 5. A more detailed version of the same complex, in which we substituted the Postnikov systems (6.1) for the 4-vertices, is presented in Fig. 6. There the horizontal morphisms χ_{in} and χ_{out} are related to the top crossing, while the vertical ones are related to the bottom crossing.

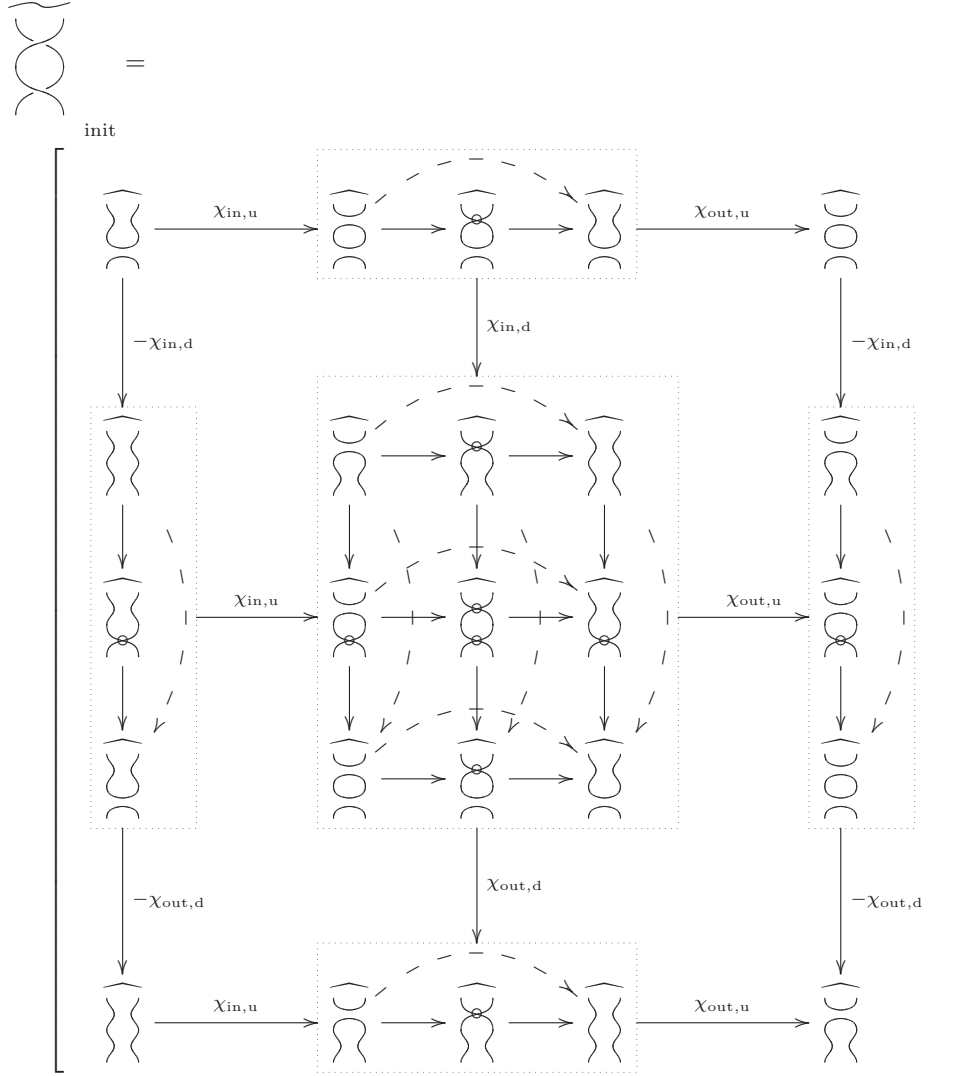


FIGURE 6. Detailed structure of the initial complex for the tangle diagram in the second Reidemeister move

In order to simplify the complex of Fig. 6, first, we simplify the central Postnikov system to the form of Fig. 4. Then the other Postnikov systems are simplified as its sub-

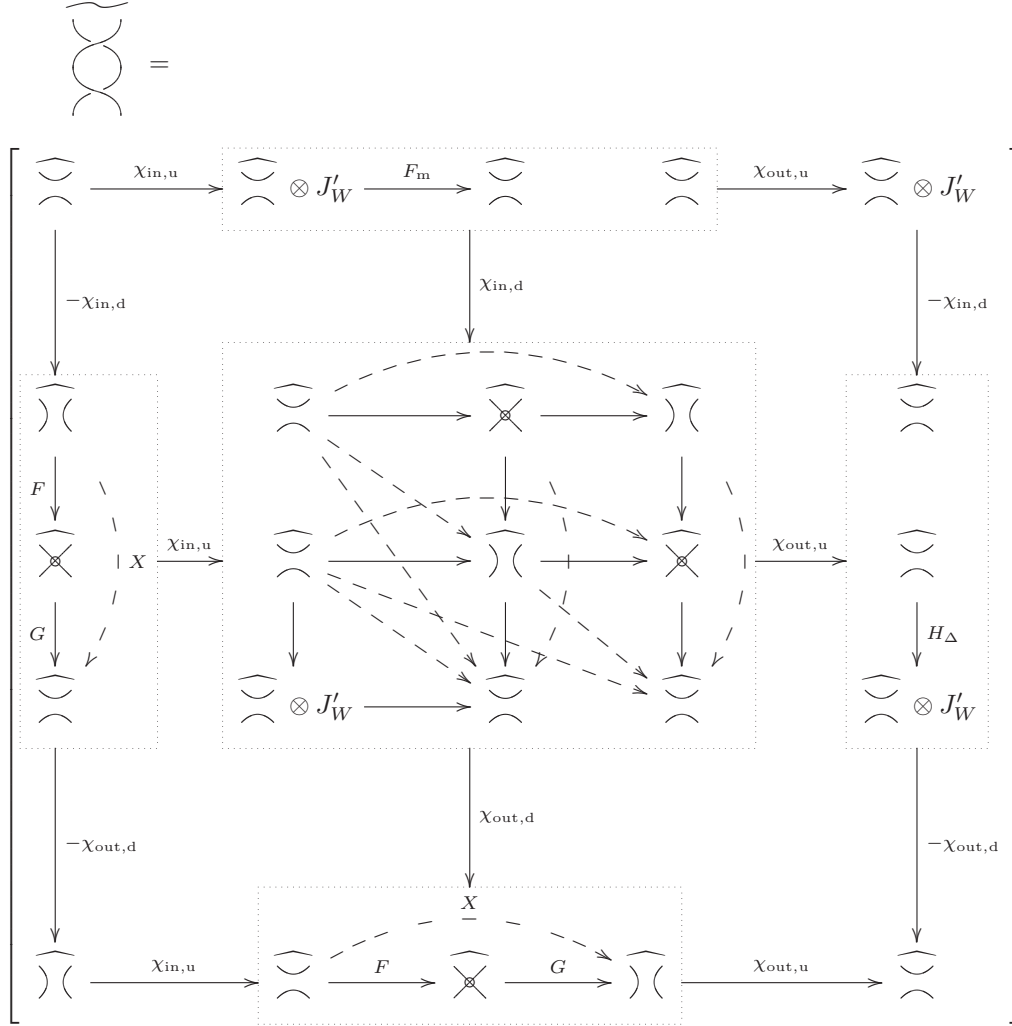


FIGURE 7. Simplified complex of the tangle diagram in the second Reidemeister move

and factorsystems. The result is the complex of Fig. 7 in which the differential is still a sum of natural morphisms relating a Postnikov system to its corner subsystems and factorsystems.

6.3.3. A complex of a link

A link \mathcal{L} is a tangle without legs. Let $\Gamma_{\mathbf{r}}$ be the closed graphs constructed by replacing the crossings of \mathcal{L} with 2-arc or 4-vertex elementary graphs according to the values of r_i . Further, let $\Gamma_{\mathbf{r};\mathbf{k}}$ be the closed graphs, constructed by replacing the 4-vertices of $\Gamma_{\mathbf{r}}$ with 2-arc graphs according to the values of k_i . Since a graph $\Gamma_{\mathbf{r};\mathbf{k}}$ is closed, it is a disjoint union of $n_{\mathbf{k}}$ circles. Hence the spaces $\hat{\Gamma}_{\mathbf{r};\mathbf{k}}$ are tensor powers of the unknot spaces $(J'_W)^{\otimes n_{\mathbf{k}}}$, and the primary morphisms of the simplified Postnikov system $\tilde{\Gamma}_{\mathbf{r}}$ are closed saddle morphisms of (4.52), which are multiplications m , if two circles coalesce into one, co-multiplications Δ , if one circle splits into two, or zero if one circle reconnects into another circle.

Since the twisted differentials in the constituent matrix factorizations $(J'_W)^{\otimes n_{\mathbf{k}}}$ are zero, particular solutions to the equations (5.34) may be chosen to be zero. Therefore the presence of secondary homomorphisms in a simplified Postnikov system $\tilde{\Gamma}_{\mathbf{r}}$ is due exclusively to non-trivial spaces $\text{Ext}^1(A_I, A_{I'})$. This means that $\tilde{\Gamma}_{\mathbf{r}}$ contains secondary morphisms of only odd length.

As an example of a link complex, let us consider the simplified version of the Hopf link complex in Fig. 2 (see Section 1; note that we have omitted the secondary homomorphisms in that diagram). All constituent graphs of its convolutions are either single or double circles, so the corresponding simplified matrix factorizations are either J'_W or $J'_W \otimes J'_W$. Now we choose the primary and secondary homomorphisms of the Postnikov system underlying the middle convolution, while the differentials of other convolutions are determined by the fact that they are convolutions of the convex cuts of the middle Postnikov system. The result is the complex depicted in Fig. 8, in which we omitted the grading shifts as well as the primary morphisms which are zero.

7. The Reidemeister moves

7.1. Excision of a contractible cone

The proof of Reidemeister move invariance will require a simplification of matrix factorizations and their complexes, which goes beyond the simplification of the Postnikov systems described in Subsection 6.3. In fact, we will need just one elementary trick, which we call *an excision of a contractible cone*. Let \mathcal{C} be a triangulated category (we have two

$$\mathcal{C}^\bullet \left(\left(\begin{array}{c} \text{Hopf link} \end{array} \right) \right) =$$

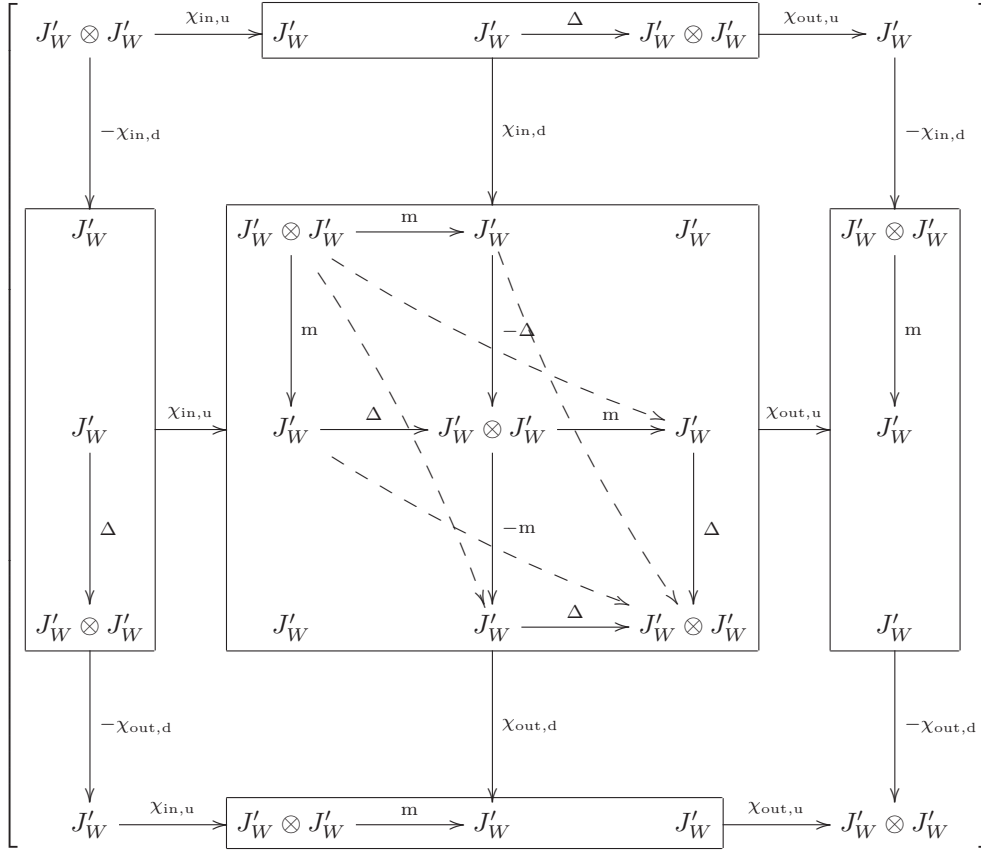


FIGURE 8. The simplified complex for the Hopf link

examples in mind: the category of matrix factorizations \mathbf{MF} and the homotopy category of their complexes $\mathbf{K}(\mathbf{MF})$). Let $[1]$ be the translation functor of \mathcal{C} .

Lemma 7.1. *Consider two objects $A, B \in \text{Ob}(\mathcal{C})$ and a morphism $A \xrightarrow{f} B$. Suppose that either A or B is a zero-object in \mathcal{C} (that is, it is contractible). Then*

$$\text{Cone}_{\mathcal{C}}(f) \simeq \begin{cases} A[1], & \text{if } B \simeq 0, \\ B, & \text{if } A \simeq 0. \end{cases} \quad (7.1)$$

In our future examples contractible objects will be the cones of the identity morphism $\text{Cone}_{\mathcal{C}}(1) \simeq 0$.

Now let us assume that \mathcal{C} is the category of matrix factorizations MF and consider two matrix factorizations $C, C' \in \text{Ob}(\text{MF})$ and two morphisms, relating them to the cone $\text{Cone}_{\text{MF}}(f)$:

$$C \xrightarrow{g} \text{Cone}_{\text{MF}}(f), \quad \text{Cone}_{\text{MF}}(f) \xrightarrow{g'} C'. \quad (7.2)$$

As an R-module, the cone is a sum

$$\text{Cone}_{\text{MF}}(f) = A \oplus B, \quad (7.3)$$

and we present the homomorphisms g and g' as a column and a row according to this decomposition:

$$g = \begin{pmatrix} g_A \\ g_B \end{pmatrix}, \quad g' = \begin{pmatrix} g'_A & g'_B \end{pmatrix}. \quad (7.4)$$

Since g and g' are matrix factorization morphisms, they commute with the twisted differential, which means that

$$dg_A = 0, \quad dg_B = -fg_A, \quad (7.5)$$

$$d'g'_B = 0, \quad d'g'_A = (-1)^{\text{deg}_{z_2}} g'_B g'_B f, \quad (7.6)$$

where

$$d = [D_A + D_B + D_C, \cdot]_s, \quad d' = [D_A + D_B + D_{C'}, \cdot]_s. \quad (7.7)$$

The first conditions indicate that g_A and g_B are matrix factorization morphisms, while the second conditions imply that g'_B and g'_A are morphisms iff

$$fg_A = 0, \quad g'_B f = 0. \quad (7.8)$$

The following lemma describes what happens to the morphisms (7.2) if one of the matrix factorizations A or B is contractible, and therefore, according to eq.(7.1), the cone of f is homotopy equivalent to either A or B .

- Lemma 7.2.** (1) *If the matrix factorization A is contractible, then $\text{Cone}_{\text{MF}}(f) \simeq B$ and*
- a. *the morphisms $\text{Cone}_{\text{MF}}(f) \xrightarrow{g'} C'$ and $B \xrightarrow{g'_B} C'$ are equivalent in the category MF;*
 - b. *there exists a morphism $h_B \in \text{Hom}_{\text{MF}}(C, B)$ such that the morphisms $C \xrightarrow{g} \text{Cone}_{\text{MF}}(f)$ and $C \xrightarrow{h_B} B$ are equivalent in MF;*
 - c. *furthermore, if g_B is a morphism (that is, $fg_A = 0$), then the morphisms $C \xrightarrow{g} \text{Cone}_{\text{MF}}(f)$ and $C \xrightarrow{g_B} B$ are equivalent in MF.*
- (2) *If the matrix factorization B is contractible, then $\text{Cone}_{\text{MF}}(f) \simeq A$ and*
- a. *the morphisms $C \xrightarrow{g} \text{Cone}_{\text{MF}}(f)$ and $C \xrightarrow{g_A} A$ are equivalent in MF;*
 - b. *there exists a morphism $h'_A \in \text{Hom}_{\text{MF}}(A, C')$ such that the morphisms $\text{Cone}_{\text{MF}}(f) \xrightarrow{g'} C'$ and $A \xrightarrow{h'_A} C'$ are equivalent in MF.*
 - c. *furthermore, if g'_A is a morphism (that is, if $g'_B f = 0$), then the morphisms $\text{Cone}_{\text{MF}}(f) \xrightarrow{g'} C'$ and $A \xrightarrow{g'_A} C'$ are equivalent in MF.*

It will be convenient to use the following abbreviated notations for the sums of graded subspaces of J'_W :

$$J'_{W;-} = y\mathbb{Q} \oplus \bigoplus_{i=1}^{2N} x^i \mathbb{Q}, \quad J'_{W;+} = y\mathbb{Q} \oplus \bigoplus_{i=0}^{2N-1} x^i \mathbb{Q}, \quad J'_{W;\pm} = y\mathbb{Q} \oplus \bigoplus_{i=1}^{2N-1} x^i \mathbb{Q}. \quad (7.9)$$

7.2. First Reidemeister move

We are going to prove Theorem 1.1 by establishing the homotopy equivalence (up to a degree shift) between the complex of a kink tangle and the matrix factorization of the 1-arc graph.

Lemma 7.3. *The following complexes are homotopy equivalent in the category $\mathbf{K}(\text{MF}_{W_2})$:*

$$\widehat{\text{XO}} \simeq \widehat{\text{J}} \{-2N-1\} \langle 1 \rangle [-1], \quad (7.10)$$

Equation (1.6) follows if we tensor-multiply both sides of eq. (7.10) by the complex associated with the tangle representing the rest of the link.

Proof. The matrix factorization $\widehat{\mathcal{X}}$ is presented initially by the complex of matrix factorizations (6.31), in which legs 2 and 4 are connected:

$$\begin{aligned} \widehat{\mathcal{X}}_{\text{init}} &= \left(\widehat{\mathcal{X}}\{-1\} \xrightarrow{\chi'_{\text{in}}} \widehat{\mathcal{X}} \xrightarrow{\chi'_{\text{out}}} \widehat{\mathcal{X}}\{-1\} \right) \langle 1 \rangle \\ &= \left(\widehat{\mathcal{X}}\{-1\} \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \boxed{\begin{array}{c} \widehat{\mathcal{X}}\{-1\} \xrightarrow{F} \widehat{\mathcal{X}}\langle 1 \rangle \xrightarrow{G} \widehat{\mathcal{X}}\{1\} \xrightarrow{(1\ 0\ 0)} \widehat{\mathcal{X}}\{-1\} \\ \vdots \\ \widehat{\mathcal{X}} \end{array}} \xrightarrow{(1\ 0\ 0)} \widehat{\mathcal{X}}\{-1\} \right) \langle 1 \rangle. \end{aligned} \quad (7.11)$$

After we simplify the constituent matrix factorizations and primary homomorphisms of the underlying complex of Postnikov systems as described in Subsection 6.3, the complex (7.11) becomes

$$\widehat{\mathcal{X}}\{1\} \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \boxed{\widehat{\mathcal{X}} \otimes J'_W\{-1\} \xrightarrow{F_m} \widehat{\mathcal{X}}\langle 1 \rangle} \oplus \widehat{\mathcal{X}}\{1\} \xrightarrow{(1\ 0\ 0)} \widehat{\mathcal{X}} \otimes J'_W\{-1\} \quad (7.12)$$

We can set the secondary homomorphisms equal to zero, because this choice satisfies the condition (5.34) and the corresponding space (5.36) for length-2 homomorphisms is trivial.

Now it remains to apply the contractible cone excision procedure in order to reduce the complex (7.12) to the *r.h.s.* of eq.(7.10). The middle convolution $\widehat{\mathcal{X}}$ in the complex (7.12), which represents the semi-closed 4-vertex matrix factorization $\widehat{\mathcal{X}}$, splits:

$$\widehat{\mathcal{X}} = \boxed{\widehat{\mathcal{X}} \otimes J'_W\{-1\} \xrightarrow{F_m} \widehat{\mathcal{X}}\langle 1 \rangle} \oplus \widehat{\mathcal{X}}\{1\}, \quad (7.13)$$

and the complex (7.12) also splits: it is a direct sum of two complexes, the first being a contractible complex

$$\widehat{\mathcal{X}}\{1\} \xrightarrow{1} \widehat{\mathcal{X}}\{1\} \quad (7.14)$$

and the second one being

$$\boxed{\widehat{\mathcal{X}} \otimes J'_W\{-1\} \xrightarrow{F_m} \widehat{\mathcal{X}}\langle 1 \rangle} \xrightarrow{(1\ 0)} \widehat{\mathcal{X}} \otimes J'_W\{-1\}. \quad (7.15)$$

The tensor product in the convolution splits

$$\widehat{\smile} \otimes J'_W = \left(\widehat{\smile} \otimes 1 \right) \oplus \left(\widehat{\smile} \otimes J'_{W;-} \right), \quad (7.16)$$

and the homomorphism F_m acts as identity on the first term in this sum (the term $i = 0$ in the sum of eq.(4.53)). Hence the cone of (7.15) can be presented in the form of a ‘double cone’, the inner one being the contractible cone of the identity homomorphism:

$$\boxed{\widehat{\smile} \otimes J'_{W;-} \longrightarrow \boxed{\widehat{\smile} \otimes 1 \xrightarrow{1} \widehat{\smile}}} \quad (7.17)$$

(here we omitted the grading shifts). We excise the contractible cone in accordance with case 2C of Lemma 7.2 and the original cone becomes

$$\boxed{\widehat{\smile} \otimes J'_W \{-1\} \xrightarrow{F_m} \widehat{\smile} \langle 1 \rangle} \simeq \widehat{\smile} \otimes J'_{W;-} \{-1\}, \quad (7.18)$$

while the whole complex (7.15) splits into a direct sum of complexes

$$\left[\left(\widehat{\smile} \otimes 1 [-1] \right) \oplus \left(\widehat{\smile} \otimes J'_{W;-} \xrightarrow{1} \widehat{\smile} \otimes J'_{W;-} \right) \right] \{-1\}. \quad (7.19)$$

The second complex in this sum is contractible and the first complex consists of only one chain module

$$\widehat{\smile} \otimes 1 [-1] \{-1\} = \widehat{\smile} [-1] \{-2N - 1\} \quad (7.20)$$

(recall that $\deg_q 1 = -2N$ in J'_W). If we substitute it for the complex in the brackets of eq.(7.11), then we get eq.(7.10). \square

In the process of proving eq.(7.10) we also proved the following formula for the semi-closed 4-vertex:

$$\widehat{\circlearrowleft} = \bigoplus_{i=0}^{2N-1} \widehat{\smile} \{-2N + 1 + 2i\} \oplus \widehat{\smile} \{-1\} \oplus \widehat{\smile} \{1\}. \quad (7.21)$$

7.3. Second Reidemeister move

Similar to the first Reidemeister move case, Theorem 1.2 follows from the next

Lemma 7.4. *The following objects are homotopy equivalent in the category $\mathbf{K}(\mathbf{MF}_{W_4})$:*

$$\begin{array}{c} \widehat{\text{---}} \\ \text{---} \\ \text{---} \\ \text{---} \\ \widehat{\text{---}} \end{array} \approx \begin{array}{c} \widehat{\text{---}} \\ \text{---} \\ \text{---} \\ \text{---} \\ \widehat{\text{---}} \end{array} . \tag{7.22}$$

Proof. We have already simplified the complex of Postnikov systems, which underlies the complex $\widehat{\text{---}}$. The result is exhibited in Fig. 7, in which the middle Postnikov system is given by Fig. 4. In order to obtain $\widehat{\text{---}}$, we apply the convolution functor to the diagram of Fig. 7, thus replacing the dotted frame boxes with solid ones.

Now it remains to excise contractible cones following the lemmas of Subsection 7.1. We will refer to the cones in (the convolution of) the complex in Fig. 7 and to the constituent matrix factorizations within the cones by pairs of indices (i, j) , as if they were entries of a matrix.

The convolutions (1,2) and (2,3) split, and the splitting matrix factorizations $\widehat{\text{---}}$ are connected by outer identity morphisms with the same matrix factorizations at the corners (1,1) and (3,3) of the complex. These pairs of $\widehat{\text{---}}$ connected by the identity morphisms form contractible cones in the category $\mathbf{K}(\mathbf{MF}_{W_4})$, and according to Lemma 7.1, they can be excised from the complex. The result is the complex in Fig. 9. The top line in this complex is similar to the complex (7.15), except that --- is replaced by $\widehat{\text{---}}$, and we apply to it a similar simplification procedure. Namely, we split the first matrix factorization of the convolution at (1,2) as

$$\boxed{\left(\widehat{\text{---}} \otimes 1 \right) \oplus \left(\widehat{\text{---}} \otimes J'_{W;-} \right) \xrightarrow{(1*)} \widehat{\text{---}}} \tag{7.23}$$

The homomorphism 1 forms a contractible cone. After its excision, the top line of the complex becomes

$$\begin{array}{ccc} \widehat{\text{---}} \otimes J'_{W;-} & \xrightarrow{\binom{0}{1}} & \left(\widehat{\text{---}} \otimes 1 \right) \oplus \left(\widehat{\text{---}} \otimes J'_{W;-} \right) \\ \downarrow * & & \downarrow -\chi_{\text{in,d}} \\ \dots & & \dots \end{array} \tag{7.24}$$

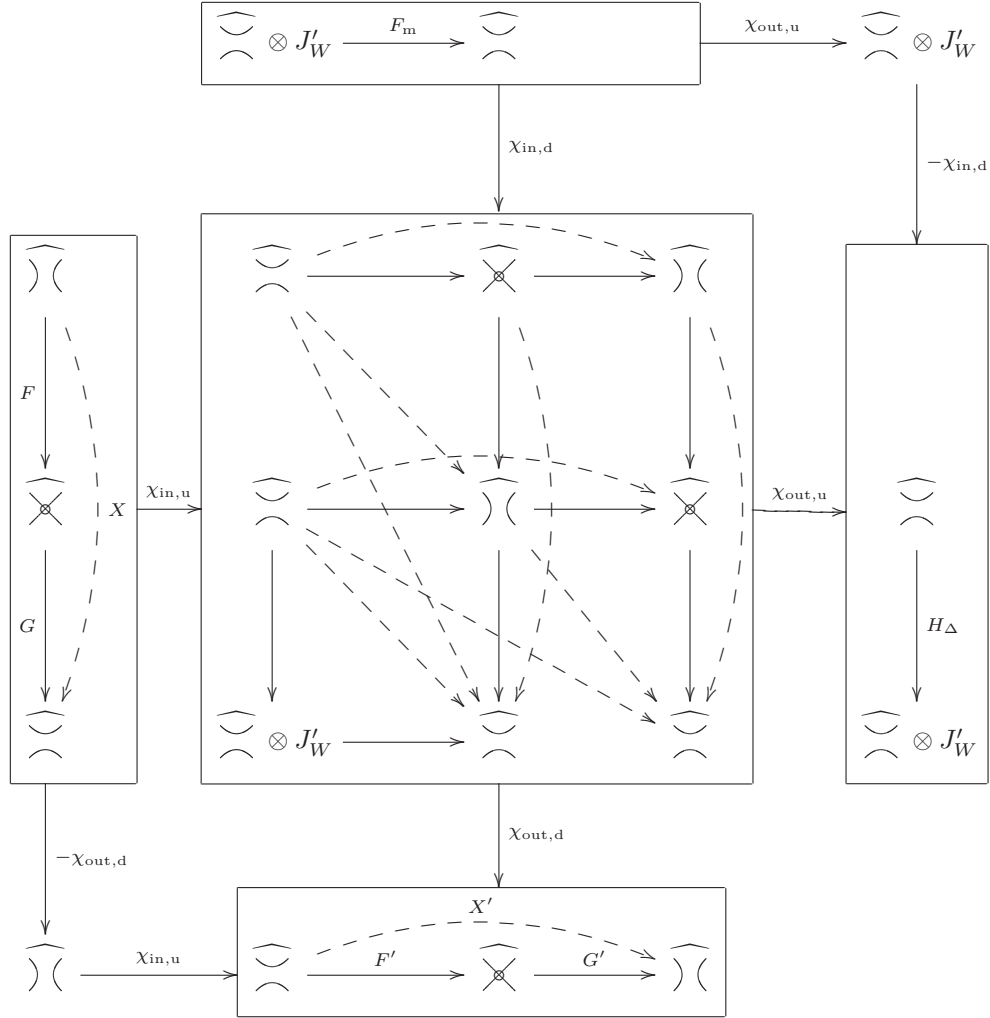


FIGURE 9. A simplified complex related to the second Reidemeister move

Now the homomorphism 1 forms a contractible cone in the category $\mathbf{K}(\mathbf{MF})$, and we excise this cone, leaving the matrix factorization $\widehat{\left(\begin{smallmatrix} \circ \\ \circ \end{smallmatrix} \right)} \otimes 1$ at the position (1,3) of the complex in Fig. 9.

Next, we simplify the middle convolution of the complex in Fig. 9. Its detailed structure is given by (the convolution of) the diagram of Fig. 4. The constituent matrix factorization

at the (3,1) entry of Fig. 4 splits:

$$\widehat{\text{C}} \otimes J'_W = \left(\widehat{\text{C}} \otimes 1 \right) \oplus \left(\widehat{\text{C}} \otimes J'_{W;\pm} \right) \oplus \left(\widehat{\text{C}} \otimes x^{2N} \right). \quad (7.25)$$

The term $-2(2N+1)1 \otimes x^{2N}$, which appears at $i=0$ in the sum of the expression (4.54) for the homomorphism H_Δ , produces a contractible cone

$$\boxed{\begin{array}{ccc} \widehat{\text{C}} & \xrightarrow{-2(2N+1)1} & \widehat{\text{C}} \\ \text{C} & & \text{C} \end{array}} \otimes x^{2N} \quad (7.26)$$

connecting the matrix factorization at the entry (2,1) in the convolution of Fig. 4 and the third term in the sum (7.25). The homomorphisms of the diagram in Fig. 4 related to the cone (7.26) are directed outwards, so the whole convolution of that diagram is a cone of type (7.1), where A is the contractible cone (7.26). Its excision modifies the homomorphism $\chi_{\text{out},u}$ according to case 1A of Lemma 7.2.

The identity homomorphism $\hat{x}^0 = 1$, which appears at $i=0$ in the sum of the expression (4.53) for the homomorphism F_m , creates another contractible cone

$$\boxed{\begin{array}{ccc} \widehat{\text{C}} & \xrightarrow{1} & \widehat{\text{C}} \\ \text{C} \otimes 1 & & \text{C} \otimes 1 \end{array}} \quad (7.27)$$

by connecting the first term in the sum (5.34) with the matrix factorization at the position (3,2) inside the middle convolution simplified by the excision of the cone (7.26). All arrows related to this cone are directed inwards, so the whole simplified convolution is a cone of the type (7.1), where B is the contractible cone (7.27). Its excision modifies the homomorphism $\chi_{\text{out},u}$ according to the case 2C of Lemma 7.2. Hence after the excision the whole complex takes the form of Fig. 10, where the homomorphism $\chi_{\text{out},u}$ acts on the constituent matrix factorization $\widehat{\text{C}} \otimes J'_{W;\pm}$ in the same way as it acted on it in the diagram of Fig. 9.

Now we turn to the convolution at the position (2,3) in the complex. After we split its bottom constituent matrix factorization as

$$\widehat{\text{C}} \otimes J'_W = \left(\widehat{\text{C}} \otimes 1 \right) \oplus \left(\widehat{\text{C}} \otimes J'_{W;\pm} \right) \oplus \left(\widehat{\text{C}} \otimes x^{2N} \right) \quad (7.28)$$

we see that the term $-2(2N+1)1$ at $i=0$ in the expression (4.54) for H_Δ forms a contractible cone

$$\boxed{\begin{array}{ccc} \widehat{\text{C}} & \xrightarrow{-2(2N+1)1} & \widehat{\text{C}} \\ \text{C} & & \text{C} \end{array}} \otimes x^{2N} \quad (7.29)$$

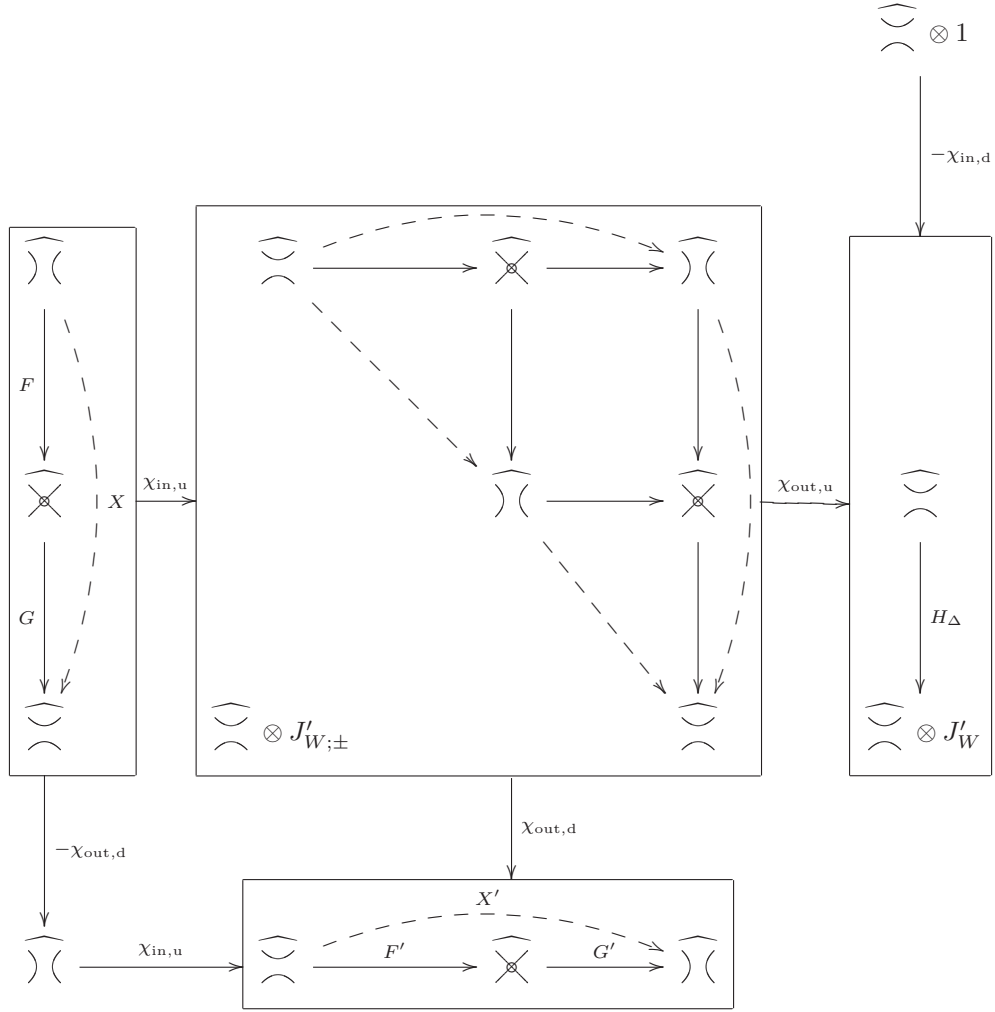


FIGURE 10. A simplified complex related to the second Reidemeister move

within this convolution. Hence it has the form (7.1) with contractible A . Its excision transforms the homomorphisms $-\chi_{in,d}$ and $\chi_{out,u}$ according to case 1C of Lemma 7.2. Hence after the excision they form contractible cones

$$\begin{array}{ccc} \widehat{\left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right)} \otimes 1 & \xrightarrow{-1} & \widehat{\left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right)} \otimes 1, & \widehat{\left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right)} \otimes J'_{W;\pm} & \xrightarrow{1} & \widehat{\left(\begin{array}{c} \text{---} \\ \text{---} \end{array} \right)} \otimes J'_W \end{array} \quad (7.30)$$

within the category $\mathbf{K}(\mathbf{MF})$ with the remaining components of the sum (7.28). After we excise them, the complex of Fig. 10 takes the form of Fig. 11.

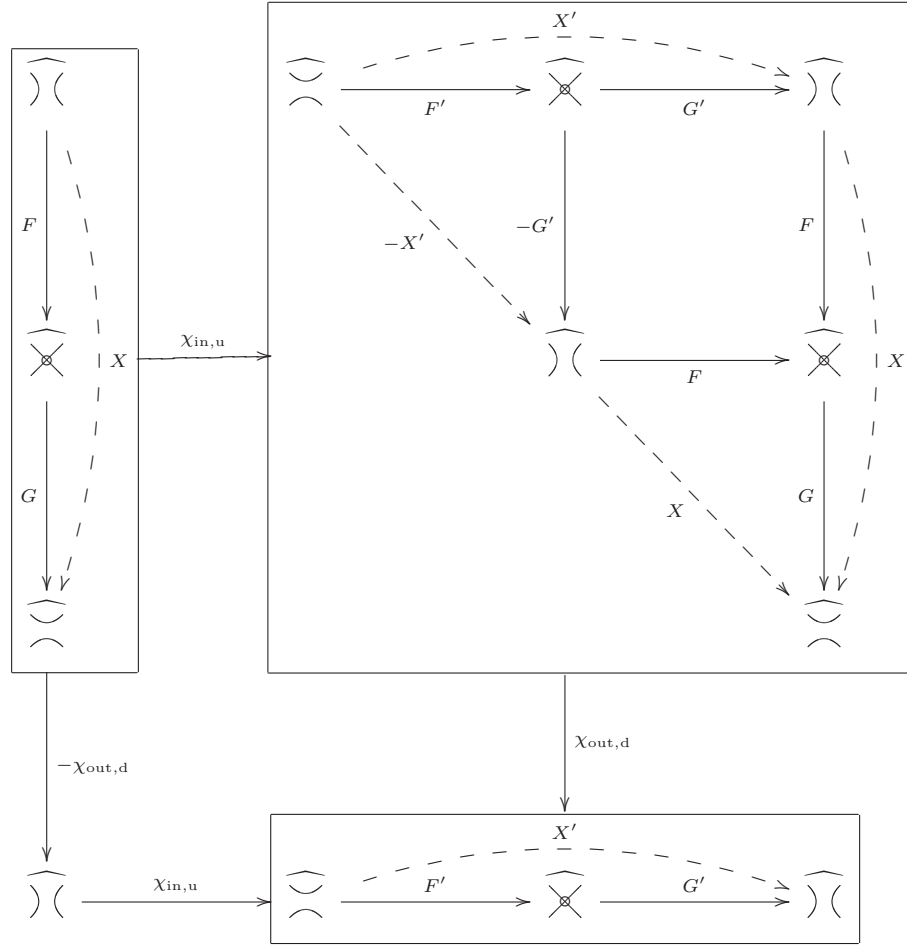


FIGURE 11. A simplified complex related to the second Reidemeister move

If we apply the homomorphism $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ to the sum of the constituent matrix factorizations at the positions (1,3) and (2,2) in the convolution (1,2) of the diagram of Fig. 11, then this convolution splits into a sum of two convolutions of the form (6.25) and (6.26).

Note that the convolutions at the positions (1,1) and (2,2) of this complex have the same form, hence they are all isomorphic to $\widehat{\times}$. Thus the complex of Fig. 11 takes the form

$$\begin{array}{ccc}
 \widehat{\times} & \xrightarrow{(*)} & \widehat{\times} \oplus \widehat{\times} \\
 \downarrow \chi_{\text{out},d} & & \downarrow (1 *) \\
 \widehat{\left. \vphantom{\times} \right\} \left. \vphantom{\times} \right\} & \xrightarrow{\chi_{\text{in},u}} & \widehat{\times}
 \end{array} \tag{7.31}$$

where $*$ denotes unspecified homomorphisms, whose precise form is not important. The components 1 of the top and right morphisms in this diagram form contractible cones, which can be excised in any order. Thus the complex $\widehat{\left. \vphantom{\times} \right\} \left. \vphantom{\times} \right\}$ is homotopy equivalent to a single matrix factorization $\widehat{\left. \vphantom{\times} \right\} \left. \vphantom{\times} \right\}$ at the position (2,1) in the complex (7.31). Let us restore

its degree shifts. This matrix factorization originates from the matrix factorization $\left. \vphantom{\times} \right\} \left. \vphantom{\times} \right\}$ at (3,1) in the diagram of Fig. 6. The latter is a tensor product of two matrix factorizations $\left. \vphantom{\times} \right\} \left. \vphantom{\times} \right\}$, the top one being the first matrix factorization in the second line of the *r.h.s.* of eq.(6.32) and the bottom one being the last matrix factorization in the second line of the *r.h.s.* of eq.(6.31). Hence this matrix factorization has no degree shifts, and we proved eq.(7.22). \square

Note that in process of establishing the homotopy equivalence (7.22), we proved the following relation between matrix factorizations (just follow the transformations of the middle matrix factorization in the convolution of the Postnikov system in Fig. 5):

$$\widehat{\left. \vphantom{\times} \right\} \left. \vphantom{\times} \right\} \simeq \left(\widehat{\left. \vphantom{\times} \right\} \left. \vphantom{\times} \right\} \otimes J'_{W;\pm} \right) \oplus \widehat{\times} \{-1\} \oplus \widehat{\times} \{1\}. \tag{7.32}$$

7.4. Virtual and semi-virtual Reidemeister moves

A diagram of a virtual graph-tangle τ is a planar graph with 1-valent vertices called legs and 4-valent vertices which are of three types: \times , \times and \times . Two edges of τ

are called virtually adjoint, if they are incident to the same virtual vertex and they are attached to it at opposite sides.

Lemma 7.5. *Suppose that the legs i and j of a virtual graph-tangle τ are connected by a sequence of virtually adjoint edges. Then*

$$\hat{\tau} \simeq \hat{\tau}' \otimes \widehat{i \text{ --- } j}, \quad (7.33)$$

where τ' is a virtual graph-tangle constructed by removing the sequence of virtually adjoint edges from τ and ‘dissolving’ the virtual vertices, which are incident to these edges.

Proof. The lemma follows easily from the definition (3.40) of the virtual vertex categorification and from the property (2.78) of 1-arc matrix factorizations, which allows us to reduce a matrix factorization of a sequence of adjacent edges into a matrix factorization of a single edge connecting the legs. \square

Proof of Theorem 1.5. This theorem is a simple corollary of Lemma 7.5.

The homotopy invariance of a virtual link complex under the first and second virtual Reidemeister moves is a particular case of eq.(7.33)

The invariance under the third virtual Reidemeister move follows from the fact that the matrix factorizations of both virtual graph-tangles are homotopy equivalent to the tensor product of three 1-arc matrix factorizations.

Finally, the invariance under the semi-virtual Reidemeister move follows from the fact that the complexes of matrix factorizations corresponding to both virtual graph-tangles are equivalent to the tensor product

$$\widehat{\begin{array}{c} \diagdown \\ \diagup \end{array}} \otimes \widehat{\text{---}}. \quad (7.34)$$

\square

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Appendix A. Virtual crossings, convolutions and the categorification of the HOMFLY-PT polynomial and of its $SU(N)$ specialization

The 2-variable HOMFLY-PT polynomial [3], [12] $J_L(q, t)$ is defined by the skein relation

$$t^{-1} J \left[\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} \right] - t J \left[\begin{array}{c} \nwarrow \nearrow \\ \nearrow \searrow \end{array} \right] = (q - q^{-1}) J \left[\begin{array}{c} \nearrow \\ \searrow \end{array} \right] \left[\begin{array}{c} \nwarrow \\ \nearrow \end{array} \right]. \tag{A.1}$$

Its $SU(N)$ specialization is defined as

$$J_L^{(N)}(t) = J_L(q, -q^{-N}). \tag{A.2}$$

A 2-arc matrix factorization corresponding to an elementary virtual crossing can be introduced in the categorification [8], [9], [13] of these link invariants. Thus both categorifications can be extended to virtual links. Moreover, an analog of the relation (1.23) between the matrix factorization of an elementary graph and the matrix factorizations of 2-arc

graphs also exists. The 2-variable and SU(N) cases are similar. First, we present the SU(N) case in details and then sketch the 2-variable case.

A.1. The SU(N) case

The basic algebra of the categorification is $\mathbb{Q}[x]$ and the basic polynomial is

$$W(x) = x^{N+1}. \tag{A.3}$$

The categorification admits a q -grading with $\deg_q x = 2$. Since $\deg_q W = 2(N + 1)$, we have to set

$$\deg_q D = N + 1 \tag{A.4}$$

on q -degrees of all twisted differentials D .

Similar to eq.(3.9), we introduce the finite differences of $W(x)$:

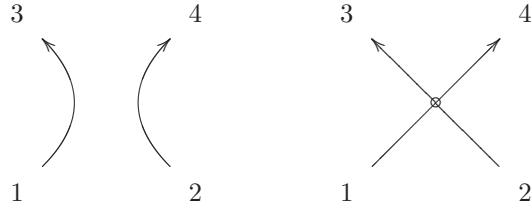
$$W(x_1, x_2) = \frac{W(x_1) - W(x_2)}{x_1 - x_2}, \quad W(x_1, x_2, x_3) = \frac{W(x_1, x_2) - W(x_1, x_3)}{x_2 - x_3}. \tag{A.5}$$

The 1-arc matrix factorization is a 1-row Koszul matrix factorization

$$1 \xrightarrow{\widehat{\quad}} 2 = K(x_2 - x_1; W(x_1, x_2)). \tag{A.6}$$

A.1.1. Virtual crossing and 2-arc matrix factorizations

For two legs oriented inwards and two legs oriented outwards there are two 2-arc graphs connecting them:



$$\tag{A.7}$$

The corresponding matrix factorizations of the polynomial

$$W_4(\mathbf{x}) = W(x_3) + W(x_4) - W(x_1) - W(x_2), \quad \mathbf{x} = x_1, \dots, x_4 \tag{A.8}$$

are 2-row Koszul matrix factorizations, which are the tensor products of matrix factorizations, corresponding to individual arcs:

$$\begin{array}{c} \widehat{\quad} \\ \curvearrowright \end{array} \begin{array}{c} \widehat{\quad} \\ \curvearrowleft \end{array} = K \begin{pmatrix} x_3 - x_1, & W(x_1, x_3) \\ x_4 - x_2, & W(x_2, x_4) \end{pmatrix}, \quad \begin{array}{c} \widehat{\quad} \\ \times \end{array} = K \begin{pmatrix} x_3 - x_2, & W(x_2, x_3) \\ x_4 - x_1, & W(x_1, x_4) \end{pmatrix}. \tag{A.9}$$

Following [8], we introduce the third matrix factorization of $W_4(\mathbf{x})$, which corresponds to the I-graph:

$$\widehat{\text{I-graph}} = \mathbf{K} \begin{pmatrix} x_3 + x_4 - x_1 - x_2, & * \\ x_3x_4 - x_1x_2, & * \end{pmatrix} \{-1\}, \quad (\text{A.10})$$

where $*$ denotes the entries whose exact form is not important for us here.

The polynomial $W_4(\mathbf{x})$ is invariant under the action of the symmetric group S_2 , permuting the legs 1 and 2 and their variables x_1 and x_2 ; hence this group acts on the category MF_{W_4} by endo-functors. In particular,

$$\hat{\sigma}_{12} \left(\widehat{\text{I-graph}} \right) = \widehat{\text{I-graph}}, \quad \hat{\sigma}_{12} \left(\widehat{\text{I-graph}} \right) = \widehat{\text{I-graph}}, \quad \hat{\sigma}_{12} \left(\widehat{\text{I-graph}} \right) = \widehat{\text{I-graph}}. \quad (\text{A.11})$$

We also need the permutation σ_{13} , switching the variables x_1 and x_3 , and the corresponding functor $\hat{\sigma}_{13}$:

$$\sigma_{13}(W_4(\mathbf{x})) = W(x_1) + W(x_4) - W(x_2) - W(x_3), \quad (\text{A.12})$$

$$\hat{\sigma}_{13} \left(\widehat{\text{I-graph}} \right) = \widehat{\text{I-graph}}, \quad \hat{\sigma}_{13} \left(\widehat{\text{I-graph}} \right) = \widehat{\text{I-graph}}. \quad (\text{A.13})$$

The matrix factorizations $\widehat{\text{I-graph}}$, $\widehat{\text{I-graph}}$ and $\widehat{\text{I-graph}}$ can be presented as the tensor products of common and proper parts. Indeed, let us introduce the proper algebra

$$\mathbf{R}_p = \mathbb{Q}[p, r, C], \quad (\text{A.14})$$

related to the 2-arc algebra $\mathbf{R} = \mathbb{Q}[\mathbf{x}]$ by the homomorphism $\mathbf{R}_p \xrightarrow{h_p} \mathbf{R}$, defined by the formulas

$$h_p(p) = x_4 - x_2, \quad h_p(r) = x_4 - x_1, \quad h_p(C) = \tilde{C}(\mathbf{x}), \quad (\text{A.15})$$

where

$$\tilde{C}(\mathbf{x}) = W(x_1, x_2, x_3) + W(x_1, x_2, x_4). \quad (\text{A.16})$$

We introduce proper matrix factorizations

$$\widehat{\text{I-graph}} = \mathbf{K}(p; rC), \quad \widehat{\text{I-graph}} = \mathbf{K}(r; pC), \quad \widehat{\text{I-graph}} = \mathbf{K}(pr; C) \{-1\} \quad (\text{A.17})$$

of the proper polynomial

$$W_{4,p} = prC \quad (\text{A.18})$$

and use the notation

$$\widehat{\gamma}_{\widehat{p}} = \widehat{h}_p(\widehat{\gamma}_p), \quad (\text{A.19})$$

where γ is a graph $\nearrow \searrow$, $\begin{array}{c} \nearrow \\ \times \\ \searrow \end{array}$ or $\begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array}$, while \widehat{h}_p is the functor (2.17), corresponding to the homomorphism h_p . Finally, we set the common matrix factorization to be

$$K_{\text{cmn}} = K(x_3 + x_4 - x_1 - x_2; A(\mathbf{x})), \quad (\text{A.20})$$

where

$$A(\mathbf{x}) = W(x_1, x_2) + (x_3 - x_4)W(x_1, x_2, x_3). \quad (\text{A.21})$$

Note that $\sigma_{12}(A(\mathbf{x})) = A(\mathbf{x})$, so

$$\widehat{\sigma}_{12}(K_{\text{cmn}}) = K_{\text{cmn}}. \quad (\text{A.22})$$

The symmetric group S_2 acts on the proper algebra R_p by permuting p and r , while leaving C intact:

$$\sigma_{12}(p) = r, \quad \sigma_{12}(r) = p, \quad \sigma_{12}(C) = C. \quad (\text{A.23})$$

The homomorphism \widehat{h}_p is equivariant with respect to the simultaneous action of S_2 on \mathbf{x} and on p, r, C . The proper polynomial $W_{4,p}$ is invariant under the action of S_2 , so this group acts on its matrix factorizations by endo-functors. In particular

$$\widehat{\sigma}_{12} \left(\widehat{\begin{array}{c} \nearrow \\ \searrow \end{array}} \right)_p = \widehat{\begin{array}{c} \searrow \\ \nearrow \end{array}}_p, \quad \widehat{\sigma}_{12} \left(\widehat{\begin{array}{c} \nearrow \\ \times \\ \searrow \end{array}} \right)_p = \widehat{\begin{array}{c} \searrow \\ \times \\ \nearrow \end{array}}_p, \quad \widehat{\sigma}_{12} \left(\widehat{\begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array}} \right)_p = \widehat{\begin{array}{c} \searrow \\ \bullet \\ \nearrow \end{array}}_p. \quad (\text{A.24})$$

Proposition A.1. *The 2-arc matrix factorizations (A.17) and the I-graph matrix factorization (A.10) factor into the tensor product of the common and proper matrix factorizations:*

$$\widehat{\gamma} \cong K_{\text{cmn}} \otimes_{\mathbb{R}} \widehat{\gamma}_{\widehat{p}} \quad (\text{A.25})$$

Proof. The application of the transformation $[2, 1]_1$ (see eq.(2.34)) to 2-arc matrix factorizations $\widehat{\begin{array}{c} \nearrow \\ \searrow \end{array}}$ and $\widehat{\begin{array}{c} \nearrow \\ \times \\ \searrow \end{array}}$ of eq.(A.9) and the application of the composition of transformation $[2]_{-1}[1, 2]_{-x_4}$ to the I-graph matrix factorization $\widehat{\begin{array}{c} \nearrow \\ \bullet \\ \searrow \end{array}}$ of eq.(A.10) makes their left columns equal to those of the corresponding matrix factorizations in the *r.h.s.* of eq.(A.25). Since the polynomials in the left columns form regular sequences, the isomorphisms (A.25) follow from Theorem 2.1. \square

Note that the 2-arc matrix factorizations can be transformed explicitly into the tensor product (A.25) by a composition of two transformations (2.34) and (2.35):

$$\hat{\gamma} \xrightarrow{[1,2]_{W(x_1, x_2, x_3)}^{[2,1]_1}} K_{\text{cmn}} \otimes_{\mathbb{R}} \hat{\gamma}_{\hat{p}}. \quad (\text{A.26})$$

A.1.2. The saddle morphism and its cone

We define the proper saddle morphism

$$F_{\hat{p}} \in \text{Ext}_{\{N-1\}}^1 \left(\widehat{\int} \widehat{\int}_p, \widehat{\int} \widehat{\int}_p \right) \quad (\text{A.27})$$

by the following commutative diagram

$$\begin{array}{ccccc} \widehat{\int} \widehat{\int}_p & & \mathbf{R}_1 \{1 - N\} & \xrightarrow{p} & \mathbf{R}_0 & \xrightarrow{rC} & \mathbf{R}_1 \{1 - N\} \\ \downarrow F_{\hat{p}} & & \searrow 1 & & \searrow -C & & \\ \widehat{\int} \widehat{\int}_p & & \mathbf{R}_1 \{1 - N\} & \xrightarrow{r} & \mathbf{R}_0 & \xrightarrow{pC} & \mathbf{R}_1 \{1 - N\} \end{array} \quad (\text{A.28})$$

The corresponding saddle morphism $F \in \text{Ext}^1 \left(\widehat{\int} \widehat{\int}, \widehat{\int} \widehat{\int} \right)$ is defined with the help of eq.(A.25) as

$$F = 1_{\text{cmn}} \otimes F_{\hat{p}}. \quad (\text{A.29})$$

We consider two other saddle morphisms, produced by the action of the functors $\hat{\sigma}_{12}$ and $\hat{\sigma}_{13}$:

$$G = \hat{\sigma}_{12}(F) \in \text{Ext}^1 \left(\widehat{\int} \widehat{\int}, \widehat{\int} \widehat{\int} \right), \quad H = \hat{\sigma}_{13}(F) \in \text{Ext}^1 \left(\widehat{\int} \widehat{\int}, \widehat{\int} \widehat{\int} \right). \quad (\text{A.30})$$

Theorem A.2. *The saddle morphism H is equal (up to a possible sign factor) to the saddle morphism η defined in Section 9 of [8].*

We need the following

Lemma A.3. *The composition of two saddle morphisms is*

$$GF \simeq -(N+1) \sum_{i=0}^{N-1} \hat{x}_1^i \hat{x}_2^{N-i-1}. \quad (\text{A.31})$$

Proof. The composition of proper saddle morphisms is presented by the diagram

$$\begin{array}{ccc}
 \begin{array}{c} \widehat{\int} \widehat{\int}_p \\ \downarrow F_p \\ \widehat{\int} \widehat{\int}_p \\ \downarrow G_p \\ \widehat{\int} \widehat{\int}_p \end{array} & \begin{array}{ccc} R_1 \{1 - N\} & \xrightarrow{p} & R_0 & \xrightarrow{rC} & R_1 \{1 - N\} \\ & \searrow 1 & & \searrow -C & \\ R_1 \{1 - N\} & \xrightarrow{r} & R_0 & \xrightarrow{pC} & R_1 \{1 - N\} \\ & \searrow 1 & & \searrow -C & \\ R_1 \{1 - N\} & \xrightarrow{p} & R_0 & \xrightarrow{rC} & R_1 \{1 - N\} \end{array} & (A.32)
 \end{array}$$

Obviously, $G_p F_p = -C 1$, so in view of eq.(A.16),

$$G_p F_p = -W(\hat{x}_1, \hat{x}_2, \hat{x}_3) - W(\hat{x}_1, \hat{x}_2, \hat{x}_4). \quad (A.33)$$

Now eq.(A.31) follows from the relations

$$\hat{x}_3 \simeq \hat{x}_1, \quad \hat{x}_4 \simeq \hat{x}_2 \quad \text{in} \quad \text{End}_{\text{MF}} \left(\widehat{\int} \widehat{\int} \right) \quad (A.34)$$

and from the formula

$$W(x_1, x_1, x_2) + W(x_1, x_2, x_2) = (N + 1) \sum_{i=0}^{N-1} x_1^i x_2^{N-i-1}. \quad (A.35)$$

□

Proof of theorem A.2. In our conventions

$$\deg_q H = \deg_q \eta = N - 1. \quad (A.36)$$

Since

$$\dim \text{Ext}_{\{N-1\}}^1 \left(\widehat{\int} \widehat{\int}, \widehat{\int} \widehat{\int} \right) = 1, \quad (A.37)$$

H is proportional to η :

$$H \simeq a\eta, \quad a \in \mathbb{Q}. \quad (A.38)$$

Applying the functor $\hat{\sigma}_{13}$ to the relation (A.31) we find that the composition of morphisms

$$\widehat{\int} \widehat{\int} \xrightarrow{H} \widehat{\int} \widehat{\int} \xrightarrow{H'} \widehat{\int} \widehat{\int}, \quad (A.39)$$

(where $H' = \hat{\sigma}_{13}(H)$) is

$$H' H \simeq -(N + 1) \sum_{i=0}^{N-1} \hat{x}_1^i \hat{x}_2^{N-i-1}. \quad (A.40)$$

The Koszul matrix factorization $K(1; prC)$ is contractible, so $\widehat{\chi}_p \oplus K(1; prC) \{1\} \simeq \widehat{\chi}_p$. \square

A.1.3. Natural morphisms and crossing complexes

The cones of eq.(A.41) are related by natural morphisms to their constituent matrix factorizations. We consider two such morphisms:

$$\widehat{\chi}_0 \xrightarrow{\chi_{\text{in}}} \boxed{\widehat{\chi}_0 \{1\} \xrightarrow{G} \widehat{\chi}_1 \{1\}} \simeq \widehat{\chi}_1 \quad (\text{A.44})$$

and

$$\widehat{\chi}_1 \simeq \boxed{\widehat{\chi}_1 \{-1\} \xrightarrow{F} \widehat{\chi}_0 \{1\}} \xrightarrow{\chi_{\text{out}}} \widehat{\chi}_0 \{-1\}. \quad (\text{A.45})$$

Theorem A.5. *The morphisms χ_{in} and χ_{out} are homotopy equivalent up to non-zero constant factors to the morphisms χ_0 and χ_1 of [8].*

We need the following

Lemma A.6. *The composition of the natural morphisms is*

$$\chi_{\text{out}} \chi_{\text{in}} \simeq \hat{x}_1 - \hat{x}_2. \quad (\text{A.46})$$

Proof. Obviously,

$$\chi_{\text{in}} = 1_{\text{cmn}} \otimes_{\mathbb{R}} \chi_{\text{in},p}, \quad \chi_{\text{out}} = 1_{\text{cmn}} \otimes_{\mathbb{R}} \chi_{\text{out},p}, \quad (\text{A.47})$$

where $\chi_{\text{in},p}$ and $\chi_{\text{out},p}$ are proper natural morphisms

$$\widehat{\chi}_p \xrightarrow{\chi_{\text{in},p}} \boxed{\widehat{\chi}_p \{1\} \xrightarrow{G_p} \widehat{\chi}_p \{1\}} \quad (\text{A.48})$$

and

$$\boxed{\widehat{\chi}_p \{-1\} \xrightarrow{F_p} \widehat{\chi}_p \{1\}} \xrightarrow{\chi_{\text{out},p}} \widehat{\chi}_p \{-1\}. \quad (\text{A.49})$$

If we replace the cones by the homotopy equivalent matrix factorization $\widehat{\text{crossing}}_p$, then the morphisms $\chi_{\text{in},p}$ and $\chi_{\text{out},p}$ take the form

$$\begin{array}{ccccc}
 \widehat{\text{crossing}}_p \{1\} & & \mathbf{R}_1 \{2-N\} & \xrightarrow{p} & \mathbf{R}_0 \{1\} & \xrightarrow{rC} & \mathbf{R}_1 \{2-N\} \\
 \downarrow \chi'_{\text{in},p} & & \downarrow -1 & & \downarrow -r & & \downarrow -1 \\
 \widehat{\text{crossing}}_p & & \mathbf{R}_1 \{2-N\} & \xrightarrow{pr} & \mathbf{R}_0 \{-1\} & \xrightarrow{C} & \mathbf{R}_1 \{2-N\} \\
 \downarrow \chi_{\text{out},p} & & \downarrow r & & \downarrow 1 & & \downarrow r \\
 \widehat{\text{crossing}}_p \{-1\} & & \mathbf{R}_1 \{-N\} & \xrightarrow{p} & \mathbf{R}_0 \{-1\} & \xrightarrow{rC} & \mathbf{R}_1 \{-N\}
 \end{array} \tag{A.50}$$

Now it is obvious that

$$\chi_{\text{out},p} \chi_{\text{in},p} \simeq -r, \tag{A.51}$$

and relation (A.46) follows, since according to eq.(A.15), $h_p(r) = x_4 - x_1$, while $x_4 \simeq x_2$ in $\text{Hom}_{\text{MF}} \left(\widehat{\text{crossing}}_p \right)$. \square

Proof of theorem A.5. In our conventions,

$$\deg_q \chi_{\text{in}} = \deg_q \chi_{\text{out}} = \deg_q \chi_0 = \deg_q \chi_1 = 0, \tag{A.52}$$

but we have shown in [8] that

$$\dim \text{Ext}_{\{0\}}^0 \left(\widehat{\text{crossing}}_p, \widehat{\text{crossing}}_p \right) = \dim \text{Ext}_{\{0\}}^0 \left(\widehat{\text{crossing}}_p, \widehat{\text{crossing}}_p \right) = 1. \tag{A.53}$$

Therefore

$$\chi_{\text{in}} = a_0 \chi_0, \quad \chi_{\text{out}} = a_1 \chi_1, \quad a_0, a_1 \in \mathbb{Q}. \tag{A.54}$$

Since $\hat{x}_2 - \hat{x}_1 \neq 0$ in $\text{Hom}_{\text{MF}} \left(\widehat{\text{crossing}}_p \right)$, eq.(A.46) implies that $a_0, a_1 \neq 0$. \square

Now, in view of the equivalences established in Theorems A.4 and A.5, we can present the categorification formulas of [8] for the elementary crossings as complexes (A.48)

and (A.49):

$$\widehat{\text{crossing}} = \left(\widehat{\text{crossing}} \left(\widehat{\text{crossing}} \left(\{1\} \xrightarrow{\chi_{\text{in}}} \boxed{\widehat{\text{crossing}} \left(\widehat{\text{crossing}} \left(\{-1\} \xrightarrow{G} \widehat{\text{crossing}} \left(\widehat{\text{crossing}} \left(\{1\} \right) \right) \right) \right) \right) \right) \right) \{-N\}[-1], \quad (\text{A.55})$$

$$\widehat{\text{crossing}} = \left(\boxed{\widehat{\text{crossing}} \left(\widehat{\text{crossing}} \left(\{-1\} \xrightarrow{F} \widehat{\text{crossing}} \left(\widehat{\text{crossing}} \left(\{1\} \right) \right) \right) \right) \xrightarrow{\chi_{\text{out}}} \widehat{\text{crossing}} \left(\widehat{\text{crossing}} \left(\{-1\} \right) \right) \right) \{N\}. \quad (\text{A.56})$$

Here we assume that the differentials χ_{in} and χ_{out} are cohomological (that is, they have the homological degree 1) and that the right terms in the complexes in parentheses have homological degree 0.

A.1.4. A categorification complex for a virtual link

The second formula of (A.9) allows us to define a categorification complex for a virtual link in the same way as we did it for ordinary links in [8]. Let L be a virtual link diagram. We split it into a disjoint union of elementary real and virtual crossings by cutting across all edges. To real crossings we associate complexes of matrix factorizations (A.55) and (A.56). To virtual crossings we associate the second matrix factorization of eq.(A.9). Finally, we take the tensor product of matrix factorizations and their complexes, thus forming a chain complex $C_{\text{SU}(N)}^\bullet(L)$ of $\mathbb{Z}_2 \times \mathbb{Z}$ -graded \mathbb{Q} -vector spaces.

Theorem A.7. *If two diagrams L and L' represent the same virtual link, then the corresponding complexes are homotopy equivalent:*

$$C_{\text{SU}(N)}^\bullet(L') \simeq C_{\text{SU}(N)}^\bullet(L). \quad (\text{A.57})$$

Proof. If the diagrams L and L' are related by a Reidemeister move, then the relation (A.57) is proved in [8]. If the diagrams L and L' are related by a virtual Reidemeister move or by a mixed move (1.27), then the relation (A.57) is obvious. \square

The graded Euler characteristic of the complex $C_{\text{SU}(N)}^\bullet(L)$ may serve as a definition of the SU(N) HOMFLY-PT polynomial for virtual links.

A.2. The 2-variable HOMFLY-PT polynomial case

Consider the categorification [9] for the HOMFLY-PT polynomial, in which we set $a = 0$ (see also [13]). Then the basic q -grading algebra is the same as in the $SU(N)$ case, but the basic polynomial is zero: $W = 0$. As a result, the matrix factorizations of the $SU(N)$ categorification are replaced by ‘inner’ complexes (we call them inner in order to distinguish them from the ‘outer’ complexes associated with the categorification of the crossings). The \mathbb{Z}_2 -grading of matrix factorizations lifts to homological \mathbb{Z} -grading of inner complexes. We refer to it as t -grading, because it generates powers of t in the formula for the graded Euler characteristic of the link categorification complex

$$\chi_{q,t}(C^\bullet(L)) = \sum_{i,j,n \in \mathbb{Z}} (-1)^{j+n} t^{2j} q^i \dim C_{\{i\}}^{n,j}(L) \tag{A.58}$$

(cf. eq.(1.10)).

Another important distinction between the categorification of the 2-variable HOMFLY-PT polynomial and its $SU(N)$ specialization is that in the former case the link diagram L should be the result of a (circular) braid closure. Suppose that L is a closure of an n -braid β , and a closed graph $\Gamma_{\mathbf{r}}$ which originates from the \mathbf{r} -resolution of the crossings of L , is the closure of the $2n$ -legged open graph $\gamma_{\mathbf{r}}$, which in turn results from the same resolution of the crossings of β . Let \widehat{h}_n be the functor (2.74) corresponding to the joining of n incoming and n outgoing legs of $\gamma_{\mathbf{r}}$, which would turn it into $\Gamma_{\mathbf{r}}$. Then in contrast to the $SU(N)$ categorification formula $\widehat{\Gamma}_{\mathbf{r}} = \widehat{h}_n(\widehat{\gamma}_{\mathbf{r}})$, we have to introduce an extra degree shift in the 2-variable case:

$$\widehat{\Gamma}_{\mathbf{r}} = \widehat{h}_n(\widehat{\gamma}_{\mathbf{r}}) \{n\} \langle -n/2 \rangle [n/2]. \tag{A.59}$$

Since $W = 0$, the condition $\deg_q D = \frac{1}{2} \deg_q W$ is no longer imposed on the differentials D of inner complexes. However, invariance under the first Reidemeister move imposes a requirement

$$\deg_q D = 2. \tag{A.60}$$

The Koszul matrix factorizations of Subsection A.1 are replaced by Koszul complexes. Let \mathbf{R} be a q -graded polynomial algebra and let $p \in \mathbf{R}$, $\deg_q p = k$. Then the Koszul complex $K(p)$ is the complex

$$\mathbf{R}_1 \{k - 2\} \xrightarrow{p} \mathbf{R}_0, \tag{A.61}$$

where the index i of R_i indicates its t -grading. The q -degree shift $\{k-2\}$ is required to satisfy the condition (A.59). For the column \mathbf{p} of polynomials $p_i \in R$, $1 \leq i \leq n$, we define

$$K(\mathbf{p}) = \bigotimes_{i=1}^n K(p_i). \tag{A.62}$$

Removing second columns in all Koszul matrix factorizations of subsection A.1 turns them into Koszul complexes and thus adapts them to the 2-variable HOMFLY-PT case. The only tricky part is the grading shifts, which have to be adjusted in accordance with eqs.(A.59) and (A.60).

The Koszul complex of the 1-arc graph is

$$1 \xrightarrow{\quad} 2 = K(x_2 - x_1). \tag{A.63}$$

The matrix factorizations of the 2-arc graphs $\widehat{\curvearrowright}$, $\widehat{\curvearrowleft}$, $\widehat{\times}$ and of the elementary 4-vertex graph $\widehat{\text{4-vertex}}$ have the form (A.25), where this time

$$K_{\text{cmn}} = K(x_3 + x_4 - x_1 - x_2), \tag{A.64}$$

while the proper algebra is $R_{\mathbf{p}} = \mathbb{Q}[p, r]$ (this time it does not include C), the proper homomorphism is

$$h_{\mathbf{p}}(p) = x_4 - x_2, \quad h_{\mathbf{p}}(r) = x_4 - x_1 \tag{A.65}$$

and the proper complexes are

$$\widehat{\curvearrowright} \quad \widehat{\curvearrowleft} \quad = K(p), \quad \widehat{\times} \quad = K(r), \quad \widehat{\text{4-vertex}} \quad = K(pr) \{-1\}. \tag{A.66}$$

The proofs are the same as in subsection A.1, except that we ignore the right columns of Koszul matrix factorizations.

The saddle morphism (A.27) is defined by the reduced version of the diagram (A.28)

$$\begin{array}{ccc} \widehat{\curvearrowright} \quad \widehat{\curvearrowleft} \quad \text{P} & & R_1 \xrightarrow{p} R_0 \\ \downarrow F_{\mathbf{p}} & & \searrow 1 \\ \widehat{\times} \quad \text{P} & & R_1 \xrightarrow{r} R_0 \end{array} \tag{A.67}$$

and it is easy to see that after the appropriate degree shifts its cone is homotopy equivalent to the complex $\widehat{\mathcal{C}}_p$, that is, isomorphism (A.42) still holds. If we tensor multiply both sides by K_{cmn} and apply the σ_{12} argument of subsection A.1, then we obtain isomorphism (A.41). The natural morphisms (A.48) and (A.49) associated with the cones (A.41) can be cast in the form

$$\begin{array}{ccc}
 \widehat{\mathcal{C}}_p\{1\} & & R_1\{1\} \xrightarrow{p} R_0\{1\} \\
 \downarrow \chi'_{\text{in},p} & & \downarrow -1 \qquad \qquad \downarrow -r \\
 \widehat{\mathcal{C}}_p & & R_1\{1\} \xrightarrow{pr} R_0\{-1\} \\
 \downarrow \chi_{\text{out},p} & & \downarrow r \qquad \qquad \downarrow 1 \\
 \widehat{\mathcal{C}}_{-1} & & R_1\{-1\} \xrightarrow{p} R_0\{-1\}
 \end{array} \tag{A.68}$$

and it is easy to verify that they coincide (up to constant factors) with the morphisms χ_0 and χ_1 of [9]. Thus we can present the elementary crossing categorification complexes of [9] in the form

$$\widehat{\mathcal{C}}_{\text{cross}} = \left(\widehat{\mathcal{C}}_p\{1\} \xrightarrow{\chi_{\text{in}}} \boxed{\widehat{\mathcal{C}}_{-1} \xrightarrow{G} \widehat{\mathcal{C}}_p\{1\}} \right) \left\langle \frac{1}{2} \right\rangle \left[-\frac{1}{2} \right], \tag{A.69}$$

$$\widehat{\mathcal{C}}_{\text{cross}} = \left(\boxed{\widehat{\mathcal{C}}_{-1} \xrightarrow{F} \widehat{\mathcal{C}}_p\{1\}} \xrightarrow{\chi_{\text{out}}} \widehat{\mathcal{C}}_{-1} \right) \left\langle -\frac{1}{2} \right\rangle \left[-\frac{1}{2} \right] \tag{A.70}$$

(fractional shifts were introduced by H. Wu [14]). Here we assume again that the differentials χ_{in} and χ_{out} are cohomological (that is, they have the homological degree 1) and that the right terms in the complexes in parentheses have homological degree 0.

Appendix B. The isomorphism between $SU(2) \times SU(2)$ and $SO(4)$ categorifications

B.1. The isomorphism theorem

A simple relation between the groups

$$SO(4) = SU(2) \times SU(2)/\mathbb{Z}_2 \tag{B.1}$$

implies a relation between the corresponding link polynomials: for a framed oriented link L

$$P_{SO(4);L}(q) = (P_{SU(2);L}(q))^2 q^{-3w(L)}. \tag{B.2}$$

Here $P_{SO(4)}$ is the $SO(4)$ specialization of the Kauffman polynomial, $P_{SU(2)}$ is the Jones polynomial considered as the $SU(2)$ specialization of the HOMFLY-PT polynomial and $w(L)$ is the writhe of L defined as

$$w(L) = \sum_{i,j=1}^{\#L} l_{ij}, \tag{B.3}$$

where $\#L$ is the number of components of L and l_{ij} are the linking numbers between the components, the self-linking numbers l_{ii} being determined by their framing. If a link diagram L has the blackboard framing, then

$$w(L) = n_+(L) - n_-(L), \tag{B.4}$$

where n_+ and n_- are the numbers of positive ($\nearrow \searrow$) and negative ($\searrow \nearrow$) crossings of L . The factor $q^{-w(L)}$ in eq.(B.2) reflects the fact that the $SO(4)$ polynomial, as defined in this paper, is covariant (eq.(1.1)) rather than invariant with respect to the first Reidemeister move.

The relation (B.2) can be categorified.

Theorem B.1. *There is a homotopy equivalence between the categorification complexes*

$$C_{SO(4)}^\bullet(L) \simeq \left(C_{SU(2)}^\bullet(L) \otimes C_{SU(2)}^\bullet(L) \right) \{ -3w(L) \} \langle w(L) \rangle [-w(L)], \tag{B.5}$$

where $C_{SU(2)}^\bullet(L)$ is the $SU(2)$ categorification complex of [7] as constructed in [8] and $C_{SO(4)}^\bullet(L)$ is the $SO(2N + 2)$ categorification complex (1.5) for $N = 1$.

B.2. The SU(2) categorification for unoriented link diagrams

In proving the relation (B.5) it is convenient to use the SU(2) complex construction which differs slightly from the prescription of [8]. Namely, we are going to construct an SU(2) complex $\tilde{C}_{\text{SU}(2)}^\bullet(L)$, which is related to the standard one by a degree shift

$$\tilde{C}_{\text{SU}(2)}^\bullet(L) = C_{\text{SU}(2)}^\bullet(L) \left\{ \frac{3}{2} w(L) \right\} \left\langle \frac{1}{2} w(L) \right\rangle \left[\frac{1}{2} w(L) \right]. \tag{B.6}$$

As eq.(B.6) suggests, the complex $\tilde{C}_{\text{SU}(2)}^\bullet(L)$ is invariant under the second and third Reidemeister moves and changes under the first Reidemeister move according to the formula

$$\tilde{C}_{\text{SU}(2)}^\bullet \left[\left| \begin{array}{c} \text{loop} \end{array} \right. \right] \simeq \tilde{C}_{\text{SU}(2)}^\bullet \left[\left| \quad \right. \right] \left\{ \frac{3}{2} \right\} \left\langle \frac{1}{2} \right\rangle \left[\frac{1}{2} \right]. \tag{B.7}$$

The main feature of the combinatorial construction of $\tilde{C}_{\text{SU}(2)}^\bullet(L)$ is that in contrast to the construction of [8] it does not require the orientation of link components and in this respect it is similar to our construction of $C_{\text{SO}(4)}^\bullet(L)$.

Let us review the matrix factorization construction of $C_{\text{SU}(2)}^\bullet(L)$ and introduce the changes that will transform it to $\tilde{C}_{\text{SU}(2)}^\bullet(L)$. The basic algebra of SU(2) categorification is $\mathbb{Q}[u]$, its q -grading defined by the condition $\deg_q u = 2$. The basic polynomial is

$$W_{\text{SU}(2)}(u) = u^3. \tag{B.8}$$

It has odd degree, hence $W_{\text{SU}(2)}(-u) = -W_{\text{SU}(2)}(u)$ and in the SU(2) case (as well as in the SU(N) case with odd N) we can adopt the same leg orientation convention, as in the SO($2N + 2$) case. Namely, we assume that all graph and tangle legs are oriented outwards, and if for the purpose of leg joining we need to switch the orientation of an i -th leg, then we change its matrix factorization by the functor (2.17) associated with the endomorphism of $\mathbb{Q}[u_i]$, which switches the sign of u_i .

We are going to review and slightly modify the categorification formulas of Appendix A in view of our new leg orientation convention. First of all, the SU(2) 1-arc matrix factorization is

$$\widehat{1 \text{ --- } 2} = K(u_2 + u_1; u_2^2 - u_2u_1 + u_1^2). \tag{B.9}$$

Next we turn to four 4-legged graphs $\widehat{\left. \right\} \left(\right)_p$, $\widehat{\left. \right\} \left(\right)_p$, $\widehat{\left. \right\} \left(\right)_p$ and $\widehat{\left. \right\} \left(\right)_p$, the latter 2-arc graph being included, because the new leg orientation convention permits it. The matrix factorizations of 4-legged graphs factor into common and proper parts according to eq.(A.25), but this time we define both parts slightly differently. Namely, we choose the polynomial A not according to the expression (A.21) at $N = 2$, but rather as

$$A(\mathbf{u}) = u_1^2 + u_2^2 + u_3^2 - u_1u_2 - u_1u_3 - u_2u_3 - u_4(u_1 + u_2 + u_3) - 2u_4^2. \quad (\text{B.10})$$

The proper algebra this time is

$$\mathbb{R}_p = \mathbb{Q}[p, q, r], \quad (\text{B.11})$$

and the proper homomorphism $\mathbb{R}_p \xrightarrow{h_p} \mathbb{R}$ is defined by the formulas

$$h_p(p) = u_2 + u_4, \quad h_p(q) = u_3 + u_4, \quad h_p(r) = u_1 + u_4, \quad (\text{B.12})$$

which are similar to eq.(3.50). The proper matrix factorizations of 4-legged graphs are

$$\widehat{\left. \right\} \left(\right)_p = K(p; 3qr), \quad \widehat{\left. \right\} \left(\right)_p = K(r; 3pq), \quad \widehat{\left. \right\} \left(\right)_p = K(q; 3pr) \quad (\text{B.13})$$

and

$$\widehat{\left. \right\} \left(\right)_p = K(pr; 3q) \{-1\}. \quad (\text{B.14})$$

The proof of the factorization formula (A.25) in Proposition A.1 is repeated *verbatim*.

The ordinary and proper matrix factorizations of 2-arc graphs have the same symmetry S_4 and S_3 as in the SO(2N + 2) categorification case (see subsection 3.4.1), so it suffices to define the saddle morphism between one pair of 2-arc graphs and the morphisms for other pairs will be dictated by the symmetry. Thus the saddle morphism is the tensor product (A.29), where the proper saddle morphism $F_p \in \text{Ext}_{\{1\}}^1 \left(\widehat{\left. \right\} \left(\right)_p, \widehat{\left. \right\} \left(\right)_p \right)$ is defined by the diagram

$$\begin{array}{ccc} \widehat{\left. \right\} \left(\right)_p & & \mathbb{R}_1 \{-1\} \xrightarrow{p} \mathbb{R}_0 \xrightarrow{3qr} \mathbb{R}_1 \{-1\} \\ \downarrow F_p & & \searrow 1 \qquad \swarrow -3q \\ \widehat{\left. \right\} \left(\right)_p & & \mathbb{R}_1 \{-1\} \xrightarrow{r} \mathbb{R}_0 \xrightarrow{3pq} \mathbb{R}_1 \{-1\} \end{array} \quad (\text{B.15})$$

This saddle morphism is equal to the $SU(N)$ saddle morphism coming from (A.28).

The application of the equivalence transformation $[1]_3$ (see (2.36)) to the matrix factorization $\widehat{\text{X}}_p$ turns it into $\widehat{\text{X}}_p \langle 1 \rangle$, hence

$$\widehat{\text{X}}_p \cong \widehat{\text{X}}_p \langle 1 \rangle, \quad (\text{B.16})$$

and we can replace $\widehat{\text{X}}_p$ by $\widehat{\text{X}}_p \langle 1 \rangle$ in the cone relations (A.41) and in categorification formulas (A.55) and (A.56). The cone relations (A.41) become

$$\begin{aligned} \widehat{\text{X}}_p \langle 1 \rangle &\simeq \boxed{\widehat{\text{X}}_p \langle -1 \rangle \xrightarrow{F} \widehat{\text{X}}_p \langle 1 \rangle} \\ &\simeq \boxed{\widehat{\text{X}}_p \langle -1 \rangle \xrightarrow{G} \widehat{\text{X}}_p \langle 1 \rangle}. \end{aligned} \quad (\text{B.17})$$

and the natural morphisms χ_{in} and χ_{out} relating the cones to constituent matrix factorizations are equal (up to a non-zero factor) to saddle morphisms. Thus if we introduce a non-oriented elementary crossing complex

$$\widehat{\text{X}}_p = \left(\widehat{\text{X}}_p \langle \frac{1}{2} \rangle \langle \frac{1}{2} \rangle \xrightarrow{F} \widehat{\text{X}}_p \langle -\frac{1}{2} \rangle \langle -\frac{1}{2} \rangle \right) \left[-\frac{1}{2} \right], \quad (\text{B.18})$$

based on the saddle morphism F and define the complex $\tilde{C}_{SU(2)}^\bullet(L)$ for an unoriented link diagram L by joining the elementary complexes (B.18) for each crossing of L . The standard $SU(2)$ categorification complex $C_{SU(2)}^\bullet(L)$ comes from the crossing complexes (A.55) and (A.56). These complexes differ from the unoriented crossing complex (B.18) by degree shifts

$$\widehat{\text{X}}_p \leftarrow = \widehat{\text{X}}_p \left\{ -\frac{3}{2} \right\} \left\langle -\frac{1}{2} \right\rangle \left[-\frac{1}{2} \right], \quad \widehat{\text{X}}_p \rightarrow = \widehat{\text{X}}_p \left\{ \frac{3}{2} \right\} \left\langle \frac{1}{2} \right\rangle \left[\frac{1}{2} \right], \quad (\text{B.19})$$

and these shifts result in the relation (B.6) between the link diagram complexes.

B.3. Proof of the isomorphism of complexes

It is convenient to use the basic SO(4) polynomial with the choice of coefficients different from that of eq.(3.1):

$$W_{\text{SO}(4)}(x, y) = 6xy^2 + 2x^3, \quad \deg_q x = \deg_q y = 2. \quad (\text{B.20})$$

Consider the algebra homomorphism

$$\mathbb{Q}[u, v] \xrightarrow{h} \mathbb{Q}[x, y], \quad h(u) = x + y, \quad h(v) = x - y. \quad (\text{B.21})$$

It turns the sum of two SU(2) polynomials into the SO(4) polynomial:

$$h(W_{\text{SU}(2)}(u) + W_{\text{SU}(2)}(v)) = W_{\text{SO}(4)}(x, y). \quad (\text{B.22})$$

Let γ and γ' be two n -legged open graphs, appearing in the unoriented version of the SU(2) categorification (that is, γ and γ' are disjoint unions of arcs and circles). Suppose that γ and γ' have the same number of legs. We index them and assign variables u_i ($1 \leq i \leq n$) to the legs of γ and v_i to the legs of γ' . Denote by h_i the homomorphism (B.21) applied to $\mathbb{Q}[u_i, v_i]$ and let \widehat{h} be the composition of all functors \widehat{h}_i . Then $\widehat{h}(\widehat{\gamma} \otimes \widehat{\gamma}')$ is a matrix factorization of the sum of SO(4) polynomials

$$\sum_{i=1}^n W_{\text{SO}(4)}(x_i, y_i). \quad (\text{B.23})$$

Introduce a shortcut notation for the SU(2) matrix factorizations:

$$\widehat{\gamma}_{\text{SU}(2)}^{\otimes 2} = \widehat{\gamma}_{\text{SU}(2)} \otimes \widehat{\gamma}_{\text{SU}(2)}. \quad (\text{B.24})$$

The following lemma leads to a quick proof of Theorem B.1.

Lemma B.2. *The functor \widehat{h} maps a tensor product of SU(2) crossing matrix factorization complexes into the SO(4) crossing matrix factorization complex:*

$$\widehat{h} \left(\widehat{\gamma}_{\text{SU}(2)}^{\otimes 2} \right) \simeq \widehat{\gamma}_{\text{SO}(4)}. \quad (\text{B.25})$$

Proof of Theorem B.1. Since the homomorphism (B.21) is an algebra isomorphism, the relation (B.25) implies the homotopy equivalence of complexes

$$\widetilde{C}_{\text{SU}(2)}^\bullet(L) \otimes \widetilde{C}_{\text{SU}(2)}^\bullet(L) \simeq \widetilde{C}_{\text{SO}(4)}^\bullet(L). \quad (\text{B.26})$$

The homotopy equivalence (B.5) follows in view of the relation (B.6). \square

Proof of Lemma B.2. The functor \widehat{h} maps 1-arc graphs into 1-arc graphs:

$$\widehat{h} \left(\widehat{\text{graph}}_{\text{SU}(2)}^{\otimes 2} \right) \cong \widehat{\text{graph}}_{\text{SO}(4)}. \quad (\text{B.27})$$

Indeed, the *l.h.s.* of this equation is a Koszul matrix factorization

$$\mathbb{K} \begin{pmatrix} x_1 + y_1 + x_2 + y_2, & * \\ x_1 - y_1 + x_2 - y_2, & * \end{pmatrix} \xrightarrow{[1]_2 1 [2]_{-1} [1]_{1/2} [1,2]_1} \mathbb{K} \begin{pmatrix} x_1 + x_2, & * \\ y_1 + y_2, & * \end{pmatrix} \quad (\text{B.28})$$

The right matrix factorization in this diagram is isomorphic to the *r.h.s.* of eq.(B.27) in view of Theorem 2.1.

The functor \widehat{h} also maps saddle morphisms to saddle morphisms:

$$\widehat{h} \left(\widehat{\text{saddle}}_{\text{SU}(2)}^{\otimes 2} \xrightarrow{F_{\text{SU}(2)}^{\otimes 2}} \widehat{\text{saddle}}_{\text{SU}(2)}^{\otimes 2} \right) = \left(\widehat{\text{saddle}}_{\text{SO}(4)} \xrightarrow{F_{\text{SO}(4)}} \widehat{\text{saddle}}_{\text{SO}(4)} \right) \quad (\text{B.29})$$

This follows from the fact that the *r.h.s.* and *l.h.s.* morphisms have the same q -degree and from the uniqueness of the saddle morphism (up to a constant factor).

Since the functor \widehat{h} maps 1-arc matrix factorizations to 1-arc matrix factorizations and saddle morphisms to saddle morphisms, then the definition (6.31) of the $\text{SO}(4)$ crossing complex $\widehat{\text{crossing}}_{\text{SO}(4)}$ indicates that it has an $\text{SU}(2) \times \text{SU}(2)$ counterpart:

$$\widehat{h} \left(\widehat{\text{crossing}}_{\text{SU}(2)}^{\otimes 2} \{1\} \xrightarrow{\chi_{\text{in}}} \widehat{\text{crossing}}_{\text{SU}(2) \times \text{SU}(2)} \xrightarrow{\chi_{\text{out}}} \widehat{\text{crossing}}_{\text{SU}(2)}^{\otimes 2} \{-1\} \right) \langle 1 \rangle = \widehat{\text{crossing}}_{\text{SO}(4)}, \quad (\text{B.30})$$

where we used a shortcut notation

$$\widehat{\text{crossing}}_{\text{SU}(2) \times \text{SU}(2)} = \boxed{ \widehat{\text{crossing}}_{\text{SU}(2)}^{\otimes 2} \{-1\} \xrightarrow{F_{\text{SU}(2)}^{\otimes 2}} \widehat{\text{crossing}}_{\text{SU}(2)}^{\otimes 2} \langle 1 \rangle \xrightarrow{G_{\text{SU}(2)}^{\otimes 2}} \widehat{\text{crossing}}_{\text{SU}(2)}^{\otimes 2} \{1\} } \quad (\text{B.31})$$

and Y is a secondary homomorphism (its choice does not impact the homotopy class of the convolution, because the condition (5.61) holds). Hence the homotopy equivalence (B.25) follows from the next lemma. \square

From now on we will be dealing only with matrix factorizations appearing in the $\text{SU}(2)$ categorification, hence we drop the indices $\text{SU}(2)$ at the graphs and morphisms.

Lemma B.3. *The following complexes of matrix factorizations are homotopy equivalent:*

$$\widehat{\times}^{\otimes 2} \langle 1 \rangle \simeq \left(\widehat{\smile}^{\otimes 2} \{1\} \xrightarrow{\chi_{\text{in}}} \widehat{\times}_{\text{SU}(2) \times \text{SU}(2)} \xrightarrow{\chi_{\text{out}}} \widehat{\smile}^{\otimes 2} \{-1\} \right) \quad (\text{B.32})$$

Proof. According to eq.(B.18) (rotated by 90°), the *l.h.s.* of eq.(B.32) can be presented as a complex

$$\widehat{\smile}^{\otimes 2} \{1\} \xrightarrow{\tilde{\chi}_{\text{in}}} \left(\widehat{\smile} \otimes \widehat{\smile} \right) \langle 1 \rangle \oplus \left(\widehat{\smile} \otimes \widehat{\smile} \right) \langle 1 \rangle \xrightarrow{\tilde{\chi}_{\text{out}}} \widehat{\smile}^{\otimes 2} \{-1\} \quad (\text{B.33})$$

where

$$\tilde{\chi}_{\text{in}} = \begin{pmatrix} F \otimes 1 \\ -1 \otimes F \end{pmatrix}, \quad \tilde{\chi}_{\text{out}} = \begin{pmatrix} 1 \otimes F & F \otimes 1 \end{pmatrix} \quad (\text{B.34})$$

and the middle matrix factorization carries the zero homological degree. Hence we will establish the relation (B.32) by proving the homotopy equivalence between the central matrix factorizations of (B.32) and (B.33)

$$\widehat{\times}_{\text{SU}(2) \times \text{SU}(2)} \simeq \left(\widehat{\smile} \otimes \widehat{\smile} \right) \langle 1 \rangle \oplus \left(\widehat{\smile} \otimes \widehat{\smile} \right) \langle 1 \rangle \quad (\text{B.35})$$

and showing that the homomorphisms χ_{in} and χ_{out} are equivalent to $\tilde{\chi}_{\text{in}}$ and $\tilde{\chi}_{\text{out}}$ respectively.

Let us replace the virtual crossing matrix factorizations $\widehat{\times}$ in the formula (B.31) for $\widehat{\times}_{\text{SU}(2) \times \text{SU}(2)}$ by the cone expression

$$\widehat{\times} \simeq \boxed{\widehat{\smile} \{-1\} \xrightarrow{F} \widehat{\smile} \{1\}} \langle 1 \rangle \quad (\text{B.36})$$

obtained by permuting legs 2 and 4 in the second equality of (B.17). Since this substitution turns the saddle morphisms of (B.31) to natural morphisms relating this cone to its

constituent matrix factorizations, the convolution in the *r.h.s.* of eq.(B.31) becomes

$$\begin{array}{c}
 \widehat{\times} \\
 \text{SU}(2) \times \text{SU}(2) \quad \simeq
 \end{array}
 \begin{array}{c}
 \boxed{
 \begin{array}{ccc}
 & & 1 \\
 & \widehat{\times}^{\otimes 2} \{-2\} \langle 1 \rangle & \xrightarrow{F \otimes 1} & \widehat{\times} \langle 1 \rangle \otimes \widehat{\times} \langle 1 \rangle \\
 & \downarrow 1 \otimes F & & \downarrow -1 \otimes F \\
 \widehat{\times}^{\otimes 2} \{-1\} & \widehat{\times} \langle 1 \rangle \otimes \widehat{\times} \langle 1 \rangle & \xrightarrow{F \otimes 1} & \widehat{\times}^{\otimes 2} \{2\} \langle 1 \rangle \\
 & & & \downarrow \\
 & & & \widehat{\times}^{\otimes 2} \{1\}
 \end{array}
 }
 \end{array}
 \begin{array}{c}
 1 \\
 \downarrow \\
 1
 \end{array}
 \tag{B.37}$$

The square in the center of this diagram represents the matrix factorization $\widehat{\times}^{\otimes 2}$ after the substitution (B.36). Note that this substitution allowed us to set the secondary homomorphism equal to zero. Let us rearrange the convolution (B.37) by assembling

some constituent matrix factorizations into ‘subcones’:

$$\begin{array}{c}
 \widehat{\times} \\
 \text{SU}(2) \times \text{SU}(2) \simeq
 \end{array}
 \begin{array}{c}
 \widehat{\left(\right)} \otimes \widehat{\left(\right)} \langle 1 \rangle \\
 \begin{array}{ccc}
 \begin{array}{c} \widehat{\left(\right)}^{\otimes 2} \\ \left\{ -2 \right\} \langle 1 \rangle \end{array} \xrightarrow{1} \begin{array}{c} \widehat{\left(\right)}^{\otimes 2} \\ \left\{ 1 \right\} \end{array} & & \begin{array}{c} \widehat{\left(\right)}^{\otimes 2} \\ \left\{ -1 \right\} \end{array} \xrightarrow{1} \begin{array}{c} \widehat{\left(\right)}^{\otimes 2} \\ \left\{ 2 \right\} \langle 1 \rangle \end{array} \\
 \begin{array}{c} \widehat{\left(\right)}^{\otimes 2} \\ \left\{ -2 \right\} \langle 1 \rangle \end{array} \xrightarrow{(F \otimes 1 \ 0)} \begin{array}{c} \widehat{\left(\right)} \otimes \widehat{\left(\right)} \langle 1 \rangle \\ \end{array} & & \begin{array}{c} \widehat{\left(\right)}^{\otimes 2} \\ \left\{ -1 \right\} \end{array} \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \begin{array}{c} \widehat{\left(\right)} \otimes \widehat{\left(\right)} \langle 1 \rangle \\ \end{array} \\
 \begin{array}{c} \widehat{\left(\right)}^{\otimes 2} \\ \left\{ -2 \right\} \langle 1 \rangle \end{array} \xrightarrow{\begin{pmatrix} -1 & 0 \\ 0 & F \end{pmatrix}} \begin{array}{c} \widehat{\left(\right)} \otimes \widehat{\left(\right)} \langle 1 \rangle \\ \end{array} & & \begin{array}{c} \widehat{\left(\right)}^{\otimes 2} \\ \left\{ -1 \right\} \end{array} \xrightarrow{\begin{pmatrix} 0 & F \\ F & 1 \end{pmatrix}} \begin{array}{c} \widehat{\left(\right)} \otimes \widehat{\left(\right)} \langle 1 \rangle \\ \end{array}
 \end{array}
 \end{array}
 \tag{B.38}$$

The cones of identity morphism are contractible. According to Lemma 7.1, their excision establishes the homotopy equivalence (B.35). It remains to establish the equivalence of morphisms $\chi_{\text{in}}, \tilde{\chi}_{\text{in}}$ and $\chi_{\text{out}}, \tilde{\chi}_{\text{out}}$. We will prove the equivalence for the first pair, since the second pair can be treated similarly. The morphism χ_{in} maps $\widehat{\left(\right)}^{\otimes 2}$ identically to the second matrix factorization in the left subcone of the convolution (B.38). Hence its equivalence to $\tilde{\chi}_{\text{in}}$ follows from the next Lemma, in which A stands for $\widehat{\left(\right)}^{\otimes 2}$ and B stands for the rest of the convolution (B.38). \square

Lemma B.4. *For a morphism between two matrix factorizations*

$$A \xrightarrow{f} B \tag{B.39}$$

consider another morphism between A and a cone

$$A \xrightarrow{(0 \ 1 \ 0)} \boxed{A \langle 1 \rangle \xrightarrow{(1 \ f)} A \oplus B} . \tag{B.40}$$

The target of this morphism is homotopy equivalent to B and the morphism itself is equivalent to f .

Proof. Consider a commutative diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{(0 \ 1 \ 0)} & \boxed{A \langle 1 \rangle} & \xrightarrow{(1 \ f)} & A \oplus B \\
 & \searrow (0 \ -1 \ f) & & \downarrow -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -f & 1 \end{pmatrix} & \\
 & & \boxed{A \langle 1 \rangle \xrightarrow{1} A} & \oplus & B
 \end{array} \tag{B.41}$$

The vertical arrow establishes an isomorphism between the upper-right cone and a sum of a contractible cone and B in the lower-right corner. After we remove the contractible cone, the morphism between the upper-left and lower-right matrix factorizations will reduce to f . \square

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